Non-Holonomic Stability Aspects of Piecewise-Holonomic Systems

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We consider mechanical systems with intermittent contact that are smooth and holonomic except at the instants of transition. Overall such systems can be non-holonomic in that the accessible configuration space can have larger dimension than the instantaneous motions allowed by the constraints. The known examples of such mechanical systems are also dissipative. By virtue of their non-holonomy and of their dissipation such systems are not Hamiltonian. Thus there is no reason to expect them to adhere to the Hamiltonian property that exponential stability of steady motions is impossible. Since non-holonomy and energy dissipation are simultaneously present in these systems, it is usually not clear whether their sometimes-observed exponential stability should be attributed solely to dissipation, to non-holonomy, or to both. However, it is shown here on the basis of one simple example, that the observed exponential stability of such systems can follow solely from the non-holonomic nature of intermittent contact and not from dissipation. In particular, it is shown that a discrete sister model of the Chaplygin sleigh, a rigid body on the plane constrained by one skate, inherits the stability eigenvalues of the smooth system even as the dissipation tends to zero. Thus it seems that discrete non-holonomy can contribute to exponential stability of mechanical systems

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1 Introduction

The stability of mechanisms is of natural interest to engineers and to students of nature's machines. The most widely studied mechanisms are mechanical systems with conservative forces and which are only constrained by workless holonomic constraints. Such systems are Hamiltonian and, amongst other properties, cannot have exponential stability in any of the configuration variables. That is, the best studied systems are not capable of the most-often desired type of stability.

As has been long known, steady motions of non-holonomic conservative systems can have exponential stability as refreshed recently, for example, in Zenkov, *et al* [15]. That is, in non-holonomic conservative systems some system variables can have stability eigenvalues with negative real parts, while all others have zero real parts. Examples include the balance and steer of uncontrolled bicycles as reviewed by Hand [6], the balance of skate boards [7], the directional stability of tricycles (e.g. [14], equivalent to the recently studied Tennessee racers [15] and some simple models of cars), the oscillations of trailers elastically coupled to cars, and the rattleback (or Celtic stone). Perhaps the simplest and most famous example is the Chaplygin sleigh, described for example in Neĭmark and Fufaev [13] (where it is spelled Căplygin). A brief analysis of the Chaplygin sleigh is reproduced here.

More recently it has been found that another class of locomotion devices also can have exponential stability in some variables. These are the passive walking machines of Tad McGeer (e.g., [11][12]) and his imitators [10] [9] [3]. Passive dynamic walkers are uncontrolled and unpowered mechanisms that balance themselves as they walk, something like how people walk, down a gentle slope. McGeer's passive walkers are dissipative in that energy is lost when collisions are made with the ground. They are also non-holonomic by the following definition of non-holonomy: the dimension of the globally accessible configuration space is larger than the dimension of the allowed incremental motions or the space of locally accessible velocities. For example, although the nonslip contact conditions do not allow a two-dimensional walking machine to slide forward, it can reach a configuration corresponding to sliding forward by a sequence of steps. The non-holonomic aspect of non-smooth contact was realized by Mariga et al [1] in the context of manipulation of rolling polyhedra. Goodwine and Burdick [2] noted the non-holonomy of legged locomotion in the context of kinematic planning strategies.

When discussing systems with both the intermittent type of non-holonomy and dissipative contact one might credit any observed exponential stability on dissipation, the usual source of stability in holonomic mechanical systems.

But might the intermittent type of non-holonomy in some sense account for the exponential stability observed in these systems? The question is somewhat subtle because dissipative collisions in the known examples of these systems make the isolation of non-holonomy difficult.

2 Discrete sister systems

In the world of smooth rigid-body mechanical systems there are only a few basic mechanically realizable non-holonomic constraints: a surface rolling on another, a curve rolling on a surface, and skates or feathers (3-D skates). Each of these has one or more nearby (non-unique) *sister* systems which are piecewise-holonomic.

A reasonable conjecture is that for a wide range of initial value problems, the motion of the smooth system and of the sister system converge as a characteristic distance in the discrete sister system goes to zero.

To illustrate the concept we briefly consider each of the four mechanical non-holonomic constraints in turn.

2.1 Surface rolling: a ball on a plane

A ball rolling without slip on a plane has the non-holonomic constraint of a surface rolling without slip on another surface. A sister system for a rolling smooth surface is a polyhedron that can only make contact at its edges or vertices, which is not allowed to bounce, which does not slip, and for which the net contact interaction is equivalent to concentrated forces at the contacting (one or two at a time) vertices, with no moments about these vertices. To uniquely define the motion one most further specify that impulsive forces only occur at a newly contacting vertex or pair of vertices.

Although I do not know of any formal studies, or even careful numerical checks, it seems likely that for many mechanics problems a rolling polyhedron would have similar motions to a rolling smooth-surface solid. That is, if one solved exactly for the motions of the smooth system and its sister for given initial conditions and a finite period of time, all positions and velocities of the system would converge as the surface mesh of the polyhedron was made finer and finer.

A precise statement of the conjecture would describe in what manner the polyhedral grid converges to the surface, how the mass distributions of the sister systems converge, and what initial conditions are allowed. But even without formal definition, such convergence is implicitly assumed by those who simulate smooth surfaces with polyhedra in dynamic simulations with contact.

2.2 Curve rolling: a disk on a plane

One could consider the constraint of a curve rolling on a surface as a limiting case of rolling between two surfaces. However it is usually treated as a separate example.

A disk rolling without slip on a plane has the constraint of a curve rolling on a surface. This classic system is often the only non-holonomic system visited by students of advanced mechanics. This system, like a ball or any axisymmetric top spinning on any surface of revolution, or like a ball or torus rolling in a horizontal prism of any cross section, has the same equations of motion going forwards or backwards. This time reversal makes non-imaginary stability eigenvalues occur in pairs that are reflected across the imaginary axis. The rolling wheel can not and thus does not have exponential stability. The wheel can have the weaker Liapunov stability, associated with imaginary stability eigenvalues, at sufficiently high rolling rates. The prevalence of this too-symmetric example of a rolling disk, and the lack of common knowledge of other systems, perhaps explains the common lack of familiarity with the possible exponential stability of non-holonomic systems.

A sister piecewise-holonomic system to the rolling disk is a rimless spoked wheel, or regular polygon, free to roll on a plane surface, but not free to slip at its contact points. The rimless wheel was compared to the rolling disk in some detail in Coleman, *et al.* [4] using perturbation calculations which were checked by numerical integration. The comparisons were only made for motions near steady rolling. For the case of the rimless wheel steady rolling was understood to mean rolling on a sloped plane with a periodic (piecewise inverted pendulum) oscillation of speed. The slope was adjusted as the number of spokes was increased, so as to maintain a constant mean rolling speed. To compare stability eigenvalues of the continuous and discrete systems, the discrete map was iterated over a given time interval.

It was found that at a given speed the stability eigenvalues of the rolling polygon approach those of the rolling wheel, as the number of spokes goes to infinity. Unfortunately this case does not prove useful for showing the role of non-holonomy in achieving exponential stability. Although the discrete system did, interestingly, have exponential stability for high enough rolling speeds, all eigenvalues tended towards pure imaginary as the number of spokes increased and the dissipation per revolution decreased. This disappearance of exponential stability with increase number of spokes is because the system's limiting sister is the smooth, conservative, non-holonomic (too symmetric) rolling wheel which has only neutral linear stability.

However, this convergence to the appropriate smooth system would also be expected for an exponentially stable bicycle, tricycle or car if its disk wheels were replaced by rimless spoked wheels. By means of this imagined replacement we can see that discrete nonholonomy, in the form of piecewise-holonomy, can contribute to stability. However these comparisons have not been made in detail as far as I know.

2.3 Skate: the Chaplygin sleigh

The skate constraint for a 2-D rigid body is that a fixed point on the body can only move relative to the plane below in a direction that is also fixed in the body. The reaction force is assumed to have no component in the direction of allowed motion. The constraint is approximately realized by an ice skate which glides easily in one direction but does not allow sideways motion. A massless rolling wheel fixed in a body also realizes this constraint.

The simplest non-holonomic mechanical system seems to be the Chaplygin Sleigh which uses this skate constraint. This system arises in various disguises as

- A braked car. One model of a skidding car is that two of the wheels skid and two roll perfectly. If the car is rigid and if the skidding wheels have *no* friction this system is identical to the sleigh.
- No hands tricycle (or Tennessee racer). If the steering mechanism of a tricycle has a vertical steer axis and finite trail (the wheel contact is not at the point where the steering axis hits the ground) and negligible steering inertia, it acts like a frictionless support. The dynamics of such a tricycle, ignoring the dynamics of the passive steering mechanism, are thus those of the Chaplygin sleigh.
- An arrow. One model of the feathers of an arrow is that they constrain that part of the arrow to move in the direction at which they are aligned. Modeling an arrow as a frictionless rigid body in two dimensions with a feather constraint reduces the arrow to the Chaplygin sleigh. One can thus take the expression 'straight as an arrow' as suggestive of the exponential stability of this non-holonomic systems or as a consequence, depending on where one takes inspiration.

At least two different piecewise-holonomic sister systems can be constructed for the skate.

1. A slot in the rigid body which slides over a sequence of pegs in the plane. Each peg starts at the front, nearer to G, end of the slot. As the body moves, the peg reaches the tail end of the slot, at which time an external agency instantly retracts the peg and then instantly inserts a new peg at the beginning of the slot.

   Although the mechanical realization is not of central interest, here are two designs for the "external agency". a) A chain on a loop on the rigid body that has bumps on the bottom. Only one bump touches the ground at a time acting as a no-slip point contact.

Just when one bump picks up another drops down. b) A massless rimless wheel with large radius is attached to a body-fixed horizontal axle. Then the rigid spokes act as a sequence of hinge points. The (non-essential) large radius is to decouple the vertical dynamics from the in-plane dynamics.

2. In the other sister system there is a peg in the body which slides in a sequence of slots in the ground. A realization of this could be the placement on the ground of a sequence of skates. Each skate is, like an exaggerated racing skate, unable to steer when in ground contact. These skates would be hinged on the body but would always be aligned with the $\hat{\mathbf{e}}_\lambda$ direction at first contact.

2.4 Feathers: arrows

The genuinely 3-D constraints imposed by one or two pairs of feathers is the constraining of the velocity of one point on a body to be tangent to a plane or fiber that rotates with the body.

Such feather systems can also be identified with sister piecewise-holonomic systems. The identifications seem to require putting rigid pegs in air and so on, and are so non-physical to be of questionable value.

3 Detailed stability comparison for the Sleigh and its sister

Although it is reasonable to conjecture that a smooth non-holonomic system and its piecewise-holonomic sister will have similar behavior it is perhaps worth making the demonstration for one example. Here I discuss what seems to be the simplest systems for which the comparison can be made, the Chaplygin sleigh and its peg-leg sister.

3.1 Chaplygin sleigh directional stability

Since this system (see the left of figure 1) is the model for our discrete sister system it is reviewed here in some detail. A rigid body in the plane has mass m with center of mass at G about which its rotary inertia is I. A distance ℓ from G is a skate constraint at C. The skate and the unit vector $\hat{\mathbf{e}}_\lambda$ are aligned with the line CG, with normal $\hat{\mathbf{e}}_n$. $\hat{\mathbf{e}}_\lambda$ and the line CG make an angle θ, measured counter-clock-wise from a fixed reference line in the plane. The skate constraint is thus that the velocity of point C on the body is $\mathbf{v}_C = v\hat{\mathbf{e}}_\lambda$ and the

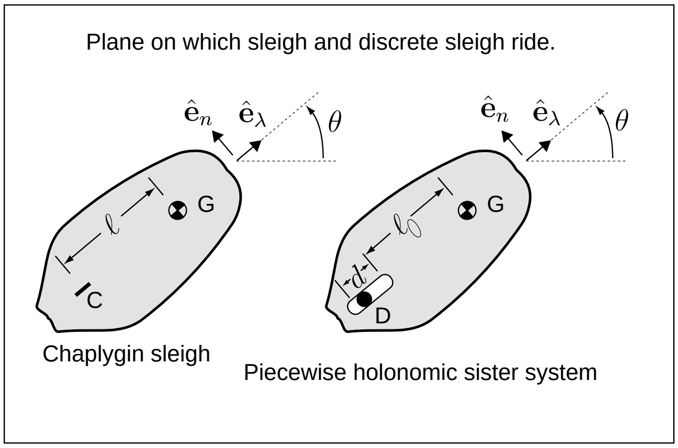

Figure 1: The Chaplygin sleigh is shown at left. It is a rigid body which is constrained to a frictionless plane. At C a skate keeps the velocity of point C along the skate. At right is a discrete holonomic sister system. A sequence of pegs fixed to the plane, slide in the slot at D.

force at point C on the body is $F\hat{\mathbf{e}}_n$. The position of point G is $\mathbf{r}_G = \mathbf{r}_C + \ell\hat{\mathbf{e}}_\lambda$. So, since $\dot{\hat{\mathbf{e}}}_\lambda = \dot{\theta}\hat{\mathbf{e}}_n$, and $\dot{\hat{\mathbf{e}}}_n = -\dot{\theta}\hat{\mathbf{e}}_\lambda$ (where $\dot{()} = d/dt()$), the velocity and acceleration of point G are given by:

$$\mathbf{v}_G = v\hat{\mathbf{e}}_\lambda + \dot{\theta}\ell\hat{\mathbf{e}}_n \qquad \text{and} \qquad \mathbf{a}_G = \dot{v}\hat{\mathbf{e}}_\lambda + v\dot{\theta}\hat{\mathbf{e}}_n + \ddot{\theta}\ell\hat{\mathbf{e}}_n - \dot{\theta}^2\ell\hat{\mathbf{e}}_\lambda. \tag{1}$$

Note that the Coriolis type term $v\dot{\theta}\hat{\mathbf{e}}_n$ only has a factor of 1 in front of it (implicitly) instead of the usual factor of 2 in holonomic problems. Linear momentum balance and angular momentum balance with respect to the point C (the non-accelerating point instantaneously coinciding with C if you like), are:

$$F\hat{\mathbf{e}}_n = m\mathbf{a}_G \qquad \text{and} \qquad \mathbf{0} = (\mathbf{r}_G - \mathbf{r}_C) \times (m\mathbf{a}_G) + I\ddot{\theta}\hat{\mathbf{e}}_z \tag{2}$$

where $\times$ is the vector cross product and $\hat{\mathbf{e}}_z$ is the normal to the plane. Substituting the kinematics of equations (1) into the equations of motion (2) and then taking the vector dot product of those equations with $\hat{\mathbf{e}}_\lambda$ and $\hat{\mathbf{e}}_z$ respectively, yields

$$\begin{aligned} \dot{v} &= \quad \ell \quad \omega^2 \\ \dot{\omega} &= -\tfrac{m\ell}{I+m\ell^2} \quad v\omega \end{aligned} \tag{3}$$

where we have defined $\omega = \dot{\theta}$. Equations (3) are two coupled first order equations for the evolution of the speed of the skate v and the angular velocity of the body ω. Dividing one equation by the other one can see that the solution curves are ellipses on the $v - \omega$ plane with the positive v axis ($\omega = 0$) attracting all solutions.

Linearizing near the straight-arrow solution $v = v_*$ and $\omega = 0$ one finds the eigenvalues 0 and $-m\ell v_*/(I + m\ell^2)$. The zero eigenvalue corresponds to the perpendicular crossing of the solution curves onto the v axis (so v is nearly constant for small perturbations). The energy conserving changes in v are of higher order than the exponentially decaying oscillations of ω governed by the negative eigenvalue. The zero eigenvalue is expected since the sleigh has a whole family of steady solutions parameterized by v.

For later reference we write the solutions of (3) to first order in Δt, not necessarily near the $\omega = 0$ solution, as

$$\begin{aligned} \Delta v &= \quad \ell\,\omega^2 \quad \Delta t \\ \Delta\omega &= -\ \tfrac{m\ell}{I+m\ell^2}\, v\omega \quad \Delta t. \end{aligned} \tag{4}$$

where $\Delta v = v(t + \Delta t) - v(t)$ and $\Delta\omega = \omega(t + \Delta t) - \omega(t)$

3.2 Piecewise-holonomic sleigh directional stability

Two types of piecewise-holonomic sister systems to the Chaplygin sleigh were described in the previous section. Because of its greater mechanical simplicity we study the first type of sister system, shown on the right part of figure (1).

The body has a slot of length d aligned with $\hat{\mathbf{e}}_\lambda$ which slides on a peg fixed in the ground at location D. The peg starts sliding in the slot with $\ell = \ell_0$ where ℓ, the distance of G to the peg, changes in time. When the end of the slot is reached and $\ell = \ell_0 + d$ the peg loses contact and a new collisional contact is made at the near end of the slot at the new D.

Analysis of the motion thus includes the smooth motion of the slotted body moving over a pin and of the collision when a new contact is made.

As for the Chaplygin sleigh, the only force on the system is $F\hat{\mathbf{e}}_n$ acting at point D. The velocity of the point on the body at D is $\mathbf{v}_D = v\hat{\mathbf{e}}_\lambda = \dot{\ell}\hat{\mathbf{e}}_\lambda$. Thus the velocity and acceleration of point G are given by

$$\mathbf{v}_G = v\hat{\mathbf{e}}_\lambda + \dot{\theta}\ell\hat{\mathbf{e}}_n \qquad \text{and} \qquad \mathbf{a}_G = \dot{v}\hat{\mathbf{e}}_\lambda + 2v\dot{\theta}\hat{\mathbf{e}}_n + \ddot{\theta}\ell\hat{\mathbf{e}}_n - \dot{\theta}^2\ell\hat{\mathbf{e}}_\lambda. \tag{5}$$

Equation (5) for acceleration is the usual polar coordinate expression for acceleration and has the usual factor of two in the Coriolis term.

The linear momentum balance equation dot product with $\hat{\mathbf{e}}_\lambda$ and the angular momentum balance equation about the stationary point D now give:

$$\begin{aligned} \dot{v} = & \quad \ell \quad \omega^2 \\ \dot{\omega} = & -2\tfrac{m\ell}{I+m\ell^2} \quad v\omega \end{aligned} \tag{6}$$

These differ from the Chaplygin governing ODEs (3) by a factor of 2 in the second equation. Solving these equations to first order in Δt we have:

$$\begin{aligned} \Delta v &= \quad \ell \quad \omega^2 \quad \Delta t \\ \Delta \omega &= -2 \quad \tfrac{m\ell}{I+m\ell^2} \, v\omega \quad \Delta t. \end{aligned} \tag{7}$$

The differential equations (6) apply until one pin is retracted and there is a collision about the new pin. The impulsive collision reaction is also in the $\hat{\mathbf{e}}_n$ direction acting at the new point D. Thus linear momentum is conserved in the $\hat{\mathbf{e}}_\lambda$ direction and angular momentum is conserved about the new point D. Denoting just before the collision with $-$ and just after with $+$ we have:

$$\begin{aligned} m\mathbf{v}_G^- \cdot \hat{\mathbf{e}}_\lambda &= m\mathbf{v}_G^+ \cdot \hat{\mathbf{e}}_\lambda \\ (\ell_0 \hat{\mathbf{e}}_\lambda) \times (m\mathbf{v}_G^-) + I\omega^- \hat{\mathbf{e}}_z &= (\ell_0 \hat{\mathbf{e}}_\lambda) \times (m\mathbf{v}_G^+) + I\omega^+ \hat{\mathbf{e}}_z. \end{aligned} \tag{8}$$

In evaluating $\mathbf{v}_G$ from the first of equation (5) we use a peg position that is a distance $\ell = \ell_0 + d$ before the collision and a distance $\ell = \ell_0$ afterwards. Direct substitution then yields the following jump conditions (exact):

$$\begin{aligned} \Delta v &= 0 \\ \Delta \omega &= \frac{\ell_0 d m \omega^-}{I + \ell_0^2 m} \end{aligned} \tag{9}$$

Where $\Delta v = v^+ - v^-$ and $\Delta\omega = \omega^+ - \omega^-$. Note that although the energy of the system decreases at this collision, the angular velocity ω increases. The center of mass speed decreases, as energy loss anticipates, because the decrease in ℓ more than compensates for the increase in ω in its effect on the first of equation (5).

Pasting together the smooth changes in v and ω from equation (7) with the jumps from (9), using $d = v\Delta t$, and systematically neglecting terms higher than first order in d or Δt we get for this system exactly the 1st order difference equations (4) of the Chaplygin sleigh. That ω decreases faster between collisions in the discrete system is made up for by its jump increase at the collision (taking $\omega > 0$ to simplify explanation).

That is, the discrete and piecewise-holonomic sister system to the Chaplygin sleigh has exactly the same governing equations as the original system, to first order in d. This is no surprise because the sister system was constructed to imitate the skate constraint.

But this convergence of the governing equations implies that the discrete system inherits the same linear stability as the continuous system. In the limit as $d \to 0$ the discrete system is always piecewise-holonomic. Also, as follows from noting the increasing gentleness of the collisions or just the form of the limiting first order equations, the discrete system tends to zero dissipation rate as $d \to 0$. That is, the stability of the discrete system cannot be attributed to its dissipation. Instead the stability must come from the non-holonomic aspect of the intermittent holonomic constraint.

This system makes more credible, perhaps, the claim made earlier that use of a rimless wheel as a wheel substitute in an exponentially stable wheeled system (e.g., a bicycle) would generate exponential stability in a piecewise-holonomic system tending to zero dissipation.

4 Some further remarks on discrete non-holonomy

In the context of smooth systems, dissipation-free non-holonomic contact conditions have several interesting features.

1. Non-holonomic constraints allow asymptotic stability of conservative systems that is not possible without the non-holonomic constraints.

2. For machines that are on a plane, say, non-holonomic constraints allow propulsion in ways that put the machine in its original configuration but moved in the plane. (Even two-Dimensional rolling might be considered non-holonomic in this context. A car moves without change of configuration by the identification of rotations mod 2π and hence a kind of non-integrability of ω.)

3. Non-holonomic constraints allow controls that have finite effect on a system's dynamics with no energy cost. Such is the case, for example, in the controlled steering of a bicycle, tricycle, or car that has no steering inertia and no mechanical trail.

Systems with intermittent contact, but that are holonomic between contacts, share some of these features.

1. The central topic of this paper has been the demonstration, by means of an example, that the non-holonomy of intermittent contact can contribute to asymptotic stability.
2. Legged locomotion is biology's use of a non-holonomic contact condition to translate a system with no net change in configuration, as discussed in [2]. (It seems that any system that can have the effect of translating a body on a level surface must either use non-holonomic constraints, of either the smooth or intermittent type, or be dissipative.)
3. The controlled breaking and engagement of contact can control mechanical systems with zero energy cost. For example Fowble and Kuo [8] used a zero energy controller to balance an otherwise unstable walking robot by controlling the location of foot contact.

5 Conclusion

Non-holonomic systems are in some ways special. Piecewise-holonomic systems that are globally non-holonomic seem to share some of this specialness. Here we have merely demonstrated that exponential stability, or at least a contribution to exponential stability, is one of the features shared by smooth non-holonomic systems and piecewise-holonomic but globally non-holonomic systems. As repeatedly noted, the demonstration was for a nearly dissipationless system arbitrarily close to a smooth system. The possible contribution of non-holonomy to exponential stability of more discontinuous intermittent contact systems, like finite-step walkers, is certainly suggested by this work. But a method of direct investigation of this contribution is unknown to me at this time.

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