Some basics of Mechanics (mostly 1 D)

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The zeroth laws of mechanics

1. mass is not ephemeral
2. time and space are as we are used to
3. the laws apply to *any* collection of mass
   (any system, any subsystem)
4. force is "the" measure of interaction between objects (systems, subsystems)

➡ *Draw free body diagrams!*

The three pillars of mechanics

1. **Geometry & kinematics** (geometry of motion): relations between

$$x, v, a, \vec{r}, \theta, \omega, \alpha, t, \ell, \ldots$$

2. **Laws of mechanics** (relations between force and acceleration), momentum balance:

$$\vec{F} = m\vec{a}, \qquad \vec{M} = \dot{\vec{H}}, \qquad \text{action and reaction}$$

3. **Material properties** = constitutive laws

$$T = k(\ell - \ell_0), \qquad T = c\dot{\ell}, \qquad \vec{F} = -c|\vec{v}|\vec{v}$$

$$F = \mu N, \qquad v^+ = -ev^-, \qquad \ldots$$

$$\Delta\ell = 0$$

The three pillars cont'd

1. **Geometry:** we live in flatland

2. **Mechanics:** Newton was right

3. **Constitutive laws:** What are things made of?

The three pillars (cont'd)

1. **Geometry:** we live in flatland 0.0000001 %
   (whether or not you can measure that well)

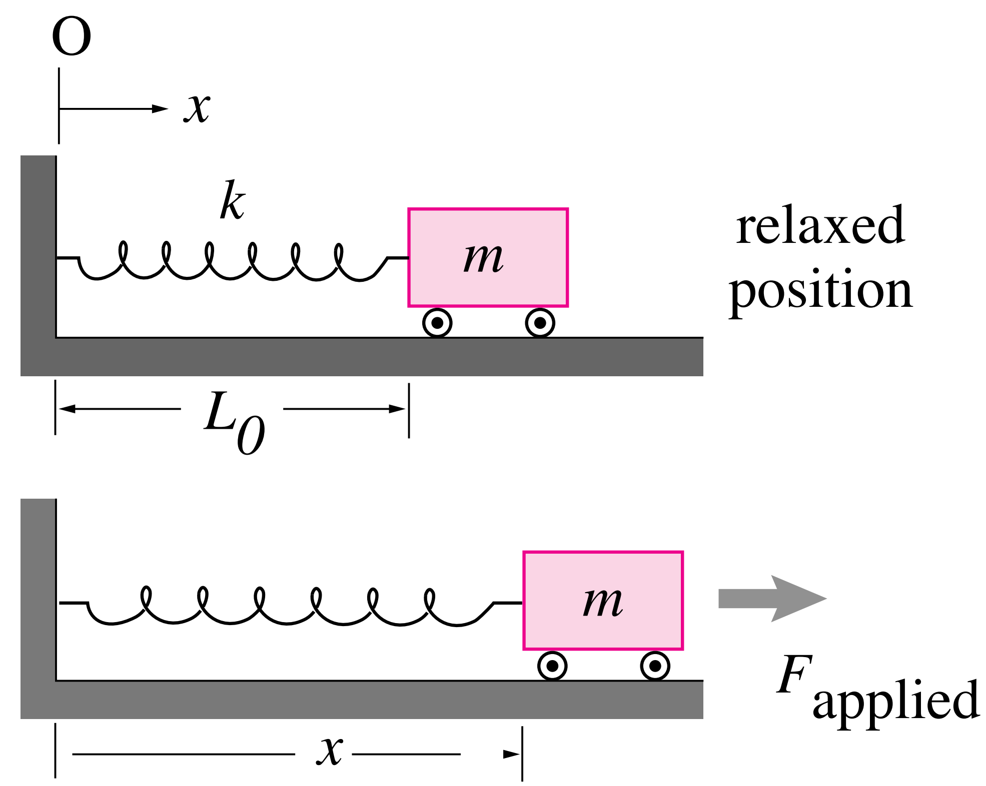
2. **Mechanics:** Newton was right 0.0000001 %
   (whether or not you can measure that well)

3. **Constitutive laws:** What are things made of?
   0.1 % (rarely, sometimes for spring constants)
   1 % (when you are lucky)
   10 % (pretty good for friction & collisions)
   50 % (not unusual)

(whether or not you can measure that well, *the equations are wrong/not accurate*)

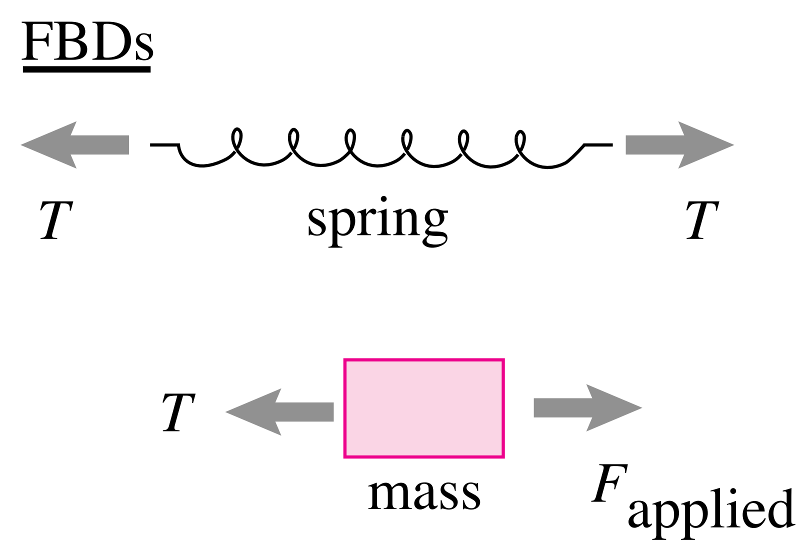
1D example: harmonic oscillator

Spring, mass and applied force



Need to pick

- generalized coordinates
- minimal coordinates
- configuration variables
- dynamic variables
- motion variables
- whatever you want to call the things that if
  - you know them and
  - you know their time derivatives then
  - you know the positions, velocities and accelerations of all points in the system (at least you know enough about them)
- *x* (q_i)



Free Body Diagrams

Think: "chainsaw" (or "scalpel"), and "deceit".

1D example: harmonic oscillator

LMB:

$$F = m\,a$$

$$F = F_{\text{applied}} - T$$

$$a = \ddot{x}$$

$$T = k\,\Delta\ell$$

$$\Delta\ell = x - \ell_0$$

"Equation of motion":

$$F_{\text{applied}} - k(x - \ell_0) = m\ddot{x}$$

Standard forms:

$$m\ddot{x} + kx = F_{\text{applied}} + k\ell_0$$

or

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -kx + F_{\text{applied}} + k\ell_0 \end{aligned}$$

or

$$\dot{z} = f(z)$$

Various theorems/facts

Impulse momentum

$$\begin{aligned} F = ma \quad \Rightarrow \quad \int F\,dt &= m \int a\,dt \\ &= m\,\Delta v \end{aligned}$$

Work energy

$$\begin{aligned} F = ma \quad \Rightarrow \quad F\,v &= m a\,v \\ &= m \frac{dv}{dt} v \\ &= m \frac{d}{dt}\left(\frac{v^2}{2}\right) \\ &= \frac{d}{dt}\underbrace{\left(m\frac{v^2}{2}\right)}_{E_k} \\ &= \dot{E}_k \\ \text{Power} &= \text{rate of increase of kinetic energy} \end{aligned}$$

$$\begin{aligned} F = ma \quad \Rightarrow \quad \int F\,dx &= m \int a\,dx \\ &= m \int \frac{dv}{dt}\frac{dx}{dt}\,dt \\ &= m \int \left(\frac{dv}{dt} v\right) dt \\ &= m \int \frac{d}{dt}\left(\frac{v^2}{2}\right) dt \\ &= \Delta \underbrace{\left(m\frac{v^2}{2}\right)}_{E_k} \\ &= \Delta E_k \end{aligned}$$

Conservation of momentum when no force

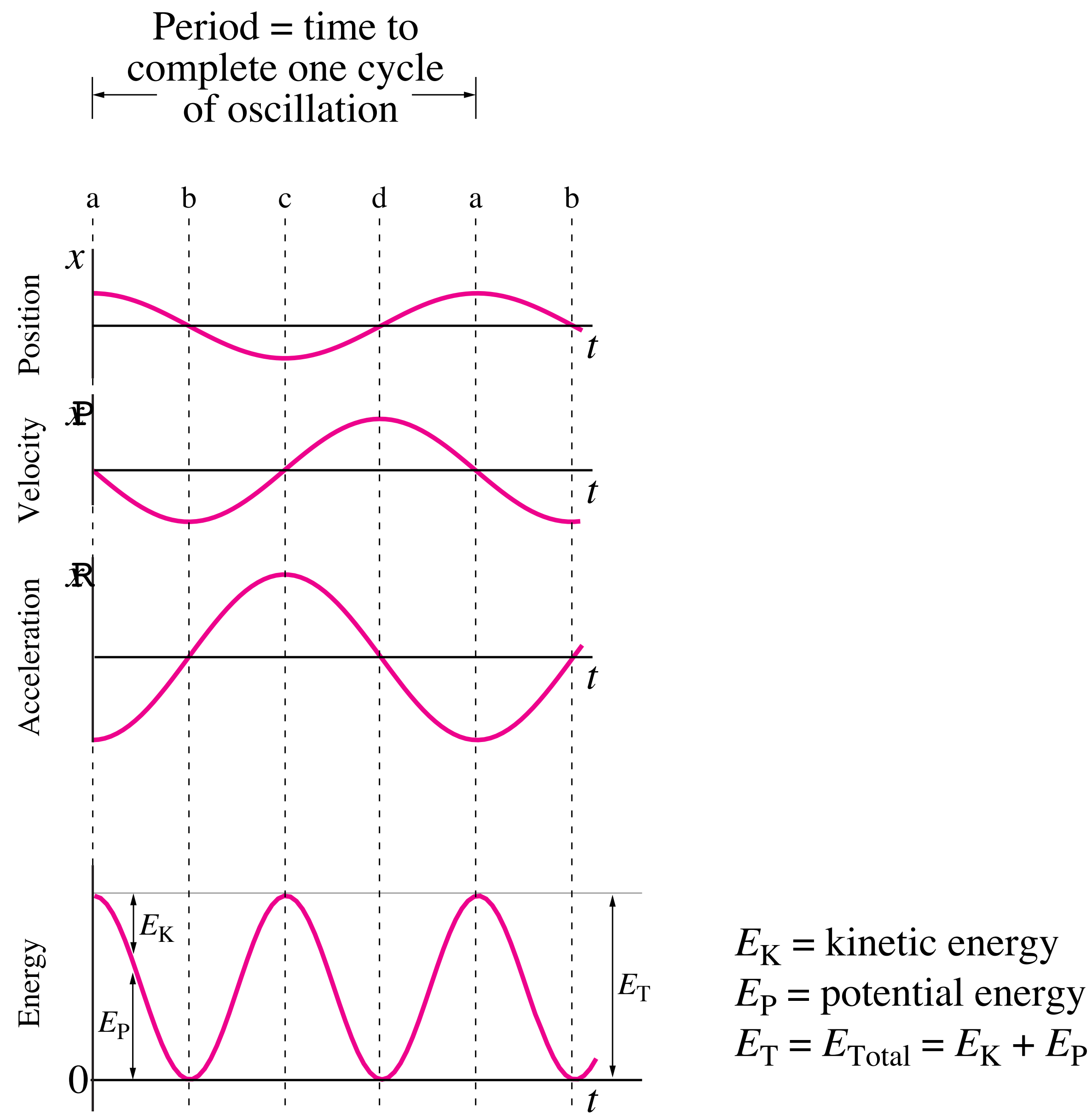
$$m\,\Delta v = 0$$

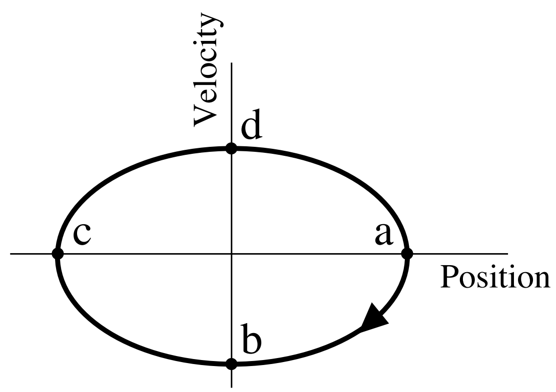
Conservation of energy when force is "conservative"

$$\text{IF} \quad F = -\frac{d}{dx}E_p(x) \quad \Rightarrow \quad F v = -\dot{E}_p \quad \Rightarrow \quad -\dot{E}_p = \dot{E}_k$$

$$\text{IF} \quad F = -\frac{d}{dx}E_p(x) \quad \Rightarrow \quad \int F\,dx = -\Delta E_p \quad \Rightarrow \quad -\Delta E_p = \Delta E_k$$

Solution of ODEs





Galilean invariance

Note: $F = m\,a$

there is no *v* in Newton's law

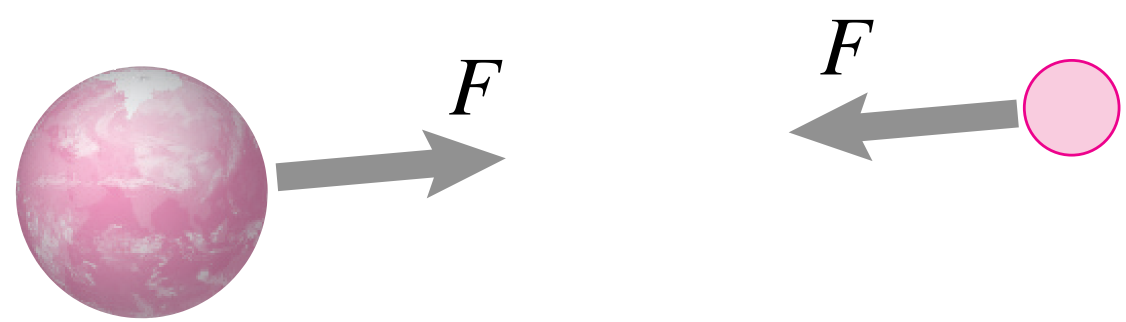
So:

all the theorems apply in a steadily translating frame
(the equations are objective, so are *t*, *d*, *m*, *F*)

But (!!):

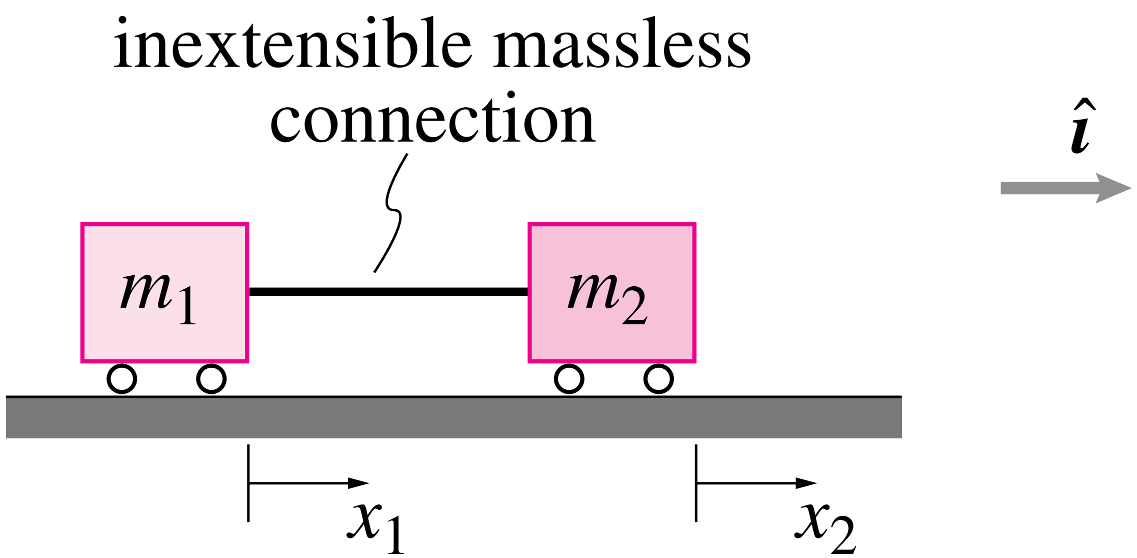
Not all of the terms are objective
(e.g., work, power, and kinetic energy
are all *frame dependent*)

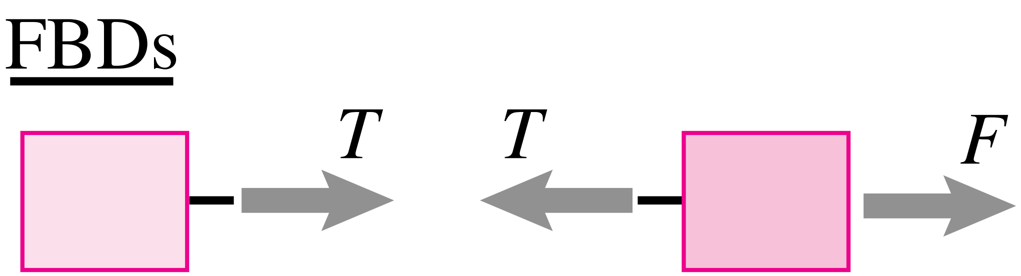
Interaction (internal to system) forces



- **Cancel** in their effect on system momentum
- **Do *not* cancel** in their effect on system energy *unless* they are ‘workless’.

Multi-bodies constrained

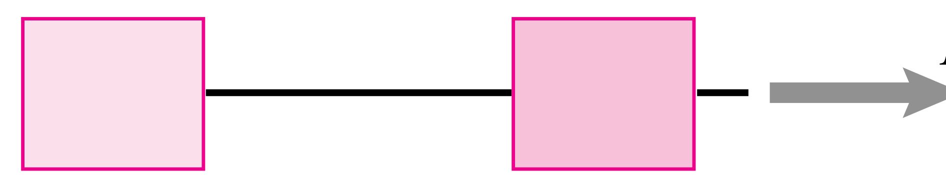




Approach I: Assemble equations for separate bodies

$$
\begin{aligned}
m_1\ddot{x}_1 &= T \\
m_2\ddot{x}_2 &= F - T \\
\ddot{x}_2 - \ddot{x}_1 &= 0
\end{aligned}
$$

Approach II: Use tricks to guide the adding and subtracting of equations to eliminate constraints



$$(m_1 + m_2)\ddot{x} = F$$