Some Simple Non-Holonomic Mechanical Systems Are Too Symmetric to Have Exponental Stability

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[Harry Dankowicz, VPI]

Also, Examples

- A quick way to see that some systems cannot be stable.
- An explanation (history of science) for why people have been slow to see the utility of non-holonomic mechanics.

Mechanical Systems

- Particles and rigid bodies
- Only conservative forces
- No explicit t dependence
- Non-holonomic or Holonomic Constraints imposed in a workless way.
- Smoothe

Particles & Rigid bodies connected to the world and each other w/ hinges, frictionless sliding, rolling, skates, ...

$$\boxed{KE + PE = \text{Constant}}$$

"Conservative" Systems

Geometric Constraints

$q_1, q_2, q_3 \ldots$ are configuration variables.

[eg.: x, y, θ, ϕ etc. of particles or rigid bodies.]

Holonomic: "Integrable"

$$F(q_1, q_2 \ldots) = 0 \quad \text{or} \quad \underbrace{f_1(\underline{q})\dot{q}_1 + f_2(\underline{q})\dot{q}_2 + \ldots = 0}$$

e.g., $x^2 + y^2 - 7 = 0$

$N = M$

e.g. $x\dot{x} + y\dot{y} = 0$

Non-Holonomic: "Non-Integrable"

$$\cancel{F(q_1, q_2, \ldots) = 0}$$

[no such thing

$$f_1(\underline{q})\dot{q}_1 + f_2(\underline{q})\dot{q}_2 + \ldots = 0$$

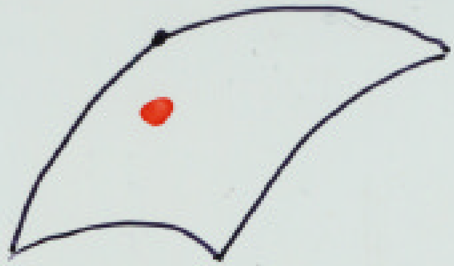
e.g., $\dot{x}\cos\theta - \dot{y}\sin\theta = 0$ $\quad N < M$

NO SUCH: $f(\cancel{x}, y, \cancel{\theta}) = 0$

Holonomic:

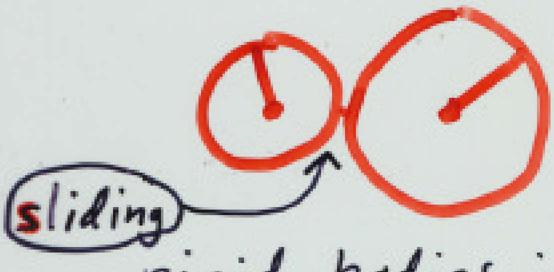
$(N=M)$

hinge
$4=4$



particle on a surface
$2=2$

particle on a wire
$1=1$



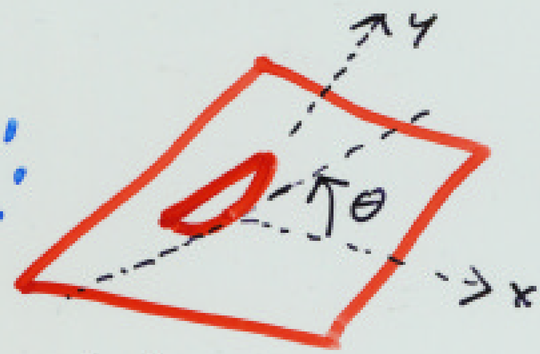
rigid bodies in sliding contact
$5=5$

N = dim. of vel. space

M = dim of config. space

Non-Holonomic:

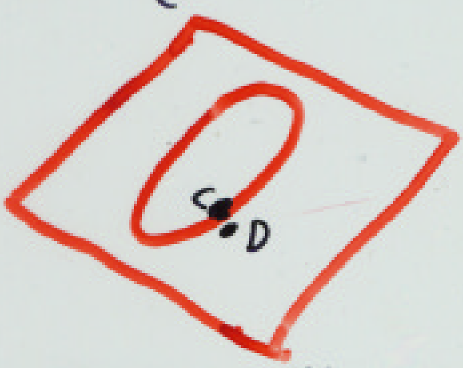
$(N<M)$



$2<3$

skate on plane

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$



Disk rolling on a plane

$\underline{v}_c = \underline{0}$

$3<5$

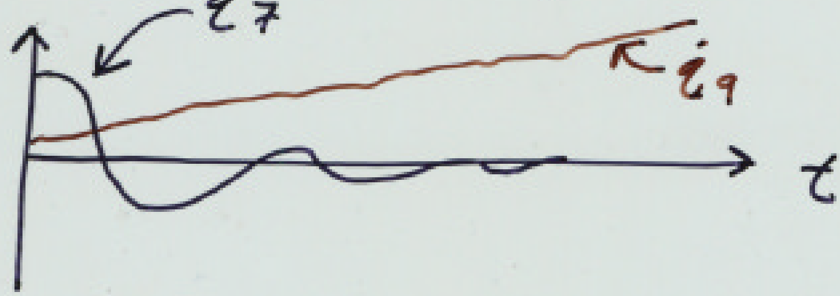
[≈ Wings, Sails, Keels]

You can get to a space of configs. bigger than the space of vels.

"parallel parking"

Exponential Stability

For all solutions near a reference solution some q_i, say q_7, goes exponentially to zero while <u>no</u> q_i or $\dot{q}_i$ goes exponentially to ∞.

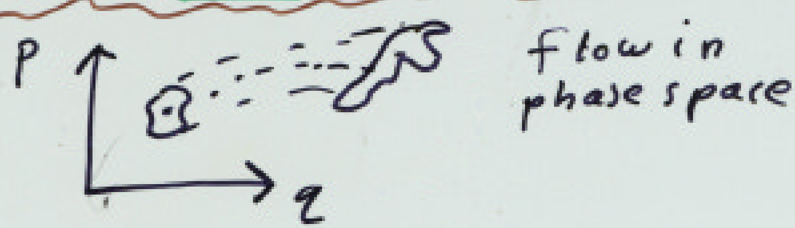


<u>Note</u>: Assume reasonable variables.

$$\boxed{\text{"}\underline{\text{Thm}}\text{": Hamiltonian systems cannot have exponential stability}}$$

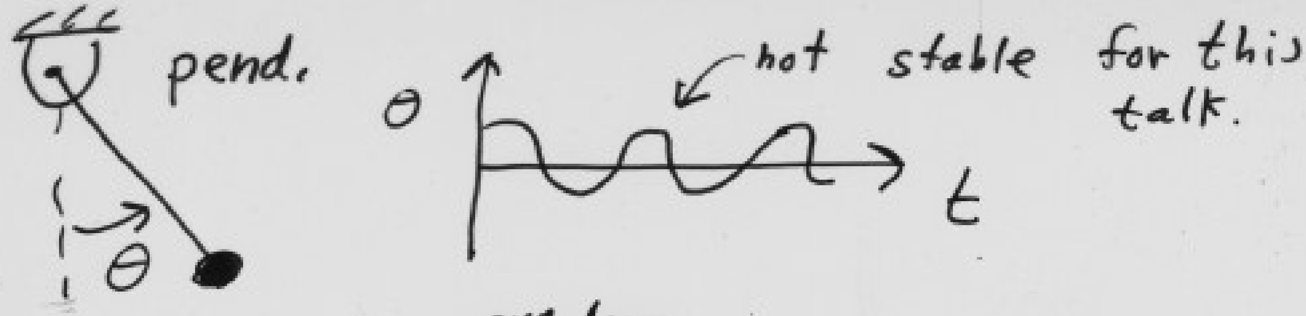
$$\dot{q} = \frac{\partial H}{\partial p}$$

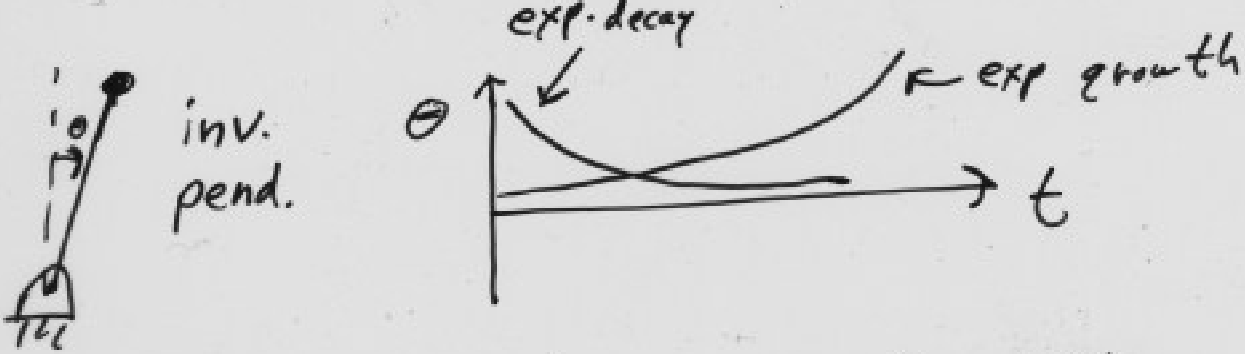
$$\dot{p} = \frac{-\partial H}{\partial q}$$

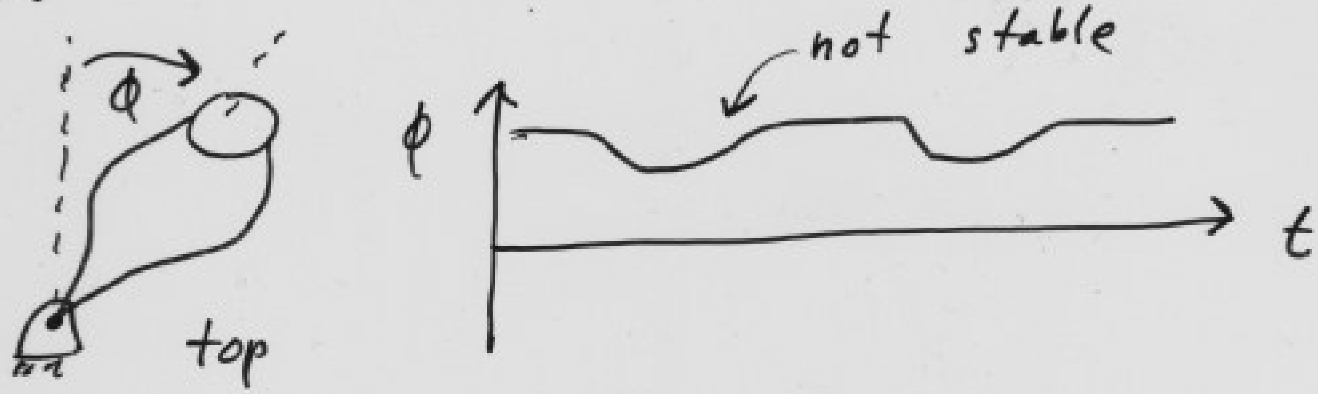


$$\underline{\nabla} \cdot \underline{v} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = \frac{\partial^2 H}{\partial p \partial q} - \frac{\partial^2 H}{\partial q \partial p} = 0$$

[To conserve Vol. if all q_7's die, something else has to blow up.]







particle sliding in a bowl

particle sliding in a trough

etc

Recall: IDEAL top <u>not</u> exponentially stable.

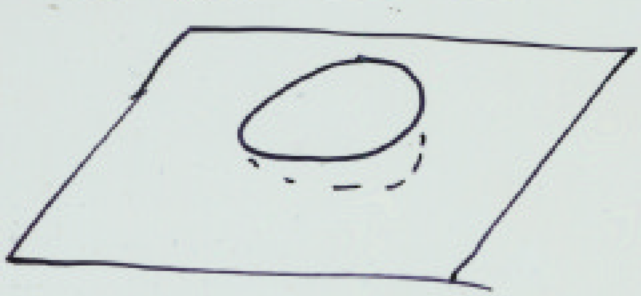
But real top is OFTEN stable.

NON-HOLONOMY DOES IT?

non-holonomic rolling contact

NON-HOLONOMY DOES IT?

Real Rolling Disk observed to be stable.



Euler Disk

NON-HOLONOMY DOES IT?

Bike: 3D

[whipple 1899
Carvallo 1901
Sommerfeld & Klein 1904]

[Jim Papadopoulos, Scott hand ≈ 1986]
(RUINA)

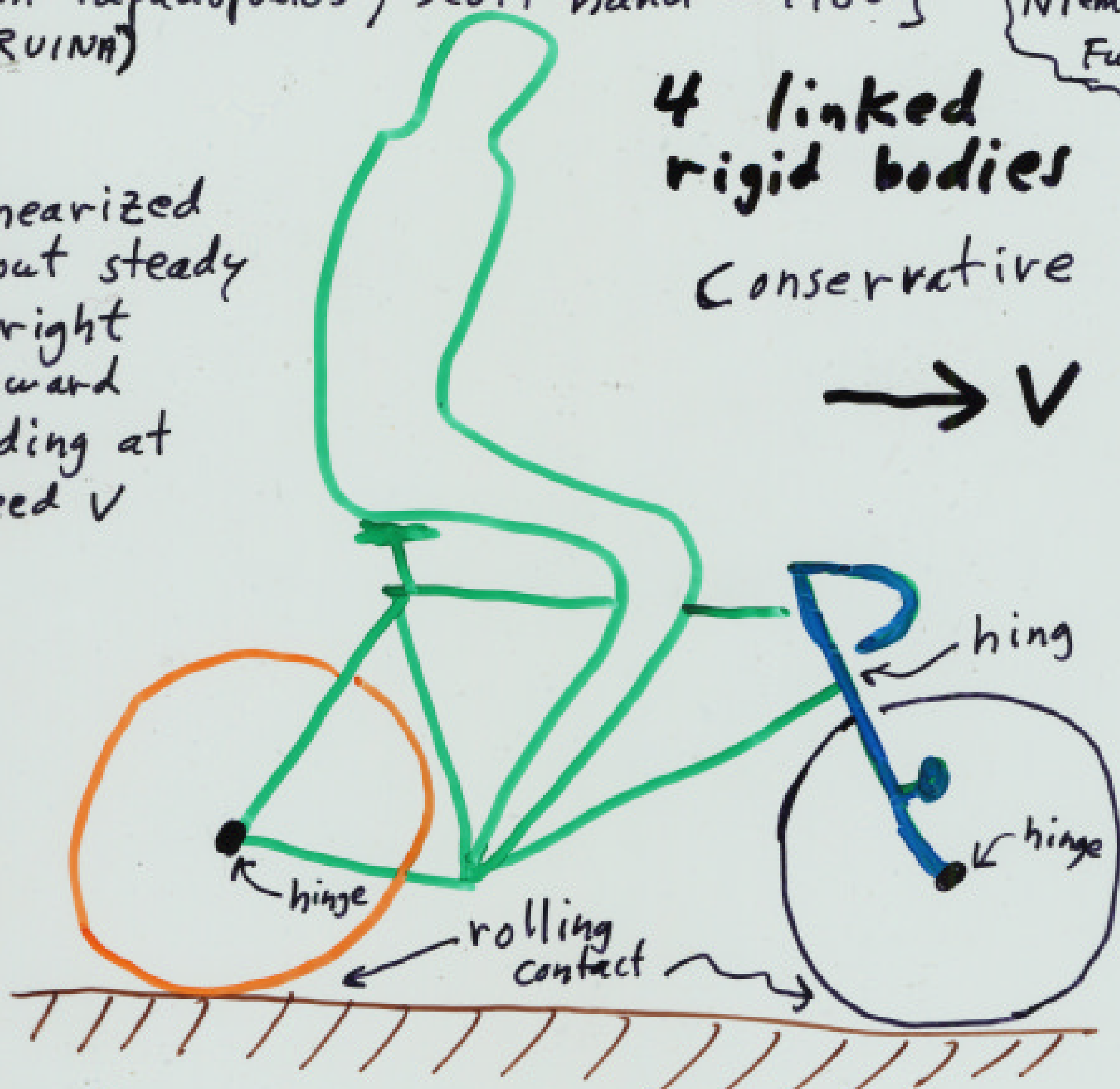
Niemark & Fufaev etc.

4 linked rigid bodies

Conservative

$\longrightarrow V$

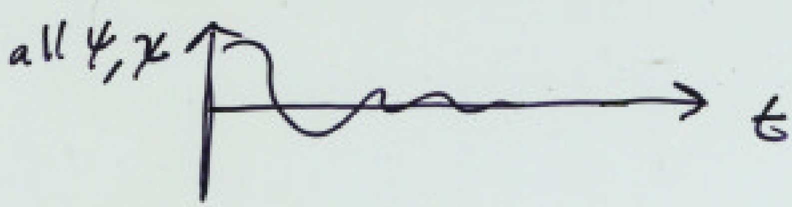
Linearized about steady upright forward riding at speed V



$$A\ddot{\chi} + B\chi + C\ddot{\psi} + D\dot{\psi} + E\psi = 0$$

$$F\ddot{\chi} + G\dot{\chi} + H\chi + I\ddot{\psi} + J\dot{\psi} + K\psi = 0$$

For realizable A, B...K there are solns. where ψ and χ are exponentially stable!

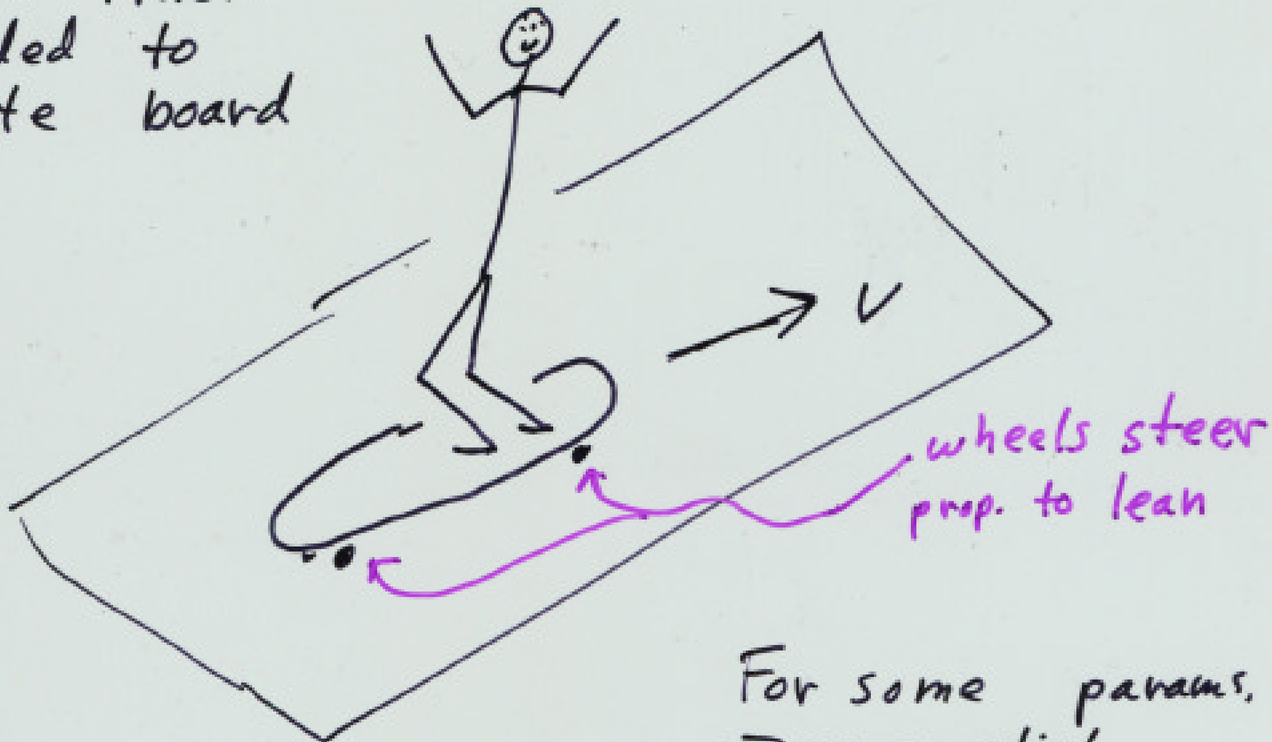


Skate board : 3D

(Monte Hubbard 1978)

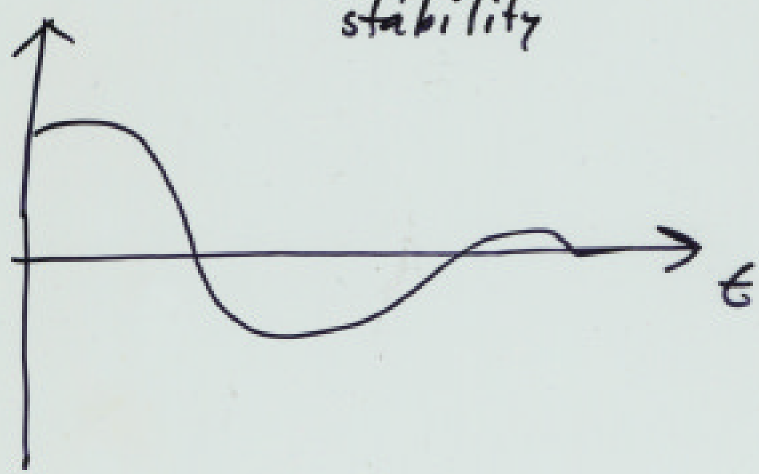
Conservative

rigid rider welded to skate board



For some params. $\Rightarrow$ exponential stability

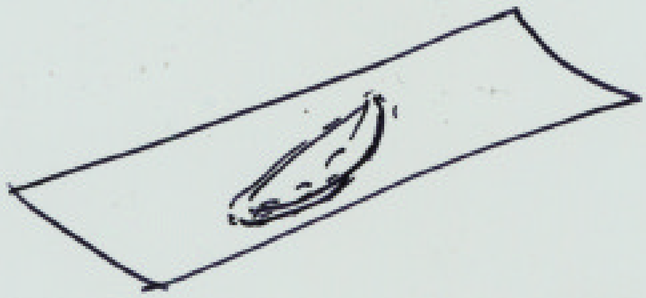
Lean vs t

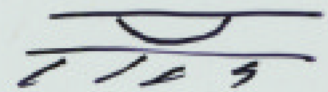


Rattleback, Celt

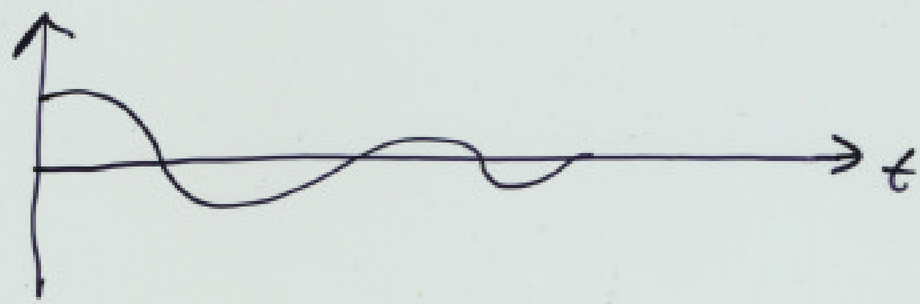
Walker '95 {well, 1895 actually}

rigid body (1), rolling on plane





for some mass dist. & contact shapes & spin rates



SO......?

1) All classical (Conservative & Hamiltonian) systems do not have exponential stability.

2) Some obscure, erratically known, conservative non-holonomic systems do have exponential stability.

[bike, skateboard, celt, trike, trailers, ...]

"...[nothing]..." -Niemark & Fufaev in NONHOLONOMIC MECHANICS

1) How much can non-holonomic contact explain stability?

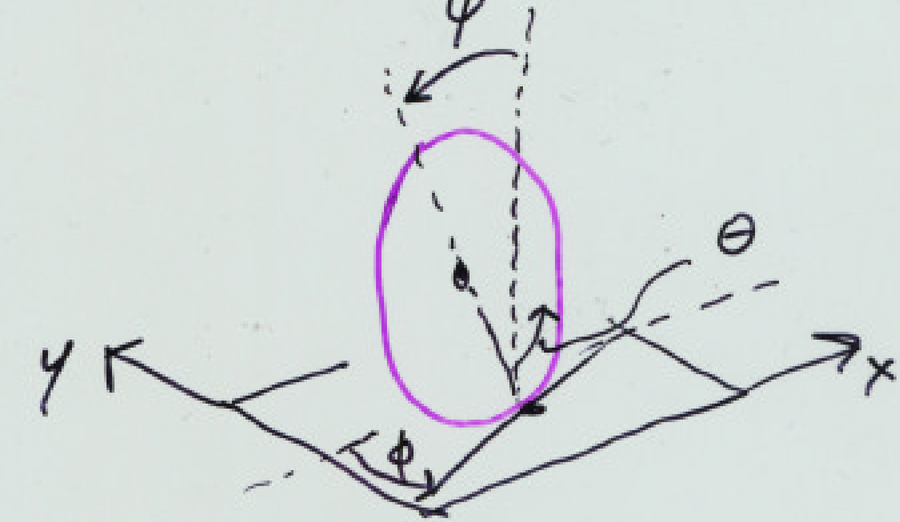
2) Why doesn't everyone know about it?

Rolling Disk:

(e.g. Goldstein)

$$\lambda^2 = \frac{1}{2J+1}$$

$$J = I/mR^2$$



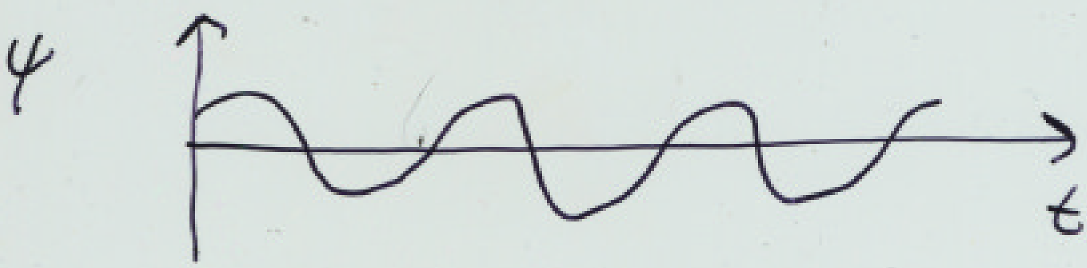
$$(1+\lambda^2)\ddot{\psi} - 2\cos(\psi)\dot{\phi}\dot{\theta} + (1+\lambda^2)\sin\psi\,\dot{\phi}^2 - 2\lambda^2\sin\psi = 0$$

$$\cos\psi\,\ddot{\phi} + 2\dot{\psi}\dot{\theta} = 0$$

$$\ddot{\theta} + \sin\psi\,\ddot{\phi} + (1+\lambda^2)\cos\psi\,\dot{\phi}\dot{\psi} = 0$$

Linearize about, say,

$\phi = \text{const}$, $\psi = 0$, $\dot{\theta} = $ big enough



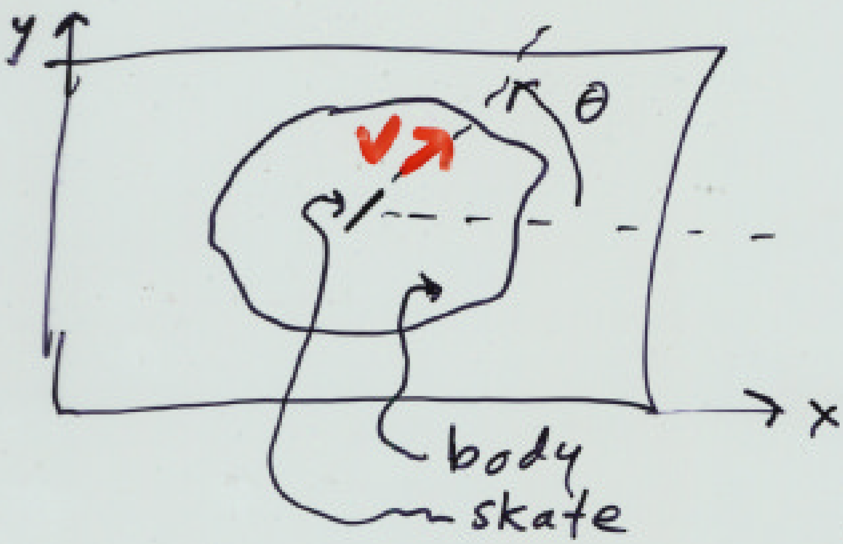
imaginary eigenvalues. $\Rightarrow$!

NO EXPONENTIAL STABILITY!

Chaplygin sleigh: V1.0

Planar rigid body moving in plane.

Skate at C.O.M.



m = mass

I = mom. of inertia.

$\dot{v} = 0$

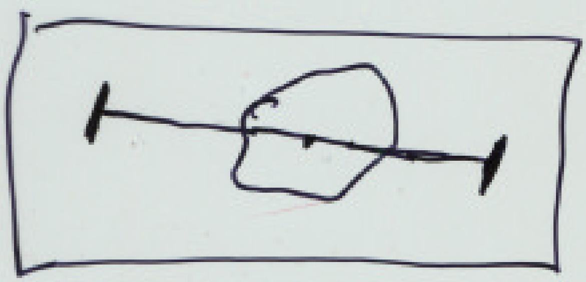
$\dot{\omega} = 0, \quad \theta = \omega t$

$\dot{x} = v\cos(\omega t)$

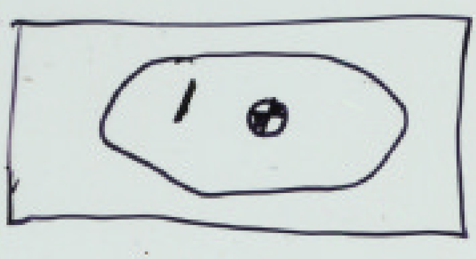
$\dot{y} = v\sin(\omega t)$

$\Rightarrow$ All motions are steady circles.

OR



OR

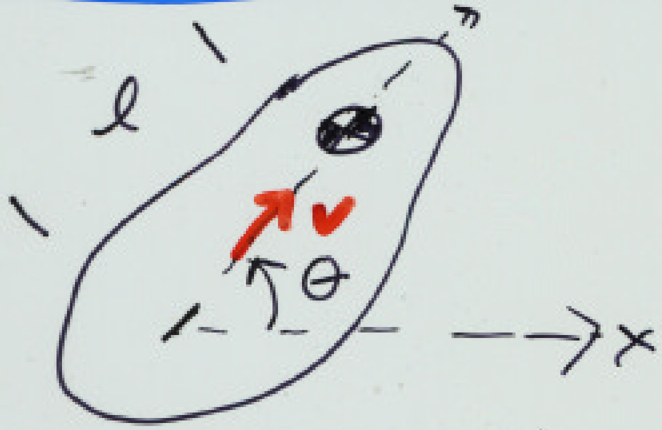


$\Rightarrow$ No exponential stability!

The two most famous examples of non-holonomis systems (DISK, skate) Don't have exponential stability.

[Maybe that's why the possible exponential stability of conservative systems is/was not so well recognized.]

Chaplygin Sleigh: V2.0



m = mass

I = moment of inertia about com

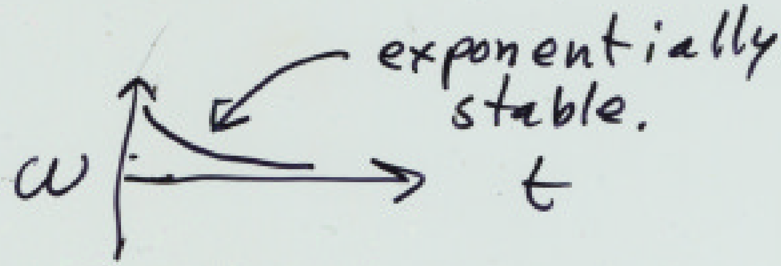
$$\dot{v} = l\,\omega^2$$

$$\dot{\omega} = -\frac{ml}{I+ml^2}\, v\omega$$

$$\begin{bmatrix} \dot{\theta} = \omega \\ \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \end{bmatrix}$$

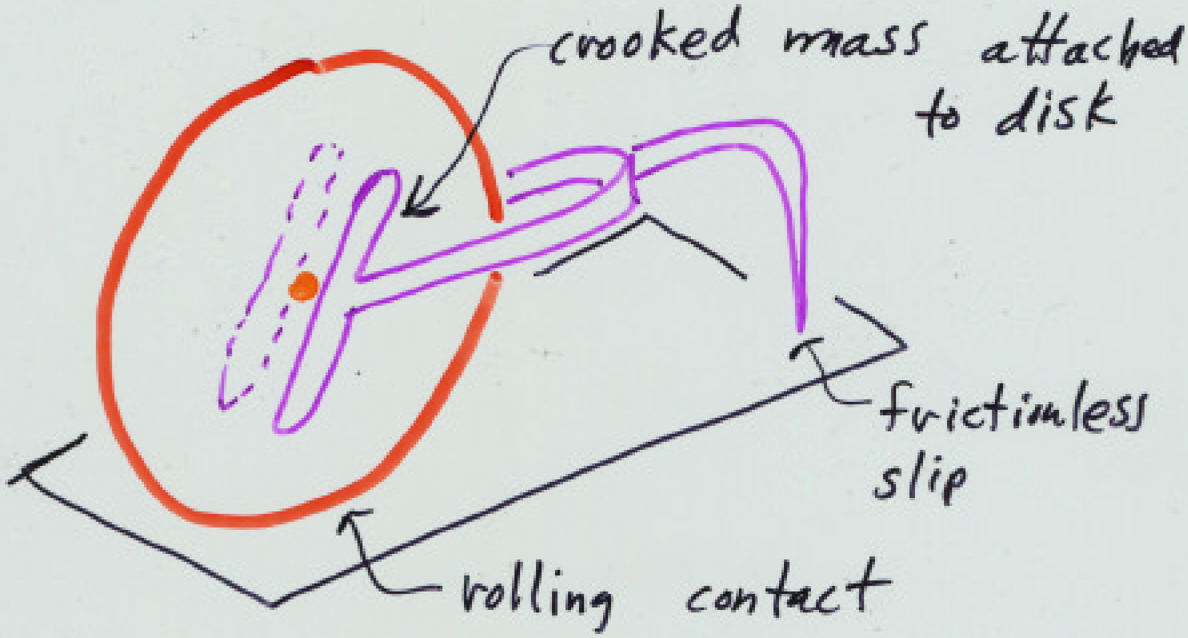
Linearize about $v = v^*$, $\omega = 0$

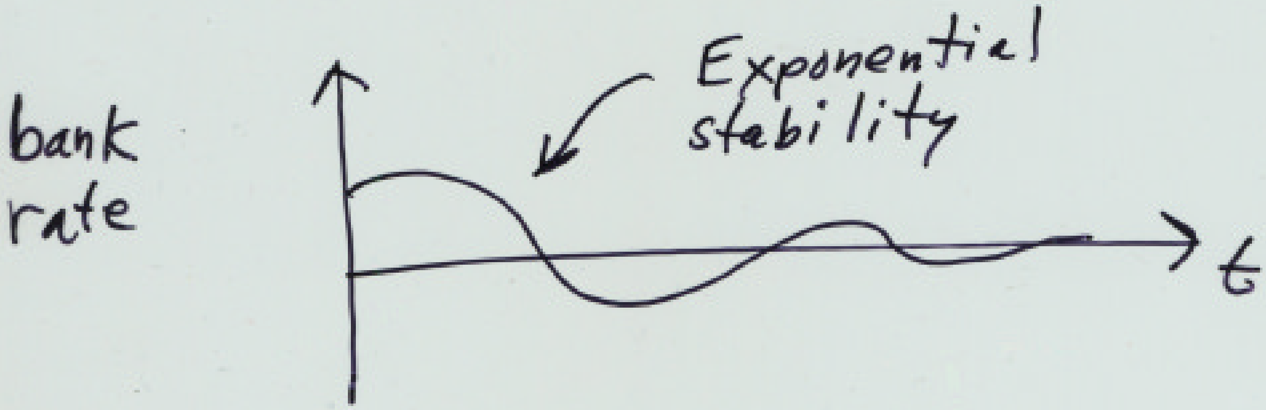
$$\Rightarrow \quad \dot{v} = 0, \quad \dot{\omega} = \left[\frac{-ml}{I+ml^2} v^*\right]\omega$$



Disk: v2

(Coleman 1997)





What is the difference between the non-holonomic systems that have exponential stability and those which do not?

Symmetry

2 Kinds

1) Time reflection

2) Space reflection

Put them together and you get a new view of the same solution.

Time reflection/reversal

$$\boxed{\text{If } \underline{q}(t) \text{ solves equations so does } \underline{q}(-t).}$$

[A movie run through a projector backwards is a legitimate motion of the system.]

TRUE for collection of particles & rigid bodies interacting w/ workless constraints,

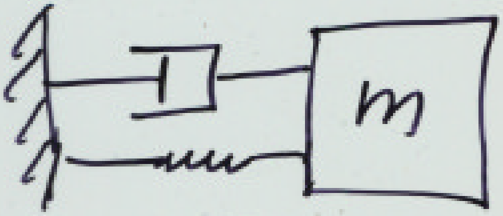
holonomic or non-holonomic

and forces dependent only on $\underline{q}$ (don't have to be conservative even).

[e.g., bike, skateboard, particle on wire, disk 1, disk 2, sleigh 1, sleigh 2, celt, every example in this talk, all conservative systems, etc.]

Why? $\underline{q}(t) \to \underline{q}(-t)$ has same <u>accel</u> at every $\underline{q}$ $\Rightarrow m\underline{a}$ balanced by same forces and same const. forces

Note: no time reversal for, say damped oscillator.



$$m\ddot{x} + c\dot{x} + kx = 0$$

say $\hat{x}(t) = x(-t)$

$$m\ddot{\hat{x}} + c\dot{\hat{x}} + k\hat{x} \stackrel{?}{=} 0$$

$$m(--\dot{x}) + c(-\dot{x}) + kx \stackrel{?}{=} 0$$

$$2c\dot{x} \stackrel{?}{=} 0$$

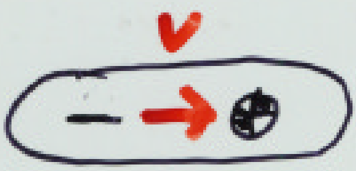
only if $c=0$

[Dashpot force depends on $\dot{q}$ not just q.]

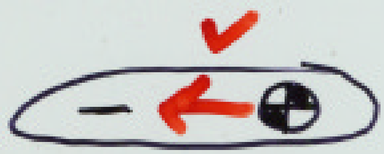
TIME REVERSAL SYMMETRY

$\Downarrow$

If a steady motion is exponentially stable the backwards steady motion is exponentially unstable.



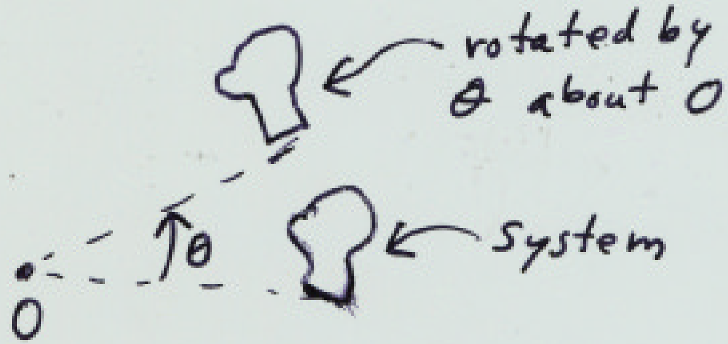
stable



unstable

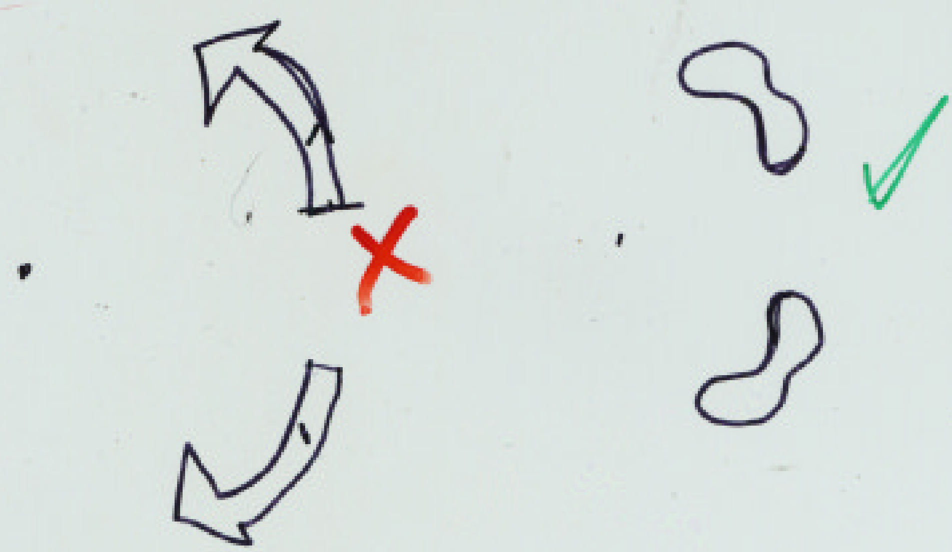
Spatial Symmetry:

say the steady solution of interest is associated w/ rotation θ about O.



ie., as system moves it traverses states which are equive to rotating each material point about O.

Spatial Symmetry $\equiv$ Replace every pt. on body w/ point at $-\theta$ is same system



A system w/ spatial symmetry w/ reflect to reflection in a symmetry direction of motion

IS

governed by the same equations when moving backwards (in reflected coordinates) as the original system moving forwards.

$\Downarrow$

If <u>an</u> exponentially decaying soln. exists for forward motion, so does it for backward motion.

Compose the two symmetries

Say a system has an exponentially decaying soln.

Time reversal $\Rightarrow$ growing soln when moving backwards

Space symmetry $\Rightarrow$ growing soln. when moving forward

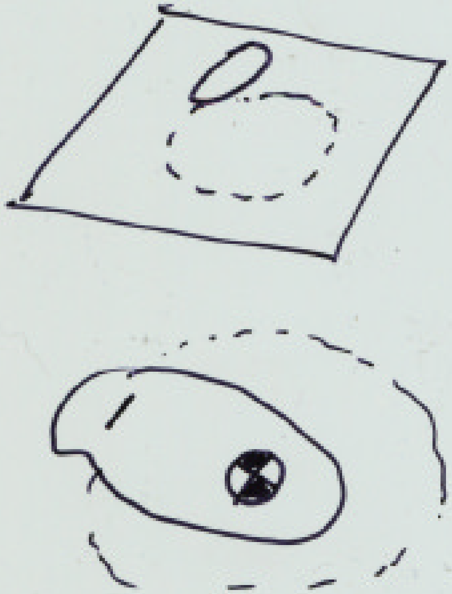
$\Longrightarrow$ $\boxed{\text{Exponential stability is not possible.}}$

(for these systems)

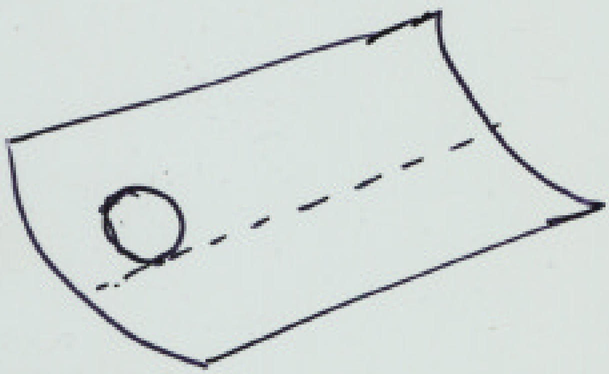

SMIKE

Examples of systems which, by inspection, can't be stable.

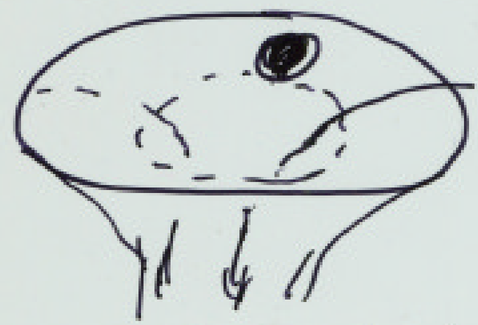
exponentially



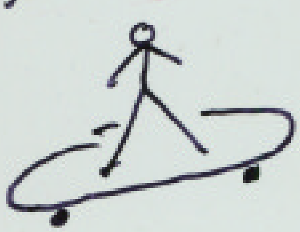
disk or torus



ball or disk in trough

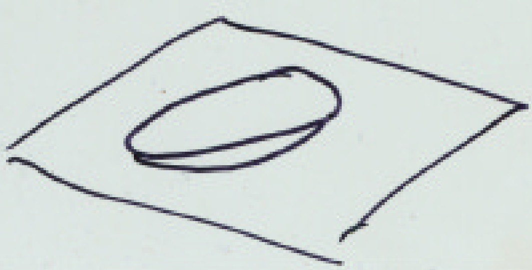


ball or disk in surf. of revolution (e.g. gravity well)



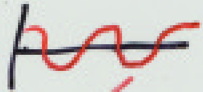
skate board rider in middle of board

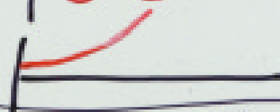
golf ball in hole, basketball on rim, symmetric celt.



Summary of Facts

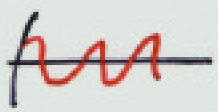
- Conservative-Holonomic Systems (Subset of Hamiltonian Systems.

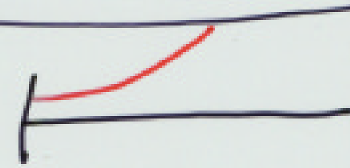
  Sometimes:  Never: 

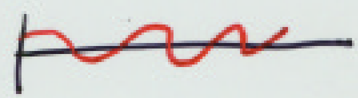
  Sometimes: 

- Conservative, non-holonomic Systems (not Hamiltonian)

  * Symmetric: Sometimes: 

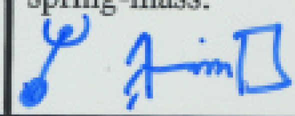
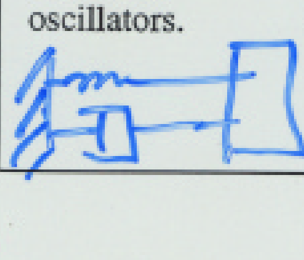
  Sometimes:  Never: 

  * Not Symmetric Sometimes: 

  Sometimes: 

  Sometimes: 

CONSTRAINTS, ENERGY, AND STABILITY

	Constraints		
	Holonomic *Integrable kinematic constraints(# of degrees of freedom equal to dimension of the configuration space);	**Nonholonomic** * Non-integrable kinematic constraints (dimension of the instantaneously accesible velocity space less than the dimension of the configuration space)	**Piecewise holonomic**
Conservative * Workless constraints; * Conservative forces; * Potential + Kinetic Energy = constant;	* Hamiltonian; * Cannot be asymptotically stable; * Simple example(s): simple pendulum, spring-mass. 	* Non-Hamiltonian; * Can be asymptotically stable; * Simple example(s): bike, skateboard, arrow w/ feathers, skate, rolling coin (not well known).	* Piecewise Hamiltonian * Stability? * Simple example(s): piecewise skate?
Non-conservative * d/dt(Pot. + Kin.) ≤0; * Friction; * Inelastic collisions; * Inelastic deformation;	* Can be asymptotically stable; * Simple example(s): Damped oscillators. 	* Can be asymptotically stable (though damping is not always stabilizing); * Simple example(s): Damped coin. 	* Can be asymptotically stable (even when statically unstable); * Example(s): 2D, 3D rimless wheel, McGeer's 2D walking machines, Tinkertoy Walker