

König, Thm., ext, work, etc.

vec. 2, 2004 ①

① $E_K = \frac{1}{2} \sum v_i^2 m_i$ [Def]

② $E_K = \frac{1}{2} m_{tot} v_G^2 + \frac{1}{2} \sum m_i v_{i/G}^2$ (König Thm.)
 $\uparrow v_G = \text{COM} = \text{center of Mass}$

$v_{i/G} = v_i - v_G$ $v_{i/G} = |v_i - v_G|$

③ $E_K = E_{KG} + E_{Krel}$

④ $\dot{E}_{KG} = \left(\sum_{\text{all}} \underline{F}_i \right) \cdot \underline{v}_G = \left(\sum_{\text{ext}} \underline{F}_j^{\text{ext}} \right) \cdot \underline{v}_G$ (Thm. from $\underline{F}^{\text{ext}} = m_{tot} \underline{a}_G$)

⑤ $\dot{E}_{Krel} = \sum_{\text{all}} \underline{F}_i \cdot \underline{v}_{i/G}$ (Thm. from syst. of particles)

(6) (a) Define: $p_b^{\text{ext}} = \text{biomech def. of ext. Power} = \sum_{\text{ext}} \underline{F}_i^{\text{ext}} \cdot \underline{v}_G$

(b) $p_b^{\text{int}} = \dots \dots \dots \text{int. Power} = \sum_{\text{all}} \underline{F}_i \cdot \underline{v}_{i/G}$
 $\leftarrow !$

(7) (a) Define $p^{\text{ext}} = \text{power + Ext. Forces} = \sum_{\text{ext}} \underline{F}_i^{\text{ext}} \cdot \underline{v}_i$

(b) $p^{\text{int}} = \dots \dots \dots \text{int. Forces} = \sum_{\text{int}} \underline{F}_i^{\text{int}} \cdot \underline{v}_i$
 $= \sum_{\text{int}} \underline{F}_i \cdot \underline{v}_{i/G}$

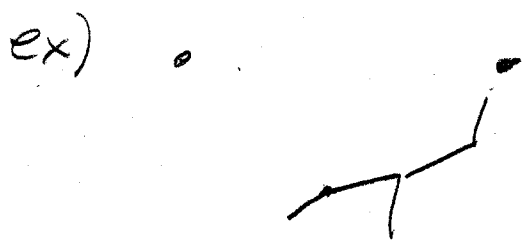
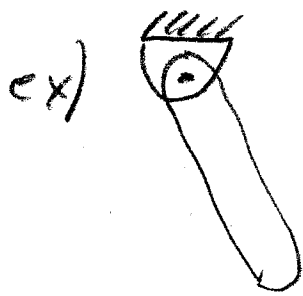
⑧ $\dot{E}_{KG} = p_b^{\text{ext}}$

⑩ $\dot{E}_K = p_b^{\text{ext}} + p_b^{\text{int}}$

⑨ $\dot{E}_{Krel} = p_b^{\text{int}}$

⑪ $\dot{E}_K = p^{\text{ext}} + p^{\text{int}}$ (work energy for system)

In systems where the concepts should apply
 P_b^{int} & P_b^{ext} give misleading answers



$$P_b^{ext} \neq 0 \quad (P^{ext} = 0)$$

↳ after gravity correction

even though no actuator work

$$P_b^{ext} \neq 0 \quad (P^{ext} = 0)$$

$$P_b^{int} \neq 0 \quad (P^{int} = 0)$$

even though no actuator work

The word "external" in "external work" implies the contrasting term "internal". Essentially the whole literature that uses these words is a mixture of

- a) wrong
- b) vague
- c) self contradictory
- d) confused & confusing

P_b^{ext} only helps insight to the extent that a particle model is useful.

P_b^{int} never ^{has} given any useful insight

Because of word choice, people often confuse P_b^{ext} & P_b^{int} with P^{ext} & P^{int} .

P_b^{int} depends on power of external forces.

On the other hand

"External Work" & "Internal Work" are old terms in biomechanics

"Ext. Work" (P_b^{ext}) is $P^{ext} = P$ for particle models

P^{ext} can be measured with external (forceplate) instrumentation.

AS/DL

$$\dot{E}_K = \overbrace{\sum_{\text{int}} \underline{F}_i^{\text{int}} \cdot \underline{V}_i}^{p_{\text{int}}} + \overbrace{\sum_{\text{ext}} \underline{F}_i^{\text{ext}} \cdot \underline{V}_i}^{p_{\text{ext}}}$$
$$= \sum \underline{F}_i^{\text{ext}} \cdot \underline{V}_G - \sum \underline{F}_i^{\text{ext}} \cdot \underline{V}_G + \sum_{\text{int}} \underline{F}_i^{\text{int}} \cdot \underline{V}_i + \sum_{\text{ext}} \underline{F}_i^{\text{ext}} \cdot \underline{V}_i$$

$$= \sum \underline{F}_i^{\text{ext}} \cdot \underline{V}_G + \sum \underline{F}_i^{\text{int}} \cdot \underline{V}_i + \underline{F}_i^{\text{ext}} \cdot \underline{V}_i / G$$

|| ← equal & opp. force pair

$$\sum \underline{F}_i \cdot \underline{V}_i / G$$

$$= \underbrace{\sum \underline{F}_i^{\text{ext}} \cdot \underline{V}_G}_{p_{\text{ext}}^b} + \underbrace{\sum \underline{F}_i \cdot \underline{V}_i / G}_{p_{\text{int}}^b}$$

DYNAMICS OF A SYSTEM OF PARTICLES

4

In the previous chapter, we discussed some of the more important principles and techniques to be used in the analysis of the motion of a single particle. When one considers the dynamics of a group of interacting particles, one may still look at individual particles, but in addition, certain over-all aspects of the motion of the system may be calculated without specifically solving for the individual motions. It is with these extensions and generalizations of previously discussed principles that this chapter will be primarily concerned. In addition, the chapter will discuss particular applications, such as the collision and rocket propulsion problems.

4-1 THE EQUATIONS OF MOTION

Consider first a system of n particles, of which three are shown in Fig. 4-1. The forces applied to a given particle may be classified as external or internal, according to their source. The *total* force on the i th particle arising from sources external to the system of n particles is designated by \mathbf{F}_i and is known as an *external force*. All interaction forces among the particles are known as *internal forces* and are designated by individual force vectors of the form \mathbf{f}_{ij} , where the first subscript indicates the particle on which the force acts and the second subscript indicates the acting particle.

From Newton's law of action and reaction, we know that the interaction forces between any two particles are equal and opposite:

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji} \quad (4-1)$$

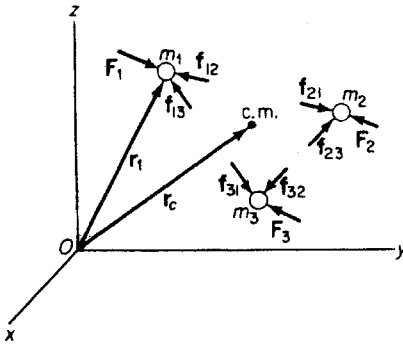


Figure 4-1 The forces acting on a system of particles.

Also, these forces are assumed to be collinear, that is, they act along the straight line connecting the particles. It can be seen that

$$f_{ii} = 0$$

in agreement with Eq. (4-1), indicating the fact that a particle cannot exert a force on itself which affects its motion.

Now let us sum all the forces acting on the i th particle. Including both external and internal forces, we can write the equation of motion in the form:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij}, \quad (i = 1, 2, \dots, n) \tag{4-2}$$

where m_i is the mass of the i th particle and \mathbf{r}_i is its position vector relative to the fixed point O . Next we sum Eq. (4-2) over all n particles, obtaining

$$\sum_{i=1}^n m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^n \mathbf{F}_i + \sum_{i=1}^n \sum_{j=1}^n \mathbf{f}_{ij} \tag{4-3}$$

But, from Eq. (4-1), we see that

$$\sum_{i=1}^n \sum_{j=1}^n \mathbf{f}_{ij} = 0 \tag{4-4}$$

since the internal forces always occur in equal and opposite pairs. Also, we note that the total mass m is

$$m = \sum_{i=1}^n m_i \tag{4-5}$$

and the *center of mass* location is given by

$$\mathbf{r}_c = \frac{1}{m} \sum_{i=1}^n m_i \mathbf{r}_i \tag{4-6}$$

The total external force is

$$\mathbf{F} = \sum_{i=1}^n \mathbf{F}_i \tag{4-7}$$

Therefore Eq. (4-3) can be written in the form

$$\mathbf{F} = m \ddot{\mathbf{r}}_c \tag{4-8}$$

This result has the familiar form of a force being equal to the product of a mass and an acceleration. It indicates that the motion of the center of mass of a system of particles is the same as if the entire mass of the system was concentrated at the center of mass and was driven by the sum of all the forces external to the system. It should be emphasized, however, that the force \mathbf{F} is calculated for the actual system of particles and not, in general, on the basis of a single particle of total mass m located at the center of mass.

4-2 WORK AND KINETIC ENERGY

We have seen that Eq. (4-8) is identical in mathematical form to the equation of motion for a single particle. It is apparent, then, that a set of principles similar to those for a single particle also applies to the motion of the center of mass of a system of particles.

Let us assume that the center of mass of the system moves from A_c to B_c under the action of the external forces \mathbf{F}_i whose sum is \mathbf{F} . Then, taking the line integral of each side of Eq. (4-8) over the path of the center of mass, we obtain a result similar to that obtained previously in Eq. (3-67). It is

$$\int_{A_c}^{B_c} \mathbf{F} \cdot d\mathbf{r}_c = \frac{1}{2} mv_c^2 \Big|_{A_c}^{B_c} \quad (4-9)$$

where v_c is the absolute velocity of the center of mass. This equation states that if one considers the sum of the external forces to be acting at the center of mass of the system, then the work done by the total external force in moving over the path of the center of mass is equal to the change in the translational kinetic energy associated with the motion of the center of mass.

Note that the work expressed by the integral on the left side of Eq. (4-9) does not include that work done by the internal forces, nor is it even the total work of the external forces. To clarify this point, let us calculate the total work done by all the forces, external as well as internal. Considering first the i th particle, the work done by all the forces acting on m_i as it moves from A_i to B_i is

$$W_i = \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij} \right) \cdot d\mathbf{r}_i \quad (4-10)$$

But, as shown in Fig. 4-2, we can express the position of the i th particle as the sum

$$\mathbf{r}_i = \mathbf{r}_c + \boldsymbol{\rho}_i \quad (4-11)$$

where $\boldsymbol{\rho}_i$ is the position vector of the i th particle relative to the center of mass. So, substituting Eq. (4-11) into (4-10) and summing over all n particles, we obtain the total work

$$W = \sum_{i=1}^n W_i = \sum_{i=1}^n \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij} \right) \cdot (d\mathbf{r}_c + d\boldsymbol{\rho}_i)$$

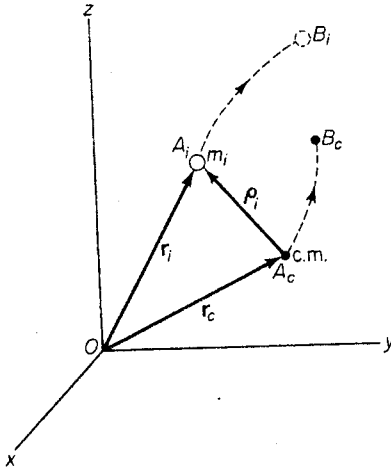


Figure 4-2 Position vectors of the center of mass and of a typical particle during an arbitrary displacement.

OR

$$W = \sum_{i=1}^n \int_{A_c}^{B_c} (\mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij}) \cdot d\mathbf{r}_c + \sum_{i=1}^n \int_{A_i}^{B_i} (\mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij}) \cdot d\boldsymbol{\rho}_i \quad (4-12)$$

where the limits on the second integral refer, in this case, to the position of the *i*th particle relative to the position of the center of mass of the system.

The first integral on the right side of Eq. (4-12) can be simplified by using Eqs. (4-4) and (4-7). We note that \mathbf{r}_c and the limits of integration are not dependent upon the summation index *i*, and therefore the summations can be carried out before the integration. The second integral cannot be simplified, since $\boldsymbol{\rho}_i$ is a function of *i*. With these changes, we obtain

$$W = \int_{A_c}^{B_c} \mathbf{F} \cdot d\mathbf{r}_c + \sum_{i=1}^n \int_{A_i}^{B_i} (\mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij}) \cdot d\boldsymbol{\rho}_i \quad (4-13)$$

Thus we see that the total work can be considered as the sum of two parts: (1) the work done by the total external force acting through the displacement of the center of mass; (2) the summation of the work done on all particles by both the external and internal forces on each particle acting through the displacement of that particle relative to the center of mass.

Now, for each particle, the principle of work and kinetic energy applies, so we can write

$$W_i = \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \Big|_{A_i}^{B_i} = \frac{1}{2} m_i (\dot{\mathbf{r}}_c \cdot \dot{\mathbf{r}}_c + 2\dot{\mathbf{r}}_c \cdot \dot{\boldsymbol{\rho}}_i + \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i) \Big|_{A_i}^{B_i}$$

where we have substituted for $\dot{\mathbf{r}}_i$ from Eq. (4-11). Summing over all particles, we find that

$$W = \sum_{i=1}^n W_i = \frac{1}{2} m v_c^2 \Big|_{A_c}^{B_c} + \dot{\mathbf{r}}_c \cdot \sum_{i=1}^n m_i \dot{\boldsymbol{\rho}}_i \Big|_{A_i}^{B_i} + \frac{1}{2} \sum_{i=1}^n m_i \dot{\boldsymbol{\rho}}_i^2 \Big|_{A_i}^{B_i} \quad (4-14)$$

where $\dot{\boldsymbol{\rho}}_i$ is the relative velocity of m_i , as viewed by a nonrotating observer

moving with the center of mass. Here we have used the notation:

$$\dot{\boldsymbol{\rho}}_i^2 = \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i$$

But we see from Eq. (4-6) that

$$\begin{aligned} m\mathbf{r}_c &= \sum_{i=1}^n m_i \mathbf{r}_i = \sum_{i=1}^n m_i (\mathbf{r}_c + \boldsymbol{\rho}_i) \\ &= m\mathbf{r}_c + \sum_{i=1}^n m_i \boldsymbol{\rho}_i \end{aligned}$$

and, consequently, that

$$\sum_{i=1}^n m_i \boldsymbol{\rho}_i = 0 \quad (4-15)$$

in agreement with the original assumption that $\boldsymbol{\rho}_i$ is measured from the center of mass. Also, of course,

$$\sum_{i=1}^n m_i \dot{\boldsymbol{\rho}}_i = 0 \quad (4-16)$$

and therefore Eq. (4-14) reduces to

$$W = \frac{1}{2} m v_c^2 \Big|_{A_c}^{B_c} + \sum_{i=1}^n \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2 \Big|_{A_i}^{B_i} \quad (4-17)$$

The right-hand side of Eq. (4-17) represents the sum of the increases in kinetic energy of the individual particles, or, in other words, the increase in the total kinetic energy of the system. Thus we can write

$$W = T_B - T_A \quad (4-18)$$

in a manner similar to Eq. (3-70) which was derived for a single particle. In this case, T_A and T_B represent the total kinetic energy of the system at the beginning and at the end, respectively, of the line integrations. It is apparent from Eq. (4-17) that the total kinetic energy is

$$T = \frac{1}{2} m v_c^2 + \sum_{i=1}^n \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2 \quad (4-19)$$

Now Eq. (4-18) is valid for an arbitrary interval. So if we consider the case where the work and the change in kinetic energy are evaluated during an infinitesimal time interval Δt , we can write

$$\Delta W = \Delta T$$

or, in the limit as Δt approaches zero, we obtain

$$\dot{W} = \dot{T} \quad (4-20)$$

Thus the rate of increase of the total kinetic energy is equal to the rate at which work is done on the system, that is, it is the instantaneous *power* associated with the external and internal forces.

Returning now to a further consideration of work and kinetic energy relationships, we find from Eqs. (4-9), (4-13), and (4-17) that

$$\sum_{i=1}^n \int_{A_i}^{B_i} \left(\mathbf{F}_i + \sum_{j=1}^n \mathbf{f}_{ij} \right) \cdot d\mathbf{p}_i = \sum_{i=1}^n \frac{1}{2} m_i \dot{\mathbf{p}}_i^2 \Big|_{A_i}^{B_i} \quad (4-21)$$

Hence the work done by the external and internal forces in moving through displacements relative to the center of mass is equal to the increase in the kinetic energy of relative motion. It is important to note that the relative velocity $\dot{\mathbf{p}}_i$ can arise from rigid body rotations in which the particle separations do not change with time, as well as in the more obvious case of changing particle separations.

Referring again to Eqs. (4-9), (4-19), and (4-21), we can summarize the results of this section as follows:

1. The total kinetic energy is equal to that due to the total mass moving with velocity of the center of mass plus that due to the motions of the individual particles relative to the center of mass. This is *König's theorem*.
2. The work done by the external forces in moving through the displacement of the center of mass is equal to the increase in the kinetic energy due to the total mass moving with the velocity of the center of mass.
3. The work done by the external plus the internal forces in moving through displacements relative to the center of mass is equal to the increase in the kinetic energy associated with the relative motions.
4. The total work done by the external and internal forces is equal to the increase in the total kinetic energy.

4-3 CONSERVATION OF MECHANICAL ENERGY

The principle of conservation of mechanical energy was developed in Sec. 3-3 for the case of a single particle. Now let us extend this principle to apply to a system of particles. First, recall from Eq. (4-8) that the motion of the center of mass is the same as though it were a particle with a mass equal to the total mass m and acted upon by the total external force \mathbf{F} . Therefore, if the total external force is derivable from a single-valued potential function involving the center of mass position only, then, by analogy to the results obtained previously for a single particle, the center of mass moves such that the energy E_c is constant.

$$E_c = T_c + V_c \quad (4-22)$$

T_c is the kinetic energy due to the translational motion of the center of mass, and V_c is the potential energy associated with the position of the center of mass. It can be seen that it is possible for conservation of the energy E_c to occur even in the case of dissipative internal forces.

For the case where the internal as well as the external forces are conservative, that is, they are derivable from a single-valued potential function

involving the coordinates only, we find that the total energy E is conserved, where

$$E = T + V \quad (4-23)$$

In this case, T is the sum of the kinetic energies of the individual particles, or it can be considered to be the sum of the portion due to the motion of the center of mass plus the portion due to motion relative to the center of mass, as in Eq. (4-19). The potential energy V is often just the sum of the potential energy due to gravity and that due to the deformations of elastic elements such as springs. In any event, the potential energy of n particles in a three-dimensional space can be written in the form:

$$V = V(x_1, x_2, \dots, x_{3n}) \quad (4-24)$$

since we assume that the system may have as many as $3n$ degrees of freedom. Thus, as in Eq. (3-89), we find that if a small increase in x_k results in a small displacement of a certain particle in a given direction, then the force

$$F_k = -\frac{\partial V}{\partial x_k} \quad (4-25)$$

acts on the given particle in the direction of increasing x_k . This is the total force and includes, in general, both internal and external forces.

For the case of a system in which both conservative and nonconservative forces are acting, one can use work and energy concepts in place of a strict conservation of energy. For example, if W_n represents the work done on the system by *nonconservative* internal and external forces in going from configuration A to configuration B , then

$$W_n = E_B - E_A \quad (4-26)$$

where the total energies E_B and E_A include potential energy terms corresponding to all the forces doing work on the system except for the nonconservative forces.

Example 4-1 Atwood's Machine

Two masses m_1 and m_2 are connected by a massless, inextensible rope which passes over a pulley, as shown in Fig. 4-3. Neglecting the mass and the bearing friction of the pulley, find the acceleration of m_1 and the tension in the rope as the system moves under the action of gravity.

Let us use the coordinates x_1 and x_2 to designate the vertical displacements of m_1 and m_2 , respectively, measured from some nominal position. Since the rope is inextensible, we see that

$$x_1 = -x_2 \quad (4-27)$$

Also, the pulley exerts no inertial or frictional forces on the rope and therefore the tension T is uniform throughout its length.

First we shall solve the problem using Newton's law of motion for each mass separately. Each mass has two forces acting on it, namely, the rope