# Discussion

*SCALING RULES (similar to rules for a rolling disk)*

<u>Mass Scaling:</u> If all the mass densities of all the parts is doubled, the equations are absolutely unaffected. Thus a motorcycle plus rider is perhaps like a heavy bicycle with a young child riding it.

<u>Speed and Length Scaling:</u> If the dimension of <u>every</u> part on the bicycle (including the rider) is multiplied by a given factor, the stability will be the same as long as the speed is correctly scaled:

$$\frac{V^2}{gL} = \text{constant},$$

where $L$ is a typical length such as wheelbase or c.m. height. A larger size is thus like a smaller $V$ — at a given speed, an otherwise standard bicycle which is too large will weave, and one which is too small will 'capsize' (i.e. lean increasingly to one side without weaving). If $V_{min}$ is a bicycle's lowest stable speed, $V_{min}^2/gL$ is a sort of 'index of merit' for bicycle design — the smaller the better. However as there is no rule for defining $L$, slight differences are only meaningful if bicycle shape varies little.

Note that a lengthened bicycle such as a tandem is similar to a lowered bicycle.

<u>Time Scaling:</u> When similar bicycles of different sizes have the appropriate velocities defined above, their stability or instability will be similar. (That is, the two fourth-order polynomials will have proportional solutions.) However, everything will happen slower with the larger bicycle: the observed period of any oscillation *when the speed is correctly scaled* is $L/V$ or $\sqrt[2]{L/g}$.

<u>Force Scaling:</u> The horizontal ground–contact forces will scale like $(FL/MV^2) = f(gL/V^2)$, which approaches a constant at sufficiently high velocities (i.e., when gravity may be ignored and an oscillation takes place over a fixed distance).

***** y many people including ourselves, and has been predicted by a large number of theoretical analyses (including those mentioned above).

In contrast, a bicycle which is <u>unstable</u> may be expected either to weave back and forth to a greater and greater extent, or to lean increasingly to one side or other. The loss of balance may take place either slowly (thus permitting rider intervention) or quickly.

Note that a <u>stable</u> bicycle need not be slow-responding or require large handlebar torques. To understand this, consider a balsa-wood pendulum hanging from a pivot: this is unquestionably stable according to the above definition, yet it may quickly and easily be displaced.

To solve for the no–hands motion of a bicycle we can eliminate <u>one</u> variable (either $\chi$ or $\psi$) from the above second-order differential equations of lean and steer, which leaves a single fourth order eqation for the <u>other</u> variable. This equation is the same whichever variable is kept, so we simply present it as a differential operator:

$$\left(M_{\chi\chi}\frac{d^2}{dt^2}+K_{\chi\chi}\right)\left(M_{\psi\psi}\frac{d^2}{dt^2}+C_{\psi\psi}\frac{d}{dt}+K_{\psi\psi}\right)-\left(M_{\psi\chi}\frac{d^2}{dt^2}+C_{\psi\chi}\frac{d}{dt}+K_{\psi\chi}\right)\left(M_{\chi\psi}\frac{d^2}{dt^2}+C_{\chi\psi}\frac{d}{dt}+K_{\chi\psi}\right)$$

or

$$A\frac{d^4}{dt^4} + B\frac{d^3}{dt^3} + C\frac{d^2}{dt^2} + D\frac{d}{dt} + E = 0 \quad ,$$

where the coefficients $B, C, D, E$ are polynomial functions of the velocity (which at any given speed are simply constants like $A$); see APPENDIX C. These coefficients are each rather complex combinations of bicycle parameters.

To study stability, we have considered several options:

— Solve for the motion of a given bicycle using a computer, and see whether it falls down. The trouble with this approach is that there are so many quantities needed to specify a bicycle — ten or twenty design parameters — that to explore non-standard designs thoroughly (say five or ten values of each parameter) might require millions or even trillions of computer runs. Even if it were practical to perform these, the results would be difficult to absorb.

— Solve the equations analytically, i.e. in terms of sines, cosines, and exponentials. Unfortunately this is extremely difficult, and the results are so messy that one can not see general truths — numbers describing a specific bicycle and specific initial motion must be substituted, leading to the same situation as above.

— Realize that we don't really care about the *particular* motion arising from a *particular* disturbance, but rather whether *all leaning and steering motions* eventually damp out or not. This property of solutions to the equations of motion can be found out from the *eigenvalues* alone, that is from the roots of the fourth-order *characteristic polynomial:*

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \; .$$

If all the roots (eigenvalues) have negative real parts, then all possible solutions will involve decaying exponentials, and $\chi$ and $\psi$ will always return to zero. The trouble is that even the roots alone are hard to find, unless we use a computer on specific numerical cases. So this is still not really promising.

# CHRONOLOGICAL COMPARISON OF THE LINEARIZED EQUATIONS OF MOTION FOR THE BASIC BICYCLE MODEL

In Chapter III of this thesis the linearized equations of motion for a Basic bicycle model were presented. This chapter chronologically compares other authors' linearized equations of motions for a bicycle to those we derived in Chapter III. When it was not possible to directly compare equations, we have tried to simplify other authors' equations to represent the linearized equations of the Basic bicycle model, and/or we simplified the equations from Chapter III to represent the model studied by the particular author. In some cases neither approach was possible due to the complexity of the other equations. For these cases we have given a brief description of the their equations, and when possible have commented on the likelihood of correctness.[1]

The purpose of comparison was to gain confidence in our derived equations and to see which studies could legitimately be used without modifications. Knowing if other authors' equations agree with ours also gives us a basis for evaluating their conclusions. It is from the results of the numerous comparisons given in this chapter, that we developed and tailored what we feel is the easiest way to correctly derive the linearized equations of motion, as presented in chapter III.

The equations of motion of a bicycle have mainly been derived either from Lagrange's equations, or using Newton's Laws on the individual rigid bodies which make up a bicycle. Chapter III describes the derivation of the linearized equations of motion using Lagrange's equations with nonholonomic constraints. Döhring [1955] derived an equiva-

[1] This chapter includes some of the conversion notations required to make our comparisons.

lent set of linearized equations using Newton's Laws. Weir [1972] gives a four degree of freedom Newtonian derivation using vector notation. To our knowledge we are the first to derive correct linearized equations of motion for a fully general Basic bicycle model with Lagrangian methods. It seems that using Lagrange's equations is a simpler approach because it eliminates the requirement of solving complicated simultaneous equations representing the force and moment balance on the 4 rigid bodies that make up the Basic bicycle model.[2] We mention however, that the Lagrangian approach suffers because many students are not exposed to Lagrangian dynamics at the undergraduate level, especially for systems with nonholonomic constraints. In this sense, the Basic bicycle model could be used to introduce the subject, having general appeal and pedagogical meaning.

On the whole, the previous literature concerning the equations of motion suffers from 3 major flaws:

1) Some derivations seem impenetrable. This results from leaving out steps, from choosing notation which is not well suited to the job, from using roundabout procedures when more direct ones are possible, from not simplifying the results afterwards, and from not explaining their physical significance. The resulting equations are often far too complicated to use, except numerically. Some equations are so long that it takes several pages just to define the coefficients. Most of these studies do not enhance the reader's understanding of bicycle motion.

2) Few or no comparisons were made to works by previous authors, so their correctness was not known, and earlier results were 'lost'. Only one author explicitly stated that he had compared his equations to a previous author's.

[2] See for example Roland [1971].

3) The models used by some authors have ignored major stability-related design parameters. Some lack steering axis tilt, have only point masses, or make other assumptions restricting distribution and location of mass.

What follows is a chronological comparison of all papers found in which equations of motion for bicycles (or motorcycles) are presented. Because we feel that failure to compare to others' equations (especially when they are cited as a reference) is inexcusable, we have noted each author's comments on previous works. In our comparisons we found that in fewer than half of the papers do the equations of motion resemble those derived in Chapter III. Of the papers discussed, we found that exactly two derived fully general and perfectly correct results (one of which was adopted by another investigator). Several more were either a little less general or had minor errors which an alert reader might catch. A number of others were too complicated to check in full (but some of them raised some questions we could not answer). Finally, several are just plain wrong.

## Results of Chronological Comparison of Linearized Equations of Motion[3]

*Bourlet, 18??*

Bourlet published a paper containing mathematical analyses of bicycle stability based on a "complicated differential equation" which is not derived or published in his treatise. His paper includes a rider controlled and "hands- off" analysis and was criticized by

[3] Comparisons to works by Whipple [1899], Carvallo [1901], Sommerfeld and Klein [1903], and Döhring [1955] were performed mainly by Dr. Jim Papadopoulos, whose results are summarized here. Some of his understanding and commentary on other comparisons are contained in other parts of this chapter.

Whipple (1899) for making assumptions about the rider that were "so artificial that the results have little interest." Because Bourlet did not publish his equations of motion no comparison of his equations is possible. Results from his analysis seem to be sketchy based on Whipple's discussion.[7]

*McGaw, 1898*

McGaw in 1898, according to Whipple (1899), discussed steady motion which Whipple claims is a special case of his analysis (see (4), page 325). Whipple noted the difference between himself and Mcgaw and attributed it to an error in the application of virtual work by McGaw. Any work done by Mcgaw is assumed then to be contained in Whipple's analysis. We have not yet looked at McGaw's paper.

*Whipple, 1899*

The first to formally derive a fully general and scholarly set of equations for the Basic bicycle model was Whipple in 1899. He treats the front and the rear parts symmetrically throughout the derivation. He derived nonlinear governing equations of motions for a Basic bicycle model with an active (leaning) rigid rider, and then linearized about the vertical equilibrium configuration. His equations of motion can be found as eq. XIV, eq. XV, and eq. XIII in his paper on pp. 321-323, but not all terms are defined. The equation are restated more clearly and explicitly in matrix form or p. 326. We also note that the figure defining some of his variables is at the end of the bound volume containing his paper.

It is most convenient to compare Whipple to Döhring [1955] since similar axis orientation is used. The equations on page 326 are in the form of the $3 \times 3$ matrix which operates

[7] See Whipple (1899), page ***.

on his variables $\phi$, $\phi'$, $\tau$, where $\lambda$ is the derivative operator $\frac{d}{dt}$. There are a few evident typos: the first term of the second row should have $\lambda^2$ not $\lambda_2$; and the third column second row should have $W\gamma$, not $W'\gamma$) .

We found his notation to be more difficult to understand than most and therefore give some details about the comparison. In his notation,

$$\phi = \frac{\psi}{\sin\theta}$$

and

$$\phi' = \frac{\psi}{\sin\theta} - Q$$

where $\psi$ is the lean of the rear frame and $Q$ is the steer angle. (In our notation the lean of the rear frame is $\chi_r$ and $\psi$ is the steer angle.) The last equation in his matrix defines $\tau$ as a function of $\phi$ and $\phi'$ and allows one to eliminate $\tau$ from the first two equations of the matrix. Doing so, one finds the first equation is in complete agreement with our lean equation when $\phi$ and $\phi'$ are written in terms of $\psi$ and $Q$. The second equation of the matrix, when it is corrected and then multiplied by $\frac{\mu\mu'}{b\cos\theta}$, we find agrees completely with Döhring's [1955] equation (31). As is explained later, Döhring's equation (31) is a linear combination of our lean and steer equation and thus Whipple's linearized equations are in complete agreement with those presented in Chapter III. His work, which is as sophisticated as almost any later investigation, was evidently done for his degree from Trinity College, Cambridge University.

Overall, the definitions of Whipple's variables are difficult to decipher and make his paper difficult to read, but his equations appear to be rigorously derived and are fully general when compared to those given in Chapter III. Whipple is critical of McGaw's

[1898] study of tricycles, and Bourlet's [pre 1896] study of bicycles, neither of which have we read.

*Carvallo, 1901*

Carvallo [1901] wrote 300 generally lucid pages on the stability of monocycles (rider inside a single wheel) and bicycles. Only the second part of the monograph, which won a prestigious prize, concerns us. In it he modifies Lagrange dynamics to deal with rolling hoops and bicycles (we were not able to tell if his method is a new way of dealing with non-holonomic constraints). We are concerned primarily with section V on no-hands stability. The equations where each term was derived are laid out on pp. 100-101, and restated in condensed form on p. 103. The equations are exactly analogous to ours, one for lean and one for steer.

Although we could not find where Carvallo said this, it appears that his bicycle has two identical heavy wheels, the rider and frame are considered a single unit, and the mass of the front assembly is at the center of the front wheel and its inertia properties are those of the wheel. This is not an unreasonable idealization if the handlebars are not heavy and are positioned *on* the steering axis as was common in designs of that day. Technically, for such a design the mass of the handlebars and straight part of fork can then be considered part of the rear frame.

We find that Carvallo's equations (for a bicycle with massless forks and handlebars) agree exactly with ours. Most quantities are defined in the text, but the reader should note that the wheel inertias are defined relative to their ground contact, i.e. $C_1$ is for spin about a diameter, $A_1$ is for lean (i.e., $A_1 = C_1 + \mu_1 R^2$), $B_1$ is for rolling about the contact point (i.e., $B_1 = I_p + \mu_1 R^2$). $S = \frac{V}{R}$ is the wheel rotation rate. Carvallo makes no reference to other works, which is not surprising as his research was evidently performed in 1898.

*Sommerfeld and Klein, 1903*

Sommerfeld and Klein (S & K) in 1903 derived the linearized equations of motions for the Basic bicycle model having all the mass and inertia of the front assembly in the front wheel (similar to Carvallo). Somewhat similar to Whipple [1899], they used a Newtonian analysis of the front and rear assembly, and treated the two parts as two trailers attached to the steering axis, deriving the linearized equations of motions using axes parallel to the steering axis. S & K refer to Whipple [1899] and Carvallo [1901] but do not say whether their equations agree.

Their equations are most easily compared to Döhring's [1955], and are found to be a correct subset of his. It is possible that S & K's slight simplification(s) to the model were due to their main interest in determining what effect the wheels as gyroscopes had on the stability (since the article is a chapter in their massive work on gyroscopes). They are critical of Bourlet [1898] (whose book we have not read).

*Bower, 1915*

Bower in 1915, without reference to any previous bicycle work, derived the linearized equations of motion for a simplified Basic bicycle model at the end of an article mainly concerning the gyroscopic effects of the engine and wheels on steady turns. His model consists of a rear frame with its center of mass above the rear contact point, having polar inertia $R_{\overline{z}\overline{z}}$ provided by two point masses, one ahead and one behind the center of mass. Two smaller masses at the same height are attached to the front assembly. Wheel inertia and caster trail are also included, but the steering axis is restricted to be vertical.

Instead of providing two second order equations for his model, he presents the governing 4th order linear differential equation (eq. (19) in his analysis), which is not convenient for comparison. The $e$ coefficient, given as equation (24) in his paper, is comparable to the determinant of the $\mathcal{K}$ matrix presented in our chapter III. Comparison indicates that Bower's equations are missing the $g\nu$ term in the $\mathcal{K}_{\chi\psi}$ coefficient of the lean equation for his simple model which confirms that his equations lack some of the effects of trail has on the bicycle. No comparison was made to Bower's coefficients $A$-$D$ for his simplified bicycle model, but casual observation indicates they also lack terms.

Looking back at his derivation it appears that his $\phi$ is our $-\psi$, and his $\theta$ is our $\chi_r$. His eqs. (15) and (16) may be added to eliminate the internal reaction $P$, thus leading to a lean equation. However, (a) he has ignored product of inertia terms (relative to the wheel contacts) which should appear multiplying his $\ddot{\psi}_1$, $\ddot{\psi}_2$; this is correct for the rear part of his simplified model, but not for the front unless trail vanishes. Also, (b) he has left out the lateral offset of the front and rear mass center from the track line due to steer angle; this too is correct for the rear part of his simplified bicycle but not for the front unless trail vanishes. (It also appears that he should have included a vertical reaction force at the steering bearing, though this would cancel when (15) and (16) are added.) Finally (c) his centrifugal forces (such a $f_1$ are in error because he assumes a steady curve due to steer angle divided by a finite wheelbase, whereas in fact even with an infinite wheelbase the rate of steer can produce path curvature of the front wheel and with nonzero trail the rate of steer also affects the yaw rate of the rear wheel. Based on these observations, it seems likely that his lean equation could apply correctly to his simplified model only when the trail is zero.

We believe the steer equation is formed by adding $(1 + \frac{c}{l})(eq.\ 17) + (\frac{c}{l})(eq.\ 18)$ to eliminate $P$ (the term multiplied by his $\ddot{\theta}$), but we have not checked this in detail.

*Pearsall, 1922*

In 1922 Pearsall, with the stated intention of extending Bower's [1915] ideas and discovering the cause of "speedmans wobble," derived a set of equations for a bicycle model somewhat similar to the Basic bicycle model presented in Chapter III. He never states precisely whether his model is restricted in any way, but for example, his equations don't include any product of inertia terms, so they are probably not general.

His technique for deriving the equations of motion was to first linearize the equations of motion of a rolling hoop and then "add on" the trailer effects due to the remaining parts of the bicycle using fairly casual arguments. While his brief verbal justifications sound valid, in fact almost no terms in the equations are exactly correct. We did not make the effort to trace his errors, but note that there may have been a major mistake in the kinematical treatment (which is not spelled out very explicitly), that is, the headings $\gamma$ and $\theta$ of the rear and front assemblies are defined relative to the track line, but then they are treated as coordinates relative to inertial space in the equations.

We compared his equation (4) to our steer equation and his equation (5) to our lean equation and found his equations significantly differ in almost every term when compared to those presented in Chapter III. His equations also disagree with Bower's model.

Pearsall does not say if he compared his equations to Bower's, and he does not refer to any other works.

*Timoshenko and Young, 1948*

In this textbook on advanced dynamics, Timoshenko and Young derived a nonlinear (large-angle) lean equation for a simplified Basic bicycle model having only a point mass in the rear part of the bicycle, and a steer angle controlled by the rider. Their model neglects wheel inertias, steering axis tilt, trail and a front mass offset from the steering axis. When linearized, we find this lean equation agrees with our lean equation simplified for an equivalent configuration.

*Döhring, 1955*

In 1955, in order to more generally analyze the stability of motorcycles and motorscooters, Döhring extended Sommerfeld and Klein's (S & K) [1903] linearized equations for the Basic bicycle model by allowing the mass distribution of the front assembly to be fully general. Just as S & K did, Döhring used Newton's Laws to derive the equations of motion in linearized form, rather than linearizing from nonlinear equations as Whipple had.

Döhring's final equations were found to be in exact agreement with those derived in Chapter III. In order to compare his equations to ours we made the following substitutions in his equations (29) and (30) of his [1955] paper,

$$\psi = \gamma \cos \sigma$$

$$\theta_1 = \theta_2 - \gamma \sin \sigma$$

where $\gamma$ is steer angle (our $\psi$) and $\theta_2$ is lean angle (our $\chi_r$). When these substitutions are made Döhring's equation (30) is exactly our lean equation. Our steer equation results from the linear combination of Döhring's equation (31) and (30). Using Döhring's notation this combination is as follows:

$$\frac{(eq.\,31)}{l} + \frac{c_1 \sin \sigma (eq.\,30)}{l} = -M_d = our\ \ M_\psi$$

Although not rigorous in how his linearizations are made, Döhring's derivation was fairly easy to follow, and offers a good physical description of the variables and equations of motion. Döhring refers to S & K, but never states explicitly how his equations compare.

*Collins, 1963*

In his 1963 University of Wisconsin Ph.D. dissertation R. N. Collins, working on a project supported by Harley Davidson Motor Company, studied a Basic bicycle model with the addition of a driving force on the rear tire and an explicit force for aerodynamic drag applied to the front fork/handlebar assembly. He derived the equations of motion using Euler's equations (Newton's Laws) for the 4 rigid bodies of the Basic bicycle model.

Collins derives nonlinear velocity and acceleration expressions for the rear and front center of mass first (see page 19 and 20 of his dissertation), and then linearizes about the vertical equilibrium position, before deriving the linearized equations of motion. By writing the drive force and aerodynamic drag force as a function of the square of the forward velocity of the motorcycle (see p. 12 in his dissertation), he alters the vertical contact forces on the front and rear wheels. By making the assumptions of no slip angle and constant velocity he has only two degrees of freedom for his model and he is therefore able to write the linearized governing equations as two coupled second order ordinary differential equations in the lean and steer angles (see p. 76 eq. (5.1) and eq. (5.2) in his dissertation). The final equations are complicated in appearance and include over 30 quantities defined in terms of motorcycle parameters. (These quantities often include previously defined quantities, which further complicates understanding of the equations.)

His equation (5.1) is not exactly the steer equation, and his equation (5.2) is not exactly the lean equation. However, if we transfer all the terms to the left hand side, and form the combination,

$$\sin\alpha[eq.(5.1)] + h_2[eq.(5.2)] = [equation\ with\ no\ \dot{\phi}\ and\ no\ M_3],$$

the result appears to be the lean equation. That is, in our notation the coefficients $\mathcal{M}_{\chi\chi}$, $\mathcal{C}_{\chi\chi}$ (which is zero), $\mathcal{K}_{\chi\chi}$ are all in agreement with those presented in Chapter III. The steering moment $M_3$, our equivalent $M_\psi$, also drops out of the equation as it should. So while the task of multiple substitution was tedious and prevented us from completely comparison of the lean equation, or even from determining what combination of his equations ought to give our steer equation, it may be that Collins resulting equations are correct.

The only potential flaw to come to light is that Collins equivalent to our $\mathcal{C}_{\chi\psi}$ term, namely

$$-(\sin\alpha K_{21} + h_2 K_{31}),$$

should probably include the angular momentum of both wheels. However, this expression appears to contain only the front moment of inertia $I_1'$, not $I_2'$.

Collins refers to the works of Sommerfeld and Klein [1903], Bower [1915], Pearsall [1922], and Döhring [1955], but never compares his equations to theirs (nor to those of Whipple [1899] or Carvallo [1901], who were cited by S & K).

*Singh, 1964*

One year later, working on the same project sponsored by Harley Davidson Motor Company, D. V. Singh's Ph.D. dissertation added tire side slip to Collins model. For reasons not stated, Singh <u>rederived</u> the equations of motion in a notation similar to Collins, with just a few modifications for tire side slip.

Singh's final equations are (6.11-d) and (6.12-d) on p. 74 of his dissertation. These equations were judged too impenetrable to compare to those in chapter III, because the coefficients are defined in terms of secondary quantities, which in turn are defined as functions of physical parameters. However, on p.49 he assumes that the tire corning forces (tire side slip) are proportional to the steer angle, which is only true for steady turns. Hence, we judge at least his treatment of side-slip (eq. 4.31) to be incorrect, though if sideslip is prevented we can't say whether or not his equations are correct.

Surprisingly, it was noted by casual review of Singh's and Collins' theses that disagreement exists in their expression for the velocity of the rear center of mass of the vehicle. This can be found on page p. 52 of Singh's dissertation eq. (4.40a-c) and p. 19 of Collin's dissertation eq. (2.13a-c). Equation 4.40(a) of Singh's dissertation has an extra term compared to 2.13(a) of Collins, and some signs appear to be different in subsequent equations, although the coordinate axes choosen in both treatments seem to be equivalent.

Though he refers to Collins and to Collins' references, Singh does not compare his equations to anyone.

*Neĭmark and Fufaev, 1967*

In 1967 Neĭmark and Fufaev (N & F) with a brief reference to Döhring [1955] derived equations of motion of the bicycle as a classical example of a nonholonomic system. In their derivation they use Lagrange's equations with nonholonomic constraints for the path of the wheels and obtain the linearized equations of motion for the Basic bicycle model. It is their derivation that our chapter III mainly follows.

The equations in their book which represent the relations between the auxiliary variables and generalized coordinates, linearized rolling constraints, kinetic energy for the rear and front part of the bicycle, potential energy of the bicycle, and equations of motion, can be found starting on p. 334 as eq. (2.10), eq. (2.15), eq. (2.26) and eq. (2.29), eq. (2.30), and eq. (2.37-38), respectively.

As mentioned in Chapter III an error is made in their formula for potential energy eq. (2.30). (The correct potential energy to second order is found in section 4 our of Appendix A.) This error results in the incorrect coefficients $a_4$, $b_3$, and $b_4$ in eqs. (2.37-38), where $gm_2d$ should be replaced by $g\left(m_2d + \frac{c_1}{c}l\right)$. In addition to these corrections the reader should note that a typographical error occurs in the $b_2$ term on p. 344 of their text (where $\frac{1}{2}$ should read $\frac{1}{c}$) and in several other terms in the description of the geometry and viscous damping expressions. Also, in deriving the nonlinear equations they present nonlinear kinematic equations which are actually incorrect because they neglect the rise and fall (pitch) of the bicycle due to variations in the steer angle. (However, the linearized versions of these equations are correct as shown in section 1 of our Appendix A. However, quadratic terms are needed to derive the correct potential energy.) By eliminating the effects of viscous damping in the steering column, and making the above corrections N & F's final equations of motion can be brought into agreement with those derived in chapter III.

N & F refer to Döhring[4] and state that their equations agree in form, but it is unlikely they meant term for term as we have found them to be in disagreement. They also refer to a Russian book by Loǐcjanskiǐ and Luré [1935] when analyzing a simplified model of an uncontrolled bicycle on p. 355. Because this reference was not available, it is not known if agreement actually exists, however it seems probable because N & F's equations become correct when simplified in this way. N & F do not mention any other bicycle-related works, although their massive reference list includes Carvallo [1901].

*Singh and Goel, 1971*

In January 1971 Singh and Goel (S & G) add steer damping to the Basic bicycle model in analyzing a Rajdoot motor scooter. In their analysis they claim to use Döhring's [1955] linearized equations of motion (which we have found to be correct) with a steering torque proportional to the the time derivative of the steer angle (viscous damping). We have not rigorously compared term by term but casual observation shows that the equations are in the same format as those of Döhring [1955].

S & G refer to Pearsall [1922], Timoshenko [1948], Döhring [1955], Collins [1963], and Singh [1964], but make no comparison to their equations of motion.

[4] See p. 361 of their text.

In August 1971 Sharp, who apparently began working on the equations of motion while at the B. S. A. motorcycle company, published a paper presenting his version of the linearized equations of motions for the motorcycle. In his Lagrangian approach rather than using the method presented by Neĭmark and Fufaev in chapter III, he explicitly allows the vertical force from the ground on the front wheel ($Z_f$) to do work on the bicycle. For this reason $Z_f$ appears in his expressions for the generalized forces. In this way he accounts for the change in potential energy of the bicycle when steered. The nonlinear equations he presents are actually only approximations for this reason.

Allowing for wheel side slip, and incorporating the work done by the vertical force on the front wheel, he derived Lagrange's equations with generalized forces at the wheels' contact with the ground. These resulted in four equations of motion, incorporating front and rear tire side forces, which govern lateral motion, yaw, roll, and steer of the motorcycle. They which appear in his paper starting at the bottom of p. 327 (no equation numbers are given). These equations are correct as far as we know.

However, when assuming that the tires have infinite stiffness (no side slip), which reduces the number of equations from four to two, an algebraic mistake and several typographical errors occur in the appendix to Sharp's 1971 paper. As a result the steer equation in Appendix II of his paper (the second equation) is incorrect. The algebraic error made by Sharp results in the incorrect cancellation of the following term (in his notation),

$$2[M_f ek + I_{fz} \cos\epsilon + M_f eb] l_1 t \ \ddot{\delta}$$

We also make note of the following typos: the $x_1^2$ in the lean equation of Appendix II should read $\dot{x}_1^2$; there is an extra parathesis in the ninth term of the fourth equation in Appendix 1 section entitled "Linear equations of motion"; the term $\dot{x}_1 \cos\epsilon\ddot{\delta}$ in the expression for $\ddot{\psi}$ in Appendix II should read $\dot{x}_1 \cos\epsilon\dot{\delta}$; $I_{fy}$ should read $i_{fy}$ in the $\dot{\phi}$ term of the steer equation of appendix 2; and finally terms involving $\frac{i_{fy}}{R_f} l_1 t \dot{x}_1 \sin\epsilon$ in the $\dot{\delta}$ term of the steer equation can be eliminated as they cancel one another.

Sharp also makes the slightly restrictive assumption that one principal axis of the center of mass moment inertia tensor of the front assembly is parallel to the steering axis. Thus the equations in his paper, when corrected, are a subset of those derived in Chapter III. Sharp refers to the work of Whipple [1899], Pearsall [1922] and Collins [1963], but does not compare his equations to theirs.

*Roland, 1971*

In 1971 Roland published a report written for the Schwinn Bicycle Company containing a extensive nonlinear computer simulation study.[5] In this report Roland derived nonlinear equations that represent the motion of a bicycle with tire side slip and rider lean. His 8 equations of motion are shown in matrix form on p. 37 of his report. Reading from the top down the first three equations represent force balance for the entire bicycle. The fourth through the sixth equations represent moment balance for the entire bicycle. The seventh equation is apparently moment balance for the frame assembly about the steer axis, which can be used to solve for the steering torque if the tire side force is eliminated. The eighth and final equation represents the rider upper-body lean degree of freedom, and

[5] This report was based on work performed for a National Commission on Product Safety research contract. See Roland [1970].

can be used to solve for the tilting moment of the bicycle on the rider when rider motion is prescribed. These equations are written so that the second time derivatives are all on the left side of the matrix equation, while all the lower order terms are on the right hand side.

Roland used axes parallel and perpendicular to the steering axis in the plane of the rear frame, and perpendicular to the rear frame. However, his report to the Schwinn bicycle company is missing an important figure describing the orientation of the body-fixed axes. This figure is contained in a later publication Mechanics and Sport [1973]. In the later publication, Roland also corrects some typos that were in the 1971 publication.

In the 1971 report, the seventh equation, the steer moment equation, is given on p. 13 as eq. (2.3.30). We took this equation and assumed $e = x''_{TF} - x''_F$ in order for it to agree with the seventh equation in the matrix on page 37. We then made simplifications to the equation to see if it agreed with our steer equation.

First we set any term multiplied by $y_F$ or $y''_F$ equal to zero. (This means there is no lateral imbalance.) We then neglected terms multiplied by the pitch rates $q$ and $q''$, which are second order effects. Next we assumed angles (and their time derivatives) to be small and let $\sin\delta = \delta$ and $\cos\delta = 1$, and cancelled any products of $p$, $p''$, $q$, $q''$, and $\delta$ and their time derivatives.

We then linearized the variables $\gamma_{12}$, $\gamma_{22}$, $\gamma_{32}$ in the same way. Terms multiplied by $\gamma_{12}$ become zero, $\gamma_{22} = m_f(rV - g\phi)$ and $\gamma_{32} = m_f g$.

The coefficient of the $\ddot{\delta}$ term seems correct, and the resulting equation appears somewhat similar to our equation (3.14), but we are not able to make the resulting equation agree completely. There is some question as to whether the comparison we are making is correct, because it is not understood if in fact Roland's equation (2.3.30) should be equivalent to our steering equation.

An equation equivalent to our lean equation has not yet been constructed from Roland's equations. However, it is probalbe that an equivalent equation would be obtained by combining the fourth and sixth equations in his matrix to represent rolling moment about the track line, setting the rider lean angle to zero, including the mass of the rider with that of the rear frame and rear wheel, setting pitching motion to zero, and setting tire side slip to zero. The lateral forces, $F_{ytr}$ and $F_{ytf}$, on the wheels can perhaps be solved for analytically using the first, fourth and sixth matrix equations or taken from other linearized equations studies. Since we have not attempted this, we are not able to judge whether his lean equation reduces to ours.

Roland refers to the works of Whipple [1899], Bower [1915], Pearsall [1922], Manning [1951], Döhring [1955], Collins [1963], Singh [1964], and also Singh and Goel [1971]. However, he makes no comparisons to their equations of motion.

*Weir, 1972*

In an appendix to his 1972 UCLA Ph.D. dissertation focusing mainly on the control and handling characteristics of motorcycles, Weir derived the equations of motion for the Basic bicycle model with a general Newtonian approach, linearizing as the derivation proceeded. Weir's final 4 equations, eq. [A-85], [A-92], [A-99], [A-108] in his analysis, represent the lateral motion, yaw, lean and steer equations of motion, like Sharp's [1971] four equations. Weir, however, did not make the simplifying assumption regarding the principal axes of the front inertia.

Weir was the only author to state explicitly that he compared his equations to another past work.[6] He compared his equations to Sharp's equations before simplification to only two nontrivial degrees of freedom. In comparing Weir's 4 equations to Sharp's four equations, we find Weir and Sharp in agreement with one another. Weir, however, is more general than Sharp, with regard to the principal axes of the front system. Weir's two equations agree with ours, as long as our nonstandard sign convention for wheel angular momentum (positive for forward rolling) is recognised.

Besides stating that comparing his equations agree with to Sharp's, Weir refers to Whipple [1899], Pearsall [1922], Döhring [1955], Singh [1964], Singh and Goel [1971], but does not compare his equations to these works.

*Singh and Goel, 1975*

In 1975 Singh and Goel presented (but did not derive) a 12th order mathematical model, for the continued analysis of the Rajdoot scooter. Instead of using Singh's [1964] equations, or Döhring [1955] equations as they did in 1971, they employ a Lagrangian formulation which appears similar to Sharp's [1971] format. The authors claim that the model used is a fully general Basic bicycle model, having in addition unsymmetric lateral mass distribution, lateral slip, aerodynamic forces, transient tire forces and moments (which account for the high order of the system), and viscous damping of the steering.

The four equations of motion presented are said to represent the lateral motion, yaw, lean, and steer equations of motion. We have not yet checked these equations for correct-

[6] See page 130 of Weir's dissertation.

ness, but they do appear similar to Sharp's [1971] four equations. Singh and Goel refer to the 1971 paper on their Rajdoot scooter and Sharp's [1971] paper but do not compare their equations.

*Sharp and Jones, 1975*

In 1975 Sharp and Jones use the equations derived by Sharp [1971] and modify it to incorporate a different tire model. As in the 1971 paper the principal axes of inertia are assumed to be parallel and perpendicular to the steering axis equations of motion. Other than this, these equations are equivalent to those in his 1971 paper, which when simplified are a subset of our equations presented in Chapter III of this thesis.

*Weir and Zellner, 1978*

Weir and Zellner later published the results of Weir's dissertation derivation in Motorcycle Dynamics and Rider Control (SP-428, 1978), but mistakenly thinking Weir's earlier derivation was wrong, they deleted a necessary term without comment. The term needing correction can be found on page 8 in the matrix equation (1), where the second row fourth column terms of the matrix should read,

$$\frac{L_{\ddot{\delta}}}{I_{xx}}s^2 + L_{\dot{\delta}}s + L_{\delta}$$

There are also some typos in equation (1) and we note the third row fourth column term should read,

$$N_{\ddot{\delta}}s^2 + N_{\dot{\delta}}s + N_{\delta}$$

and finally the fourth row fourth column term should read,

$$T_{\ddot{\delta}}s^2 + T_{\dot{\delta}}s + T_\delta$$

Because of these typographical errors we recommend using Weir's dissertation for any comparison of equations or results.

*Gobas, 1978*

Using a technique which he calls the Boltzman Hamel method, in 1978 Gobas derived a linearized set of equations very similar in form to Neĭmark and Fufaev [1967]. Gobas' equations, (1.4) in his paper, incorporate the forward acceleration of the bicycle, $\dot{V}$. Setting $\dot{V}$ terms to zero and comparing, we think the lean equation may be correct, but in the steer equation the coefficient to the $\chi_r$ term seems to be in disagreement with the equations in our Chapter III. The variable $b$ is not defined in the paper but we suspect that it is equivalent to our $\nu$.

Gobas refers to Neĭmark and Fufaev, but does not compare equations.

*Adiele, 1979*

In his 1979 Master's thesis Adiele, focusing on design optimization and performance evaluation of two-wheeled vehicles, derived nonlinear equations of motion for the Basic bicycle with tire side slip using the method of generalized active and inertia forces. His model is linearized for small steer angles and results in 4 equations representing the lateral motion, yaw, lean, and steer equations of motion on page 33 of his thesis.

His equation representing lateral motion, lean, steer, and yaw (in that order) are present in matrix form on pages 22-24 of his thesis. His variable $V$ is our $\dot{X}_r$, $\lambda$ is our $\chi_r$, $\theta$ is our $\psi$, and $r$ is our $\dot{\theta}_r$. Because his equations resembled Sharp's [1971] four equations, we expanded Adiele's matrix, linearized his equations and compared to the equations in Sharp's Appendix I.

The results show that Adiele's equations are in error, missing several terms compared to Sharp and having several sign errors. However, by allowing the front mass to be zero his equations are nearly correct.

Adiele refers to Roland [1971], but does not compare equations.

*Lowell and Mckell, 1982*

In 1982 Lowell and Mckell, with ad hoc arguments similar in style to Pearsall [1922] derive a set of linearized equations for a Basic bicycle model with a point mass in the rear part, some steering inertia, but no front mass, and no tilt of the steering axis. When compared to our equations simplified for this case, we find there is significant disagreement. Several terms have been neglected in both the lean and steer equation, however, the terms presented are correct. The neglected terms are significant, as a bicycle with vertical steering axis and positive trail should show divergent instability at all speeds ($E < 0$), whereas in fact they find oscillation about a steady turn ($E = 0$).

We find the only way to make their equations correct is to use them for a bicycle with zero gyroscopic effects and zero trail.

Lowell and McKell refer to Timoshenko and Young [1948], Grey [1918], and Pearsall [1922] but only state (correctly) that their lean equation agrees with Timoshenko's when simplified. No other comparisons are made.

**Conclusions**

Of the 20 sets of equations discussed in this chapter only 3 sets (Döhring [1955], Singh and Goel's [1971] adaptation of these, and Weir assume zero angular momentum and zero trail terms. [1972]) agreed exactly with those we presented in chapter III of this thesis. (The slip angle condition had to be set to zero in Weir's equations.) Five others simply had minor errors, or were not as general (Whipple [1899], Carvallo [1901], Sommerfeld and Klein [1903], Timoshenko and Young [1948], and Sharp [1971]). Three (Collins [1963], Singh [1964], and Roland [1972]) were to difficult to evaluate, though we have definite reservations about the first two. The remaining eight were missing terms, or disagreed in other ways (we did not check Singh and Goel [1975].

Other works which derived the linearized equations of motion, but whose comparison results are not presented here, are Eaton [1973] and Psiaki [1979]. Eaton's derivation was not noticed until late in this thesis's progress. Psiaki derived very dense nonlinear equations and then linearizes. His equations were judged not worth the effort to sort out.

Guo [1979] performed a nonlinear analyses but did not linearize, so we did not compare to his equations. Psiaki stated he found numerical agreement with Collins, and Guo referred to Neĭmark and Fufaev but made no comparison with them.

Other scientists have studied various aspects of bicycle behavior without deriving equations of motion. Rankine [1869] described steering phenomenology, and discussed the relation between sinusoidal steering motion and the resulting sinsoidal leaning. Sharp [1896] derived the steer torque in a steady turn.[7] Jones [1970] incorrectly approximated the steer torque for a leaning bicycle. Man and Kane [1979] studied only steady turning.

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[7] See eq. 6 on page 231 of his book. His result is nearly correct except that it neglects the straightening effect of centrifugal force on the mass-center of the front assembly.

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