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Why K? High order singularities and small scale yielding

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Abstract. Singular terms in the crack tip elastic stress field of order $\sigma \sim r^{-3/2}, r^{-5/2}, \ldots$ are often neglected, thus rationalizing the use of the K field, $\sigma \sim r^{-1/2}$, as the dominant term for fracture mechanics. We find the common explanation for neglecting the more singular terms in the series solution for the crack tip stress field unsatisfying. Further, the more singular terms *are* non-zero and are needed to understand the energetics of fracture, i.e, J and $\mathcal{G}$. Given that the singular terms are generally present, the rationale for the validity of the small scale yielding assumption (the basis of linear elastic fracture) is more subtle than any argument which depends on the *elimination* of terms with stress $\sigma \sim r^{-3/2}, r^{-5/2}, \ldots$. Our explanation for the validity of small scale yielding is as follows. First, with or without small scale yielding, the stress field outside of the nonlinear zone does contain more singular terms. In the limit as the nonlinear zone at the crack tip shrinks to zero size (SSY) we show that the $r^{-1/2}$ term in the Williams expansion dominates both the more singular and the non-singular terms in an annular region somewhat removed from this zone. Further, in this limit the magnitude of the $\sigma \sim r^{-1/2}$ term is almost entirely determined by tractions on the outer boundary. Our theory and examples are for representative problems in mode III anti-plane shear fracture. We expect, however, that the general results also apply to mode I and mode II fracture.

1. Introduction

Central to Linear Elastic Fracture Mechanics (LEFM) is the concept of the stress intensity factor introduced by Irwin [1]. This concept is commonly introduced by finding the asymptotic stress field at the tip of a planar crack using the complete Williams expansion [2a,b]. For a crack loaded under plane strain mode I conditions, the stress expansion in the neighbourhood of a crack tip has the form

$$\begin{aligned}\sigma_{ij} &= \sum_{m=-\infty}^{\infty} a_m r^{m/2} f_{ij}^{(m)}(\theta) \\ &= \cdots + a_{-3} r^{-3/2} f_{ij}^{(-3)}(\theta) + a_{-2} r^{-1} f_{ij}^{(-2)}(\theta) + \\ &\quad + \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} f_{ij}^{(-1)}(\theta) + a_o + a_1 r^{1/2} f_{ij}^{(1)}(\theta) + \cdots, \qquad (1)\end{aligned}$$

where (r, θ) is a local polar coordinate system at the crack tip and where $K_{\mathrm{I}} \equiv a_{-1}\sqrt{2\pi}$ is called the mode I stress intensity factor. The higher order singularities $r^{-5/2}, r^{-3/2}, r^{-2}$ etc. in (1) are usually eliminated from consideration, e.g. by Williams [2a], by one or another 'physical' arguments.

However, we find these arguments unsatisfying in that they are not rationally based, as we will discuss later. They are also questionable on the basis of their incorrect conclusion, that the more singular terms in the Williams expansion are in general not present. The presence of the more singular terms will be discussed in subsequent sections.

The central subject of our paper is not whether or not singular terms and non-singular terms always exist in the elastic field outside the nonlinear zone: *they do*. Our main point is this: given that these terms *do* exist, how is one to justify the small scale yielding approximation and how accurate is it? *Why and when does the $Kr^{-1/2}$ term in the stress field effectively dominate all other terms?*

The original motivation for this work was twofold:

(1) Many people incorrectly believe that the more singular terms do not exist. The existence of these terms is not pointed out in textbooks and usually not in the fracture literature, either. We wanted to highlight the presence of these terms and also show the flaws in the common arguments for their rejection.
(2) Since many researchers in and around fracture mechanics (e.g. [8]) and ourselves had doubts about either the dominance of the K field or the justification for this dominance, we wanted to find a more satisfying explanation.

Our hope is that this paper will help better justify SSY for both students and researchers.

OUTLINE OF THIS PAPER

In the next section we critique the common reasons for neglecting the more singular terms in the stress field expansion. We then review the concept of small scale yielding and the energetics of crack growth in the context of the existence of these terms.

We then set up a model mode III fracture problem with a nonlinear inner zone Ω. We use this model to demonstrate that the higher order singular terms in (1) generally do exist in a physical problem where Ω has finite but possibly small dimension when compared to relevant specimen geometry. We also show the relationship of these higher order terms to the underlying physical quantities (such as the dimension of Ω). These calculations lead to explicit conditions for satisfaction of the small scale yielding assumptions. Furthermore, we show how the full series given by (1) approaches the usual series (2) for any fixed (x, y) as the dimension of Ω goes to zero, assuming some smoothness in the inelastic behaviour in the crack tip region. Finally we discuss the energy release rate, weight functions, and matched asymptotics.

For mathematical simplicity and ease of presentation we restrict our analysis throughout to the special case of antiplane shear deformation where there is only one nontrivial displacement $w(x, y)$. We expect that a more complicated analysis would lead to similar results for mode I and mode II fracture.

2. Examination of classical reasons for ignoring higher singularities

The usual arguments for throwing away more singular terms in the stress field expansion are these.

(i) The strain energy in the region of the crack tip must be bounded.
(ii) The displacement in the region of the crack tip must be bounded.
(iii) Uniqueness of elastic solutions is lost if higher order singular solutions are allowed.
(iv) The solution for the stress field in the vicinity of an elliptical hole, in the limit as the aspect ratio goes to infinity, does not have such higher order singular terms.

These are all flawed reasons. For example, (i) and (ii) above are violated by many singular solutions in linear elasticity that are used to represent physical phenomena. Such solutions include: a line load on the surface of a half space, a point load in a full space, and a dislocation. All of these solutions have unbounded strain energy and the first two also have unbounded displacement. The displacement and energy singularities are not problematic in these solutions, however, because the solutions are not assumed to be applicable all the way to the singular point. Rather, the solution is used only at distances that are large compared to the region over which the model (a load with no spatial extent and linear elastic behaviour) is invalid. Likewise with fracture mechanics. Since no real material remains linearly elastic at arbitrarily large stresses, and no material can bear infinite stress or strains, there must be some region Ω surrounding the crack tip where the material behaviour deviates significantly from that of linear elasticity. Inside this region, the process zone, the material behaviour is such that the displacements and strain energy are bounded; outside this region, where linear elasticity is accurate, there is no singularity. Thus solutions that would have infinite displacement or strain energy if evaluated at the crack tip are not used at the crack tip. That is, the standard arguments about bounded diplacement or strain energy do not legitimately rule out the highly singular terms because these arguments make use of the linear elastic solution in a region where the solution is *a priori* known not to be valid.

The uniqueness assumption (iii) is also not well justified. Since the strain energy is bounded outside the process zone Ω, the uniqueness condition (iii) *is* satisfied in the elastic region $\mathcal{D}$ which consists of the material outside Ω, whether or not terms with high order stresses $\sigma \sim r^{-3/2}, r^{-5/2}, \ldots$, are in the series description of the elastic field. But whether or not an entire fracture problem has a unique solution then depends on the material behaviour inside the process zone. Uniqueness, if assumed to be valid for unknown material models in the process zone, is an additional postulate. Uniqueness of solutions could be added to the list of pure elastic-fracture assumptions, but it does not follow from the basic elasticity equations.

Although one way of getting a flat crack is as the limit of an infinite aspect ratio elliptical hole (iv), a real crack is not necessarily well described by this limit. A mathematically sharp crack, if that is what one chooses to study, may be found as the limit of any number of non-singular fields. Many such limits lead to high order singularities in the limit of zero nonlinear zone size. For example, a cohesive zone model with both tensile and compressive stresses which go appropriately to infinity as the cohesive zone is reduced in size will lead to a traction free crack solution that *does have* high order singularities.

Thus, the question remains: do the higher order 'singular' terms in (1) exist outside the process zone Ω? In other words, since arguments eliminating singular terms are unsatisfying, should the entire series (1) be used outside the inelastic region Ω? Further, if the entire series is needed outside Ω, what does happen to the highly singular terms of this series as the dimension of Ω goes to zero, as the crack appears increasingly like a mathematically sharp crack?

Small Scale Yielding

The importance of these questions is that they underlie the concept of Small Scale Yielding (SSY). Here is a description by Rice [3] of the SSY assumption and its basic role in LEFM.

> 'The utility of elastic stress analyses lies in the similarity of near crack tip stress distributions for all configurations. Presuming deviations from linearity to occur only over a region that is small compared to geometrical dimensions (small scale yielding), the elas-

tic stress-intensity factor controls the local deformation field. This is in the sense that two bodies with cracks of different size and with different manners of load application, but which are otherwise identical, will have identical near crack tip deformation field if the stress intensity factors are equal. Thus, the stress intensity factor uniquely characterizes the load sense at the crack tip in situations of small scale yielding, and criteria governing crack extension for a given local load rate, temperature, environment, sheet thickness (where plane stress fracture modes are possible), and history of prior deformation may be expressed in terms of stress intensity factors...'

One naturally assumes that if the stresses are accurately known on the boundary of a region that enclosed the crack tip, then the full (nonlinear, finite deformation, discontinuous etc.) force and deformation fields inside the region are determined. That is, one assumes that some kind of uniqueness assumption does hold for the crack tip material. The small scale yielding assumption for a given body is that it has an interior surface Γ, somewhat removed from the non-elastic crack tip zone, on which the traction is accurately given by that of the K field. That the K field describes the stress field in some region, to some reasonable degree of accuracy, is known as K dominance. Whether a given accuracy in the K field description at some radius is sufficient for reasonably accurately determining the nonlinear behaviour near the crack tip is a question that we do not address here. In order for two crack tip regions to be effectively loaded by the same far field boundary conditions, any pair of bodies of identical material composition for which K is used to characterize fracture should have interior surfaces at the same distance from the crack tip where the stress field is K dominated. Thus, the small scale yielding concept depends on the following two conditions:

(*) the K field is dominant over other terms of (1), both non-singular and more singular, in some annular region surrounding the crack tip outside Ω.

(**) Specimens which are compared using the small scale yielding assumption must have regions of K field dominance which have non-zero intersection. That is, a radius r_k exists so that the K field dominates the stress field on all such specimens at r_k. This situation is illustrated schematically for two specimens in Fig. 1.

The condition (*) requires some explanation. If the higher order singular terms do exist outside the inelastic zone Ω, how can the K field, which is less singular than these terms, dominate the near tip field? Indeed, this quandary makes the neglect of the higher singular terms appealing. For example, the expression for the stress field outside the inelastic zone (which is assumed to be very small compared with typical specimen dimensions) is often assumed to be of the form:

$$\sigma_{ij} = (2\pi)^{-1/2} K_I r^{-1/2} f_{ij}(\theta, 0) + a_1 r^{1/2} f_{ij}(\theta, 1)$$
$$+ \cdots + a_m r^{m-(1/2)} f_{ij}(\theta, m) + \cdots, \tag{2}$$

where the $f_{ij}(\theta, m)$ are functions of θ that do not depend on the geometry of the body so long as it has a flat crack. Some researchers have attempted to compute the amplitude of these higher order terms, i.e., $a_1, \ldots, a_m \ldots$ for $m > 0$ so as to improve on, or to confirm the SSY description of the near tip stress field (e.g. [4–7]). In fact, however, if the entire series (1) is valid outside the inelastic zone Ω, any attempt to estimate or calculate a_m 's in (2) is not only futile but incorrect as infinitely many terms with $m < -1$ are thus neglected.

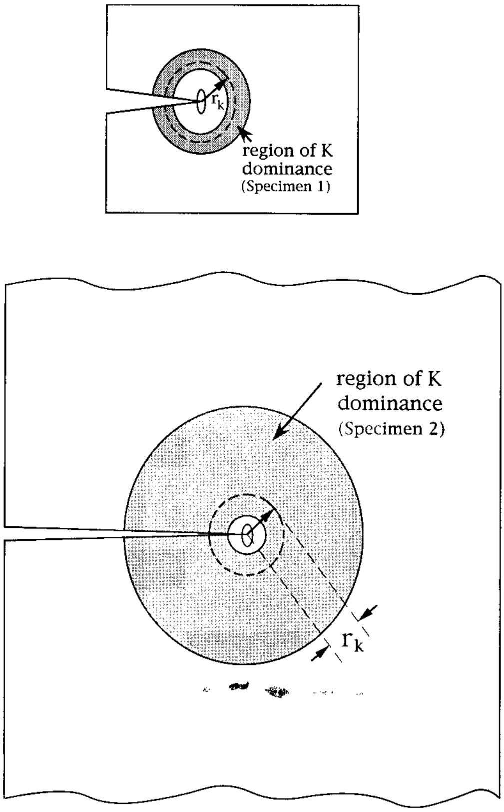

Fig. 1. Two different specimens (I and II) made of the same material are shown. The region in which the K field dominates is shaded in both cases. In order to assume that the two specimens have the same fracture behaviour that is characterized by K there must be a radius r_k that is in the region of K dominance for both specimens. The small elliptical regions at the crack tips represent a region where linear elasticity does not apply.

It is also interesting to note that the usual 'asymptotic' statement of the far field boundary condition of the SSY problem [3] is:

$$\sigma_{ij} \sim K_{\mathrm{I}} r^{-1/2} f_{ij}(\theta, 0) \quad \text{as } r \to \infty, \tag{3}$$

where one has implicitly assumed that appropriate rescaling of the spatial variables has taken place (e.g. the process zone Ω has a radius of unity under the rescaling). To improve the accuracy of the description of the inner field in Ω, it would seem more appropriate to replace (3) by

$$\sigma_{ij} \sim K_{\mathrm{I}} r^{-1/2} f_{ij}(\theta, 0) + g_{-1} r^{-3/2} f_{ij}(\theta, -1)$$
$$+ g_{-2} r^{-5/2} f_{ij}(\theta, -2) + \cdots \quad \text{as } r \to \infty. \tag{4}$$

Whatever the case may be, it does not make sense to use (2) as a far field boundary condition since terms like $g_1 r^{3/2} f_{ij}(\theta, m)$ dominate the K_{I} term in the far field as $r \to \infty$, thus violating the SSY assumption.

SMALL SCALE YIELDING, J, AND $\mathcal{G}$

It is a well known result in LEFM, assuming steady state conditions in some annular region around the crack tip, that the energy release rate $\mathcal{G}$ is equal to the path independent integral J [3]. Under SSY conditions, J is related to the stress intensity factors by

$$J = (1 - \nu^2)(K_{\rm I}^2 + K_{\rm II}^2)/E + K_{\rm III}^2/(2G), \tag{5}$$

where E is the Young's modulus, ν the Poisson's ratio and G the shear modulus. It is well known that (5) is not valid if the SSY condition is violated. However, if one believes that the expansion (1) is complete it follows that the exact calculation of J must be determined by all the coefficients a_m, whether or not small scale yielding is satisfied. However, if one neglects terms a_m with $m < 0$, a short calculation shows that J is calculated exactly by (5). How then is (5) considered as only approximate when SSY is not applicable? Either (5) is exact whether or not there is small scale yielding (and it is not), or the exact calculation of J depends on the inclusion of terms a_m with $m < 0$ (it does).

3. Anti-plane shear formulation

Figure 2 shows a crack in a finite specimen loaded under antiplane shear condition. Assume the nonelastic zone Ω surrounding the crack tip which is located at $x = 0, y = 0$ does not completely engulf the crack length. Let (r, θ) be a polar coordinate system attached to the crack tip. In this paper the term nonelastic zone Ω means that zone within which one or more of the usual assumptions regarding small strain linear elasticity break down. Outside Ω the linear elastic assumptions are assumed to be exactly satisfied. The problem is shown schematically in Fig. 2a, where C_1 and C_2 are circular boundaries enclosing the crack tip region. Between the circles, C_1 and C_2, the material is purely linearly elastic. Thus we may consider the purely elastic problem as shown in Fig. 2b. The radius of C_1, ρ, is chosen to be as small as possible with Ω still being contained in C_1. That is ρ is the outermost radius of Ω. The radius of C_2 is R, which is outside C_1, and is chosen to be as large as possible and still be contained in the specimen. The traction τ_r on these boundaries is assumed to be bounded so that

$$\tau_r(R, \theta) = f(\theta) = \tau_a F(\theta) \quad \text{(outer boundary)}, \tag{6a}$$

$$\tau_r(\rho, \theta) = h(\theta) = \tau_0 H(\theta) \quad \text{(inner boundary)}, \tag{6b}$$

where τ_a and τ_0 are reference stresses defined by the maximum values of f and h in $[-\pi, \pi]$. We shall assume that each of these prescribed tractions $f(\theta)$ and $h(\theta)$ are separately self-equilibrated. Physically, one may think of τ_a as a scalar measure of the level of applied stress on the boundary of the specimen and τ_0 as a scalar measure of the inelastic stress at the outer boundary of the nonelastic zone Ω.

For any actual specimen and confined nonlinear zone our replacement problem is exactly coincident with the solution to the original problem. The original singular and possibly nonlinear problem is reduced to the solution of a bounded linear problem. This is the central idea which allows our analysis to proceed.

Our approach is to first obtain the stress field inside the annulus region $A = \{(r, \theta) | \rho < r < R \;\&\; -\pi < \theta < \pi\}$. The governing equation in A, assuming linear isotropic elasticity, is

$$r^{-1}(r\phi_{,r})_{,r} + r^{-2}\phi_{,\theta\theta} = 0, \tag{7}$$

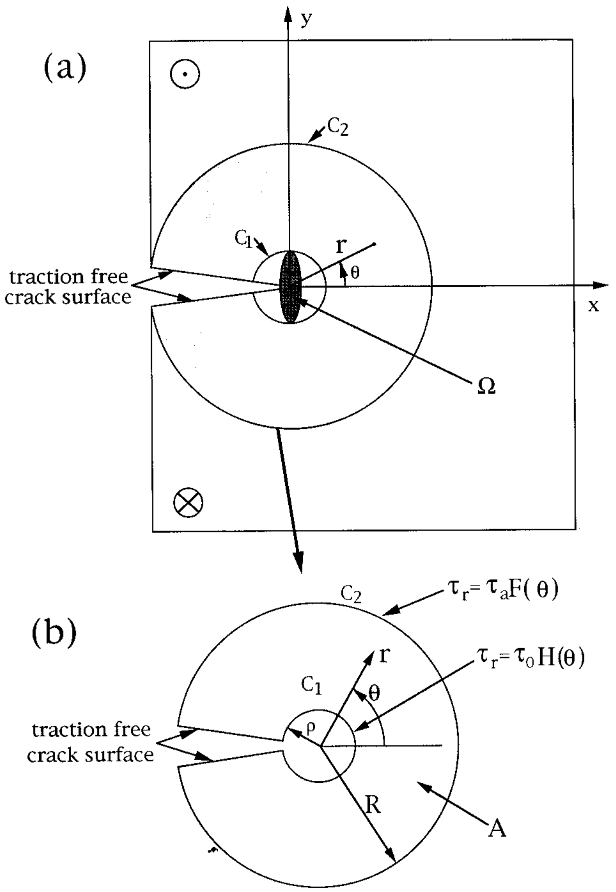

Fig. 2. (a) shows a specimen with two circles drawn. The inner circle C_1 is the smallest circle that can be drawn that totally encloses the nonlinear processes near the crack tip. The region of nonlinear processes is denoted Ω. The material behaviour outside C_1 is homogeneous, and linearly elastic. The radius of the outer circle C_2 is R and is chosen so that $R > \rho$ and as close as possible to the specimen dimensions. (b) shows our replacement for the speciment of Figure 1a. A new specimen is considered in the region A that has boundaries at C_1 and C_2. The tractions on these circular boundaries are the same as those occurring in the original specimen at the same locations. The tractions $\tau_r = \tau_o H(\theta)$ and $\tau_r = \tau_a F(\theta)$ are prescribed on the circles C_1 and C_2, respectively. A polar coordinate system (r, θ) is set up so that its origin is at the center of the inner circle, the 'crack tip'. Since we are considering only a mode III crack the only traction on these boundaries is τ_r.

where a comma (,) denotes partial derivative and ϕ is the stress function defined by

$$\tau_r = -r^{-1}\phi_{,\theta} \quad \text{and} \quad \tau_\theta = \phi_{,r}. \tag{8}$$

Equation (7) is solved subject to the boundary conditions (6) and the traction free condition on the crack faces, i.e.

$$\tau_r(r, \theta = \pi \text{ or } -\pi) = 0 \text{ for } \rho < r < R. \tag{9}$$

4. Anti-plane shear elastic solution

Closed form solutions of (7) subjected to the boundary conditions (6) and (9), are given by a sum of four fields labelled AI (antisymmetric, inner surface traction free), AO (antisymmetric, outer surface traction free), SI (symmetric, inner) and SO(symmetric, outer surface traction free)

$$\tau_i = \tau_a \cdot (\tau_i^{\mathrm{AI}} + \tau_i^{\mathrm{AO}} + \tau_i^{\mathrm{SI}} + \tau_i^{\mathrm{SO}}), \quad \text{where} \quad i = r, \theta. \tag{10}$$

The antisymmetric solutions are of greatest interest. They are

$$\begin{aligned} \tau_r^{\mathrm{AI}} &= \sum_{n=0}^{\infty} b_n^I (r/R)^{(2n-1)/2}[1 - (\rho/r)^{2n+1}] \sin\{(2n+1)\theta/2\}, \\ \tau_\theta^{\mathrm{AI}} &= \sum_{n=0}^{\infty} b_n^I (r/R)^{(2n-1)/2}[1 + (\rho/r)^{2n+1}] \cos\{(2n+1)\theta/2\}, \end{aligned} \tag{11a}$$

$$\begin{aligned} \tau_r^{\mathrm{AO}} &= \sum_{n=0}^{\infty} \mu^{-1} \epsilon^{(2n+3)/2} b_n^{II} (r/R)^{(2n-1)/2}[1 - (R/r)^{2n+1}] \sin\{(2n+1)\theta/2\}, \\ \tau_r^{\mathrm{AO}} &= \sum_{n=0}^{\infty} \mu^{-1} \epsilon^{(2n+3)/2} b_n^{II} (r/R)^{(2n-1)/2}[1 + (R/r)^{2n+1}] \cos\{(2n+1)\theta/2\}, \end{aligned} \tag{11b}$$

with

$$\epsilon = \rho/R, \mu = \tau_a/\tau_o,$$

and

$$b_n^I = F_n/[1 - \epsilon^{2n+1}] \quad \text{with} \quad F_n = (1/\pi) \int_{-\pi}^{\pi} F(\theta) \sin\{(2n+1)\theta/2\}\, d\theta,$$

and

$$b_n^{II} = -H_n/[1 - \epsilon^{2n+1}] \quad \text{with} \quad H_n = (1/\pi) \int_{-\pi}^{\pi} H(\theta) \sin\{(2n+1)\theta/2\}\, d\theta. \tag{11c}$$

The displacement w and the stresses $\tau_i^{\mathrm{SI}}, \tau_i^{\mathrm{SO}}$ due to symmetric loading are given in the Appendix.

By definition (11c), b_n^I , b_n^{II}, are order one terms (they are $4/\pi$ or less) since $F(\theta) \leqslant 1$, and $H(\theta) \leqslant 1$. $\tau_i^{\mathrm{AI}} + \tau_i^{\mathrm{SI}}$ is the stress field if traction is applied on the outer boundary (6a), and no traction is applied on the inner boundary

$$\tau_r(\rho, \theta) = 0, \tag{12}$$

whereas $\tau_i^{\mathrm{AO}} + \tau_i^{\mathrm{SO}}$ is the stress field if traction is applied on the inner boundary (6b), and no traction is applied on the outer boundary

$$\tau_r(R, \theta) = 0. \tag{13}$$

In the antisymmetric case, the series solution consists of terms that are fractional powers of r and there is displacement discontinuity across the crack faces. In the symmetric case the series solution consists of terms that are integer powers of r and there is no displacement discontinuity across the crack face.

The results above can be written in a more compact form by noting that the function $F(z) = f + iGw$ is analytic where $i = (-1)^{1/2}$, $z = x + iy$ and f is the usual stress function [3]. If we define $\tau(z) = \tau_y + i\tau_x = F'(z)$, then (11) can be rewritten as

$$\tau(z) = \tau_a(\tau^{\rm AI} + \tau^{\rm AO} + \tau^{\rm SI} + \tau^{\rm SO}), \tag{14a}$$

where

$$\tau^{\rm AI} = \sum_{n=0}^{\infty} b_n^I (z/R)^{(2n-1)/2} + \sum_{n=0}^{\infty} \varepsilon^{(2n-1)/2} b_n^I (\rho/z)^{(2n+3)/2}, \tag{14b}$$

$$\tau^{\rm AO} = \sum_{n=0}^{\infty} b_n^{II} \epsilon^{(2n+3)/2} \mu^{-1} (z/R)^{(2n-1)/2} + \sum_{n=0}^{\infty} \epsilon^{(2n+3)/2} \mu^{-1} b_n^{II} (R/z)^{(2n+3)/2} \tag{14c}$$

$$\tau^{\rm SI} = \sum_{n=1}^{\infty} (-i) c_n^I (z/R)^{(n-1)} - \sum_{n=1}^{\infty} (-i) \epsilon^{(n-1)} c_n^I (\rho/z)^{(n+1)}, \tag{14d}$$

$$\tau^{\rm SO} = \sum_{n=1}^{\infty} (-i) \epsilon^{(n+1)} \mu^{-1} c_n^{II} (z/R)^{(n-1)} - \sum_{n=1}^{\infty} (-i) c_n^I (\rho/z)^{(n+1)}. \tag{14e}$$

The coefficients c_n^I, c_n^{II} are defined by (A2) in the Appendix. The b_n^I , b_n^{II} are the same as before.

The results (10)–(11) or (14) above establish that all the terms involving higher order singularities (i.e., terms in the second sum in (14b–d)) are present in the annular region A. Note that the coefficient of the 0th term in (11a,b) is related to the mode III stress intensity factor $K_{\rm III}$ by:

$$K_{\rm III} = \frac{\overbrace{(2\pi)^{1/2} \tau_a F_o R^{1/2}}^{K_{\rm III}^{\rm applied}} \{1 - (H_o/F_o)\mu^{-1}\epsilon^{3/2}\}}{(1-\epsilon)} \tag{15}$$

where $K_{\rm III}^{\rm applied} = (2\pi)^{1/2} \tau_a F_o R^{1/2}$ is the usual stress intensity factor, namely, the strength of the $r^{-1/2}$ stress singularity in a specimen with a perfectly sharp structureless crack. Recall that, H_o and F_o are order one terms and μ^{-1} is approximately the ratio of the stress in the nonlinear zone to the applied traction on the outer boundary. From (A2), the symmetric terms do not contribute to the stress intensity factor.

Since we have not yet explicitly made any assumptions about material behaviour in Ω, τ_o and ρ and thus μ and ϵ are as yet unrelated.

If SSY is valid, or approximately valid, the size of the non-elastic zone must be governed by $K_{\rm III}$ and τ_o. A dimensional argument thus requires that ρ be proportional to $K_{\rm III}^2/\tau_o^2$. But since $K_{\rm III}$ is approximately proportional to $R^{1/2}\tau_a$ we have

$$\epsilon = C\mu^2, \tag{16}$$

where C is a positive dimensionless function of ϵ and is of order one. This means that

$$K_{\rm III} = \frac{K_{\rm III}^{\rm applied}\{1 - C\epsilon\}}{(1-\epsilon)}, \tag{17}$$

where constants of order one were absorbed in C. Only if the non-elastic zone is small compared with typical specimen dimensions, i.e., $\epsilon = \rho/R \ll 1$, is the local stress intensity factor K_{III} well approximated by the applied stress intensity factor $K_{\mathrm{III}}^{\text{applied}}$. Thus a necessary condition for SSY is that

$$\epsilon = \rho/R \ll 1. \tag{18}$$

Since it is common in the literature to neglect terms that are even function of θ, we consider first only the antisymmetric case where all displacements are odd functions of θ (i.e., we assume that the applied load and the material response are antisymmetric with respect to the crack line).

SIZE OF TERMS IN THE WILLIAMS EXPANSION

Let us examine the order of magnitude of each term of the series (14b) for a point in the annular region Ω' defined by

$$\Omega' = \{(r,\theta) \quad \text{with} \quad \epsilon^{\lambda_2}R < r < \epsilon^{\lambda_1}R, \quad -\pi < \theta < \pi\} \tag{19}$$

where λ_1 and λ_2 are real numbers such that

$$0 < \lambda_1 < \lambda_2 < 1 \tag{20}$$

Any point z in Ω' has the property $|z| = r = \epsilon^{\lambda}R$ for some $0 < \lambda < 1$. In other words, λ is a logarithmic measure of distance from the crack tip. The solution (11) or (14) can now be written in terms of λ instead of r. For the stress τ^{AI} ((14b), in the case of a traction free hole with loads on the outer boundary, C_2 given by (6a)), the solution is

$$\begin{aligned}\tau^{\mathrm{AI}} &= \sum_{n=0}^{\infty} b_n^I(\epsilon^{\lambda})^{(2n-1)/2} + \sum_{n=0}^{\infty} \epsilon^{(2n-1)/2} b_n^I (\epsilon^{1-\lambda})^{(2n+3)/2} \\ &= \cdots + \underbrace{b_1^I\epsilon^{\lambda/2}}_{r^{1/2}} + \underbrace{b_o^I\epsilon^{-\lambda/2}}_{r^{-1/2}} + \underbrace{b_o^I\epsilon^{-1/2+3(1-\lambda)/2}}_{r^{-3/2}} + \cdots.\end{aligned} \tag{21}$$

Examination of the terms in (21) reveals that the $r^{-1/2}$ term has the smallest power in ϵ. In other words, for any fixed λ in the interval (0,1), the K field, or the $r^{-1/2}$ term, dominates over all other terms in Ω' (singular and non-singular) in the series (14b) in the limit as $\epsilon \to 0$. By fixing λ we are keeping to a region that is removed both from the outer boundary and from the crack tip. Likewise, for (14c), the arrangement is

$$\begin{aligned}\tau^{\mathrm{AO}} &= \sum_{n=0}^{\infty} b_n^{II}\epsilon^{(2n+3)/2}\mu^{-1}(\epsilon^{\lambda})^{(2n-1)/2} + \sum_{n=0}^{\infty} \epsilon^{(2n+3)/2}\mu^{-1} b_n^{II}(\epsilon^{-\lambda})^{(2n+3)/2} \\ &= \cdots + \underbrace{b_1^{II}\mu^{-1}\epsilon^{5/2+\lambda/2}}_{r^{1/2}} + \underbrace{b_0^{II}\mu^{-1}\epsilon^{3/2-\lambda/2}}_{r^{-1/2}} + \underbrace{b_0^{II}\mu^{-1}\epsilon^{3/2-3\lambda/2}}_{r^{-3/2}} + \cdots.\end{aligned} \tag{22}$$

Note that in the second series (14c), the $r^{-3/2}$ term is always the dominant term in Ω' as $\epsilon \to 0$. To determine the dominant term of the sum of these two series, i.e., the full solution τ, we must compare

$$\epsilon^{-\lambda/2} \quad \text{and} \quad \mu^{-1}\epsilon^{3(1-\lambda)/2}. \tag{23}$$

Thus, the K field is dominant in Ω' if and only if

$$\epsilon^{(-1+\lambda)}\mu\epsilon^{-1/2} = \epsilon^{(-1+\lambda)}(\tau_a/\tau_o)\epsilon^{-1/2} \gg 1. \tag{24}$$

Although in our formulation τ_a and $\epsilon \equiv \rho/R$ can be considered as independent, for a given specimen μ is actually a function of ϵ. If we assume our specimen satisfies the condition (24), then, for sufficiently small ϵ, the K_{III} term ($n = 0$) dominates *every* term of the series solution of τ in Ω' since λ is in $(0, 1)$! In other words, the higher order singular terms (e.g. $r^{-3/2}$) are subdominant in Ω' because the amplitudes of the singularities are small; whereas the non-singular terms corresponding to $\sigma \sim r^{1/2}$, $\sigma \sim r^{3/2}$ etc. are subdominant in Ω' because Ω' is sufficiently close to the crack tip. We can use (24) to restate the small scale yielding condition.

Namely, λ in $(0, 1)$ must exist so that (24) is satisfied.

If one includes terms that are even functions of θ , the arrangement is

$$\begin{aligned}\tau^{\mathrm{SI}} &= \sum_{n=1}^{\infty}(-i)c_n^I(\epsilon^{\lambda})^{(n-1)} - \sum_{n=1}^{\infty}(-i)\epsilon^{(n-1)}c_n^I(\epsilon^{1-\lambda})^{(n+1)} \\ &= \cdots + \underbrace{c_1^I}_{r^0} + \underbrace{c_1^I\epsilon^{2(1-\lambda)}}_{r^{-2}} + \cdots.\end{aligned} \tag{25}$$

Likewise, for (14e), the arrangement is

$$\begin{aligned}\tau^{\mathrm{SO}} &= \sum_{n=1}^{\infty}(-i)\epsilon^{(n+1)}\mu^{-1}c_n^{II}(\epsilon^{\lambda})^{(n-1)} - \sum_{n=1}^{\infty}(-i)c_n^I(\epsilon^{1-\lambda})^{(n+1)} \\ &= \cdots + \underbrace{c_1^{II}\mu^{-1}\epsilon^2}_{r^0} + \underbrace{c_1^{II}\mu^{-1}\epsilon^{2(1-\lambda)}}_{r^{-2}} + \cdots.\end{aligned} \tag{26}$$

Clearly, the addition of the symmetric terms does not change our previous conclusion i.e., (25) since $\epsilon^{-\lambda/2} \gg c_1^I$ and $\mu^{-1}\epsilon^{3(1-\lambda)/2} \gg c_1^{II}\mu^{-1}\epsilon^{2(1-\lambda)}$.

If (24) is satisfied, a stronger statement can be made from dimensional considerations. If the K term ($\sigma \sim r^{-1/2}$) is dominant, then dimensional consideration would imply that

$$\rho = O[(K_{\mathrm{III}}^A/\tau_o)^2].$$

Using $K_{\mathrm{III}}^A = (2\pi)^{1/2}\tau_a F_o R^{1/2}$, the statement of SSY in dimensionless form is therefore

$$\mu^2 = C^{-1}\epsilon, \tag{27a}$$

where C is a constant of order one defined by (16). Equation (27a), together with the condition

$$\epsilon = \rho/R \ll 1, \tag{27b}$$

is the usual statement of SSY. Thus, we conclude that if (27a,b) are satisfied, then the K_{III}^A term is indeed dominant in Ω'. We now address the second condition (**) in the Introduction, that the specimen have overlapping regions where the $r^{-1/2}$ field is dominant. This condition is satisfied since Ω' includes all λ in $(0, 1)$ so that as long as (27a,b) is satisfied, there is still K dominance. However, it should be noted that, the smaller the ϵ, the more accurate is the K field and it is dominant over a larger region. In other words, the region of dominance Ω_{ϵ_1}

for specimens with a larger value of $\epsilon = \epsilon_1$ is included in the region of dominance of Ω_{ϵ_2} of the specimen with the smaller value of $\epsilon = \epsilon_2$. Finally, it should be noted that the series given by (14b) and (14c) is absolutely and uniformly convergent in A, so that the K field in Ω' is dominant over the sum of all the other terms!

5. An estimate of the error of the SSY assumption

The error made by the SSY assumption can be obtained using (14). Let us first ignore those terms in the series expansion that are even functions of θ. Let $z = \epsilon^{\lambda}$ be an arbitrary point in Ω'. Then

$$|\tau^I - b_o^I\epsilon^{-\lambda/2}|/|b_o^I\epsilon^{-\lambda/2}| < \sum_{n=1}^{\infty} e_n^I(\epsilon^{n\lambda}) + \sum_{n=0}^{\infty} e_n^I\epsilon^n\epsilon^{(n+1)(1-\lambda)} \tag{28}$$

where $e_n^I = |b_n^I/b_o^I|$. Note that e_n^I is an order one quantity and is in general a monotonic decreasing function of n for most $F(\theta)$. Let the maximum of e_n^I be denoted by e_I for all n. We anticipate $e_I \approx 1$ in practical situations. Equation (28) can be written as

$$\begin{aligned}\frac{|\tau^{\mathrm{AI}} - b_o^I\epsilon^{-\lambda/2}|}{|[b_o^I\epsilon^{-\lambda/2}]|} &< e_I\left[\sum_{n=1}^{\infty}(\epsilon^{n\lambda}) \quad + \quad \sum_{n=0}^{\infty}\epsilon^n\epsilon^{(n+1)(1-\lambda)}\right] \\ &= e_I\left[[\epsilon^{\lambda}/(1-\epsilon^{\lambda})] + \epsilon^{(1-\lambda)}/[1-\epsilon^{(2-\lambda)}]\right] \\ &\sim e_I\left[\epsilon^{\lambda} + \epsilon^{(1-\lambda)}\right].\end{aligned} \tag{29}$$

Similarly, using (14c), we have

$$\begin{aligned}|\tau^{\mathrm{AO}}/[b_o^I\epsilon^{-\lambda/2}]| &< \mu^{-1}\epsilon^{1/2}[\epsilon^{\lambda} + \epsilon^{(1-\lambda)}]\sum_{n=0}^{\infty} e_n^{II}\epsilon^{n(1-\lambda)} \\ &\sim e_{II}\mu^{-1}\epsilon^{1/2}[\epsilon^{\lambda} + \epsilon^{(1-\lambda)}],\end{aligned} \tag{30}$$

where $e_n^{II} = |b_n^{II}/b_o^I| = O(1)$ and $e_{II} = \max(e_n^{II})$. Combining (29) and (30) and using the SSY assumption (27a), an upper bound for the error E made by the SSY assymption is

$$E \sim b_1\epsilon^{\lambda} + b_2\epsilon^{(1-\lambda)}, \tag{31}$$

where the b_i's are constants of order 1. Note that the minimum error is of order $\epsilon^{1/2}$ and occurs at $\lambda \approx 1/2$ or $r \approx \rho^{1/2}R^{1/2}$ – the geometric mean of the plastic zone size and the specimen dimension! Furthermore, at this distance, i.e., $r \approx \rho^{1/2}R^{1/2}$, the error made by ignoring the $r^{1/2}$ term and the $r^{-3/2}$ terms are of the same order of $\epsilon^{1/2}$.

If we include the even terms in the series expansion, i.e., from (14d), we have

$$\begin{aligned}|\tau^{\mathrm{SI}}/[b_o^I\epsilon^{-\lambda/2}]| &< m_I\epsilon^{\lambda/2}\left[\sum_{n=1}^{\infty}(\epsilon^{(n-1)\lambda}) \quad + \quad \sum_{n=1}^{\infty}\epsilon^{(n-1)}\epsilon^{(n+1)(1-\lambda)}\right] \\ &= m_I\epsilon^{\lambda/2}\left[[1/(1-\epsilon^{\lambda})] + \epsilon^{2(1-\lambda)}/[1-\epsilon^{(2-\lambda)}]\right] \\ &\sim m_I\epsilon^{\lambda/2}\left[1 + \epsilon^{2(1-\lambda)}\right],\end{aligned} \tag{32}$$

where m_I is the maximum of $|c_n^I/b_o^I|$ for all n . Note that m_I is an order one quantity. Similarly, using (14e), we have

$$|\tau^{SO}/[b_o^I\epsilon^{-\lambda/2}]| < \mu^{-1}\epsilon^{\lambda/2}m_{II}\left[\epsilon^2+\epsilon^{2(1-\lambda)}\right]$$
$$\sim m_{II}\mu^{-1}\epsilon^{\lambda/2}\epsilon^{2(1-\lambda)}, \qquad (33)$$

where m_{II} is the maximum of $|c_n^{II}/b_o^I|$ for all n . Combining (29–33), an upper bound for the error E made by the small scale yielding assumption is

$$E \sim b_1\epsilon^{\lambda/2} + b_2\epsilon^{(1-\lambda)}, \qquad (34)$$

where the b_i's are constant of order 1. Note that the minimum error is of order $\epsilon^{1/3}$ and occurs at $\lambda \approx 2/3$ or $r \approx \rho^{2/3}R^{1/3}$. The error in the case of general loading can therefore be much greater than the case where all the even terms vanish.

6. Two examples

Before we attend to the question of the energy release rate, we consider the following two special cases of a mode III crack under antisymmetric loading:

(a) $\tau_o = 0$
(b) $w(r=\rho,\,\theta) = 0$

The first case (a) corresponds to a traction free hole (e.g. the material in Ω behaves like soft jello). The second case (b) corresponds to the other limit where the material in Ω is rigid (e.g. the nonlinear zone behaves like a much stiffer material). The actual behaviour of real material in Ω might lie roughly somewhere in between these two limiting cases.

The exact solution corresponding to case (a) has already been found and is given by τ_i^{AI} (14b). The exact solution of case (b) is found to be

$$\tau_r^{\mathrm{rigid}} = \tau_a\sum_{n=0}^{\infty} b_n(r/R)^{(2n-1)/2}[1+(\rho/r)^{2n+1}]\sin\{(2n+1)\theta/2\}, \qquad (35a)$$

$$\tau_\theta^{\mathrm{rigid}} = \tau_a\sum_{n=0}^{\infty} b_n(r/R)^{(2n-1)/2}[1-(\rho/r)^{2n+1}]\cos\{(2n+1)\theta/2\}, \qquad (35b)$$

where b_n is given by:

$$b_n = F_n/[1+\epsilon^{2n+1}] \qquad (35c)$$

in which F_n is given by (11c). In terms of z, $\tau(z) = \tau_y + i\tau_x$ is found to be

$$\tau^{\mathrm{rigid}} = \tau_a\left[\sum_{n=0}^{\infty} b_n(z/R)^{(2n-1)/2} - \sum_{n=0}^{\infty} b_n\epsilon^{(2n-1)/2}(\rho/z)^{(2n+3)/2}\right]. \qquad (35d)$$

The amplitude of the $r^{-1/2}$ term for case (a) and case (b) is $\tau_aF_0R^{-1/2}/[1-\epsilon]$ and $\tau_aF_0R^{-1/2}/[1+\epsilon]$, respectively. In the case of the hole, the local stress intensity factor

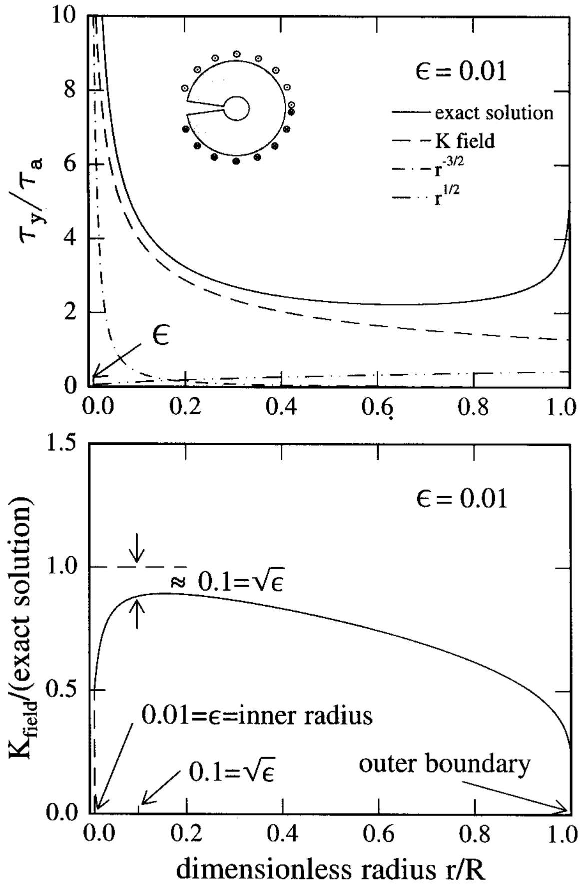

Fig. 3. The shear stress τ_y directly ahead of the crack tip is plotted for the case of the traction free hole from $r/R = \epsilon$ to $r/R = 1$ in Fig. 3a for $\epsilon = 0.01$ from the exact solution from (14b) (solid). The applied outer traction is a step function of θ. Also shown are the K field, i.e., the $\sigma \sim (r^{-1/2})$ term (dashed), the $(r^{-3/2})$ term (dot dash), and the $(r^{1/2})$ term (dot dot dash) in the Williams expansion. Figure 3b plots the ratio of the K field to the exact solution given by (14b). This figure shows that the solution for this particular problem is consistent with the general results that the K field is most accurate at $r =\approx \sqrt{\epsilon}R = 0.1$ where the fractional error of the K field compared to the exact solution is of order $\sqrt{\epsilon} = 0.1$.

increases by the amount $\epsilon\tau_a F_0 R^{-1/2}/[1-\epsilon] = \epsilon K_{\mathrm{III}}^A/[1-\epsilon]$ whereas for the case of the rigid plug, the local stress intensity factor decreases by the amount $\epsilon K_{\mathrm{III}}^A/[1+\epsilon]$. In both cases, the change in stress intensity factor is $O(\epsilon)$. Indeed, the two series solutions are practically identical except for a sign change and following the previous argument, we again can easily verify the dominance of K in Ω' for small ϵ.

Equations (14b) and (14c) imply that, for any fixed z in Ω', the solution approaches to the linear elastic solution with all higher order singular terms vanishing as ϵ approaches 0, However, the convergence to (2) is not uniform, since all terms of these series are of equal order when $r = \rho$.

To illustrate the ideas expounded above, we plot $\tau_y(r, \theta = 0)$ for the traction free hole where the traction on the outer boundary is τ_a sign(θ) or $F(\theta) = \mathrm{sgn}(\theta)$ from $r/R = \epsilon$ to $r/R = 1$ for the case of $\epsilon = 0.01$ (Fig. 3a). The K field, i.e., the $(r^{-1/2})$ term, the $(r^{-3/2})$ term and the $(r^{1/2})$ term in the Williams expansion are also shown in Fig. 3a. It is clear that the contribution from the $(r^{-3/2})$ term is as important as the $(r^{1/2})$ term in the region where the K field is dominant (e.g., $r/R \approx \epsilon^{1/2}$). In particular, as one approaches the nonelastic region, the $(r^{-3/2})$ term is of the same order as the $(r^{-1/2})$ term. In Figure 3b, we plot the ratio of the $(r^{-1/2})$ to the exact solution (11a). As predicted by our analysis, the minimum error is of order $\sqrt{\epsilon}$ and occurs at $r/R \approx \sqrt{\epsilon} = 0.1$.

7. J-Integral

The J-integral can be evaluated using (14) or (11). Since J is path independent outside Ω, a circular path is used. Note that if all the higher order singular terms are excluded, the value of the J-integral is always equal to

$$J = K_{\rm III}^2/(2G). \tag{36}$$

This is because the remaining nonsingular terms cannot contribute to the J-integral since these terms and their products with any other terms in the series (including the K terms) are bounded as $r \to 0$. Using the path independence of J and shrinking the radius of the circular path to zero gives (36). Equation (36) can also be verified by direct computation. On the other hand, if the entire series is used; terms like $r^{-3/2}$ and $r^{1/2}$ can cross multiply giving rise to terms with r^{-1} which lead to nontrivial contributions to J. After some computation using (11) and the definition of J, we obtained

$$J = (\pi R\tau_a^2/G)(d_o)^2 + 2\sum_{n=0}^{\infty} d_{n+1}e_n, \tag{37}$$

where the coefficients d_n and e_n are defined by (A4). If we recall $K_{\rm III}^A = \pi\tau_a F_o R^{1/2}$, then $J = K_{\rm III}^2/(2G)$ if and only if $\epsilon = 0$. Note that the even terms do not contribute to the J-integral.

Equation (37) shows that there are two additional contributions to the energy release rate due to the existence of the non-elastic zone. The first one is due to the change in the local stress intensity factor which is of order $O(\epsilon)$. The second one, also of order $O(\epsilon)$, is due to the interaction of the nonsingular terms and the higher order singular terms.

8. Weight function

The weight function method was introduced by Bueckner [9] to determine the stress intensity factors in cracked bodies that are linearly elastic. The weight functions are universal functions for a given crack configuration and body geometry. Once found, the stress intensity factors induced at the crack tip by any surface tractions can be computed using the weight functions and quadratures. Recently, higher order weight functions have been developed to evaluate the coefficients of the non-singular terms of the series expansion with the implicit assumption that terms in the series expansion of the stresses in actual crack bodies that are more singular than $r^{-1/2}$ do not exist near the crack tip [10]. For example, it is commonly believed that the inclusion of the non-singular terms permits a more accurate interpretation of stress data obtained at finite distances from the crack tip [5, 6, 7, 10]. However, the analysis we have presented indicates that the sole inclusion of the non-singular terms does not necessarily improve the accuracy of the stress field near the crack tip in real materials. Is it possible,

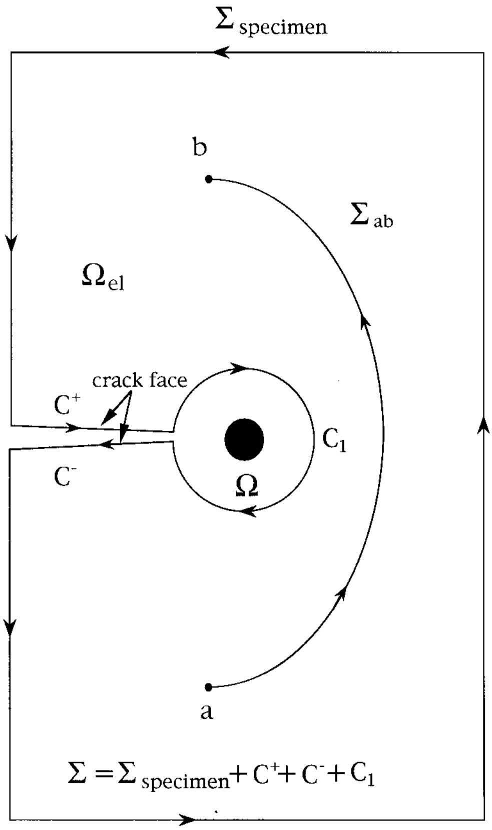

Fig. 4. A cracked body containing a non-elastic zone denoted by Ω. The material outside Ω is assumed to be linearly elastic, homogeneous, isotropic and is denoted by Σ_{el}. The traction-free crack faces are denoted by C^+ and C^- respectively. The curve Σ_{specimen} coincides with the specimen boundary. Σ_{ab} is a smooth curve in Σ_{el} with end points a and b. It will eventually be identified with the circular path C_1 enclosing the non-elastic zone from Fig. 1. The end points a and b for this special case correspond to points on the lower and upper crack face, respectively. Together the curves $\Sigma_{\text{specimen}} + C^+ + C^- + C_1$ make up a closed curve Σ outside the non-elastic zone that is used in (38).

therefore, to develop 'weight functions' to determine the coefficients of the singular and non-singular terms of the series representation of the stresses near the crack tip in a real material? It turns out that such a development is possible but the universality of the weight function method is lost in the process. Our derivation below follows the same approach as the work of Bueckner [9] and Sham [10]. As before, we restrict our analysis to that of a mode III crack. Consider the loading system in Fig. 4. For any closed curve Σ whose interior lies outside the non-elastic zone Ω, the reciprocal theorem, in the absence of body force, is

$$\int_{\Sigma} w_1 T_2 \, \mathrm{d}s = \int_{\Sigma} w_2 T_1 \, \mathrm{d}s, \tag{38}$$

where $(w_i, T_i), i = 1, 2$ are the displacement and traction for two elastic states and s is the arc length of the curve Σ. The traction T is related to the 'stress vector' $\boldsymbol{\tau} = (\tau_x, \tau_y)$ by $T = \boldsymbol{\tau} \cdot \mathbf{n}$ where $\mathbf{n}$ is the unit normal of Σ. Let $F_i(z)$ be the analytic functions defined by

$F_i(z) = f_i + iGw_i$. It can be shown that, for any open curve Σ_{ab} in Ω_{el} (see Fig. 4), where Ω_{el} is the material outside Ω, the functional $H(\Sigma_{ab})$ defined by

$$H(\Sigma_{ab}) = \int_{\Sigma_{ab}} w_1 T_2 \, \mathrm{d}s - \int_{\Sigma_{ab}} w_2 T_1 \, \mathrm{d}s \tag{39}$$

can be expressed in terms of F_i as

$$\begin{aligned} H(\Sigma_{ab}) &= (1/G)\Bigg(\Im\left[\int_{\Sigma_{ab}} F_2(z) F_1'(z)\,\mathrm{d}z\right] \\ &\quad - [\Im\{F_1(b)\}\Re\{F_2(b)\} - \mathrm{Im}\{F_1(a)\}\Re\{F_2(a)\}]\Bigg), \end{aligned} \tag{40}$$

where $\Re()$ and $\Im()$ are the real and imaginary parts of (), respectively. Let (w_1, T_1) be the elastic state in Ω_{el} caused by the traction T_1 acting on the boundary of the cracked body Σ_{specimen}. Note that the traction free boundary condition on the crack faces is satisfied by this elastic state and that $F_1'(z)$ has the form

$$F_1'(z) = \tau_1(z) = \tau_{1y} + i\tau_{1x} = \sum_{m=-\infty}^{\infty} a_n z^{(2n-1)/2}, \tag{41}$$

where we have assumed that the stress state is antisymmetric with respect to the x-axis so that the coefficients a_n are real. The subscript 1 in τ_{1y} indicates that τ_{1y} is the y component of the stress vector of elastic state 1. Let F_2 be given by

$$F_2(z) = \tau_2(z) = \tau_{2y} + i\tau_{2x} = A_k z^{(2k-1)/2}, \tag{42}$$

where A_k is a real constant and k is an integer. Note that F_2 satisfies the traction free boundary condition on the crack faces. Let Σ in (38) be identified with $\Sigma_{\text{specimen}} + C^+ + C^- + C_1$ in Fig. 4 and Σ_{ab} be identified with C_1, then (38) becomes

$$\begin{aligned} H(C_1) &= (1/G)\Im\Big(\int_{C_2} F_2(z) F_1'(z)\,\mathrm{d}z\Big) \\ &= \int_{\Sigma_{\text{specimen}}} w_2 T_1 \,\mathrm{d}s - \int_{\Sigma_{\text{specimen}}} w_1 T_2 \,\mathrm{d}s, \end{aligned} \tag{43}$$

where we have used the condition that $\Re(F_2) = 0$ on the crack faces and (40). However, $H(C_1) = (1/G)\Im(\int_{C_1} F_2(z) F_1'(z)\,\mathrm{d}z)$ can be easily computed using contour integration and is equal to $2\pi A_k a_{-k}/G$ so that (43) becomes

$$2\pi A_k a_{-k}/G = \int_{\Sigma_{\text{specimen}}} w_2 T_1 \,\mathrm{d}s - \int_{\Sigma_{\text{specimen}}} w_1 T_2 \,\mathrm{d}s. \tag{44}$$

Equation (44) shows that the coefficients a_k can be found by quadrature in terms of the 'weight function' $F_2 = A_k z^{(2k-1)/2}$ once the displacement and traction on the boundary of the specimen of the elastic state 1 is known. It is interesting to note that the coefficients of the singular terms are identified with positive integers k whereas the coefficients of the non-singular terms are identified with negative integers k. Note that the elastic fields corresponding to F_2 for negative k's have unbounded energy (if the crack tip region is included). It should be noted that our definition above of 'weight function' is not the same as that given by Bueckner [9] and Sham [10]. Their weight function is constructed using an $F_2(z)$ (Bueckner defined

$F_2(z)$ as the Fundamental field) that also satisfies the traction free boundary condition on the specimen boundary Σ_{specimen}. Specifically, for the problem above, Bueckner's $F_2(z) = A_k z^{(2k-1)/2} + F^*(z)$ where $F^*(z)$ is a regular field without body forces and traction on the crack faces and k is negative. Furthermore, it is chosen so that $F_2(z)$ satisfies the traction free boundary condition on the specimen boundary Σ_{specimen}. A field is regular if the series representation of F does not contain any terms that give rise to unbounded energy as r approaches zero. The difference between our derivation and theirs is that they are dealing with an idealized material that is linearly elastic all the way to the crack tip. This, together with the assumption of bounded energy, justifies their assumption that $F_1(z)$ is a regular field. To be more specific, let us consider a special case where the specimen boundary Σ_{specimen} is circular so that $\Sigma_{\text{specimen}} = C_2$ in Fig. 2. In this case the fundamental field of Bueckner F_2 is

$$F_2 = A_k z^{(2k-1)/2} - A_k R^{(2k-1)} z^{-(2k-1)/2} \tag{45}$$

which satisfies the traction free boundary condition on $r = R$. Note that, if F_1 is a regular field so that k in (41) is a non-positive integer, then

$$2\pi A_k a_{-k} = G \int_{\Sigma_{\text{specimen}}} w_2 T_1 \, \mathrm{d}s. \tag{46}$$

The second integral in (44) vanishes since the traction due to the fundamental field F_2 (45) on the specimen boundary is zero. However, F_1 in our case is not regular, in the sense that terms more singular then the square root singularity exist due to the presence of the nonelastic region Ω. If we use F_2 defined by (45) instead of that defined by (42), we will have

$$2\pi A_k(a_{-k} - a_{k-1}) = G \int_{\Sigma_{\text{specimen}}} w_2 T_1 \, \mathrm{d}s, \tag{47}$$

so that there are two unknowns a_{-k} and a_{k-1} and one equation. By excluding the term $F^*(z)$ in the definition of our fundamental field, we have been able to solve for a_k, with the disadvantage that the boundary term $\int_{\Sigma_{\text{specimen}}} w_1 T_2 \, \mathrm{d}s$ in (44) no longer vanishes. In other words, the displacement of state 1 must be computed on the external boundary if traction boundary condition is prescribed for this state. Thus the universality of the weight function method is lost.

The traditional weight functions allow the calculation of the K field and of the non-singular terms in the Williams expansion from the tractions on the boundary of the specimen with the assumption that the singular terms and the crack-tip part of the K field are negligible. The weight functions we have presented require that both the traction and the displacement on the boundary of the specimen be known. From this argument, the weight functions allow the determination of the complete Williams expansion and thus the complete elastic field up to the nonlinear zone.

9. Connection with asymptotic matching

Edmunds and Willis have provided a systematic refinement of the SSY approximation of a finite mode III crack in an elastic perfectly plastic material using the method of matched asymptotics [11]. Their goal was to provide explicit, highly accurate approximate solutions for crack problems, not to justify the dominance of a single (K) term in the asymptotic expansion. The important point is that Edmunds and Willis did include both the singular and non-singular terms in their inner and outer expansions. Here we shall give an elementary description of their matched asymptotic approach [11] in the context of the exact solution of the rigid plug (i.e., (35)) given above. It should be noted that the small parameter ϵ used by Edmund and Willis [11] is equivalent to $\epsilon^{1/2}$ used in this work. Let us introduce the following dimensionless variables:

$$\hat{z} = z/R = (x + iy)/R, \tag{48a}$$

$$\hat{w} = (\tau_a R/G) w(r/R, \theta), \tag{48b}$$

$$\hat{\tau} = \tau/\tau_a, \tag{48c}$$

$$\hat{r} = r/R. \tag{48d}$$

In terms of these normalizations, the rigid plug (the 'inelastic zone') is of radius $\epsilon \ll 1$. The outer region is $\hat{r} \gg \epsilon$. The outer solution in this region is obtained by choosing a fixed $\hat{z}$ outside the boundary layer, that is, $\epsilon \ll \hat{r} < 1$. The difference between the exact solution and the outer solution is small as long as $\hat{r} \gg \epsilon$. Since the material is linearly elastic outside $\hat{r} > \epsilon$, we have, for any fixed $\hat{z}$ outside the inelastic zone

$$\hat{\tau}_{\text{out}} = \sum_{m=-\infty}^{\infty} c_n (\hat{z})^{(2n-1)/2}, \tag{49}$$

where $\hat{\tau}_{\text{out}}$ denotes the stress field in the outer region. Note that all the singular terms are included as $\hat{z}$ is a fixed point outside the inelastic zone. In this approach one first finds an outer solution required to satisfy the outer boundary condition on $\hat{r} = 1$, i.e., $\hat{\tau}_r(\hat{r} = 1, \theta) = F(\theta)$. A simple calculation shows $\hat{r} = 1$ is satisfied if and only if

$$c_n - c_{-(n+1)} = F_n, \quad n = 0, 1, 2, 3, \ldots, \tag{50}$$

where F_n is defined by (11c). When carrying out this procedure one anticipates, assuming no unusual behaviour in the plastic zone, that the coefficients $c_n, n = -1, -2, \ldots,$ vanish as $\epsilon \to 0$. In other words, for any fixed $\hat{z}$ in the outer region, the outer solution converges to the usual elastic solution with bounded energy as $\epsilon \to 0$. Although many of the terms are expected to be small, at this point in the calculation none of the c_n are known. Only the combinations $(c_n - c_{-(n+1)})$ are known.

The inner solution is obtained by considering the inner limit in which $\hat{z} = \epsilon\eta$ with η fixed and finite as $\epsilon \to 0$. Here η is the inner variable and is of order one near the inelastic zone. The inner solution is required to satisfy the displacement boundary condition on $\hat{r} = \epsilon$, i.e., $\hat{w}(\hat{r} = \epsilon, \theta) = 0$. Since the material is linearly elastic outside $\hat{r} > \epsilon$, the stress $\hat{\tau}_{in}$ is exactly

$$\hat{\tau}_{in} = \sum_{n=-\infty}^{\infty} d_n \eta^{(2n-1)/2}, \tag{51}$$

where $\hat{\tau}_{in}$ denotes the inner solution. A simple calculation shows that the rigid plug boundary condition is satisfied if and only if

$$d_n = -d_{-(n+1)}, n = 0, 1, 2, 3, \ldots. \tag{52}$$

Physically, we anticipate that the coefficients $d_n, n = 0, 1, 2, \ldots$ vanish as $\epsilon \to 0$. That is, the coefficients of the positive power terms are small if ϵ is small. The vanishing of the negative power terms in the outer solution and of the positive power terms in the inner solution are not incompatible because the variables have been scaled.

To determine the coefficients c_n and d_n, one matches the outer and the inner solution in an overlap region which is defined by the intermediate limit $\hat{z} \to 0, \eta = \hat{z}/\epsilon \to \infty, \epsilon \to 0$. For example, $\hat{z} = \epsilon^{\lambda}, 0 < \lambda < 1$ satisfies this criterion. Matching $\hat{\tau}_{\text{out}}$ and $\hat{\tau}_{\text{in}}$ in this overlap region implies that c_n and d_n, $n = \cdots, -2, -1, 0, 1, 2, 3\ldots$, satisfy

$$d_n = c_n \epsilon^{(2n-1)/2} \tag{53a}$$

which gives

$$c_n - F_n = -d_n \epsilon^{(2n+3)/2} \quad n = 0, 1, 2, 3, \ldots, \tag{53b}$$

so that

$$c_n = F_n/[1 + \epsilon^{(2n+1)}], \quad n = 0, 1, 2, 3, \ldots, \tag{53c}$$

$$d_n = F_n \epsilon^{(2n-1)/2}/[1 + \epsilon^{(2n+1)}] \quad n = 0, 1, 2, 3, \ldots. \tag{53d}$$

The matching in this case is performed for all orders and the overlap region is the entire elastic region $\epsilon < \hat{r} < 1$ since the geometry we used is highly idealized (e.g. the boundary of the nonelastic zone is a circle of known radius) so that the matched solution is exact and is the same as that given by (35).

In general, the inner and outer solutions are expanded in terms of asymptotic series as in [11]. The inner expansion in many fracture mechanics problems is almost impossible to determine due to material or geometric nonlinearities in the nonlinear zone. Also, the boundary of the nonlinear zone is not known *a priori*. The matching procedure is much more complicated than what we have presented above. Furthermore, the extent of the overlap region may vary with the order of the perturbation theory. This simple analysis also points to an important fact, a series expansion without the more singular terms is not complete as the boundary condition outside the nonelastic zone can not be exactly satisfied.

What we have shown previously in this paper is that for small ϵ, there is an overlap region suitable for asymptotic matching where the K field dominates all other terms. This motivates the usual SSY assumption in which the K field is used as the far field boundary condition. In the language of matched asymptotics this is equivalent to setting all the coefficients c_n in (50) to zero, except c_0, when matching with the inner solution.

Besides the difference in the focus of the work of Edmunds and Willis [11] and ours, it should be noted that although the method of matched asymptotics is a powerful technique, mathematical justification of its applicablity is possible only in the limit of $\epsilon \to 0$, though very often asymptotic solutions provide a good approximation even when ϵ is relatively large, as demonstrated by [11]. Furthermore, asymptotic series are usually divergent and mathematical justification of the matching procedure is still in its infancy (although it works almost all the time). In this work, our description of the solution outside the nonelastic zone is exact for all $0 < \epsilon < 1$ and is valid for a wide range of material behaviour inside the nonelastic zone. However, we have not attempted, as others [11, 13, 14] have done, to relate the behaviour of the material inside the nonlinear zone to that of the elastic solution outside this zone since we will then have to specify the material behaviour in the nonelastic zone.

10. Discussion

It should be mentioned that existence of the higher order singular terms of the form $r^{-3/2}$, $r^{-5/2}, \ldots$, had been pointed out by the earlier work of Edmunds and Willis [11] and is implicit in a few of the exact solutions of any fracture mechanics problem with a nonelastic zone near the crack tip (e.g. the Dugdale model [12] and the problem of a mode III crack in an elastic-plastic material [13, 14]).

The above analysis indicates that the regular terms and the higher order 'singular' terms are of equal importance. For example, (31) implies that at $\lambda = \frac{1}{2}$, that is, $r \approx \epsilon^{1/2}R$, where the K field is most accurate, the errors introduced by ignoring the $r^{-3/2}$ and $r^{1/2}$ terms are of the same order. If a different choice of λ is selected (e.g. $\lambda = \frac{3}{4}$), then the K field is no longer the best approximation. Under such circumstances, one may attempt to include the next non-singular term (i.e. the term $r^{1/2}$) to the K field to improve the description of the stress field. However, the analysis we have presented shows that there is no guarantee that such an approximation is an improvement over the choice of $\lambda = \frac{1}{2}$. Also, if one attempts to further improve the near tip stress description by including the next nonsingular term (i.e. $r^{3/2}$) to the K field, the approximation will be inconsistent since it is likely that the next dominant term in the actual series representation of the stress field is not the term $r^{5/2}$ but rather the term $r^{-3/2}$.

Equation (22) points to an important fact, i.e., in problems involving residual crack-tip stresses with no applied external loads, the dominant term outside the nonlinear zone is the $r^{-3/2}$ term and not the $r^{-1/2}$ term. This is the term which coincides with the dominant term of the weight-function for the elastic body.

About ten years ago when we did the bulk of this research we discussed these issues with many people. The reactions we got were generally of three types. One set of people, experts in the field, were well acquainted with our issues and seemed to feel that the questions were near trivial. They felt that since small scale yielding is in fact often an accurate approximation, ignorance about subtleties in its rationale does not harm its usefulness. A second set of people seemed closer to ourselves in sentiment. They either doubted the reasoning behind or the correctness of SSY and felt in need of more rigorous justification of the type we hope our paper supplies, at least in part.

Finally, some people needed more convincing. They felt that the higher order singular terms *do not* exist. So they could not accept our central question: given that the more singular terms *do* exist, why can they ever be neglected and why does the $Kr^{-1/2}$ term dominate?

11. Conclusions

We have been critical of the classical physical reasons for neglecting the terms in crack solutions with stresses more singular than $r^{-1/2}$ in the stress field. We have found that, although these terms do exist, we expect them to be dominated by the K field in a region that is far from the inner nonlinear zone if the nonlinear zone is small in spatial extent. The K term dominates the non-singular terms in a region far (inwards) from the outer boundary. Fortunately for the happy continued use of linear elastic fracture mechanics, these two regions are often expected to have some overlap where the K field dominates all other terms.

Somewhat disatisfying to us is that our success in justifying the dominance of the K term is based entirely on mathematical reasoning. We have not been able to find a simple physical-like argument to replace the unsatisfying 'physical' arguments about finite displacement, strain energy and so on.

Here are the main results.

1. The stress field outside the plastic zone under SSY is of the form:

$$\begin{aligned}\sigma_{ij} &= \sum_{m=-\infty}^{\infty} a_m r^{m/2} f_{ij}^{(m)}(\theta) \\ &= \cdots + a_{-3} r^{-3/2} f_{ij}^{(-3)}(\theta) + a_{-2} r^{-1} f_{ij}^{(-2)}(\theta) \\ &\quad + \frac{K_{\rm I}}{\sqrt{2\pi r}} f_{ij}^{(-1)}(\theta) + a_o + a_1 r^{1/2} f_{ij}^{(1)}(\theta) + \cdots \end{aligned} \tag{1}$$

   so that the higher order singular terms cannot be neglected. Any attempt to improve on the K field should include the possible effects of the higher order singular terms as well as the non-singular terms.
2. One precise quantitative statement of SSY is:
   $\epsilon = O(\rho/(K_{\rm III}^A/\tau_a)^2) \ll 1,$
   where ρ is the characteristic dimension of the plastic zone. All specimens satisfying this condition are guaranteed to have a region Ω' inside which the K field computed using the outer boundary conditions is dominant. The region of dominance Ω' for specimens with a larger value of ϵ is included in the region of dominance of Ω' of the specimen with the smaller value of ϵ.
3. The SSY approximation is most accurate at $r \approx \rho^{1/2} R^{1/2}$, the geometric mean of the plastic zone size and the specimen dimension.
4. The path independent J-integral can be determined exactly in terms of the coefficients of the singular and the non-singular terms.

We expect that the results presented here for mode III fracture can be extended to the case of plane strain, though the calculations would be more complicated.

Appendix: symmetric solutions

$$\begin{aligned}\tau_r^{\rm SI} &= \sum_{n=0}^{\infty} c_n^I (r/R)^{(n-1)}[1 - (\rho/r)^{2n}]\cos\{n\theta\}, \\ \tau_\theta^{\rm SI} &= \sum_{n=0}^{\infty} c_n^I (r/R)^{(n-1)}[1 + (\rho/r)^{2n}]\sin\{n\theta\},\end{aligned} \tag{11c}$$

$$\begin{aligned}\tau_r^{\rm SO} &= \sum_{n=0}^{\infty} \mu^{-1}\epsilon^{(n+1)} c_n^{II} (r/R)^{(n-1)}[1 - (r/R)^{2n}]\cos\{n\theta\} \\ \tau_\theta^{\rm SO} &= \sum_{n=0}^{\infty} \mu^{-1}\epsilon^{(n+1)} c_n^{II} (r/R)^{(n-1)}[1 + (r/R)^{2n}]\sin\{n\theta\},\end{aligned} \tag{A1}$$

where

$\epsilon = \rho/R$

$$c_n^I = -F_n^E/[1-\epsilon^{2n}] \quad \text{with} \quad F_n^E = (1/\pi)\int_{-\pi}^{\pi} F(\theta)\cos\{n\theta\}\,\mathrm{d}\theta,$$
$$c_n^{II} = H_n^E/[1-\epsilon^{2n}] \quad \text{with} \quad H_n^E = (1/\pi)\int_{-\pi}^{\pi} H(\theta)\cos\{n\theta\}\,\mathrm{d}\theta. \qquad \text{(A2)}$$

The displacement field w is found to be

$$\begin{aligned}(G/R\tau_a)w(r,\theta) = & \sum_{n=0}^{\infty} D_n(r/R)^{(2n+1)/2}\sin\{(2n+1)\theta/2\} \\ & + \sum_{n=0}^{\infty} E_n(r/R)^{(2n+1)/2}\sin\{(2n+1)\theta/2\} \\ & - \sum_{n=1}^{\infty} D_n^E(r/R)^{(n)}\cos\{n\theta\} \\ & - \sum_{n=1}^{\infty} E_n^E(r/R)^{(n)}\cos\{n\theta\}, \end{aligned} \qquad \text{(A3)}$$

where

$$\begin{aligned} D_n &= 2d_n/(2n+1) && \text{with } d_n = b_n^I + \epsilon^{(2n+3)/2}\mu^{-1}b_n^{II}, \\ D_n^E &= d_n^E/n && \text{with } d_n^E = c_n^I + \epsilon^{(n+1)}\mu^{-1}c_n^{II}, \\ E_n &= 2e_n/(2n+1) && \text{with } e_n = b_n^I\epsilon^{(2n+1)} + \epsilon^{(2n+3)/2}\mu^{-1}b_n^{II}, \\ E_n^E &= e_n^E/n && \text{with } e_n^E = c_n^I\epsilon^{(2n)} + \epsilon^{(n+1)}\mu^{-1}c_n^{II}. \end{aligned} \qquad \text{(A4)}$$

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