## **A Particle Collision Model for Calculating the Energetic Cost of Legged Locomotion** Andy Ruina, Cornell TAM (&MAE)

with

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Some support from N\$F.

Read all about it:

J. Theor. Biol., **237**, Issue 2, Pages 170-192, Nov 2005

Nature, **439**, Pages 72-75, Jan 2006.

TAM, Cornell Feb 06 "Dynamic Walking", May 06



**General Goal:** Understand coordination choices of animals (including people).

# **General Approach:** Assume the principle of maximum laziness.



Animals move in such a way as to minimize their use of energy: Your great great ... great ... ... great grandmother's sister ... . . . died (young).

# **Some issues/caveats:**

- **Selection does not mean, exactly, optimization.**
- **X** There is selection for other things,
	- e.g., speed, weight bearing, IQ, sharp teeth,...
- Energy use per unit \_\_\_\_\_\_\_ ?
- **X** How to calculate relation between motion and
	- energy use (we use: cost = muscle work)
- **X** Math 191, optimization is inherently inaccurate.

### Performance Optima tend to be insensitive to control parameters.





When work is substantial, energy use is roughly proportional to work.

# Assume a spherical horse...

- That's too hard
- Make it a small sphere, a particle
- Massless legs





**Leg Work W:**  

$$
W = \int dW = \int P dt = \int \mathbf{F} \cdot \mathbf{v} dt = \int F \dot{\ell} dt = \int F d\ell
$$

#### Minimizing work at fixed v and d finds solutions which spend most time with *Power* = 0: *l = 0 or F=0* **Fig. 1 Srinivasan & Ruina .**<br>.



#### gets close to a  $\mathbf{L}$ <u>u</u> Continuous solution gets close to an *x* impulsive solution as numerical grid gets finer.





= v<sup>−</sup> *·* P<sup>∗</sup> + *|*P∗*|* Chemical *<sup>E</sup>*˙ <sup>=</sup> <sup>4</sup> *·*(Mechanical power) *W* = *E* = *E* = *E* = *m* = *E* # *<sup>|</sup>*v+*<sup>|</sup>* February 13, 2006 Relate impulse and work (2 ways)

!*dt* =

The im |
| ulse is  $\mathsf{The}$ 9 = v<sup>−</sup> *·* P<sup>∗</sup> + *|*P∗*|* The impulse is I. Net change:<br>The impulse is **II.** Integrate:

 $\frac{1}{\sqrt{2}}$ 

$$
\mathbf{P}^* = \int_{t_1}^{t_2} \mathbf{F} dt = \hat{\lambda} \int_{t_1}^{t_2} F dt
$$

Impulse momentum *t*1  $A$ 

 $m\mathbf{y}^+ = m\mathbf{v}^- + \mathbf{P}^*$ 

 $=$   $m$ 

$$
(net) Work Energy
$$
\n
$$
W = \Delta E = \frac{m}{2} (|\mathbf{v}^+|^2 - |\mathbf{v}^-|^2)
$$
\n
$$
= \mathbf{v}^- \cdot \mathbf{P}^* + |\mathbf{P}^*|^2 / (2m).
$$

.<br>.-

 $+$  **P** 

Calculate the (net) work *<sup>W</sup>* <sup>=</sup> <sup>∆</sup>*<sup>E</sup>* <sup>=</sup> *<sup>m</sup>* <sup>2</sup> <sup>−</sup> *<sup>|</sup>*v−*<sup>|</sup>* <sup>2</sup>*/*(2*m*)*.* (1)  $\mathbf{r}$  +  $\mathbf{r}$  + F*dt* = λˆ *Fdt* (2) The partial impulse  $P, 0 < p < 1$ , is  $\mathbf{P}(t) \equiv$  $\int_0^t$ *t*1  $\equiv \int_{a} \mathbf{F}(t')dt' = p\mathbf{P}^*$  $W$  $\int d$ " *<sup>t</sup>*<sup>2</sup>  $W = \int dW = \int \mathbf{v}$ (v<sup>−</sup> + P*/m*  $\begin{matrix} 0 & \longrightarrow \\ & \vee \end{matrix}$  ${\bf P}/m\big)$ %&'( F*dt*  $\int_0^1 (\mathbf{v}^- + p\mathbf{P}^*/m) \cdot \mathbf{P}$  $= \mathbf{v}^- \cdot \mathbf{P}^* \int_0^1 d\mathbf{p}$ *<sup>W</sup>* <sup>=</sup> <sup>∆</sup>*<sup>E</sup>* <sup>=</sup> *<sup>m</sup> <sup>|</sup>*v+*<sup>|</sup>* = v<sup>−</sup> *·* P<sup>∗</sup> + *|*P∗*|* <sup>2</sup>*/*(2*m*)*.* (1) " *<sup>t</sup>*<sup>2</sup> F*dt* = λˆ  $\mathcal{L}$  $\longrightarrow$  (2) " *<sup>t</sup>*  $dt' = p\mathbf{P}^*$ "  $dW =$  $\int_{}^{t_{2}}$ *t*1  $\mathbf{v} \cdot \mathbf{F} dt$ =  $\int^{\mathbf{P}^*}$ 0  $({\bf v}^-+{\bf P}/m)$  $\begin{array}{c}\n\diagup \\ \diagdown \\ \mathbf{v}\n\end{array}$ ) *· d*P  $\sum_{\mathbf{F}dt}$ F*dt*  $\int_0^1$  $\overline{0}$  $(\mathbf{v}^- + p\mathbf{P}^*/m) \cdot \mathbf{P}^* dp$  $\int_0^1$  $\overline{0}$ *dp*  $\begin{bmatrix} 1 \end{bmatrix}$ + P<sup>∗</sup> *·* P<sup>∗</sup> *m*  $\int_0^1$  $\overline{0}$ *pdp*  $\begin{array}{r} \hline 1/2 \end{array}$  $^{2}/(2m).$  =  $\mathbf{v}^{-} \cdot \mathbf{P}^{*} + |\mathbf{P}^{*}|^{2}/(2m)$  $\epsilon$ |
|
| |
|
| ์<br>ว alla.<br>F  $|\mathbf{v}^{+}|^{2} - |\mathbf{v}^{-}|^{2}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$ <sup>2</sup>*/*(2*m*)*.* (1)  $Fdt$   $\mathbf{p}(t) = \int_0^t \mathbf{F}(t)dt$ where the period of force application is between  $\epsilon$  and  $\epsilon$  and  $\epsilon$  and  $\epsilon$  and  $\epsilon$  is between  $\epsilon$  is and  $\epsilon$  is and  $\epsilon$  is an  $\epsilon$  is a en die bestied van die bestied<br>Die bestied van die bestied va **.** —<br>।<br>।

Calculate the (net) work (net) Work Energy = v<sup>−</sup> *·* P<sup>∗</sup> + *|*P∗*|* Chemical *<sup>E</sup>*˙ <sup>=</sup> <sup>4</sup> *·*(Mechanical power)  $\mathbf{r}$  +  $\mathbf{r}$  + F*dt* = λˆ The partial impulse  $P, 0 < p < 1$ , is  $\mathbf{P}(t) \equiv$  $\int_0^t$ *t*1  $\equiv \int_{a} \mathbf{F}(t')dt' = p\mathbf{P}^*$  $W$  $\int d$ " *<sup>t</sup>*<sup>2</sup>  $W = \int dW = \int \mathbf{v}$ (v<sup>−</sup> + P*/m*  $\begin{matrix} 0 & \longrightarrow \\ & \vee \end{matrix}$  $\int_0^1 (\mathbf{v}^- + p\mathbf{P}^*/m) \cdot \mathbf{P}$  $= \mathbf{v}^- \cdot \mathbf{P}^* \int_0^1 d\mathbf{p}$ " *<sup>t</sup>*<sup>2</sup> F*dt* = λˆ " *<sup>t</sup>*  $dt' = p\mathbf{P}^*$ "  $dW =$  $\int_0^t 2$ *t*1 =  $\int^{\mathbf{P}^*}$ 0  $({\bf v}^-+{\bf P}/m)$  $\begin{array}{c}\n\diagup \\ \diagdown \\ \mathbf{v}\n\end{array}$  $\int_0^1$  $\overline{0}$  $(\mathbf{v}^- + p\mathbf{P}^*/m) \cdot \mathbf{P}^* dp$  $\int_0^1$  $\overline{0}$ *dp*  $\begin{bmatrix} 1 \end{bmatrix}$  $^{2}/(2m).$  =  $\mathbf{v}^{-} \cdot \mathbf{P}^{*} + |\mathbf{P}^{*}|^{2}/(2m)$ The im 6 *·* |
| ulse is  $\mathsf{The}$  $\epsilon$ |
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| |
|
| !*dt* = ์<br>ว alla.<br>F  $m\mathbf{y}^+ = m\mathbf{v}^- + \mathbf{P}^*$ *E*  $\overline{L}$  **Mork 2**  $\frac{1}{2}$  Work Energy  $W = \Delta E = \frac{m}{2}$  $|\mathbf{v}^{+}|^{2} - |\mathbf{v}^{-}|^{2}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$   $\mathbf{v}^{-}$ Impulse momentum *W* = *E* = *E* = *E* = *m* = *E* 9 # *<sup>|</sup>*v+*<sup>|</sup>* February 13, 2006 = v<sup>−</sup> *·* P<sup>∗</sup> + *|*P∗*|* <sup>2</sup>*/*(2*m*)*.* (1)  $\int_0^t$ *t*1  $\mathbf{F} dt = \boldsymbol{\hat{\lambda}}$  $\int_0^t$ *t*1  $Fdt$   $\mathbf{p}(t) = \int_0^t \mathbf{F}(t)dt$ where the period of force application is between  $\epsilon$  and  $\epsilon$  and  $\epsilon$  and  $\epsilon$  and  $\epsilon$  is between  $\epsilon$  is and  $\epsilon$  is and  $\epsilon$  is an  $\epsilon$  is a *t*1  $A$ *dW* = v *·* F*dt* (v<sup>−</sup> + P*/m* ) *· d*P  $W = \Delta E = \frac{m}{2} (|\mathbf{v}^+|^2)$  $\overline{\phantom{a}}$  $\mathbf{1}$  $\mathbf{P}^* = \int^{t_2} \mathbf{F} dt = \hat{\lambda} \int^{t_2} F dt$ en die bestied van die bestied<br>Die bestied van die bestied va **.** —<br>।<br>।  $\frac{1}{\sqrt{2}}$  $=$   $m$ .<br>.- $+$  **P** 2  $(|\mathbf{v}^+|^2 - |\mathbf{v}^-|^2)$  $=$   $v^- \cdot P^* + |P^*|$ The impulse is I. Net change:<br>The impulse is **II.** Integrate:

Relate impulse and work (2 ways)

 $\mathbf{P}^{*}=% \begin{bmatrix} \omega_{11} & \omega_{12} & \ldots & \omega_{1n-1} \ \omega_{21} & \omega_{22} & \ldots & \omega_{2n-1} \end{bmatrix}% ,$ 

 $\bigg($ 

0

 $\mathsf{n}$ 

*<sup>W</sup>* <sup>=</sup> <sup>∆</sup>*<sup>E</sup>* <sup>=</sup> *<sup>m</sup>*

*<sup>W</sup>* <sup>=</sup> <sup>∆</sup>*<sup>E</sup>* <sup>=</sup> *<sup>m</sup>*

= v<sup>−</sup> *·* P<sup>∗</sup> + *|*P∗*|*

<sup>2</sup> <sup>−</sup> *<sup>|</sup>*v−*<sup>|</sup>*

 $\mathcal{L}$ 

*<sup>|</sup>*v+*<sup>|</sup>*

 ${\bf P}/m\big)$ 

%&'( F*dt*

+ P<sup>∗</sup> *·* P<sup>∗</sup>

*m*

 $\mathbf{v} \cdot \mathbf{F} dt$ 

) *· d*P

 $\sum_{\mathbf{F}dt}$ F*dt*

 $\int_0^1$ 

 $\overline{0}$ 

 $\begin{array}{r} \hline 1/2 \end{array}$ 

*pdp*

<sup>2</sup>*/*(2*m*)*.* (1)

*Fdt* (2)

<sup>2</sup>*/*(2*m*)*.* (1)

 $\longrightarrow$  (2)

Net Work in "Collision" is<br>
Met Work in "Collision" is

positive (generated) work - negative (absorbed) work.

$$
\Delta E = -E_a + E_g
$$

## where Ea = must be constructed to the East of the E<br>Sumptions: More assumptions:

- Leg is close to vertical
- $\begin{array}{c} \begin{array}{c} \end{array} \ \end{array}$ • Motion is close to horizontal
- E<sup>a</sup> is the energy absorbed in leg shortening and E<sup>g</sup> is • Speed is close to constant

### One shallow angle collision:



## Change in energy in collision:  $\Delta E = -E_a + E_g$  $\sim$  done equal to the amount previously absorbed. It is in the amount previously absorbed. It is



#### Passive Walking and rimless wheel (rolling polygon) Passive Walking and rimless wheel (rolling

Simplest model of passive-dynamic walking  $s$  implest model of  $\begin{array}{ccc} & & & s \end{array}$ 



#### Rimless wheel **with a collision of the collision of the right-**(rolling polygon)  $\sim$





## r = 0, eg = 0 Energetic cost of taking one step

$$
E_m = b\phi^2 v^2 m/2
$$

inefficiency, about 4

## Similarly for running.

(b) Passive running downhill  $(b)$ **SIMIIArly for runing<br>
(b) Passive running** 

f

$$
\sqrt{1 - E_m} = b\phi^2 v^2 m/2
$$

shevsky (1948). Consider running as a point mass colli-

(step length)  $/$  2(leg length)

and the mentioned in Tucker (1975) and a strike (1985) and a strike (1985) and a strike (1975) and a strike (1975)  $\frac{1}{2}$  $\mathcal{M}$  and  $\mathcal{M}$  are collisional cost of walking can be constructed as  $\mathcal{M}$  $\overline{\mathbf{v}}$  and  $\overline{\mathbf{v}}$ etc. We take the nominal (nearly constant) forward speed with gravitational energy supply. Balanced with gravitational energy supply.

#### Hodograph: trajectory of tip of velocity vector



### from constant *v* circle

#### Energy saving trick for walking: then land on leading leg *(eg = -1)* B *V***+** C old stance leg collision reduction factor i  $= J$ <br>=  $E_m / (bmv^2\phi^2/2)$  $\phi/2$ path **i** 11 i *1*  $\phi/2$ ii *1/4* new stance leg *V* **-** A

Pushoff with trailing leg *(eg = +1)*

#### Energy saving trick for running: then push off *(net eg = 0)* B *V***+** C collision reduction factor *= J* i path  $\phi/2$ ii i *1* ii *1/4*  $\phi/2$ new stance leg *V* **-** A Absorb first

Pseudo-elastic collision (no real elasticity).

#### Energy saving trick for running: then push off *(net eg = 0)* B *V***+** C collision reduction factor *= J* i path  $\phi/2$ ii i *1* ii *1/4*  $\phi/2$ new stance leg *V* **-** A Absorb first

Pseudo-elastic collision (no real elasticity).

Two morals:

 1) A sequence of collisions uses less energy than a single collision (for given deflection angle) II) A pseudo-elastic collision uses less energy than a plastic collision.

### Horse gallop: Seems to use both systems: Ba-duh-dump ba-duh-dump



#### Relation between brachiation and galloping



#### Relation between brachiation and galloping





#### Relation between brachiation and galloping



#### Somewhat odd result: csull.

 An infinite number of infinitely small collisions, each "orthogonal" to the path, tends to **perfectly elastic**, no matter the nature of the individual collisions (plastic or generative or in-between). er the n Turriudal Colli<br>Itween)



#### "Simultaneous" collisions **Colligions and the sequential and the net collision is**  $^{16}$ Cimultango aug" collisione **butulated by considing**  $\overline{\phantom{a}}$ i. *P1 \* P2 \* t* Lis<sup>"</sup> collision



$$
m\mathbf{v}^{+} = m\mathbf{v}^{-} + \mathbf{P}_{1}^{*} + \mathbf{P}_{2}^{*}
$$

$$
W = \Delta E = \frac{m}{2} ((v^{+})^{2} - (v^{-})^{2})
$$

$$
W_1 = \int dW_1 = \int_{t_1}^{t_2} \mathbf{v} \cdot \mathbf{F}_1 dt
$$
\n
$$
= \int_0^{\mathbf{P}_1^*} (\mathbf{v}^- + (\mathbf{P}_1 + \mathbf{P}_2)/m) \cdot d\mathbf{P}_1
$$
\n
$$
= \int_0^1 (\mathbf{v}^- + (p\mathbf{P}_1^* + q\mathbf{P}_2^*)/m) \cdot \mathbf{P}_1^* dp
$$
\n
$$
= \mathbf{v}^- \cdot \mathbf{P}_1^* \int_0^1 dp + \frac{\mathbf{P}_1^* \cdot \mathbf{P}_1^*}{m} \int_0^1 pdp
$$
\n
$$
+ \frac{\mathbf{P}_1^* \cdot \mathbf{P}_2^*}{m} \int_0^1 qdp
$$
\n
$$
= \mathbf{v}^- \cdot \mathbf{P}_1^* + |\mathbf{P}_1^*|^2/(2m) + (\mathbf{P}_1^* \cdot \mathbf{P}_2^*)s_o/m
$$

$$
W_1 \geq 0 : \mathbf{v}^- \cdot \hat{\lambda}_1 \geq 0 \text{ and } \mathbf{v}^+ \cdot \hat{\lambda}_1 \geq 0
$$
  

$$
W_2 \leq 0 : \mathbf{v}^- \cdot \hat{\lambda}_2 \leq 0 \text{ and } \mathbf{v}^+ \cdot \hat{\lambda}_2 \leq 0
$$

= × • <del>P</del>∗  $W_2 = \mathbf{v}^- \cdot \mathbf{P}_2^* + |\mathbf{P}_2^*|^2 / (2m) + \mathbf{P}_1^* \cdot \mathbf{P}_2^* (1 - s_o) / m$  $W_2 = {\bf v}^- \cdot {\bf P^*_2} + |{\bf P^*_2}|^2/(2m) + {\bf P^*_1} \cdot {\bf P^*_2}(1-s_o)/m$ 

 $\frac{1}{\sqrt{2}}$ 

#### "Simultaneous" collisions **Colligions and the sequential and the net collision is**  $^{16}$ Cimultango aug" collisione **butulated by considing**  $\overline{\phantom{a}}$ i. *P1 \* P2 \* t* Lis<sup>"</sup> collision



$$
m\mathbf{v}^{+} = m\mathbf{v}^{-} + \mathbf{P}_{1}^{*} + \mathbf{P}_{2}^{*}
$$

$$
W = \Delta E = \frac{m}{2} ((v^{+})^{2} - (v^{-})^{2})
$$

$$
W_1 = \int dW_1 = \int_{t_1}^{t_2} \mathbf{v} \cdot \mathbf{F}_1 dt
$$
\n
$$
= \int_0^{\mathbf{P}_1^*} (\mathbf{v}^- + (\mathbf{P}_1 + \mathbf{P}_2)/m) \cdot d\mathbf{P}_1
$$
\n
$$
= \int_0^1 (\mathbf{v}^- + (p\mathbf{P}_1^* + q\mathbf{P}_2^*)/m) \cdot \mathbf{P}_1^* dp
$$
\n
$$
= \mathbf{v}^- \cdot \mathbf{P}_1^* \int_0^1 dp + \frac{\mathbf{P}_1^* \cdot \mathbf{P}_1^*}{m} \int_0^1 pdp
$$
\n
$$
+ \frac{\mathbf{P}_1^* \cdot \mathbf{P}_2^*}{m} \underbrace{\int_0^1 qdp}_{s_o}
$$
\n
$$
= \mathbf{v}^- \cdot \mathbf{P}_1^* + |\mathbf{P}_1^*|^2/(2m) + (\mathbf{P}_1^* \cdot \mathbf{P}_2^*)s_o/m
$$

$$
W_1 \geq 0 : \mathbf{v}^- \cdot \hat{\lambda}_1 \geq 0 \text{ and } \mathbf{v}^+ \cdot \hat{\lambda}_1 \geq 0
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W_2 \leq 0 : \mathbf{v}^- \cdot \hat{\lambda}_2 \leq 0 \text{ and } \mathbf{v}^+ \cdot \hat{\lambda}_2 \leq 0
$$

$$
W_2 = \mathbf{v}^- \cdot \mathbf{P_2^*} + |\mathbf{P_2^*}|^2 / (2m) + \mathbf{P_1^*} \cdot \mathbf{P_2^*} (1 - \mathbf{s_o}) / m
$$

#### "Simultaneous" collisions **Colligions and the sequential and the net collision is**  $^{16}$ Cimultango aug" collisione **butulated by considing**  $\overline{\phantom{a}}$ i. *P1 \* P2 \* t* Lis<sup>"</sup> collision



 $W_2 \leq 0$  :  $\mathbf{v}^- \cdot \hat{\lambda}_2 \leq 0$  and  $\mathbf{v}^+ \cdot \hat{\lambda}_2 \leq 0$  $k = \frac{1}{2}$  for the case of a single impulse (Eqn. 36 weight) (Eq. 36 weight) :  $\mathbf{v} \cdot \lambda_2 \leq 0$  and

= × • <del>P</del>∗  $W_2 = \mathbf{v}^- \cdot \mathbf{P_2^*} + |\mathbf{P_2^*}|^2/(2m) + \mathbf{P_1^*} \cdot \mathbf{P_2^*}(1 - s_o)/m$  $W_2 = {\bf v}^- \cdot {\bf P^*_2} + |{\bf P^*_2}|^2/(2m) + {\bf P^*_1} \cdot {\bf P^*_2}(1 - \overline{s_o})/m$ 

fore force in another direction climbs above zero, then the The work depends on the order parameter s<sub>0</sub>.  $\Delta$  by successive use of  $\Delta$  by successive use of  $\Delta$ **Application to walking** For walking we take P<sup>∗</sup> W = 1200 An the impulse from the trailing order from the trailing order from the trailing order for the trailing order for ther distinguished limit. We can 2<br>2<br>2 de 2012 ≠ (v For the cases of the cases we can assume that the cases we can assume that the cases impulse only does with one sign. For definiteness  $\Lambda$  $\overline{\phantom{a}}$  $\frac{1}{2}$ er parameter s<sub>0.</sub> .<br>other dictinguiched limit ouler gischigaished minu. W = ∆E → 2n the order For the cases of main interest we can assume that each impulse only does with one sign. For definiteness of  $\mathsf{Ano}$ <sup>2</sup> is totally absorbto which the collision is simultaneous collision in the collision of the collision is simultaneous (see Fig. 9  $\mu$  and  $\kappa$  is  $\sigma$ . aer distinguished limit as would be the case impulses proportions of the impulses propor-The work depends on the order parameter so. Another distinguished limit.

# Back to walking





# Back to walking



#### Even within collisional/rolling model, energetics is sensitive to details.









# Punchline



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#### Even within collisional/rolling model, energetics is sensitive to details.

