J. Pap. Part II ?

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Governing Equations

Lagrange's Equations for the Basic Bicycle Model

We now introduce the Lagrange function L,

$$L = KE_t^+ - PE_t^+$$

where KE_t^+ and PE_t^+ (the kinetic and potential energy which contribute to Lagrange's equations) are given by equations (3.6) and (3.8). In order to derive Lagrange's equations we also introduce the Lagrangian operator L_q,

$$\mathrm{L}_q = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}$$

where q represents the generalized coordinates Y_r, ϕ_r, ϕ_f, X_r, θ_r, χ_r, and ψ.

What then is L_q operating on L? As is explained by Goldstein [1980], the L_q operating on L equals the generalized force on the system. In this case with the help of figure 3.9, the generalized forces, represented by the letter Q, can be written to first order as follows,[11]

$$\mathrm{L}_{Y_r}(L) = Q_{Y_r} = -D_f - D_r \tag{3.9a}$$
$$\mathrm{L}_{\phi_r}(L) = Q_{\phi_r} = D_r a_r \tag{3.9b}$$
$$\mathrm{L}_{\phi_f}(L) = Q_{\phi_f} = D_f a_f \tag{3.9c}$$
$$\mathrm{L}_{X_r}(L) = Q_{X_f} = F_f + F_r \tag{3.9d}$$
$$\mathrm{L}_{\theta_r}(L) = Q_{\theta_r} = -F_f c_w \tag{3.9e}$$
$$\mathrm{L}_{\chi_r}(L) = Q_{\chi_r} = M_{\chi_r} \tag{3.9f}$$
$$\mathrm{L}_{\psi}(L) = Q_{\psi} = M_{\psi} + c_f F_f \tag{3.9g}$$

where,

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$$Q_{Y_r} = -D_f - D_r = m_t \ddot{Y}_r \tag{3.10a}$$
$$Q_{\phi_r} = D_r a_r = C_r \ddot{\phi}_r \tag{3.10b}$$
$$Q_{\phi_f} = D_f a_f = C_f \ddot{\phi}_f \tag{3.10c}$$
$$Q_{X_r} = F_f + F_r = m_t \ddot{X}_r - m_t \bar{l}_t \ddot{\theta}_r + m_t \bar{h}_t \ddot{\chi}_r - m_f d \ddot{\psi} \tag{3.10d}$$
$$Q_{\theta_r} = -c_w F_f = -m_t \bar{l}_t \ddot{X}_r + T_{zz}\ddot{\theta}_r + T_{yz}\ddot{\chi}_r + H_t \dot{\chi}_r + F''_{\lambda z}\ddot{\psi} - H_f \sin\lambda \dot{\psi} \tag{3.10e}$$
$$\begin{aligned} Q_{\chi_r} = M_{\chi_r} &= m_t \bar{h}_t \ddot{X}_r + T_{yz}\ddot{\theta}_r - H_t \dot{\theta}_r + T_{yy}\ddot{\chi}_r - g m_t \bar{h}_t \chi_r + F'_{\lambda y}\ddot{\psi} \\ &\quad - H_f \cos\lambda \dot{\psi} + g\nu\psi \end{aligned} \tag{3.10f}$$
$$\begin{aligned} Q_{\psi} = M_{\psi} + c_f F_f &= -m_f d \ddot{X}_r + F''_{\lambda z}\ddot{\theta}_r + H_f \sin\lambda \dot{\theta}_r + F'_{\lambda y}\ddot{\chi}_r \\ &\quad + H_f \cos\lambda \dot{\chi}_r + g\nu\chi_r + F'_{\lambda\lambda}\ddot{\psi} - g\nu \sin\lambda \psi \end{aligned} \tag{3.10g}$$

[11] In figure 3.9 M_{ψ} is an internal torque to the Basic bicycle model imposed by the rider on the steering axis and reacted by the rear frame. In our free body diagram we have assumed the component of the reaction moment on the rear frame, say M_{θ_r}, is negligible.

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THE FOUR CANONICAL DIFFERENTIAL EQUATIONS IN CANONICAL FORM (see page 23 of hand copy of this paper)

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(1) **FOR THE WHOLE BIKE: the total X-force required to effect the lateral acceleration of all mass points in a general motion = the sum of applied X-forces.**

$$m_t\ddot{X} + m_t\bar{h}_t\ddot{\chi} - m_t\bar{l}_t\ddot{\theta} - m_f d\ddot{\psi} = \mathcal{F}_{Xr} + \mathcal{F}_{Xf}$$

LHS:

- the lateral acceleration of m_t due to lateral acceleration of the rear contact, no lean or yaw
- the X acceleration of m_t due to yawing (only) of the whole bike
- the X acceleration of m_t due to lean (only) of the whole bike
- the X acceleration of m_f due to steering only

RHS: sum of forces in X-direction

(2a) **FOR THE WHOLE BIKE: the total χ-moment (about the heading line of the rear assembly), required when accelerating mass-points laterally in a general motion, = sum of χ moments of external forces about the same line.**

$$m_t\bar{h}_t\ddot{X} + T_{yy}\ddot{\chi} + T_{yz}\ddot{\theta} + F'_{\lambda y}\ddot{\psi} - H_t\dot{\theta} - H_f\cos\lambda\dot{\psi} = gm_t\bar{h}_t\chi - gm_f d\psi - c_f(m_t g\frac{\bar{l}_t}{c_w})\psi$$

$$= gm_t\bar{h}_t\chi - g\nu\psi \ ,$$

where for convenience we have defined $\nu = (m_f d + m_t\frac{l_t}{c_w}c_f)$.

LHS:

- the y (or χ) moment required to accelerate the center of mass of the whole bike laterally
- the y-moment required for angular acceleration ($\ddot{\chi}$, $\ddot{\theta}$) of the whole bicycle about the rear contact point
- the y-moment required for angular acceleration of the front assembly about the steering axis $\vec{\lambda}$ (remember that the contacts can slip sideways, and that X, χ, θ are being held fixed
- the gyroscopic moment (from wheel spin angular momenta H_r, H_f) required for yawing of the whole bike about a vertical axis
- the gyroscopic y-moment required for the precession about the vertical ($\dot{\psi}\cos\lambda$) of the front wheel

RHS:

— the moment of gravity force acting at the c.m. of the total bicycle which is leaned only (Fig.A8a)

— the moment of gravity force acting at the center of mass of the front assembly, for a non-leaning bicycle (Fig.A8b)

— the moment of vertical front contact force $(m_t g\bar{l}_t/c_w)$ which is offset due to steer only (Fig.A8b)

— the steer moment $\mathcal{M}_\psi$ does not contribute because $-\mathcal{M}_\psi$ also acts on the bicycle. The forces $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$ do not contribute because they are in the ground plane and so have no moments about the rear-assembly heading line.

(2b) <u>FOR THE WHOLE BIKE:</u> the total θ-moment (about the z-axis through the rear contact P_r, which for small angles is equivalent to a *vertical* axis) required when accelerating mass-points laterally in a general motion = sum of moments of external forces about the same line.

$$-m_t\bar{l}_t\ddot{X} + T_{zy}\ddot{\chi} + T_{zz}\ddot{\theta} + F''_{\lambda z}\ddot{\psi} + H_t\dot{\chi} - H_f\sin\lambda\dot{\psi} = -c_w\mathcal{F}_{Xf}$$

LHS:

— the z (or θ) moment required to accelerate the center of mass of the whole bike

— the z-moment required for angular acceleration $(\ddot{\chi},\ \ddot{\theta})$ of the whole bicycle about the rear contact point

— the z-moment required for angular acceleration of the front assembly about the steering axis $\vec{\lambda}$

— the gyroscopic moment required for tipping of the whole bike about the heading line of the rear assembly

— the gyroscopic z-moment required for the precession about the horizontal $(-\dot{\psi}\sin\lambda)$ of the steered front wheel

RHS: moment of $\mathcal{F}_{Xf}$ about the z axis.

(3) <u>FOR THE FRONT ASSEMBLY ONLY:</u> the total ψ-moment about the steering axis $\vec{\lambda}$ required for a general bicycle motion = sum of external moments about the same axis.

$$-m_f d\ddot{X} + F'_{\lambda y}\ddot{\chi} + F''_{\lambda z}\ddot{\theta} + F'_{\lambda\lambda}\ddot{\psi} + H_f(\dot{\chi}\cos\lambda + \dot{\theta}\sin\lambda)$$

$$= \mathcal{M}_\psi + c_f\mathcal{F}_{Xf} - g(m_f d + m_t\tfrac{\bar{l}_t}{c_w}c_f)\chi + g\sin\lambda(m_f d + m_t\tfrac{\bar{l}_t}{c_w}c_f)\psi$$

$$= \mathcal{M}_\psi + c_f\mathcal{F}_{Xf} - g\nu\chi + g(\sin\lambda)\nu\psi\ ,$$

(where ν is defined in (2a) above).

LHS:

— ψ-moment required to support lateral acceleration of the front assembly

— Moments about $\vec{\lambda}$ axis required when the front assembly is given angular acceleration about the χ (y) axis and θ (z) axis. Because inertia tensors about any point are symmetric matrices, the moment about one axis required for angular acceleration about another is the same as the moment about the second required for angular accleration about the first.

— the moment about the steering axis required for angular acceleration of the front assembly about that axis (the coefficient is the polar moment of inertia)

— the moment about the steering axis required for precession ($\dot{\chi}\cos\lambda + \dot{\theta}\sin\lambda$) about an axis in the plane of the bicycle which is perpendicular to the steering axis.

RHS:

— the steering moment $\mathcal{M}_\psi$, and the moment about the steering axis of the horizontal force, are easy to see. (Fig.A9a)

— when the bicycle is leaned only, the vertical reaction force at the front contact and the vertical gravitational force on m_f both have components proportional to χ which are perpendicular to the plane of the bike. These forces act on lever arms c_f and d. (Fig.A9b)

— when the bicycle is steered only, these two forces are displaced from the plane of the bicycle, and no longer pass through the steering axis. Resolve them initially into components perpendicular and parallel to the steering axis; then when they are displaced, it is easy to see that only the components initially *perpendicular* to the steering axis (which are multiplied by $\sin\lambda$) exert moments, with lever arms equal to their lateral displacements ψd and ψc_f. (Fig.A9c)

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Constraints

REDUCED EQUATIONS OF MOTION

Equations $1, 2a, 2b, 3$ are true whatever horizontal forces $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$ act at the wheel contacts, and in particular they are true if the forces are just right to prevent the wheels from side-slipping relative to their instantaneous headings.

[Need a discussion of side slip here......]

For the rear wheel, zero side slip is expressed by the equation $\dot{X} = -V\theta$. (See Fig.A10a.) For the front contact, an analogous relation is needed in terms of the X-co-ordinate and heading of the front wheel: $\dot{X}_f = -V\theta_f$. To write this in terms of the variables describing the motion, we need $X_f = X - c_w\theta + c_f\psi$ for the X-co-ordinate of the front contact P_f (Fig.A10b), and $\theta_f = \theta + \psi\cos\lambda$ for the heading of the front wheel

(Fig.A10c). ($\cos\lambda$ comes in because a small rotation of the front wheel about the steering axis can be considered a sum of small rotations about horizontal and vertical axes — and only the latter changes the front wheel's heading.)

With these relations, we can express such quantities as $\dot{\theta}$, $\ddot{\theta}$ and $\ddot{X}$ in terms of and ψ and $\dot{\psi}$. By subtracting the rear-wheel constraint from the front-wheel constraint, we obtain

$$-c_w\dot{\theta} + c_f\dot{\psi} = V\psi\cos\lambda \ ,$$

which may be solved for $\dot{\theta}$ to give

$$\dot{\theta} = \frac{c_f}{c_w}\dot{\psi} + V\frac{\cos\lambda}{c_w}\psi \ .$$

This may be differentiated once to give $\ddot{\theta}$:

$$\ddot{\theta} = \frac{c_f}{c_w}\ddot{\psi} + V\frac{\cos\lambda}{c_w}\dot{\psi} \ .$$

Finally, the rear constraint relation may be differentiated once, and $\dot{\theta}$ may be replaced:

$$\ddot{X} = -V\dot{\theta} = -V\frac{c_f}{c_w}\dot{\psi} - V^2\frac{\cos\lambda}{c_w}\psi$$

******* ** ** **

After substituting these relations into Eqs. 1,2,3, to eliminate $\dot{\theta}, \ddot{\theta}$ and X, we have four equations but apparently only two variables. However, demanding that the horizontal contact forces should prevent the wheel sideslip means that these forces are no longer freely selectable, but must be exactly the right magnitude at every instant. In essence, they are the two remaining variables.

Since we are concerned at present only with the *motion*, the most convenient thing is to rearrange the equations so that the unkown forces do not appear in two of them; these two will allow us to solve for the unknowns χ, ψ.

Equation (2a) (with X and θ eliminated) is already in that form; because it dealt with moments about a line in the ground it will be called the <u>lean equation</u>. For the other equation we simply eliminate $\mathcal{F}_{Xf}$ from (2b) and (3), and leave $\mathcal{M}_\psi$ on the right hand side; this is called the <u>steer equation</u>. (Evidently equation (1) is not needed, unless we wish to find $\mathcal{F}_{Xr}$.)

We write these two equations in the form:

$$M_{\chi\chi}\ddot{\chi} + M_{\chi\psi}\ddot{\psi} + C_{\chi\psi}\dot{\psi} + K_{\chi\chi}\chi + K_{\chi\psi}\psi = 0 \text{ , the lean equation*}$$

(note that there is no $C_{\chi\chi}\dot{\chi}$ term); and

$$M_{\psi\chi}\ddot{\chi} + M_{\psi\psi}\ddot{\psi} + C_{\psi\chi}\dot{\chi} + C_{\psi\psi}\dot{\psi} + K_{\psi\chi}\chi + K_{\psi\psi}\psi = \mathcal{M}_\psi \text{ , the steer equation.}$$

* If we had allowed flexible training wheels (say) to help support the <u>rear assembly</u> against leaning, the lean equation would have to have the supporting moment $\mathcal{M}_\chi$ on the right hand side.

The coefficients to the lean equation are

$$
\begin{aligned}
M_{\chi\chi} &= T_{yy} \\
M_{\chi\psi} &= F'_{\lambda y} + \frac{c_f}{c_w} T_{yz} \\
C_{\chi\chi} &= 0 \\
C_{\chi\psi} &= -\left(H_f \cos\lambda + \frac{c_f}{c_w} H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w} m_t \overline{h}_t\right) \\
K_{\chi\chi} &= -\,g m_t \overline{h}_t \\
K_{\chi\psi} &= g\nu - H_t V \frac{\cos\lambda}{c_w} - V^2 \frac{\cos\lambda}{c_w} m_t \overline{h}_t
\end{aligned}
$$

and the coefficients to the steer equation are:

$$
\begin{aligned}
M_{\psi\chi} &= F'_{\lambda y} + \frac{c_f}{c_w} T_{yz} \\
M_{\psi\psi} &= F'_{\lambda\lambda} + 2\frac{c_f}{c_w} F''_{\lambda z} + \frac{c_f^2}{c_w^2} T_{zz} \\
C_{\psi\chi} &= H_f \cos\lambda + \frac{c_f}{c_w} H_t \\
C_{\psi\psi} &= V\left(\frac{\cos\lambda}{c_w} F''_{\lambda z} + \frac{c_f}{c_w}\left(\frac{\cos\lambda}{c_w} T_{zz} + \nu\right)\right) \\
K_{\psi\chi} &= g\nu \\
K_{\psi\psi} &= -\,g\nu \sin\lambda + V H_f \frac{\sin\lambda \cos\lambda}{c_w} + V^2 \frac{\cos\lambda}{c_w}\nu
\end{aligned}
$$

Note that most coefficients are functions of velocity. In fact the angular momentum H_f for the front wheel typically could be written as $H_f = V(J_f/a_f)$, where a_f is front wheel radius and J_f is front wheel polar moment of inertia; and similarly for the rear. However there is also the possibility of adding independent high-speed gyros to the bicycle, in which case H_f and/or H_r might be constant, or a negative multiple of speed, etc.

When developed in this form, the equations display a degree of symmetry.

Section 2: Derivation of Constraint Relations for the Basic Bicycle Model

In chapter III, equations (3.2a-e) were used to simplify the Lagrange's equations and said to be derived from the constraint relations on the front and rear contact points motion for the Basic bicycle model. This section shows how the derivation of equations (3.2a-e) follows from simplifying the constraints that exist on a bicycle with thin rigid disks as wheels.

For the Basic bicycle model we have assumed the tires to be part of the wheels which are assumed thin rigid disks. This implies infinitely stiff tires, so no side-slip angle can exist on our Basic bicycle model. That is, the direction of the velocity of the contact point is defined by the intersection of the plane of the first wheel and the ground plane. This direction is referred to as the instantaeous direction that the bicycle is headed for the Basic bicycle model.

In addition, we assume enough friction exists between the thin rigid disks and the ground so that there is no relative motion between the point of contact of the rigid wheel and the ground. That is, there is no sliding of the wheel on the surfaces of the road. In the practical sense this could be caused by oil on the pavement or loose gravel.

Based on the above assumptions, 4 nonlinear kinematic rolling constraints exist for the Basic bicycle model. These constraints are nonholonomic and can be added only after developing Lagrange's equations.[4] The constraint equations relate the velocity of the rear and front contact point velocity to their respective heading in the ground plane and respective wheel rotations. Writing these in their nonlinear form for the rear contact point velocity we have,

$$\dot{Y}_r = a_r \dot{\phi}_r \cos\theta_r \tag{A.2a}$$

$$\dot{X}_r = -a_r \dot{\phi}_r \sin\theta_r \tag{A.2b}$$

where $\dot{\phi}_r$ is the angular velocity of rotation of the rear wheel in its own plane, that is, the spin rate. Similarly for the front contact point,

$$\dot{Y}_f = a_f \dot{\phi}_f \cos\theta_f \tag{A.2c}$$

$$\dot{X}_f = -a_f \dot{\phi}_f \sin\theta_f \tag{A.2d}$$

where $\dot{\phi}_f$ is the spin rate of the front wheel. These equations (and all others in this chapter unless otherwise indicated) assume the sign convention used in chapter III.

Equations (A.2a-d) represent 4 nonholonomic constraints imposed on the Basic bicycle model which has seven generalized coordinates: X_r, Y_r, θ_r, ψ, χ_r, ϕ_r, ϕ_f. Hence, for the given assumptions, the bicycle has three degrees of freedom.[5] However, as a consequence of linearizing the equations of motion, for the case of linearized equations of motion in the derivation it is shown that $\dot{Y}_r$ is constant to first order. Hence, for the linearized model, only two degrees of freedom exist.

What follows is the linearization and simplification of the nonlinear nonholonomic constraints. As will be shown, as a result of the linearization 2 constraints become holonomic and 2 remain nonholonomic.

By assuming small angles of rotation

the 4 nonlinear nonholomic constraints reduce to four linear nonholonomic constraints as follows,

$$\dot{Y}_r = a_r \dot{\phi}_r \tag{A.3a}$$

$$\dot{X}_r = -\dot{Y}_r \theta_r \tag{A.3b}$$

[4] This is what the author interprets from Goldstien [1980].

[5] See Neimark and Fufaev [1967].

$$\dot{Y}_f = a_f \dot{\phi}_f \tag{A.3c}$$
$$\dot{X}_f = -\dot{Y}_f \theta_f \tag{A.3d}$$

Eliminating the auxiliary variables we can simplify these expressions.

Using equation (3.1b), (A.3c) becomes,

$$\dot{Y}_f = \dot{Y}_r = a_f \dot{\phi}_f \tag{A.4}$$

Taking the time derivative of equation ($A.1a$) and substituting equation ($A.3b$) for $\dot{X}_r$,

$$\dot{X}_f - \dot{X}_r = -c_w \dot{\theta}_r + c_f \dot{\psi} \tag{A.1a}$$

$$\dot{X}_r = \dot{X}_f + c_w \dot{\theta}_r - c_f \dot{\psi} = -\dot{Y}_r \theta_r \tag{A.5}$$

Substiting ($A.1d$) into ($A.5$) and cancelling $\dot{X}_f$,

$$\theta_r = \theta_f - \psi \cos\lambda \tag{A.1d}$$

$$\dot{X}_f + c_w \dot{\theta}_r - c_f \dot{\psi} = -\dot{Y}_r(\theta_f - \psi \cos\lambda) \tag{A.6a}$$

$$c_w \dot{\theta}_r - c_f \dot{\psi} = \dot{Y}_r \psi \cos\lambda \tag{A.6b}$$

Rewriting equations $A.3a$, $A.3b$, $A.6b$, and $A.3c$ we have the four linear constraints expressed in terms of the generalized coordinates,

$$\dot{Y}_r = a_r \dot{\phi}_r \tag{A.7a}$$
$$\dot{Y}_r = a_f \dot{\phi}_f \tag{A.7b}$$
$$\dot{\theta}_r = \frac{c_f}{c_w} \dot{\psi} + \psi \dot{Y}_r \frac{\cos\lambda}{c_w} \tag{A.7c}$$
$$\dot{X}_r = -\dot{Y}_r \theta_r \tag{A.7d}$$

As is shown in Chapter III the time derivatives of these constraint equations is sometives required. Taking their time derivatives we have,

$$\ddot{Y}_r = a_r \ddot{\phi}_r \tag{A.7e}$$
$$\ddot{Y}_r = a_f \ddot{\phi}_f \tag{A.7f}$$
$$\ddot{\theta}_r = \frac{c_f}{c_w} \ddot{\psi} + \dot{\psi} \dot{Y}_r \frac{\cos\lambda}{c_w} \tag{A.7g}$$
$$\ddot{X}_r = -\ddot{Y}_r \theta_r - \dot{Y}_r \dot{\theta}_r = -\ddot{Y}_r \theta_r - \frac{c_f}{c_w} \dot{Y}_r \dot{\psi} - \psi \dot{Y}_r^2 \frac{\cos\lambda}{c_w} \tag{A.7h}$$

Equations (B.7a-b) are used to simplify Lagrange's equations by reducing the number of degrees of freedom in the final equations. And as is shown in chapter III $\ddot{Y}_r$ is zero to first order so we have eliminated this term in the expression presented in chapter III.

Note that the lean angle χ_r is not present in these relations, and they are not dependent on the radii of the wheels.
******

Rolling constraints

In general a bicycle has four nonholonomic contraints relating the motion of the location of the rear and front contact point in the inertial XY reference plane, to the orientation and rotation rate of the wheels. These relations are referred to as rolling constraints. We can specify these rolling constraints due to our assumptions of no 'sliding' (skidding) and no 'side slip' (due to tire deformation). These relations can be linearized according to our assumption of only small deviations from the equilibrium motion, because they are added to the problem only *after*the Lagrange equations are developed. The relevant first-order results, which are used later to develop the equations of motion, and which are derived in more detail in section 2 of appendix A, can be expressed as follows,

$$\ddot{Y}_r = a_r\ddot{\phi}_r \tag{3.2a}$$
$$\ddot{Y}_r = a_f\ddot{\phi}_f \tag{3.2b}$$

$$\dot{\theta}_r = \frac{c_f}{c_w}\dot{\psi} + \psi\dot{Y}_r\frac{\cos\lambda}{c_w} \tag{3.2c}$$

$$\ddot{\theta}_r = \frac{c_f}{c_w}\ddot{\psi} + \dot{\psi}\dot{Y}_r\frac{\cos\lambda}{c_w} + \psi\ddot{Y}_r\frac{\cos\lambda}{c_w} \tag{3.2d}$$

$$\ddot{X}_r = -\ddot{Y}_r\theta_r - \dot{Y}_r\dot{\theta}_r = -\ddot{Y}_r\theta_r - \frac{c_f}{c_w}\dot{Y}_r\dot{\psi} - \psi\dot{Y}_r^2\frac{\cos\lambda}{c_w} \tag{3.2e}$$

Later equations (3.2a,b) are used to prove that for small disturbances, $\ddot{Y}_r = 0$ to first order, so it can be eliminated from (3.2d,e).

Originally the number of generalized coordinates is seven. By implementing the constraint equations (3.2a-e), four of which are independent, the variables θ_r, X_r, ϕ_r, and ϕ_f will be eliminated from Lagrange's equations, and thus the system will be left with only three generalized coordinates: χ_r, the lean angle, ψ, the steer angle, and Y_r, the coordinate that locates the rear wheel along the Y axis. These three generalized coordinates represent the three degrees of freedom for the linearized Basic bicycle model. However, as just mentioned $\ddot{Y}_r$ is zero to first order and therefore $\dot{Y}_r$ becomes a constant to first order.[7] Thus because Y_r is not present in the final equations there are only *two nontrivial* degrees of freedom for the linearized model. Mathematically this means that two second order differential equations, or one fourth order differential equation, represent the motion of the system.

In general, these constraints should be added to the problem only after Lagrange's equations have been found. Also, note that the generalized coordinate χ_r is not present

[7] This is one of the subtle points we referred to earlier and is discussed more in section 6 of appendix A.

in the rolling constraint relations and that the relations are not dependent on the wheel radii.

Elimination of χ_r and θ_r from equations (3.12a,b)

Substituting the remaining rolling constraints (3.2c-e) (where $\dot{Y}_r$ has been set equal to a constant V and $\ddot{Y}_r = 0$),

$$\dot{\theta}_r = \frac{c_f}{c_w}\dot{\psi} + \psi V \frac{\cos\lambda}{c_w} \tag{3.2c}$$

$$\ddot{\theta}_r = \frac{c_f}{c_w}\ddot{\psi} + \dot{\psi} V \frac{\cos\lambda}{c_w} \tag{3.2d}$$

$$\ddot{X}_r = -V\dot{\theta}_r = -\frac{c_f}{c_w}V\dot{\psi} - \psi V^2 \frac{\cos\lambda}{c_w} \tag{3.2e}$$

into equations (3.12a,b), the variables $\dot{\theta}_r$, $\ddot{\theta}_r$, and $\ddot{X}_r$ can be eliminated. Thus we form two coupled second order linear differential equations with constant coefficients.
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REDUCED EQUATIONS OF MOTION

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Representation of the Equations of Motion

Equations (3.14) and (3.15) can be expressed either as two second order differential equations or one fourth order differential equation. For the case of two second order differential equation it is common to write the equations in matrix form as follows,

$$\left[\overbrace{\begin{pmatrix} \mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi} \\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi} \end{pmatrix}}^{\mathcal{M}} D^2 + \overbrace{\begin{pmatrix} \mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi} \\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi} \end{pmatrix}}^{\mathcal{C}} D + \overbrace{\begin{pmatrix} \mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi} \\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi} \end{pmatrix}}^{\mathcal{K}}\right] \begin{pmatrix} \chi_r \\ \psi \end{pmatrix} = \begin{pmatrix} M_{\chi_r} \\ M_{\psi} \end{pmatrix}$$

where $\mathcal{M}$, is the mass matrix , $\mathcal{C}$ is the damping matrix, $\mathcal{K}$ is the stiffness matrix, and D is the differential operator. Expanding $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{K}$ we have, [see Hand thesis page 28 & 29 for defs. we need TEX file]

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi} \\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi} \end{pmatrix} = \begin{pmatrix} need & need \\ need & need \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi} \\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi} \end{pmatrix}$$
$$= \begin{pmatrix} need & need \\ need & need \end{pmatrix}$$

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi} \\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi} \end{pmatrix} = \begin{pmatrix} need & need \\ need & need \end{pmatrix}$$

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Augmented Basic Bicycle Equations

Applications of the Governing Equations

The equations of bicycle motion, which govern rear-assembly lean-angle χ and steer angle ψ, are:

$$M_{\chi\chi}\ddot{\chi} + K_{\chi\chi}\chi + M_{\chi\psi}\ddot{\psi} + C_{\chi\psi}\dot{\psi} + K_{\chi\psi}\psi = \mathcal{M}_{\chi} \quad \text{(the lean equation)},$$

and

$$M_{\psi\chi}\ddot{\chi} + C_{\psi\chi}\dot{\chi} + K_{\psi\chi}\chi + M_{\psi\psi}\ddot{\psi} + C_{\psi\psi}\dot{\psi} + K_{\psi\psi}\psi = \mathcal{M}_{\psi} \quad \text{(the steer equation)}.$$

$\mathcal{M}_{\psi}$ is the steering moment exerted by the rider (or by some device attached to the *rear assembly*), and the tipping (or supporting) moment $\mathcal{M}_{\chi}$ is normally zero.

By solving, or studying the stability of, these equations, a variety of topics may be studied. Here, several are described very briefly, to give the reader an idea of the equations' potential usefulness.

Steering Moment Given as Function of Motion

If the steering moment $\mathcal{M}_{\psi}$ is given as a function of bicycle motion (for example, by a 'balance — controller' which applies a steering moment depending on the bicycle's lean), this may be incorporated in the analysis by modifying the steer equation. Then the lean equation and the modified steer equation can be used simultaneously to determine the behaviour of χ and ψ. This approach can be used to evaluate balancing strategies or controller designs.

On the other hand, we may want to study the effect of controlling the steer angle as a function of bicycle lean. If ψ is given in terms of χ, the lean equation must be used to solve for χ. If desired, the solution may be inserted into the steer equation to find the resulting steer moment.

No-Hands Stability

After setting the steering moment to zero, we may use both equations simultaneously to study no-hands motion and stability (the primary focus of this report).

Steer Moment Given as Function of Time

The heading of a stable bicycle may be affected (and thus in some sense controlled) by applying steering moments. The steering produced by instantaneous or ramped application of a steer moment can be deduced by solving the equations of motion with the given $\mathcal{M}_\psi(t)$ on the right-hand-side, under the assumption that $\psi, \dot{\psi}, \chi, \dot{\chi} = 0$ initially. Similarly, if the wind tends to tip the bicycle, the time-dependent tipping moment $M_\chi(t)$ must be used in the lean equation.

Skateboards

Skateboards have axles designed so that the wheels steer by an amount proportional to the lean angle. Since both the front and the rear wheels of a skateboard steer, to apply the bicycle analysis it is necessary to define an imaginary 'non-steering rear-wheel contact point', P_r^{im}, as the correct origin for defining moments of inertia, etc. If the front and rear axles steer exactly opposite amounts, P_r^{im} is half-way between the front and rear contact points (any trail is ignored for simplicity). More generally, for a given lean angle imagine that there is a series of wheels along the length of the skateboard, whose steer angles vary linearly between the actual values at the front and back. Then P_r^{im} is at the imaginary wheel whose steer angle is zero (this might be off the skateboard altogether). Then the wheelbase (i.e. the distance to the front contact) and other important quantities are calculated with respect to P_r^{im}, not with respect to the actual rear contact.

Once the appropriate equation coefficients have been provided, the analysis is performed on the lean equation after setting the steer angle ψ proportional to the lean angle χ (cf. *"Steer Angle Given"*, above).

Tricycles

We can study the behaviour of tricycles by ignoring the lean equation (which should have the supporting moment $\mathcal{M}_\chi$ on the right hand side), and employing the steer equation with $\chi = 0$. If $\psi(t)$ is given, this equation delivers the steering torque $\mathcal{M}_\psi$ required; and if we set $\mathcal{M}_\psi = 0$, it determines hands-off stability, and allows us to find $\psi(t)$ if desired. Once $\psi(t)$ is either specified or found, the lean equation gives the supporting moment $\mathcal{M}_\chi$ supplied by the rear wheels.

This tricycle analysis can also be used to investigate whether a shopping cart will tend to travel straight, when rolling freely either forwards or backwards.

Lateral Forces on Wheels

Bicycle wheels, especially in the traditional large size, are rather weak laterally — side loads cause high spoke stresses and may even lead to collapse. The equations of motion can be used to find these forces in any given motion; perhaps the most interesting case is when the rider varies the steer angle rapidly, or applies a large steer torque. In this case we first solve for the motion as above so that both $\chi(t)$ and $\psi(t)$ are known. Then we use equations (1) and (2b) to solve for the horizontal forces $\mathcal{F}_{Xr}$ and $\mathcal{F}_{Xf}$, in terms of χ and ψ and their derivatives. The lateral force at each wheel is not just the horizontal force, but because of lean also includes a component of the vertical force. So at the rear the lateral force is $\mathcal{F}_{Xr} - \chi(gm_t\frac{c_w - l_t}{c_w})$, and at the front it is $\mathcal{F}_{Xf} - (\chi - \psi\sin\lambda)(gm_t\frac{l_t}{c_w})$.

The above topics can be treated in a straightforward way with essentially no modifications to the equations of motion. To study subjects such as the following requires more thought, and often a significant alteration of the equations. The greatest complication arises when new *degrees of freedom* are introduced: ways mass can move independent of the four rigid bodies which define the basic bicycle.

Finite Fore and Aft Forces

This concerns the effects of finite forces acting mostly in the forward or backwards direction. In a small steering and leaning motion, these may slightly shift their direction or point of application in such a way as to enter the lean and steer equations.

Aerodynamic 'Lift' Forces in Still Air

When a 'flat' object moves relative to the air, often the largest aerodynamic forces are 'lift' forces perpendicular to its relative motion (a horizontal force is still called lift). There is some concern that a covered (disk) wheel or covered bicycle might generate such forces in small steering motions, which would destroy stability. The idea is perhaps that if the front wheel turns a little, the lift forces will cause it to turn even more (and also apply a tipping moment to the bicycle).

While we have not worked out whether these effects destroy self-stability, at least it seems likely that any instability arising from this still–air 'viscous damping' will occur slowly, and may not be too important for a rider-controlled bicycle. However sidewinds — steady, or especially sudden — will produce large disturbing forces.

Tire Phenomena

— tire deformation under maneuvering forces gives rise to 'sideslip', 'camber thrust', etc. (raising the pressure reduces the effect).

— *bicycle* tire behaviour may be well modelled by a series of springy fingers. (Note: a tire's vertical stiffness apparently is

— Is tire 'sideslip' an important factor in bicycle dynamics? Analytical estimates of tire 'sideslip' behaviour, and of the forces which cause it, suggest that tire deformation phenomena may be much more important for motorcycles in their typical riding conditions than for bicycles in theirs (and also that sideslip may be insignificant in steady turns, because the force of the ground is nearly in the plane of the wheel). On the other hand, wheel (and bicycle) flexibility appears to be an important factor in bicycle shimmy. The question is whether tire behaviour is important in bicycle shimmy conditions.

— Brief instructions for incorporating tire behaviour and wheel flexibility in the equations of motion — and the order of equation which results from each choice. For best understanding, perhaps it is preferable to retain simplicity by modelling the front tire only.]

Rider Body Articulation

[Not yet written. Riders may both bend and shift sideways at the waist; are both important separately? Number of equations; sketch how to derive them.]

Frame Flexibility

[Not yet written. 'Torsional' bicycle flexibility may be important for the study of shimmy — in contrast to wheel flex, it involves the gyroscopic effects of wheel tilt. How to derive equations. Possibility of developing simplified equations which represent high-frequency oscillations alone?]
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What to do with the Equations of Motion

Once the equations of motions have been derived they can be used in various type analyses. Among them are:

1) Given arbitrary χ_r and ψ we can calculate the required moments M_{χ_r} and M_ψ.

2) If $M_{\chi_r} = 0$ we can solve for the behavior of χ_r for a prescribed ψ (i.e. we can define a controller $\psi(\chi_r)$ and analyze stability).

3) Solve for M_ψ (given that $M_{\chi_r} = 0$ for and ψ.

4) Given $M_{\chi_r} = 0$ and $M_\psi = 0$ we can solve for the equation of motion and analyze the bicycle's self stability.

5) Passive mechanisms such as gyroscopes, springs, and dampers can be easily added to the equations of motion to analyze their effect on the bicycle
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[PRIMITIVE BICYCLE GOES HERE]
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The Meaning of Bicycle Stability

For the stability of the bicycle we are really concerned with the stability of the variables representing the lean and the steer degrees of freedom, χ_r and ψ, respectively. That is, after the bicycle system is perturbed, θ_r and $\dot{X}_r$, can take on new nonzero equilibrium positions and the system, for all practical purposes, can still be considered stable. We lack mentioning the other generalized coordinates' time derivatives $\dot{Y}_r$, $\dot{\phi}_r$, and $\dot{\phi}_f$, which do not effect the linearized equations of the Basic bicycle model. Note however, these variables can also take on new equilibrium positions. Hence, when we discuss bicycle stability we apply the definition of dynamic stability only to the lean and steer (χ_r and ψ) degrees of freedom.

We therefore define bicycle stability in the following way:

A bicycle is stable, if, after a very small disturbance from its vertical straight-ahead equilibrium motion it asymptotically approaches a vertical straight-rolling configuration in the lean and steer degrees of freedom, χ_r and ψ, respectively.

Before going on, we point out that just because a bicycle design configuration is found to be stable does not necessarily imply that a rider would or should desire it more than an unstable bicycle design configuration. We emphasize that mathematically based definitions of what is more stable compared to what a rider feels is 'more comfortable' or 'easier to ride' could differ substantially.

Stability Analysis Techniques

Historically, stability of the vertical straight ahead (upright) equilibrium configuration of the bicycle and motorcycle has been studied in four ways: analysis of the eigenvalues and eigenvectors of the system; numerical integration of the equations of motion and study of the solutions; application of the Routh-Hurwitz criteria; or experimental observations of bicycle behaviour. Each approach has its merits and drawbacks, as we will now explain.

First, using the system of equations, or the characteristic fourth order polynomial derived in chapter III, one can determine the eigenvalues (the roots to the fourth order equation), and eigenvectors (mode shapes) of the system which can be used to calculate the natural frequencies and mode shapes of the system representing a bicycle traveling at a particular speed. If, in such an analysis, any of the eigenvalues have positive real parts, the solution to the system will grow away from the equilibrium in time and the system is unstable (at least based on the linearized equations). Complex eigenvalues represent oscillatory solutions whose real parts determine whether the amplitude of the oscillations will grow or decay in time. Thus eigenvalue-eigenvector analyses allow for various design configurations to be compared by numerically evaluating their eignevalues. From this we can determine which design configuration is mathematically more or less stable. Traditionally, this has been done by saying the more negative the real part of the eigenvalue, the more stable the system. It is from the eigenvalue-eigenvector stability analyses that the terms capsize, weave and wobble modes have been adopted to described the motion of the bicycle or motorcycle at various speeds.

A second method used to study stability is to numerical integration of the equations of motion using a digital or analog computer. Plots of the response (the solution) to various inputs can be used to quantitatively and qualitatively characterize the stability of the system, and/or changes in design parameters (just as with the eigenvalue-eigenvector analysis.). This method can also be used to verify stable equilibrium motion(s), if any exist.

One of the major problems associated with numerical studies is the verification of the equations themselves. Because nonlinear equations are generally more complicated to solve for than linear equations, we believe the probability of mathematical error is higher with nonlinear models (and quite possibly little further understanding is gained). Thus, although the computer may be powerful enough to solve nonlinear equations, the results should be reviewed with caution until the nonlinear equations have been verified in some way.[7]

A third method of evaluating bicycle stability is to apply the Routh-Hurwitz criteria. This method allows for the stability of an equilibrium configuration to be determined based on the *coefficients* of the fourth order polynomial. It determines whether any of the eigenvalues (roots to the fourth order polynomial) have real positive parts, *without* actually solving for them. This technique yields criteria which directly lead to analytical expressions linked to the stability of the system. Qualitative statements can be made by comparing various design configurations stability regions. Quantitative statements can be formulated from analytical expressions linked to stability.

The Routh-Hurwitz criteria are limited in the quantitative aspect, in that, it does

[7] We note when other's equations were compared to those derived in chapter III that only 2 of 18 agreed with our resulting equations.

not give any exact measure of *how* stable a *particular* design configuration is relative to another.

This paper does not discuss the experimental methods used in analyzing bicycle stability or their results. The interested reader is referred to the work of Kondo [1955], Kageyama [1962], Kondo [1962], Fu [1966], Roland [1970], Jones [1970], and Eaton [1971].

Discussion of Analysis Techniques

SPECIAL CASES

Note: in some of these cases, a bicycle which is not (self-)stable may still be balanced very easily by a rider

— A bicycle (+rider) with *standard* dimensions and mass–distribution can *never* be (self–)stable if the gyroscopic effects of the front and rear wheels are somehow canceled. But this is not to say that all bikes must have gyroscopic effects to be stable.

— A 'primitive' bicycle with a vertical steering axis, symmetry of the front assembly about the steering axis, and an arbitrary distribution of mass on the rear assembly (see Fig.1a), is slightly unstable: unless the rear assembly has enough high-up mass in front of the steering axis (as when the rider is leaned forward, and the front wheel bears most of his weight; Fig.1b). In the case that we checked, the computer showed that it can be stabilised by making the trail slightly positive, or by moving the front-assembly center of mass ahead of the steering axis (while leaving the front c.m. inertia tensor with a vertical principal axis). By combining these effects, the bicycle may be stabilised even when the trail is somewhat negative, as long as the front–assembly center of mass is sufficiently far forward of the steering axis (Fig.1c). The gyroscopic effect of the front wheel is essential to the stability of this vehicle, because in its absence C and D will always be negative. On the other hand, too large a gyroscopic influence from the rear wheel will keep RH negative.

— An isolated rolling wheel (see Fig.2a for the equivalent bicycle) is not quite stable, because it never uprights itself from a steady turn, and it never stops oscillating to either side of its average lean angle. However, a wheel with a downwards tilted bar attached [at or just below the axle] (Fig.2b) does stop oscillating above a certain speed.

— Perhaps motivated by the example of a furniture caster, many authors have suggested the importance of trail to stability, however with little solid evidence. Trail (or rather *mechanical trail,* the perpendicular distance behind the steering axis of the front contact point) does strongly affect the handlebar torque required in a given maneuver, partly because in a rapid steering motion it will make the rear assembly yaw more, but mostly because any ground-contact force component which is perpendicular to the wheel exerts a turning moment (with mechanical trail as the lever-arm) which is partly resisted by the rider's hands. Moderate such moments are probably important, as feedback for steering, or to allow the rider to control the bicycle by forces rather than displacements (control of the steering angle would have to be precise at high speeds, and such control would prevent the bicycle from contributing to the balancing task). However large moments would make steering a real effort. So we looked at the stability of a bike with a massless front assembly, no gyroscopic effects (as if it is sliding on skates), and some steering-axis tilt and trail (Fig.3a). Surprisingly for us, such a configuration can be stable for a huge speed range. The trail must be positive, but it is best if it is small. Finally, the mass of the rider must be stretched out along a forwards–leaning line (which, however, must not lean as far as the line from the rear contact P_r to the center of mass m_t). A crude model made with furniture casters (Fig.3b) was indeed stable, though perhaps not to the extent of the theoretical model, which predicted stability from a very slow velocity up to infinity. Despite this tremendous stability, such a design would require only tiny handlebar torques to steer it.

— The lean equation shows that a bicycle can be stabilised if the steer angle is controlled to be proportional to the lean:

$$\psi \approx -\mathrm{k}_0 \chi \ .$$

Skateboard wheels actually steer according to this rule, so some results of their stability analysis can be found easily.

— Tricycles don't have the same balancing problems as two-wheeled vehicles, but they are still reputed to be hard to control in some circumstances. We may analyse their behaviour by assuming that a bicycle has an adequate supporting moment $\mathcal{M}_\chi$ holding it upright ($\chi = 0$), and using the steer equation to evaluate stability of no-hands riding or the torque required in a steady turn.