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BICYCLE STEERING DYNAMICS AND SELF–STABILITY: A SUMMARY REPORT ON WORK IN PROGRESS

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[*** Please note that this is just a draft, and may contain serious errors. Square brackets such as these denote unfinished sections, or questions which need to be cleared up. ***]

ABSTRACT

Moving bicycles often balance themselves (and recover from disturbances) without rider assistance, a feature that bicycle designers might want to take into account.

However, while the property of self–stability has been studied often (for an ideal bicycle), not all of these investigations appear to be correct; and most authors have tended to solve complicated equations numerically to evaluate stability in specific cases, without providing guidelines on how to estimate or improve stability.

The goal of this research was not to perform more realistic (i.e. detailed) analyses or simulations, but rather to develop a good grasp of the effects of important design features. We hoped to understand linear self–stability of an ideal bicycle in the same way that one understands small oscillations of an ideal mass-spring-damper system.

Towards this end, we have developed a simple notation and a short derivation of the linearised equations of motion, and can express self–stability conditions in terms of a few algebraic inequalities relating a bicycle's design parameters. However, the sought-for understanding has been only slightly achieved. Some special unorthodox bicycle designs are discussed to demonstrate the scope of the methods.

TABLE OF CONTENTS

I. INTRODUCTION

Though modern bicycles steer and balance remarkably well, how and why they do this is still something of a mystery. Common knowledge of this matter is incomplete at best, and perhaps also inconsistent. Similar-appearing bicycles can have various 'feels', for example that of 'twitchiness'. Old problems such as self–excited shimmy, and new ones such as crosswind-sensitivity of front disk wheels, have not really been solved. Standard rules of thumb appear not to work well for radically different designs.

The principal concerns which moved us to undertake this study include:

— Lack of clear confirmed *explanations* for bicycle steering phenomena (self–stability, feel, shimmy ...)
— Lack of *design guidelines* for achieving (or preventing) particular behaviour
— Uncertainty about complex analyses in the literature

To address these matters, we felt an important first step would be to derive, confirm, and explore the governing equations for a reasonably complex idealised bicycle. This would permit us to see effects of any given trail, head angle, weight distribution, etc.; and to build intuition by studying the properties of a variety of 'simplified' bicycles. It seemed very important to try to understand this idealised bicycle well: if it is a good model, this will help produce the answers we are seeking; and if the model turns out to be partly or wholly inapplicable, good understanding will help pinpoint its shortcomings, and give a start in developing and analysing a more realistic model.

We studied the motion of an ideal moving bicycle which is only slightly perturbed from upright straight-ahead rolling. The bicycle has rigid knife-edge wheels; a rigid rear-frame including rigidly attached immobile rider; and a rigid steerable 'front-frame' (fork+stem+bars), on which the rider may or may not exert a steering torque. Rolling resistance, bearing resistance, and aerodynamic forces are all neglected, so no power is needed for steady forward motion, and the rider is assumed not to be pedalling.

In exploring this model, which encompasses arbitrary frame geometry and mass distributions, we aimed to *develop equations of motion,* which could be used to predict such things as lean angles, steer torque, and lateral forces on wheels, in any given maneuver. We planned to *compare our equations* against many published derivations, so that other users of the literature would waste less time. Most importantly, we hoped to *understand self–stability* of an uncontrolled bicycle: simple experiments and published analyses show that bicycles can be self–stable in a limited speed range, but with little explanation of why this should be so, or how it may be achieved by design. Such understanding might not resolve many bicycle handling problems, but it seems like an important first step (bicycles which are very unstable probably take extra skill and effort to ride; bicycles which are adequately stable may free the rider to attend to other things).

We believe this work will make it easier to study related problems: augmented bicycle models (with tire 'slip' angles and scrub torques, frame elasticity, aerodynamic forces, rider lean, acceleration/braking); rider perceptions, preferences, and control strategies; and so forth.

Following is a summary of our results to date in following this plan. It includes an explanation of how we have used the equations of motion to evaluate stability, and reviews our progress to date in understanding the stability of bicycles (and other simple balancing devices). The equations themselves are described and derived (non-rigorously) in APPENDIX A. In APPENDIX B it is briefly suggested how tire models, aerodynamic forces, rider control actions, etc. may be added to the equations. APPENDIX C contains some general results from studying the formulas which determine stability.

This report draws heavily on *Comparisons and Stability Analysis of Linearized Equations of Motion for a Basic Bicycle Model,* the 1987 M.Sc. Thesis of R. Scott Hand, Department of Theoretical and Applied Mechanics, Cornell University. Hand describes notation in detail, derives the equations of motion very carefully, and compares them to many others in the literature. In this preliminary report I have not attempted to cite many others who have worked on the topics discussed here: a separate comprehensive review is tentatively planned.

II. OVERVIEW OF EQUATIONS OF MOTION

From APPENDIX A we may take the two equations of motion governing rear-assembly rightwards lean-angle χ, and front-assembly leftwards steer angle ψ (dots denote time derivatives):

$$M_{\chi\chi}\ddot{\chi} + K_{\chi\chi}\chi + M_{\chi\psi}\ddot{\psi} + C_{\chi\psi}\dot{\psi} + K_{\chi\psi}\psi = 0 \quad \text{(the lean equation)},$$

and

$$M_{\psi\chi}\ddot{\chi} + C_{\psi\chi}\dot{\chi} + K_{\psi\chi}\chi + M_{\psi\psi}\ddot{\psi} + C_{\psi\psi}\dot{\psi} + K_{\psi\psi}\psi = \mathcal{M}_{\psi} \quad \text{(the steer equation)}.$$

The M, C, K coefficients are defined in detail in APPENDIX A; they are all constants at any given bicycle velocity V. $\mathcal{M}_{\psi}$ is the steering moment exerted by the rider; and if we had allowed spring-mounted training wheels (say) to help support the rear assembly against leaning, the lean equation would have to have the supporting moment $\mathcal{M}_{\chi}$ on the right hand side.

APPENDIX B contains a list of ways these equations of motion may be applied.

Before using these equations of motion, it is important to pause and consider what they may be good for.

Is the derivation correct? We believe it is. First, each term makes sense physically. Also, as mentioned in APPENDIX A, we get the same results from a more rigorous Lagrangian development. Finally, we have a high level of confidence because of comparisons to derivations in the literature which employ other approaches. Our equations are entirely consistent with those of Whipple (1899, with typographical corrections), Carvallo (1901), Sommerfeld & Klein (1904), Döhring (1955), Neimark & Fufaev (1967, with corrected potential energy), R. Sharp (1971, with a minor algebraic correction), and Weir (1972, his Dissertation, not his paper). (Many other authors' equations disagree with these and each others'. An important comparison yet to be made is with the equations of Roland (1973) and Rice (1976).)

Are all the significant physical phenomena included? Unfortunately, probably not. For example, the points where the wheels contact the ground are not simple rigid-body contacts — their 'scrubbing' and 'side-slipping' behaviour is more complex:

— I have had the experience of replacing worn tires with new ones, and immediately thereafter feeling much less stable in high-speed turns.

— The equations of motion of an ideal rolling disk predict unceasing oscillation, whereas one can easily observe that a real disk or isolated bicycle wheel straightens up and stops oscillating. [If the disk equations are supplemented with a scrubbing torque (due to twisting about a vertical axis), these oscillations die away.]

In fact, motorcycle-tire behaviour figures heavily in motorcycle modelling (though this may have to do with their lower pressures, and the greater masses and speeds involved). Further, frame or wheel elasticity may be important:

— Some real bicycles *shimmy,* at a frequency which appears close to that of torsional oscillation of a stationary bicycle, i.e. lateral oscillation of the head tube when the seat and the wheel-contacts are held fixed.

Factors such as these are undoubtedly important for some aspects of bicycle behaviour, but we may hope not for all— observations or experiments are needed to determine the limits of applicability of our equations. Even if the equations are found to be largely irrelevant for typical bicycles, perhaps the methods of developing and studying them may be applied usefully to subsequent versions.

Assuming for the present that they are adequate to describe *some* aspects of bicycle behaviour, what do these equations say about bicycle design? The real difficulty here is that we don't have many well-defined design requirements. (For example, no-one has said "a good bicycle must have a particular variation of lean angle with time, $\chi(t)$, when a given torque $\mathcal{M}_{\psi}$ is briefly applied to the handlebars", or "in a given maneuver, the lateral force acting on the front wheel must not exceed a certain structurally safe value".) One of the simpler topics to look at, which appears at least *somewhat* relevant to what a rider might feel and like (or dislike) is self–stability. We are concentrating on this until better questions are formulated.

Our idea is that the more unstable a bicycle is, the more skill and concentration it will require of the rider, and the less forgiving of momentary inattention it will be. On the other hand, a bicycle which *assists* in the balancing task should free the rider to focus on traffic conditions, or on pedalling hard.

The precise problem we are studying is *uncontrolled (no-hands, rigid-rider) motion: when does a bicycle balance itself?* In that case we must set $\mathcal{M}_\psi = 0$ (i.e. no steering moment), and simultaneously solve the above system of two equations to describe the motion resulting from any disturbance.

III. DETERMINATION OF NO-HANDS STABILITY (SELF-STABILITY)

By stability, we mean just one thing: that the moving bicycle will *automatically* return to an upright configuration after any moderate disturbance, or after the handlebars are released in a steady turn. Such behaviour has been observed experimentally by many people including ourselves, and has been predicted by a large number of theoretical analyses (including those mentioned above).

In contrast, a bicycle which is unstable may be expected either to weave back and forth to a greater and greater extent, or to lean increasingly to one side or other. The loss of balance may take place either slowly (thus permitting rider intervention) or quickly.

Note that a stable bicycle need not be slow-responding or require large handlebar torques. To understand this, consider a balsa-wood pendulum hanging from a pivot: this is unquestionably stable according to the above definition, yet it may quickly and easily be displaced.

To solve for the no–hands motion of a bicycle we can eliminate one variable (either χ or ψ) from the above second-order differential equations of lean and steer, which leaves a single fourth order eqation for the other variable. This equation is the same whichever variable is kept, so we simply present it as a differential operator:

$$\left(M_{\chi\chi}\frac{d^2}{dt^2}+K_{\chi\chi}\right)\left(M_{\psi\psi}\frac{d^2}{dt^2}+C_{\psi\psi}\frac{d}{dt}+K_{\psi\psi}\right)-\left(M_{\psi\chi}\frac{d^2}{dt^2}+C_{\psi\chi}\frac{d}{dt}+K_{\psi\chi}\right)\left(M_{\chi\psi}\frac{d^2}{dt^2}+C_{\chi\psi}\frac{d}{dt}+K_{\chi\psi}\right)=0$$

or

$$A\frac{d^4}{dt^4}+B\frac{d^3}{dt^3}+C\frac{d^2}{dt^2}+D\frac{d}{dt}+E=0 \ ,$$

where the coefficients B, C, D, E are polynomial functions of the velocity (which at any given speed are simply constants like A); see APPENDIX C. These coefficients are each rather complex combinations of bicycle parameters.

To study stability, we have considered several options:

— Solve for the motion of a given bicycle using a computer, and see whether it falls down. The trouble with this approach is that there are so many quantities needed to specify a bicycle — ten or twenty design parameters — that to explore non-standard designs thoroughly (say five or ten values of each parameter) might require millions or even trillions of computer runs. Even if it were practical to perform these, the results would be difficult to absorb.

— Solve the equations analytically, i.e. in terms of sines, cosines, and exponentials. Unfortunately this is extremely difficult, and the results are so messy that one can not see general truths — numbers describing a specific bicycle and specific initial motion must be substituted, leading to the same situation as above.

— Realize that we don't really care about the *particular* motion arising from a *particular* disturbance, but rather whether *all leaning and steering motions* eventually damp out or not. This property of solutions to the equations of motion can be found out from the *eigenvalues* alone, that is from the roots of the fourth-order *characteristic polynomial:*

$$As^4+Bs^3+Cs^2+Ds+E=0 \ .$$

If all the roots (eigenvalues) have negative real parts, then all possible solutions will involve decaying exponentials, and χ and ψ will always return to zero. The trouble is that even the roots alone are hard to find, unless we use a computer on specific numerical cases. So this is still not really promising.

— Recall that one needn't solve for the eigenvalues to know whether their real parts are all negative: the Routh-Hurwitz tests may be applied to the characteristic polynomial, and if it passes we are assured that the bicycle is stable. Conversely, if the polynomial fails any of the tests, the bicycle must be unstable. The tests are:

$$A, B, C, D, E > 0$$

and

$$C - A\left(\frac{D}{B}\right) - E\left(\frac{B}{D}\right) > 0 \text{ (for brevity, this quantity will be called } RH\text{)}.$$

A, B, C, D, E and RH are all simple enough (compared to formulas for the eigenvalues) that one may hope to recognise the effects of various parameters without extensive numerical studies.

This last approach is the one we have adopted. Perhaps what distinguishes our work from other correct analyses is that we have persisted in trying to simplify and use these tests to understand what makes a bicycle stable. However, while we have made some headway, the algebraic complexity of the general expressions has kept the rate of progress slow.

IV. SPECIAL CASES, AND COMPUTER EVALUATION

The above-mentioned Routh-Hurwitz tests appeared to be the only method which could evaluate stability in terms of simple functions of the bicycle parameters. The potential value of such formulas is that one could in principle, by looking at the formulas, see how bicycles must or must not be built to achieve stability, and what could be done to any given design to improve its stability. Sufficiently clear formulas can provide a kind of understanding which tables of numbers or pages of plots cannot.

A little progress has been made along these lines (see APPENDIX C), and more is expected, but the stability formulas are complicated enough that such progress has been slow. So we have also made use of two other strategies: One is to *study simplified configurations* which lead to less complicated formulas, and attempt to build up an 'intuition'. The other is to *use a computer* to evaluate the stability formulas numerically for any particular design, and also to explore the effects on stability of changing each parameter one-at-a-time. Below are briefly summarised some of the results and conclusions.

SPECIAL CASES

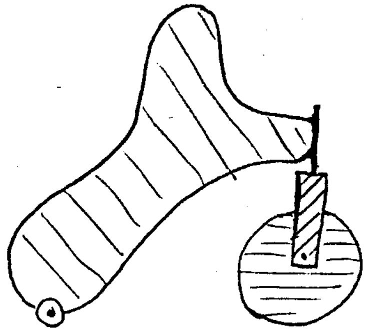
Note: in some of these cases, a bicycle which is not (self-)stable because $E < 0$ may still be balanced very easily by a rider (see APPENDIX C).

— A bicycle (+rider) with *standard* dimensions and mass-distribution can *never* be (self-)stable if the gyroscopic effects of the front and rear wheels are somehow canceled. (D can be written in the form $d_1V + d_3V^3$, but in this case d_3 vanishes, so D has the sign of d_1 which is normally negative.) But this is not to say that all bikes must have gyroscopic effects to be stable.

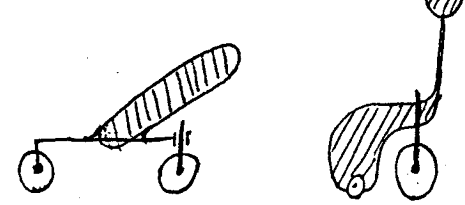
— A 'primitive' bicycle with a vertical steering axis, symmetry of the front assembly about the steering axis, and an arbitrary distribution of mass on the rear assembly (see Fig.1a), is slightly unstable: $E = 0$, and $RH < 0$ unless the rear assembly has enough high-up mass in front of the steering axis (as when the rider is leaned forward, and the front wheel bears most of his weight; Fig.1b). In the case that we checked, the computer showed that it can be stabilised by making the trail slightly positive, or by moving the front-assembly center of mass ahead of the steering axis (while leaving the front c.m. inertia tensor with a vertical principal axis). By combining these effects, the bicycle may be stabilised even when the trail is somewhat negative, as long as the front-assembly center of mass is sufficiently far forward of the steering axis (Fig.1c). The gyroscopic effect of the front wheel is essential to the stability of this vehicle, because in its absence C and D will always be negative. On the other hand, too large a gyroscopic influence from the rear wheel will keep RH negative.

— An isolated rolling wheel (see Fig.2a for the equivalent bicycle) is not quite stable, because it never uprights itself from a steady turn, and it never stops oscillating to either side of its average lean angle.*

* An actual isolated wheel does straighten up and does stop oscillating, presumably because of scrubbing friction with the road. But this phenomenon can not be used to explain the stability of the friction-free bicycles considered here.



1a. A simple 'primitive' bicycle is unstable



1b. Rear-assembly mass which is high and forwards of the steering axis can cure 'weaving' instability, but the bicycle won't straighten up from a turn.

1c. A primitive bicycle can be stabilised by a positive trail and/or a front-assembly c.m. which is forward of the steering axis (while retaining a vertical ppal. axis of inertia). In particular, a sufficiently forward c.m. can compensate for a slightly <u>negative</u> trail.

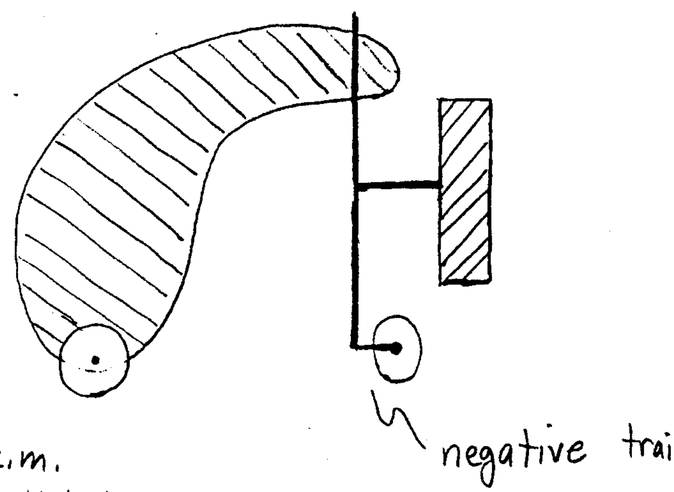


Fig. 1 A 'primitive' bicycle with vertical steering axis, and mass symmetry about the steering axis (⇒ vertical ppal. axis of the front-assembly c.m. inertia tensor)

However, a wheel with a downwards tilted bar attached [at or just below the axle] (Fig.2b) does stop oscillating above a certain speed, perhaps because a leaning acceleration tends to leave behind the top of the bar, thus causing a steer acceleration to the side of the lean. [The addition of arbitrary mass to the rear frame appears simply to change this speed somewhat.] [The vehicle can be induced to straighten up out of a turn by moving its mass center slightly ahead of the steering axis or giving it a slight positive trail, as above. (What about forward-tilted steering axis?)] For this design gyroscopic effects are crucial to stability, because in their absence D will be negative.

— Perhaps motivated by the example of a furniture caster, many authors have suggested the importance of trail to stability, however with little solid evidence.** So we looked at the stability of a bike with a massless front assembly, no gyroscopic effects (as if it is sliding on skates), and some steering-axis tilt and trail (Fig.3a). Surprisingly for us, such a configuration can be stable for a huge speed range. The trail must be positive, but it is best if it is small. [Causes instability if large?] Also the point labeled U in the figure should be slightly behind the front contact point P_f, which means that the steering axis must tilt backwards just a slight amount depending on the trail. Finally, the mass of the rider must be stretched out along a forwards–leaning line (which, however, must not lean as far as the line from the rear contact P_r to the center of mass m_t). A crude model made with furniture casters (Fig.3b) was indeed stable, though perhaps not to the extent of the theoretical model, which predicted stability from a very slow velocity up to infinity. Despite this tremendous stability, such a design would require only tiny handlebar torques to steer it.

— The lean equation shows that a bicycle can be stabilised if the steer angle is controlled to be proportional to the lean:

$$\psi \approx -k_0\chi \ .$$

Skateboard wheels actually steer according to this rule, so some results of their stability analysis (see Appendix B) will be mentioned briefly. [Not written.]

— Tricycles don't have the same balancing problems as two-wheeled vehicles, but they are still reputed to be hard to control, especially on a crowned road. We may analyse their behaviour by assuming that a bicycle has an adequate supporting moment $\mathcal{M}_\chi$ holding it upright ($\chi = 0$), and using the steer equation to evaluate stability of no-hands riding or the torque required in a steady turn (see Appendix B). [Not written.]

Scott Hand deserves the credit for demonstrating that stability is possible for both a gyro-free bicycle and a negative-trail bicycle.

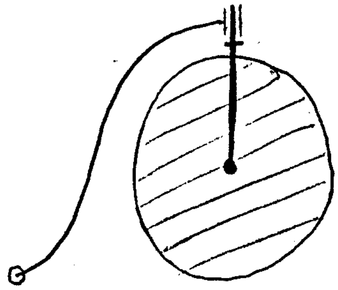
COMPUTER EVALUATION

When a bicycle's design is complex enough it is convenient to use a computer, to evaluate the Routh–Hurwitz stability quantities A, B, C, D, E and RH for any velocity, or even to calculate the entire stable speed range. This use of a computer has the same drawbacks as the numerical techniques mentioned above in *DETERMINATION OF NO-HANDS STABILITY,* namely that one can never prove general truths numerically, and that many trials are required to establish 'trends'. Also, the other numerical techniques give more information, such as the motion involved in instability. However the method we have adopted does appear to have advantages in ease of programming and speed of computation.

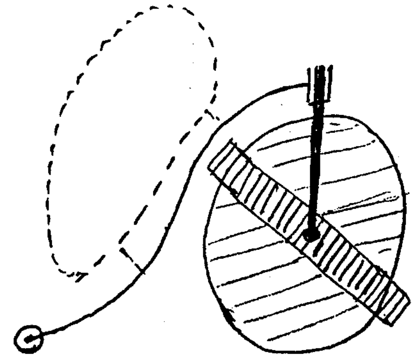
Our rudimentary programs for an IBM PC are of three types:

(a) *Defining the parameters of the bicycle, and constructing the velocity–polynomials of the six Routh–Hurwitz stability tests.*

** Trail (or rather *mechanical trail,* the perpendicular distance behind the steering axis of the front contact point) does strongly affect the handlebar torque required in a given maneuver, partly because in a rapid steering motion it will make the rear assembly yaw more, but mostly because any ground-contact force component which is perpendicular to the wheel exerts a turning moment (with mechanical trail as the lever-arm) which is partly resisted by the rider's hands. Moderate such moments are probably important, as feedback for steering, or to allow the rider to control the bicycle by forces rather than displacements (control of the steering angle would have to be precise at high speeds, and such control would prevent the bicycle from contributing to the balancing task). However large moments would make steering a real effort.



2a. An isolated wheel treated as a bicycle (all mass in wheel) is not quite stable — it never stops wobbling, and never straightens up from a turn.



2b. A downwards-tilted bar suppresse oscillation [and mass on the rea frame appears not to prevent this Shifting the bar forwards slight ought to induce straightening-up.

Fig. 2 Considering a bicycle as a <u>front</u> <u>wheel</u> with modifications

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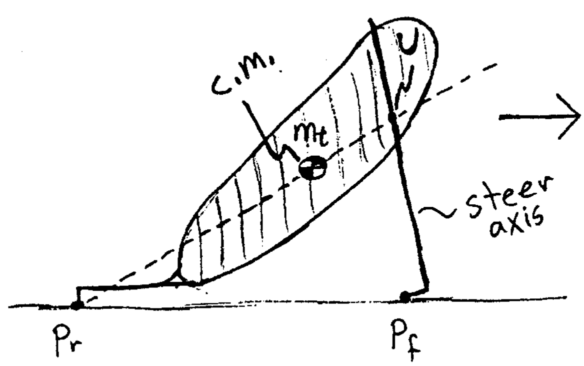
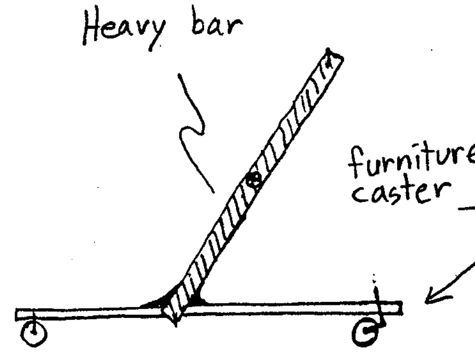


3a. <u>No</u> front mass, <u>small</u> backwards tilt of steering axis, <u>small</u> mechanical trail. The line from P_r through m_t must lean forwards <u>more</u> than rider; and the point U in which it intersects steer axis must be slightly <u>behind</u> P_f.



3b. Crude physical model with <u>moderate</u> front mass and trail was fairly stable.

Fig. 3 A "furniture caster" or "skate" bicycle (with negligible front-assembly mass) can be <u>very</u> <u>stable</u>, but still easy to steer. It appears much more difficult to attain such non-gyroscopic stability with a massive front assembly (but S. Hand achieved this with a forwards-tilted steering axis).

The idea here was to create a sequence of simple programs, which each would produce an output file to be used as input by the next level. The first program might take a file F_1 containing wheel sizes and masses, rider size and mass and body orientation, seat position, handlebar-bag mass, head angle and fork rake, etc.; and produce another file F_2 of the complete set of 17 independent parameters used in our analyses to define a bicycle. The next program would take F_2 and calculate F_{CO}, the coefficients of the various powers of velocity in the Routh–Hurwitz stability tests (see APPENDIX C).
This structure would allow the user to specify either the bicycle's physical design, or the more abstract defining-parameters; and a default F_1–file containing typical values would help the user to avoid mistakes when unfamiliar with the orders of magnitude involved. It would of course be possible to produce further files F_3, F_4 etc. containing intermediate built-up quantities such as the lean and steer equation coefficients. This could be quite informative as to the quantities which most strongly influence subsequent calculations; but it might be difficult to alter their values without violating fundamental physical restrictions (mass is positive, etc.) most easily checked at the F_2 level, or without making construction difficult (check at the F_1 level).

(b) *Using the polynomials of F_{CO}, find the sign of each stability quantity over a range of speeds — or, simply solve for the velocity range (if any) in which all are positive, and note which define the limiting velocities.*

With program types (a) and (b), one can either experiment with bicycle designs to see which are more stable, or one can use them as subroutines in another program, e.g.:

(c) *Iteration to a more stable design.*
Starting with an initial file at the F_1 level or the F_2 level, use the programs of (a) and (b) to evaluate the stable speed range. Next, make a small change to each of the input values in turn, run (a) and (b), and note by how much the stable speed range is either widened or narrowed in each case (to do this for all the parameters takes only a few seconds). Based on these results, choose a more stable design which is slightly altered from the initial case, and repeat the process. When this approach works, it will produce a succession of increasingly stable designs.

We have initial versions of programs which work along these lines [(c) has barely been used], which could perhaps be released for experimental purposes.

V. STEADY-STATE EFFECTS OF DISTURBANCES AND MISALIGNMENT

[Perhaps this section should be condensed or deleted — at any rate it needs revision.]

Qualitative Introduction

A bicycle which is stable at a given speed will still be stable when subjected to a moderate disturbing force or misalignment; however its steady equilibrium configuration will then generally involve some amount of leaning and turning. Some insight into a bicycle's response to steady forces or misalignment may prove useful.

One important concept can be illustrated by a rider travelling along a straight road. If wind blows to the right, the rider must lean left, into the wind, in order to balance while still riding straight. This behaviour of leaning into the wind corresponds to the fact that a leaned bicycle *wants* to lean *further*. On the other hand, if the bicycle is held up by spring-mounted training wheels rather than balanced by steering, then the wind would make the rider lean to the right: a *supported* bicycle wants to *return to the upright orientation* when leaned. The difference is between a maximum energy configuration in the first case (usually unstable but now dynamically stabilised); and a stable minimum energy configuration in the second case.

The steering of a *stable* bicycle has a similar property, as mentioned in APPENDIX C. That is, if the the handlebars are steered *while the bicycle is maintained in a balanced condition* (at rest, or in motion at the lean angle necessary to balance in the resulting turn), a stable bicycle wants to steer further.* This means that to balance a moderate imposed steering torque, the handlebar must turn opposite to it, i.e. *fight*

* Presumably, this is so that a bicycle steered in a steady turn, when released, will steer even sharper, and upright itself; at any rate, when $E > 0$ (which is required for stability) a bicycle has this steer torque.

against it. [In some sense, a stable bicycle is at a maximum of 'energy' (slow proportional loading) with respect to both steer and lean, not the more common minimum of stable static equilibrium.]

Qualitative Effects of Misalignment

A stable bicycle which is misaligned (with offset or tilted wheels), or imbalanced (mass or force offset from the center-planes of the rear or front assemblies), will not naturally travel upright and straight-ahead. Some interesting issues include the steering torque and lean angle needed to ride it in a straight line; and the equilibrium curve it takes up if the rider exerts no steering moment.

To understand the straight–line case is relatively simple: (a) turn the handlebars until the wheel traces are parallel; (b) tilt the bicycle until the overall center of mass is above the line joining the wheel contacts; (c) evaluate the steer moment needed to balance the vertical forces of the ground contact and the front assembly weight, if these do not pass directly through the steering axis.

Example: if the front wheel is angled or offset so it contacts the ground to the right of the rear-assembly center plane, then the bicycle is not perfectly balanced when vertical, and must be leaned slightly to the right. (For a standard bicycle, the effect of this small lean on the steering moment is small.) Since the steering axis is tilted backwards and the trail is positive, the vertical ground force tends to turn the handlebars to the left, so the rider must apply a steering moment tending to turn them to the right. Alternately, when riding no-hands the rider should incline his or her body to the left side of the bicycle frame, which adds rightward steering moment by tilting the frame to the right, and accomplishes the same end.

It is not so easy to say what stable curve the bicycle will maintain if the handlebars are released, because of the gyroscopic effects of the wheels which are constantly changing their heading. But if these effects may be ignored in comparison to centrifugal effects (as is the case for standard bicycles with a normal amount of trail), the following reasoning seems to work: (a) balance the stationary bicycle as above, for straight riding; (b) whatever direction the handlebars want to turn, slowly turn them in the *OPPOSITE* direction, while keeping the bicycle *balanced,* until no handlebar torque is required (i.e. the bicycle reaches its maximum gravitational energy); (c) since the wheels do not have parallel traces, the bicycle will follow a turn when moving, and thus will have to lean (but in the absence of gyroscopic torques, will keep exactly the same steer angle). So in the example above, the bicycle will turn to the right.

Effect of Steer Torque

These ideas may be discussed more concretely by using the steady (non-derivative) terms in the equations of motion. These relate the rightwards lean angle χ of the rear assembly, and the leftwards steer angle ψ, to the lean moment $\mathcal{M}_\chi$ tending to tip the rear assembly rightward, and the steer moment $\mathcal{M}_\psi$ tending to turn the handlebars leftward, by the K-coefficients:

$$K_{\chi\chi}\chi + K_{\chi\psi}\psi = \mathcal{M}_\chi$$

$$K_{\psi\chi}\chi + K_{\psi\psi}\psi = \mathcal{M}_\psi$$

Here $K_{\chi\chi}$ and $K_{\psi\chi}$ are constants, while $K_{\chi\psi}$ and $K_{\psi\psi}$ both typically depend on velocity.

If we consider a bicycle travelling at a given speed which is *in perfect lean equilibrium,* (i.e. $\mathcal{M}_\chi = 0$), then the lean equation tells us that for every choice of steer angle there is an appropriate lean angle χ_{bal}:

$$\chi_{\text{bal}} = \frac{-K_{\chi\psi}}{K_{\chi\chi}}\psi \,.$$

$K_{\chi\chi}$ is always a large negative constant. $K_{\chi\psi}$ is typically positive at very low speeds, but at normal speeds it is always negative (unless the bicycle has a high-speed backwards-spinning flywheel), and proportional to the velocity squared. If χ_{bal} is used in the steer equation, we find

$$\mathcal{M}_{\psi\text{bal}} = \frac{(K_{\chi\chi}K_{\psi\psi} - K_{\psi\chi}K_{\chi\psi})}{K_{\chi\chi}}\psi = \frac{\text{Det}(K)}{K_{\chi\chi}}\psi \,,$$

where $\mathcal{M}_{\psi\text{bal}}$ is the steering moment required to traverse a steady equilibrium turn with steer angle ψ. This, not $K_{\psi\psi}$, is the steady-turn "steering stiffness" felt by the rider. Now the determinant of the K matrix is just E, the constant term from the characteristic polynomial, and this must be positive for the bicycle to be self–stable,* so we may conclude that $\mathcal{M}_{\psi\text{bal}}$ and ψ have opposite signs in a stable bicycle. That is, the rider must restrain the handlebars, which try to accentuate any turn. (Because of gyroscopic effects, Det(K) for a conventional bicycle typically becomes *negative* between 10–20 mph. The handlebars then try to *center* themselves when held in a steady turn, and the uncontrolled bicycle is no longer stable.)

If a constant (say, right-steering) moment $\bar{\mathcal{M}}_\psi$ is simply applied**, and the bicycle is stable at that particular speed, eventually it will end up steered to the left! The bicycle initially steers to the right (this can be shown by looking at only the second-derivative terms in the equations of motion), which [always?] creates a leftwards lean and causes the front wheel to steer to the left to maintain balance, so the front wheel finally ends up at the *leftwards* steering angle appropriate to balance the rightwards $\bar{\mathcal{M}}_\psi$.

By carefully choosing bicycle parameters, it is possible to arrange for Det(K) to be constant at all speeds, in which case the equilibrium steer angle for a given steer moment will be independent of speed, and the lean angle will be proportional to V^2. Alternately, it is possible to develop a non-standard bicycle design for which Det(K) will be strictly proportional to V^2, so that the steer angle for a given steer moment will be smaller at higher speeds, and the resulting lean angle will be independent of speed.

Notice that a new stable condition is not reached by simply controlling the variation of ψ (which changed in a complicated way to get the bicycle into the turn), nor is it even possible to fix its final value (holding the steering angle fixed prevents the bicycle from making small steering corrections to balance itself, so it becomes absolutely unstable). However, prescribing a *steering torque* leaves the front assembly free to move and stabilise the bicycle. This suggests that radio-controlled steering of a riderless bicycle may be accomplished quite easily: simply allow a stable bicycle to perform the balancing task, and use the radio control to produce small imbalances or steering torques which will cause the bicycle to enter steady turns in the desired directions. (For example, by sliding one end of a taut rubber band along the handlebars to the right while the other end is attached to the seat, a right-steering moment will be produced, causing the bicycle to turn left. Surprisingly, it appears that if the sliding took place at the appropriate rate, the rubber band would always *assist* it, so that negligible servo power would be needed.) Stabilising a bicycle by appropriately controlling the *steer angle* is a more difficult task.

Effect of Lean Torque

When no steering torque is applied to the handlebar, the effect of $\bar{\mathcal{M}}_\chi$ (which may come from rider upper-body lean, or bicycle misalignment, or from a side force acting at some point on the rear assembly above the rear-wheel contact) may be discussed in a similar way. A stable bicycle will balance itself at some angle of lean and some angle of steer, but will a moment tending to tip the bicycle to the right cause a lean to the left, or to the right? Will the bicycle steer left or right, or go straight? Using the equations of motion as above, we may find

$$\chi = \frac{K_{\psi\psi}}{\text{Det}(K)}\bar{\mathcal{M}}_\chi \ , \ \psi = -\frac{K_{\psi\chi}}{\text{Det}(K)}\bar{\mathcal{M}}_\chi \ .$$

In contrast to the steering's behaviour of turning opposite to $\bar{\mathcal{M}}_\psi$, in theory either sign is possible for χ*, and its magnitude will usually depend on speed. A standard bicycle at normal speeds has $K_{\psi\psi} > 0$, so a rightwards tipping moment causes rightward leaning and steering. Therefore, when riding no-hands, the rider's body should be leaned to the right side of the bicycle in order to produce a steady rightwards curve. Since $K_{\psi\chi}$ is constant, ψ will depend on speed unless Det(K) is also made constant.

One option which would display leaning behaviour more in line with steering behaviour is to ask that the bicycle continue to travel straight, i.e. $\psi = 0$, by choosing $K_{\psi\chi} = 0$. Since Det(K) $= K_{\chi\chi}K_{\psi\psi}$ in this

* Typically not important when the bicycle is rider-controlled, but essential when it is uncontrolled.

** A simple method of providing a right-steering moment is to wrap a string clockwise around the steering axis, and tension it with a taut rubber band attached to the bicycle seat.

* If subjected to a rightwards–tipping force, a bicycle travelling *straight* would have to lean to the left. However, if it is turning right, the lean may also be to the right, just not to the extent of a balanced bike in the same turn.

case, it is essential that $K_{\psi\psi} < 0$ (i.e. the steering axis is tilted forward!); and it can be seen that

$$\chi = \frac{\bar{\mathcal{M}}_\chi}{K_{\chi\chi}} .$$

Since χ is negative and speed-independent, the bicycle then always leans opposite to $\bar{\mathcal{M}}_\chi$ by an amount independent of the speed.

If the bicycle is designed in this fashion (so that lean moment produces no steady turning), we can then see that when a moment is applied the bicycle first leans in the sense of the moment; then it evidently steers towards the side of the lean so much that it reverses the lean; and finally a lean is developed which balances the applied moment. This interesting design, however, removes the rider's ability to steer the bicycle by leaning his upper body!

[What is $\bar{\mathcal{M}}_\psi$ for a bicycle going straight with nonzero $\bar{\mathcal{M}}_\chi$? $\bar{\mathcal{M}}_\psi/\bar{\mathcal{M}}_\chi = K_{\psi\chi}/K_{\chi\chi}$, so a traditional bike with $K_{\psi\chi} > 0$ will try (initially) to steer to the left if $\bar{\mathcal{M}}_\chi$ is tending to tip it to the right.]

Effect of Sidewind (e.g. with disk wheels)

If the wind blows on a bicycle moving stably without rider control, it will generally end up following a steady curve into or away from the wind; which would be preferable? One reasonable goal might be for the bicycle to end up travelling straight, and leaned into the wind. Alternately, since the transient process of adjusting to the wind may well change the heading of the bicycle, it may be best to select a path curvature aimed at cancelling this heading change, so that the rider has longer to respond.

Assume that a sidewind has the effect of a lateral force acting at some point on the front assembly, plus another acting at some point on the rear assembly. These wind forces will create some values of $\mathcal{M}_\chi$ and $\mathcal{M}_\psi$ in the equations. The lean moment $\mathcal{M}_\chi$ arises from the height of the forces above the ground, which both tend to tip the bicycle away from the wind. The steer moment $\mathcal{M}_\psi$ comes from both the front and the rear forces, and is best understood in terms of front and rear 'fixed lines' — lines which do not move when an upright stationary bicycle is steered (i.e. handlebars are turned) without leaning* (Fig.4). The 'fixed line' for the rear is just the vertical line above the rear contact P_r: when the handlebars are turned the upright rear assembly pivots slightly about that line, so a side force on that line will tend to tip the bicycle but not steer it. (A force on the rear assembly ahead of the 'fixed line' will cause the handlebars to turn, typically *away* from the force.) The 'fixed line' for the front assembly is the line from the front contact P_f to P'' (the point at which the steering axis intersects the rear 'fixed line'). A side force on the front assembly which acts somewhere on that line will tend to tip the bicycle but not steer it. A force ahead of the 'fixed line' will tend to make the bicycle steer away from the force. With a front disk wheel, it seems likely that the side force on the front will contribute most of $\mathcal{M}_\psi$.

One design option is to add a trailing aerodynamic surface to the front assembly, to make the front force act just behind the front 'fixed line' (assuming the rear force acts in front of the rear 'fixed line'), so there is no net steer moment $\mathcal{M}_\psi$. Then the effect of the wind reduces to a pure leaning moment, as discussed in the previous section.

More generally, if the wind causes both $(\mathcal{M}_\chi)_W$ and $(\mathcal{M}_\psi)_W$, and we want a steady response with $\psi = 0$, we must choose

$$\frac{K_{\psi\chi}}{K_{\chi\chi}} = \frac{(\mathcal{M}_\psi)_W}{(\mathcal{M}_\chi)_W}$$

(by adjusting either the bicycle design or the relative magnitudes and application points of the wind forces). The left hand side is independent of speed, and ideally so is the right. Knowing that we want the bicycle to lean into the wind, we might arrange to steer the bicycle *away* from the wind initially, so the equilibrium lean will be achieved more rapidly. This corresponds to making

$$\frac{\ddot{\psi}_W}{\ddot{\chi}_W} = \frac{(\mathcal{M}_\psi)_W M_{\chi\chi} - (\mathcal{M}_\chi)_W M_{\psi\chi}}{(\mathcal{M}_\chi)_W M_{\psi\psi} - (\mathcal{M}_\psi)_W M_{\chi\psi}}$$

* Masses placed on the front and rear 'fixed lines' leave the full steer equation absolutely unchanged, and hence should have little effect on bicycle handling if small enough or low enough to leave the lean equation nearly unchanged.

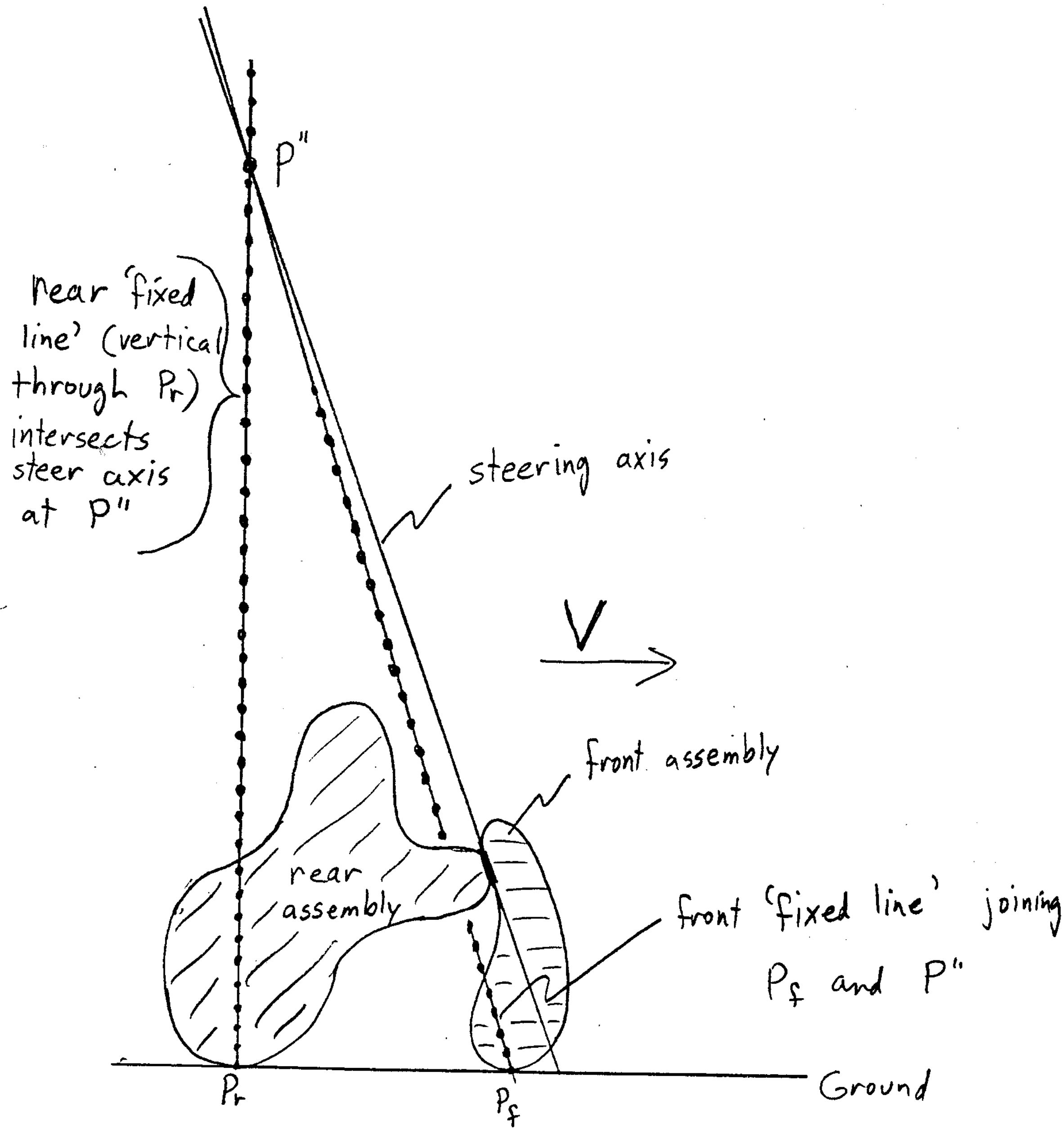


Fig. 4 'Fixed lines' for the front and rear assemblies — points on these lines do not move if the bicycle is <u>steered</u> ($\psi \neq 0$) while the rear is sufficiently negative.

[If crosswind forces prove to be too great (as might be the case for a fully–enclosed bicycle travelling at high speeds), perhaps aerodynamic compensation may be used to advantage. For example, an 'angle of attack' vane could perhaps adjust trailing flaps to make the lift coefficient vanish at the given angle of attack. Or, some surfaces might be permitted to swing around relative to the direction of travel, in order to cancel the angle of attack.

Might a fully enclosed bicycle benefit from coupling the wind forces to the steering?]

Towing a Bicycle

Towing forces might be considered a kind of disturbance. A few qualitative suggestions only will be made, pending a more complete analysis.

One aspect of towing is simpy that of affecting the direction. A brief 'tug' on a string from one side of a bicycle will generally produce a change in direction, which may or may not be towards the 'tugger', depending on where the string is attached.

A more automatic control process might be to exert a restoring force proportional to the bicycle's deviation from a desired path, as a fairly steady 'disturbance' which will tend to turn the bicycle towards the path. Based on earlier discussion, if we can produce a steering moment $\mathcal{M}_\psi$ tending to turn the handlebars away from the desired path, then the bicycle will eventually steer towards it. An elastic string fastened to the front assembly *behind* the 'fixed line' is able to produce such a moment. However we cannot conveniently attach the string at ground level, so its force will also produce a tipping moment $\mathcal{M}_\chi$. On a conventional bicycle this tipping moment also contributes to steering in the appropriate direction. [Isn't there some reason it's bad to attach the string to the rear assembly — what happened in Scott's experiments? Also, what about an inextensible string controlling displacement, not force?]

When a bicycle is pulled forward by a steady towing force, a full analysis is needed. However, qualitatively it appears that if the towing force is again applied to the front assembly, behind the 'fixed line', it will help stabilise the bicycle by making E more positive; and it will also provide the function of keeping the bicycle aligned with the towing vehicle. (Such towing of a stable bicycle has been demonstrated successfully). In order to keep the steering moment small when the towing force is great, it may be worthwhile to attach the string to a flexible support, so that the offset from the 'fixed line' (i.e. the moment-arm) is reduced when the towing force is large.

APPENDIX A

BICYCLE EQUATIONS OF MOTION

Note: to be able to follow the main aspects of this derivation, the reader should be familiar with concepts from intermediate-level dynamics, such as moments and products of inertia, angular velocity vectors, etc.

In order to describe the bicycle, its motion, and the forces acting on it, various quantities are defined in Figs. A1–A5.* (Figures may be found at the end of this Appendix.) Many related parameters, though useful, have been omitted to shorten this summary. It turns out, for the small lean and steer angles considered here, that the point where each wheel touches the ground does not move significantly around the circumference of the wheel. So for *geometric* purposes, the wheels may be replaced by fixed contact points on the bicycle frame and fork (i.e. wheel radius is not important). It is convenient, when studying geometrical questions, to represent the bicycle by the *skeleton frame* of Fig.A2.

This skeleton frame will be used to describe the accelerations of mass centers and the forces of ground contacts and gravity, but we also need certain *mass distribution* properties of the bicycle (Figs.A3–A5). Dynamically, it turns out that the ideal bicycle we are studying can be treated as just *two* rigid bodies, joined together at the steering axis. That is, for rotational accelerations $\ddot{\chi}$, $\ddot{\theta}$, $\ddot{\psi}$ (see Fig.A2) the torques may be found from the moment-of-inertia tensors of the front and rear assemblies *with the wheels considered non-rotating.* The actual rotations of the wheels produce gyroscopic effects, which may be incorporated by assuming that there is a quantity of angular momentum H_r "attached" to the rear assembly, and a quantity H_f "attached" to the front assembly. These are considered *positive* when the bicycle rolls forward.

Of course, to find **T** of Fig.A4, and $\mathbf{F}', \mathbf{F}''$ of Fig.A5, it is necessary to have center-of-mass inertia tensors for front and rear rigid bodies, plus the yz co-ordinates of front and rear mass centers (y is along the ground, and z is perpendicular to it in the plane of the frame).

Actually, the full inertia tensors $\mathbf{F}', \mathbf{F}''$ of Fig.A5 are not needed. What are actually required are:

— the polar moment of inertia about the steering axis, (that is the torque about the steering axis required for unit angular acceleration about that axis, written $(\mathbf{F}' \cdot \vec{\lambda}) \cdot \vec{\lambda} = F'_{\lambda\lambda} = m_f d^2 + F_{\overline{yy}} \sin^2 \lambda - F_{\overline{yz}} \sin 2\lambda + F_{\overline{zz}} \cos^2 \lambda$)

— the torque about the steer axis required for unit angular acceleration about the y axis, and vice versa (written $(\mathbf{F}' \cdot \vec{\lambda}) \cdot \mathbf{j} = F'_{\lambda y} = -m_f \overline{h}_f d - F_{\overline{yy}} \sin \lambda + F_{\overline{yz}} \cos \lambda$)

— the torque about the steer axis required for unit angular acceleration about the z axis, and vice versa (written $(\mathbf{F}'' \cdot \vec{\lambda}) \cdot \mathbf{k} = F''_{\lambda z} = m_f (c_w + \bar{l}_f) d - F_{\overline{yz}} \sin \lambda + F_{\overline{zz}} \cos \lambda$).

Here, $F_{\overline{ij}}$ is the center-of-mass inertia tensor of the front assembly.

DYNAMICS EQUATIONS

Assuming both wheels touch the ground, the configuration of the bicycle, and the position of every mass-point on it, are completely determined by: Y and X (position of rear contact P_r); θ (heading) and χ (lean) of the rear assembly; ψ (steer of the front assembly relative to the rear assembly); and the amount of rotation of both wheels. (See Fig.A6.) It turns out that for the case considered (small deviations from upright coasting along the Y axis of a level plane), that the forward velocity is essentially constant (like a constant-speed motorcycle), and essentially equal to dY/dt. Thus $Y = Vt$, and the wheels each rotate at a nearly constant rate (each rotation corresponding to a definite interval in Y).

Our concern is with the lateral motion of the bicycle (such as leaning); so we take V and H_r, H_f as constants and investigate only those motions which involve lateral (X) displacements of mass-points from the YZ plane. The variables which concern us are X, θ, χ, ψ, which all will be considered very small, in order to simplify the analysis.

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* Seventeen parameters are used to define the bicycle mechanically: The front and rear assemblies each need 7 quantities: their mass, three center-of-mass inertia tensor components, two center-of-mass co-ordinates relative to the wheel contact point, and the ratio of spin angular velocity to forward speed. In addition to these 14, we need the head angle, and the distances from the steering axis to each contact.

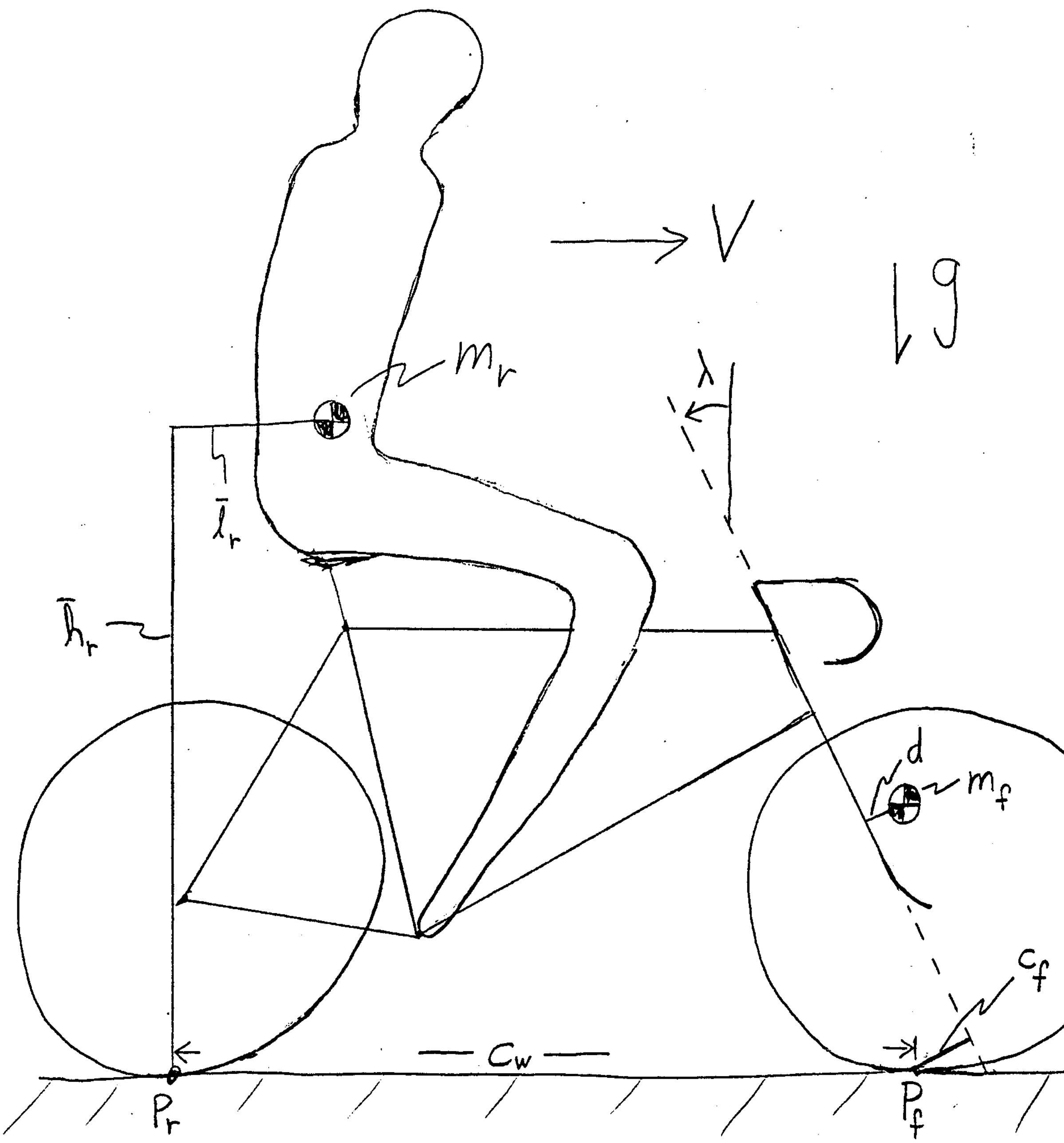


Fig.A1: Bicycle Terminology

m_r is total mass of frame + rigidly attached rider + rear wheel ("rear assembly")
$\bar{l}_r, \bar{h}_r$ are co-ordinates in the rear assembly of its c.m. (relative to P_r).
m_f is total mass of fork, handlebar, and front wheel ("front assembly")
P_r, P_f are rear and front contact points
λ is tilt of steering axis back from vertical (90 degrees minus head angle)
d is perpendicular distance of m_f (forward) from the steering axis
c_f is perpendicular distance of P_f (backward) from steering axis ("mechanical trail")
c_w is wheelbase
V is forward velocity, treated as a constant
g is gravity

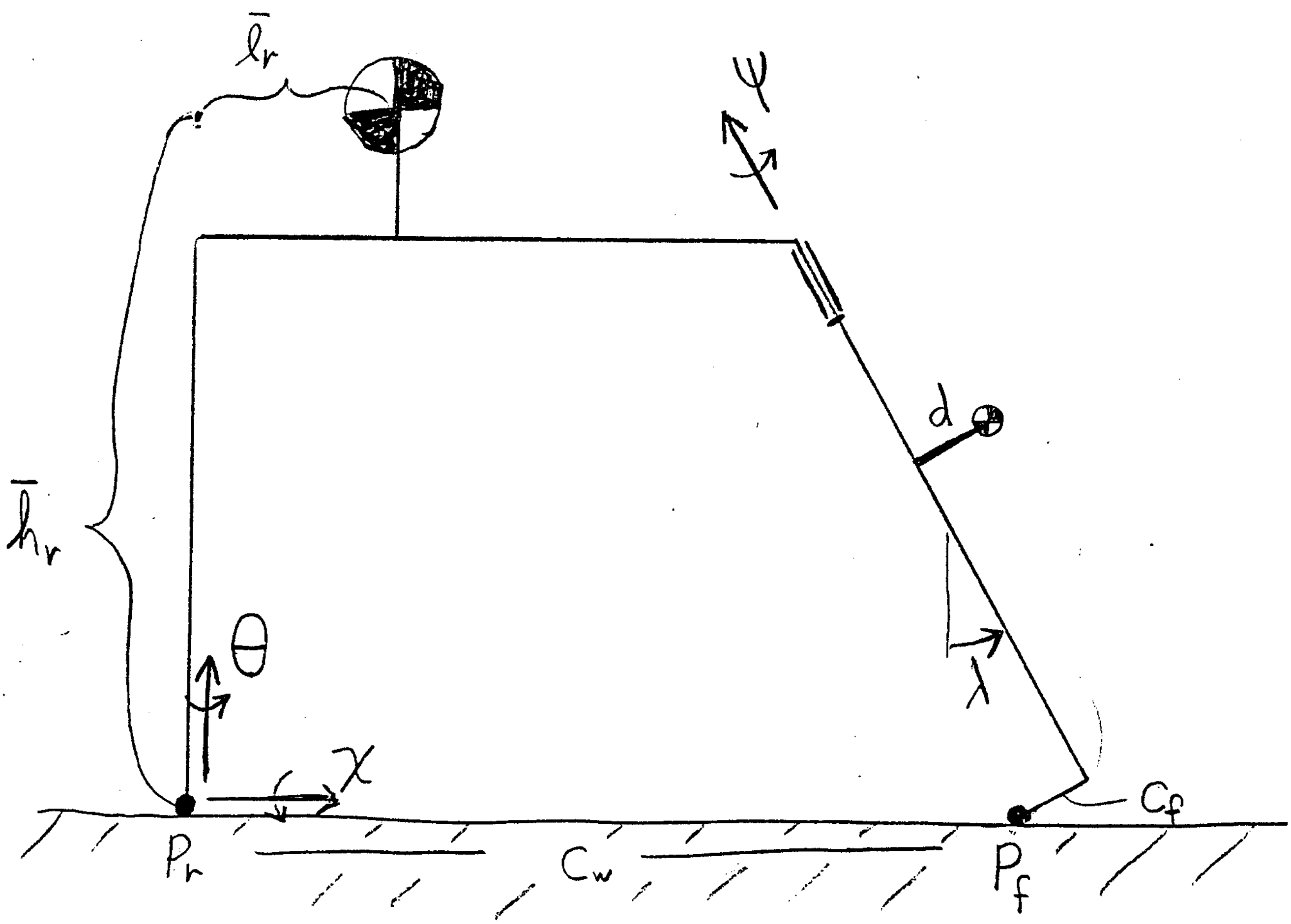


Fig.A2: Skeleton Frame
χ is the lean to the right of the rear assembly (rotation about horizontal axis)
θ is the heading of the rear assembly (rotation about a vertical axis)
ψ is the angle of steer of the front assembly *relative to the rear assembly*

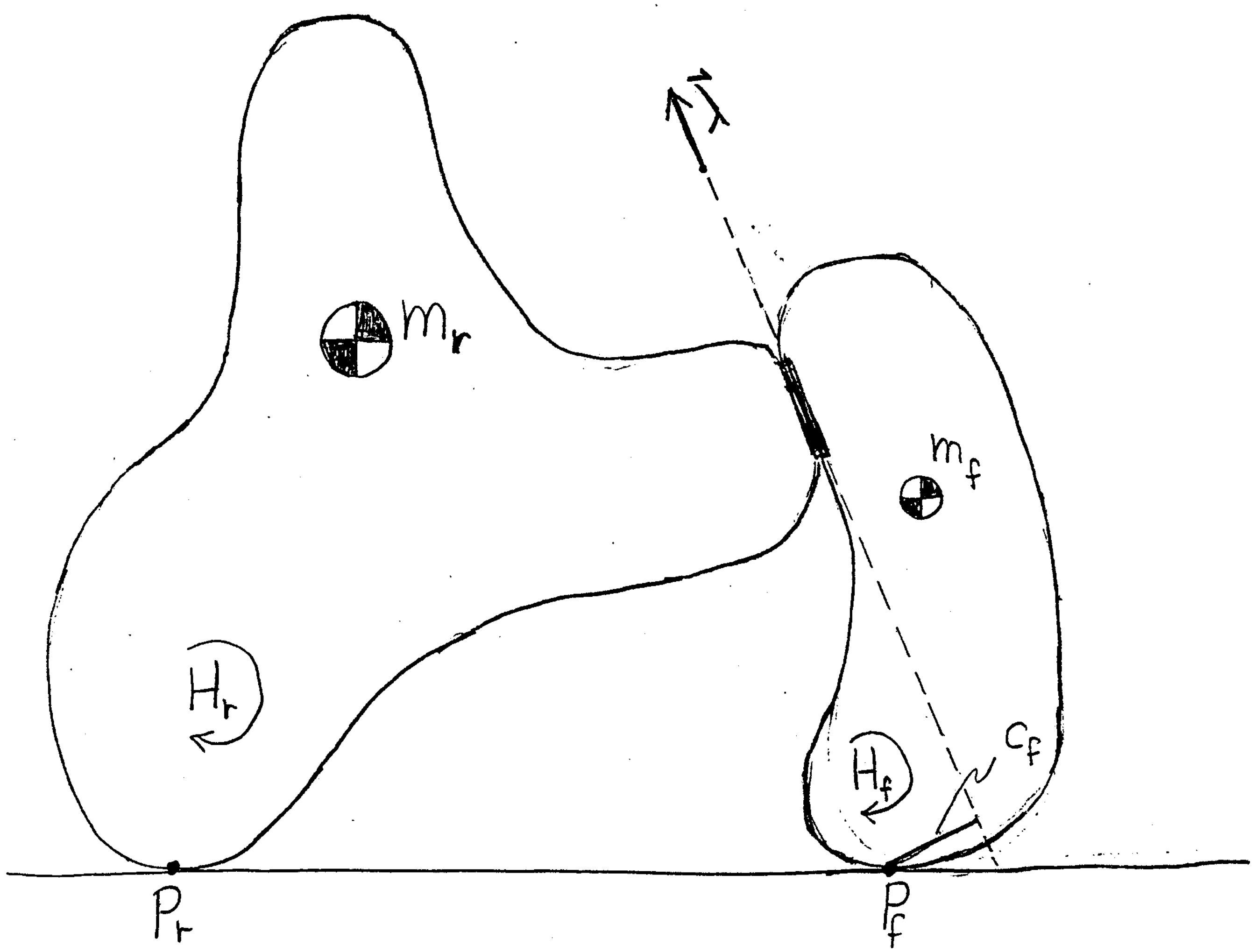


Fig.A3: Rigid Body Model
λ is a unit vector along the steering axis
H_r, H_f are angular momenta attached to rear and front assemblies
(which are treated as rigid bodies, i.e. without separately turning wheels)

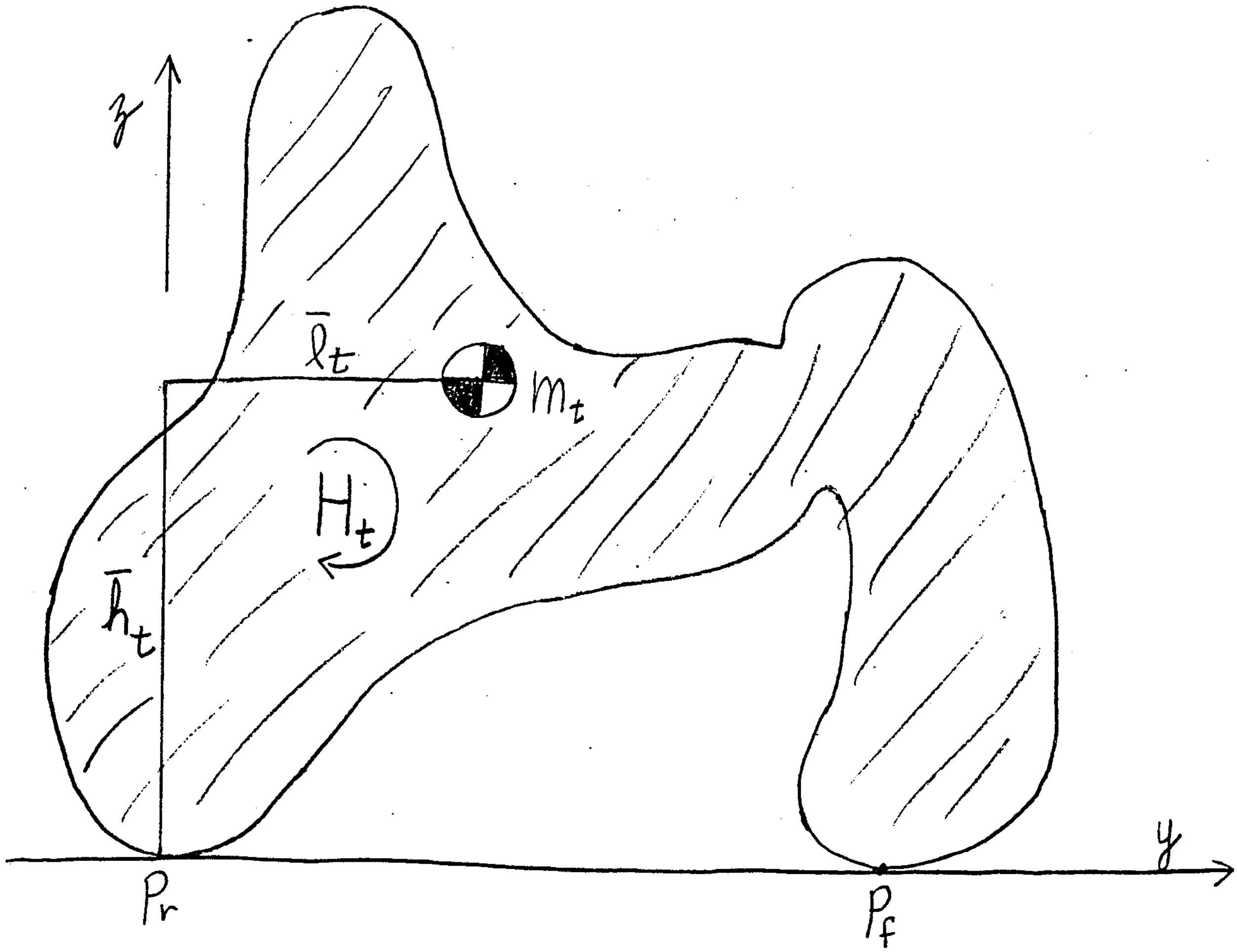


Fig.A4: Mass Distribution for Total Bicycle + Rider System
For the whole bike + rider system treated as *one* rigid body (steering locked):
m_t is total mass
yz are axes fixed to rear assembly at P_r
$\bar{l}_t, \bar{h}_t$ are position of overall c.m. in yz axes
$H_t = H_r + H_f$ is total spin angular momentum of wheels
T is moment of inertia tensor for whole bike relative to P_r (calculated in yz axes)

$$\begin{vmatrix} T_{yy} & T_{yz} \\ T_{zy} & T_{zz} \end{vmatrix}$$

In terms of the center-of-mass inertia tensor $R_{\overline{ij}}$ at m_r for the rear assembly, and the center-of-mass inertia tensor $F_{\overline{ij}}$ at m_f for the front assembly, we have $T_{yy} = m_r \bar{h}_r^2 + R_{\overline{yy}} + m_f \bar{h}_f^2 + F_{\overline{yy}}$, $T_{zz} = m_r \bar{l}_r^2 + R_{\overline{zz}} + m_f (c_w + \bar{l}_f)^2 + F_{\overline{zz}}$, and $T_{yz} = -m_r \bar{h}_r \bar{l}_r + R_{\overline{yz}} - m_f \bar{h}_f (c_w + \bar{l}_f) + F_{\overline{yz}}$.

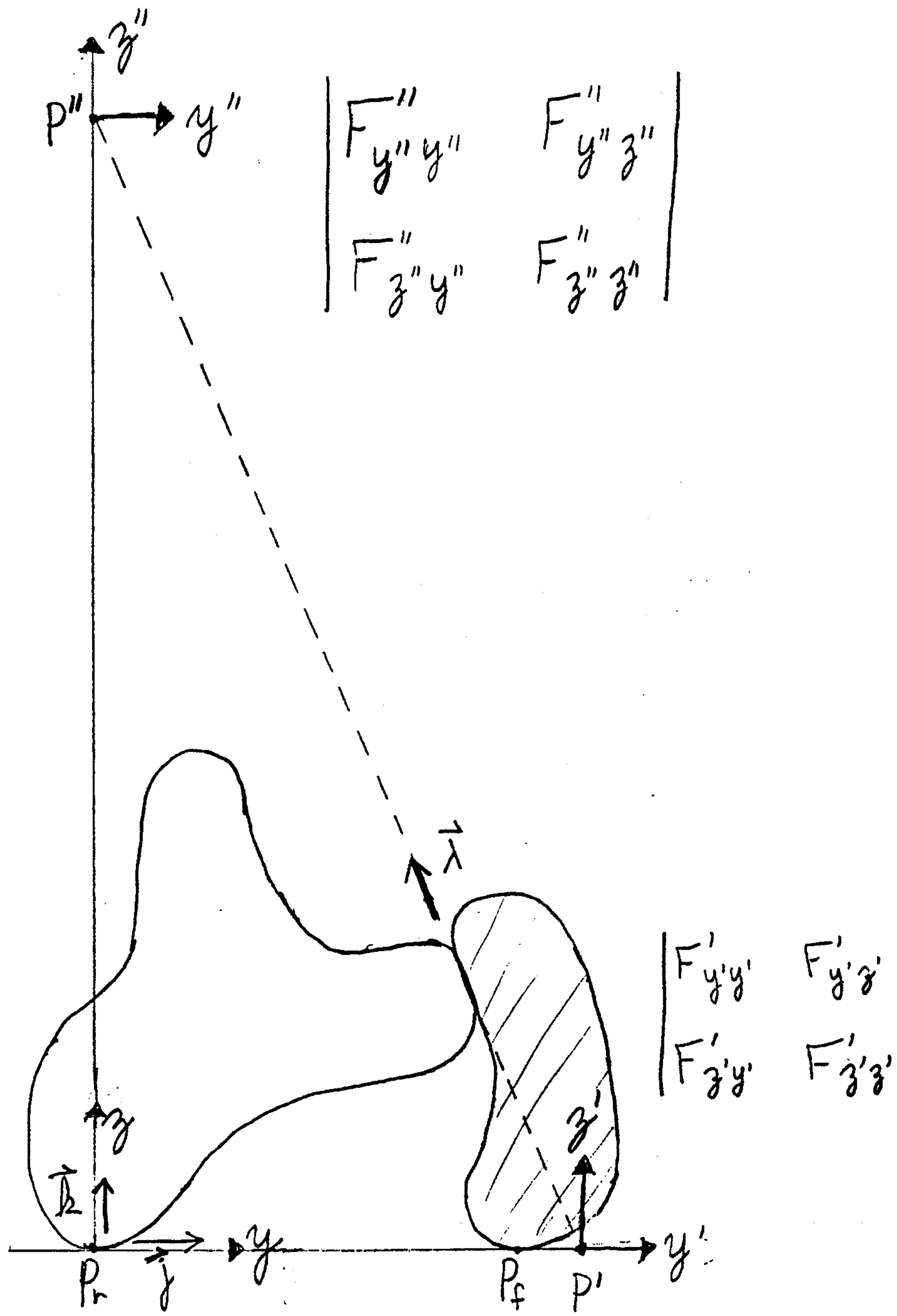


Fig.A5: Relevant Inertia Properties of Front Assembly Alone

P' is point in yz system where steering axis intersects y axis

P'' is point in yz system where steering axis intersects z axis

$y'z'$ are axes parallel to yz at P'

$y''z''$ are axes parallel to yz at P''

j, **k** are unit vectors along y, z

F' is moment of inertia tensor of the front assembly evaluated in $y'z'$. (the steering is assumed to be set straight ahead)

F'' is the moment of inertia tensor of the front assembly evaluated in $y''z''$ (steering set straight ahead)

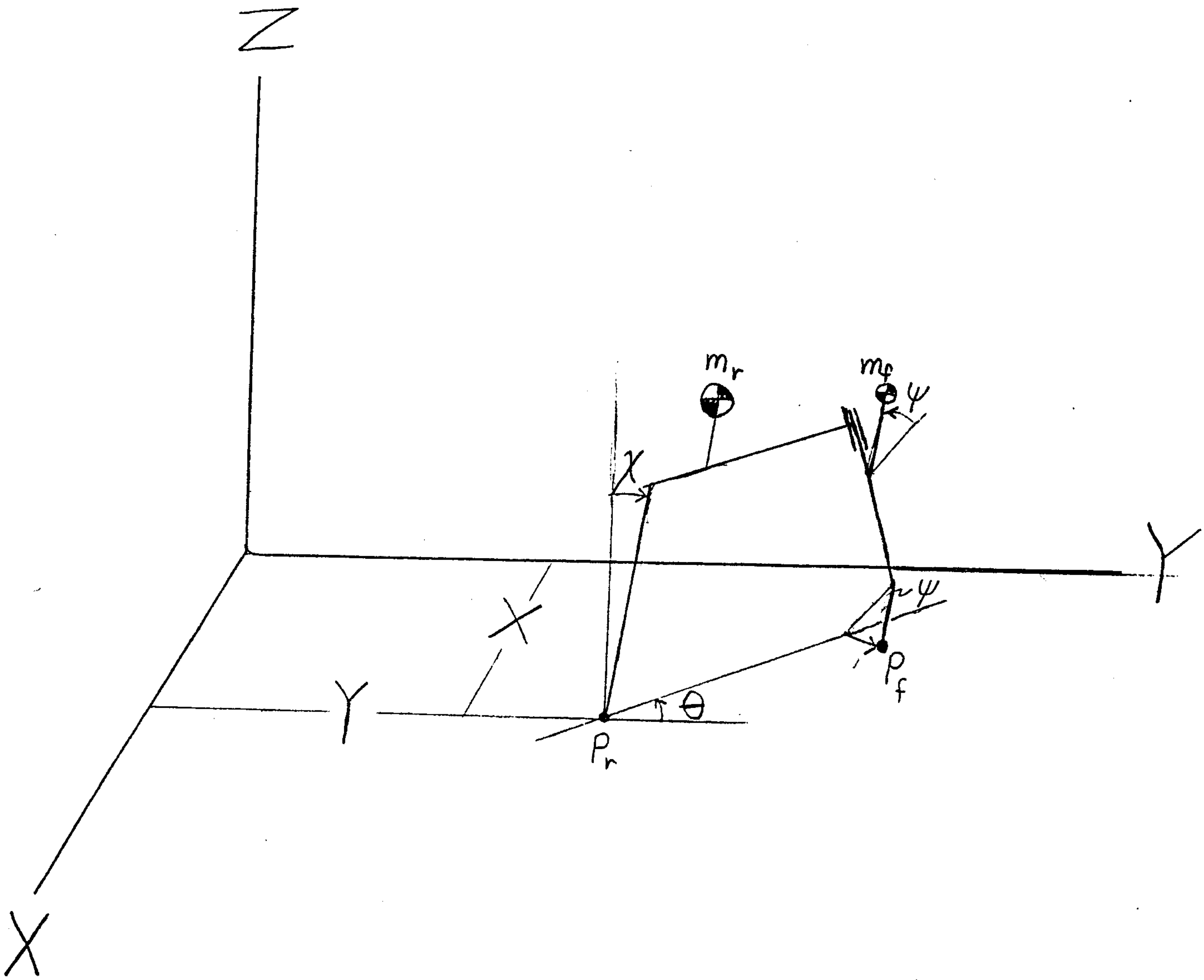


Fig.A6: Picture of skeleton frame in general configuration
X is displacement of P_r from Y-axis
χ is lean to right of rear assembly
θ is heading of rear wheel
ψ is steer to left of front assembly relative to rear assembly

To find the equations of motion we adopt the point of view that the bicycle is *on ice* as far as lateral wheel motion is concerned, but that there are some as yet unkown *side forces* ($\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$) acting at the front and rear contacts. We will find the equations of motion for the four lateral degrees of freedom (X, θ, χ, ψ) under the influence of these horizontal contact forces (temporarily assumed known), the vertical contact forces, the downwards force of gravity acting on the various masses, and the steering torque (see Fig.A7). Then we will use the conditions that front and rear contact points are each constrained to move in the direction of wheel heading (that is the wheels have zero slip angles), to eliminate θ and X. Finally, we will reduce the equations to two <u>not</u> involving the unknown forces, and two which <u>do</u> involve them. This will leave us with two equations in the variables χ, ψ (and the steering moment) which may be solved to find the motion, and two more equations into which we could insert the now-known χ, ψ to find $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$.

In the following equations we are looking at the forces or moments required to accelerate all the mass particles on the bike; and are setting these equal to the forces or moments of gravity and of the ground (both horizontal and vertical), and the steering moment applied by the rider to the front assembly (and negatively to the rear assembly).

We are performing an approximate linearised analysis, in which sines and cosines of small angles are approximated, and products of small quantities are neglected. (In this summary the linearising approximations are justified only casually if at all, but the results agree with more careful and rigorous linearisations performed by ourselves and others.) A linearised analysis can be considered "exact" if the motions are small enough; the only question is how small this is. For many systems, the linearised analysis is qualitatively and even quantitatively accurate for surprisingly large motions.

The approach which has been followed below for each variable is to put on the left hand side (LHS) the conjugate force or moment required when accelerating the various mass-points on the bike in an arbitrary motion. One way to think about this is to put forces $-ma$ at each point of the bike, and find the appropriate reaction force or moment.* (For example, if a bicycle is being accelerated to the right by an acceleration $\ddot{X}$, a force $m_t\ddot{X}$ is considered to act at the center of mass, in the $-X$ direction. If one were holding the bike upright by a moment at P_r, the magnitude of that moment would be $m_t\bar{h}_t\ddot{X}$.) Since we are trying to develop linear equations, we can legitimately consider the effects of each variable by itself while holding the others to zero — i.e., there are no interaction effects.

(1) <u>FOR THE WHOLE BIKE:</u> the total X-force required to effect the lateral acceleration of all mass points in a general motion = the sum of applied X-forces.

$$m_t\ddot{X} + m_t\bar{h}_t\ddot{\chi} - m_t\bar{l}_t\ddot{\theta} - m_f d\ddot{\psi} = \mathcal{F}_{Xr} + \mathcal{F}_{Xf}$$

LHS:

— the lateral acceleration of m_t due to lateral acceleration of the rear contact, no lean or yaw
— the X acceleration of m_t due to yawing (only) of the whole bike
— the X acceleration of m_t due to lean (only) of the whole bike
— the X acceleration of m_f due to steering only

RHS: sum of forces in X-direction

(2a) <u>FOR THE WHOLE BIKE:</u> the total χ-moment (about the heading line of the rear assembly), required when accelerating mass-points laterally in a general motion, = sum of χ moments of external forces about the same line.

$$m_t\bar{h}_t\ddot{X} + T_{yy}\ddot{\chi} + T_{yz}\ddot{\theta} + F'_{\lambda y}\ddot{\psi} - H_t\dot{\theta} - H_f\cos\lambda\dot{\psi} = gm_t\bar{h}_t\chi - gm_f d\psi - c_f(m_t g\frac{\bar{l}_t}{c_w})\psi$$

* According to Ruina, this approach is exactly equivalent to using conservation of linear X-momentum for the whole bike in (1); conservation of moment of momentum for the whole bike about forward and vertical axes whose origin instantaneously coincides with the rear contact in (2a) and (2b); and conservation of moment of momentum of the front assembly about a line coinciding with the steering axis in (3).

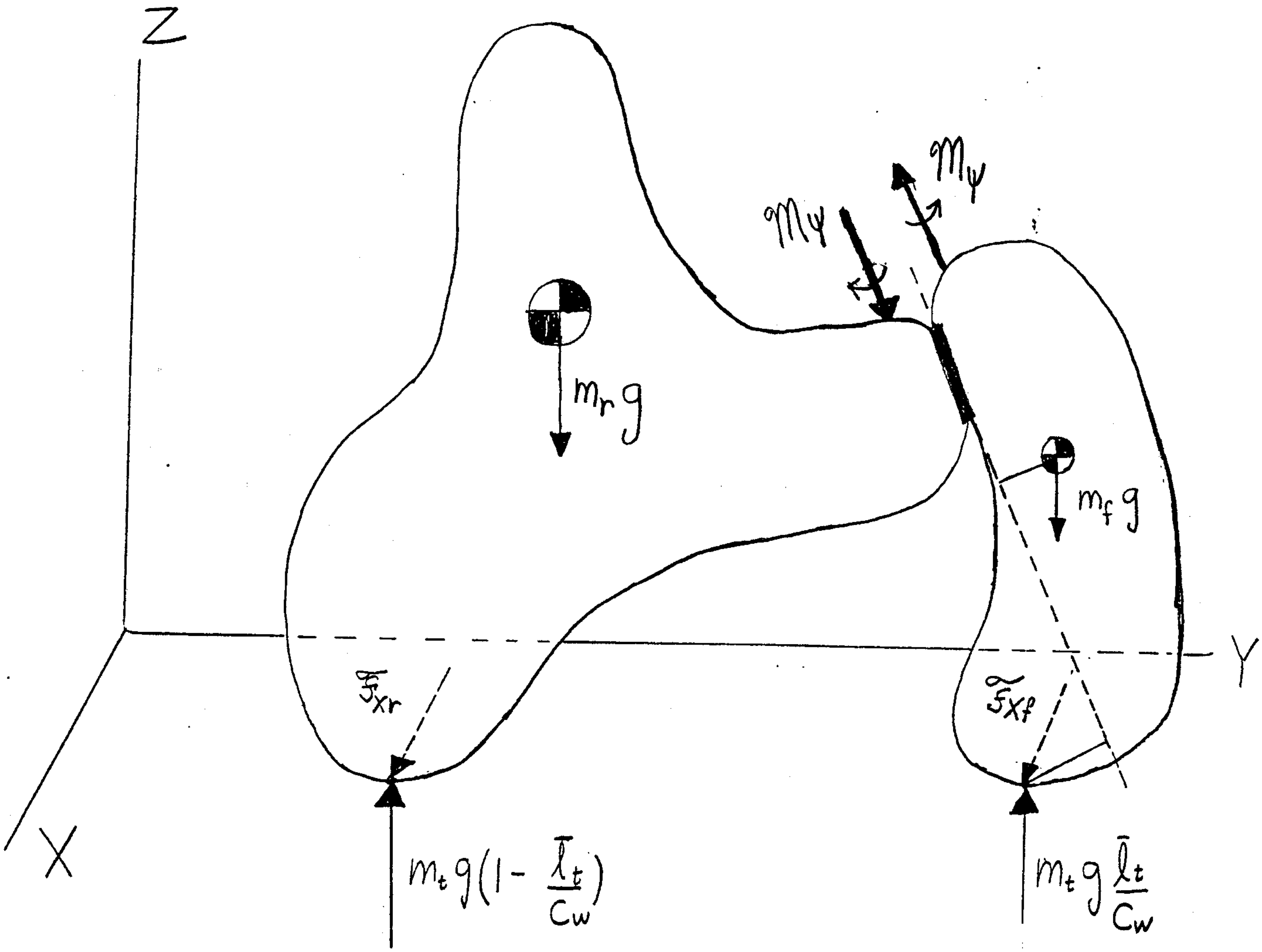


Fig.A7: Forces acting on bicycle (for small angles):
$m_r g$ acts at c.m. of 'rear assembly' (which includes rider)
$m_f g$ acts at c.m. of front assembly
$m_t g \frac{l_t}{c_w}$ acts vertically up at P_f
$m_t g(1 - \frac{l_t}{c_w})$ acts vertically up at P_r
$\mathcal{F}_{Xf}$ is essentially in the positive X-direction at P_f
$\mathcal{F}_{Xr}$ is essentially in the positive X-direction at P_r
$\mathcal{M}_\psi$ is a moment about $\vec{\lambda}$ which acts on the front assembly. The reaction moment $-\mathcal{M}_\psi$ acts on the rear assembly

$$= gm_t\bar{h}_t\chi - g\nu\psi \ ,$$

where for convenience we have defined $\nu = (m_f d + m_t \frac{l_t}{c_w} c_f)$.

LHS:

— the y (or χ) moment required to accelerate the center of mass of the whole bike laterally
— the y-moment required for angular acceleration ($\ddot{\chi}$, $\ddot{\theta}$) of the whole bicycle about the rear contact point
— the y-moment required for angular acceleration of the front assembly about the steering axis $\vec{\lambda}$ (remember that the contacts can slip sideways, and that X, χ, θ are being held fixed
— the gyroscopic moment (from wheel spin angular momenta H_r, H_f) required for yawing of the whole bike about a vertical axis
— the gyroscopic y-moment required for the precession about the vertical ($\dot{\psi}\cos\lambda$) of the front wheel

RHS:

— the moment of gravity force acting at the c.m. of the total bicycle which is leaned only (Fig.A8a)
— the moment of gravity force acting at the center of mass of the front assembly, for a non-leaning bicycle (Fig.A8b)
— the moment of vertical front contact force ($m_t g\bar{l}_t/c_w$) which is offset due to steer only (Fig.A8b)
— the steer moment $\mathcal{M}_\psi$ does not contribute because $-\mathcal{M}_\psi$ also acts on the bicycle. The forces $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$ do not contribute because they are in the ground plane and so have no moments about the rear-assembly heading line.

(2b) <u>FOR THE WHOLE BIKE:</u> the total θ-moment (about the z-axis through the rear contact P_r, which for small angles is equivalent to a *vertical* axis) required when accelerating mass-points laterally in a general motion = sum of moments of external forces about the same line.

$$-m_t\bar{l}_t\ddot{X} + T_{zy}\ddot{\chi} + T_{zz}\ddot{\theta} + F''_{\lambda z}\ddot{\psi} + H_t\dot{\chi} - H_f\sin\lambda\dot{\psi} = -c_w\mathcal{F}_{Xf}$$

LHS:

— the z (or θ) moment required to accelerate the center of mass of the whole bike
— the z-moment required for angular acceleration ($\ddot{\chi}$, $\ddot{\theta}$) of the whole bicycle about the rear contact point
— the z-moment required for angular acceleration of the front assembly about the steering axis $\vec{\lambda}$
— the gyroscopic moment required for tipping of the whole bike about the heading line of the rear assembly
— the gyroscopic z-moment required for the precession about the horizontal ($-\dot{\psi}\sin\lambda$) of the steered front wheel

RHS: moment of $\mathcal{F}_{Xf}$ about the z axis.

(3) <u>FOR THE FRONT ASSEMBLY ONLY:</u> the total ψ-moment about the steering axis $\vec{\lambda}$ required for a general bicycle motion = sum of external moments about the same axis.

$$-m_f d\ddot{X} + F'_{\lambda y}\ddot{\chi} + F''_{\lambda z}\ddot{\theta} + F'_{\lambda\lambda}\ddot{\psi} + H_f(\dot{\chi}\cos\lambda + \dot{\theta}\sin\lambda)$$

$$= \mathcal{M}_\psi + c_f\mathcal{F}_{Xf} - g(m_f d + m_t\frac{l_t}{c_w}c_f)\chi + g\sin\lambda(m_f d + m_t\frac{l_t}{c_w}c_f)\psi$$

$$= \mathcal{M}_\psi + c_f\mathcal{F}_{Xf} - g\nu\chi + g(\sin\lambda)\nu\psi \ ,$$

(where ν is defined in (2a) above).

LHS:

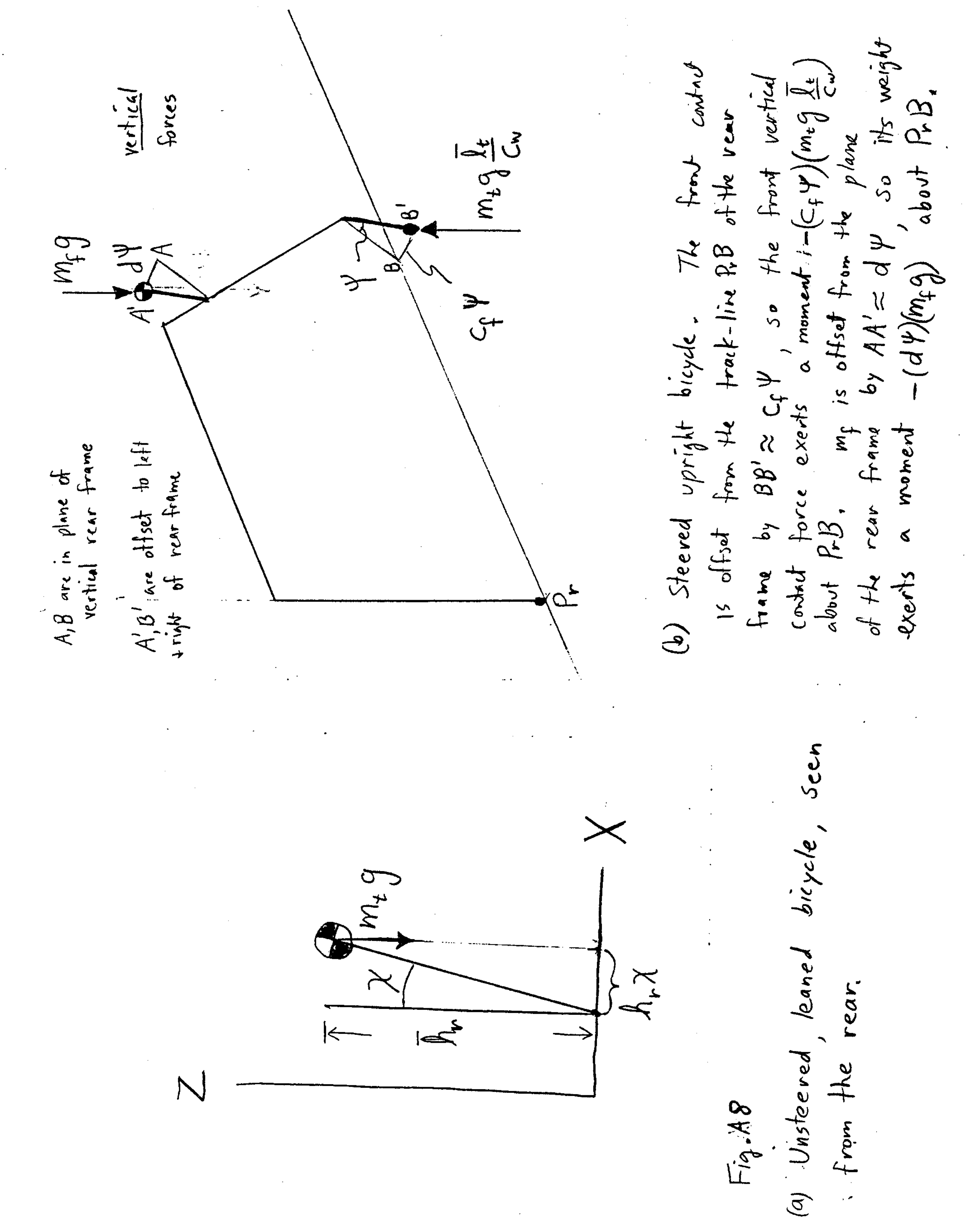


Fig.A8: Pictures showing (a) unsteered leaned bicycle from rear; and (b) steered upright bicycle for which front contact is offset from the track line of the rear, so that $m_f g$ is offset by the distance $d\psi$, and $m_t g \bar{l}_t / c_w$ (the vertical contact force at P_f) is offset by $c_f \psi$

— ψ-moment required to support lateral acceleration of the front assembly
— Moments about $\vec{\lambda}$ axis required when the front assembly is given angular acceleration about the χ (y) axis and θ (z) axis. Because inertia tensors about any point are symmetric matrices, the moment about one axis required for angular acceleration about another is the same as the moment about the second required for angular accleration about the first.
— the moment about the steering axis required for angular acceleration of the front assembly about that axis (the coefficient is the polar moment of inertia)
— the moment about the steering axis required for precession ($\dot{\chi}\cos\lambda + \dot{\theta}\sin\lambda$) about an axis in the plane of the bicycle which is perpendicular to the steering axis.

RHS:
— the steering moment $\mathcal{M}_\psi$, and the moment about the steering axis of the horizontal force, are easy to see. (Fig.A9a)
— when the bicycle is leaned only, the vertical reaction force at the front contact and the vertical gravitational force on m_f both have components proportional to χ which are perpendicular to the plane of the bike. These forces act on lever arms c_f and d. (Fig.A9b)
— when the bicycle is steered only, these two forces are displaced from the plane of the bicycle, and no longer pass through the steering axis. Resolve them initially into components perpendicular and parallel to the steering axis; then when they are displaced, it is easy to see that only the components initially *perpendicular* to the steering axis (which are multiplied by $\sin\lambda$) exert moments, with lever arms equal to their lateral displacements ψd and ψc_f. (Fig.A9c)

REDUCED EQUATIONS OF MOTION

Equations $1, 2a, 2b, 3$ are true whatever horizontal forces $\mathcal{F}_{Xr}$, $\mathcal{F}_{Xf}$ act at the wheel contacts, and in particular they are true if the forces are just right to prevent the wheels from side-slipping relative to their instantaneous headings.

For the rear wheel, zero side slip is expressed by the equation $\dot{X} = -V\theta$. (See Fig.A10a.) For the front contact, an analogous relation is needed in terms of the X-co-ordinate and heading of the front wheel: $\dot{X}_f = -V\theta_f$. To write this in terms of the variables describing the motion, we need $X_f = X - c_w\theta + c_f\psi$ for the X-co-ordinate of the front contact P_f (Fig.A10b), and $\theta_f = \theta + \psi\cos\lambda$ for the heading of the front wheel (Fig.A10c). ($\cos\lambda$ comes in because a small rotation of the front wheel about the steering axis can be considered a sum of small rotations about horizontal and vertical axes — and only the latter changes the front wheel's heading.)

With these relations, we can express such quantities as $\dot{\theta}$, $\ddot{\theta}$ and $\ddot{X}$ in terms of and ψ and $\dot{\psi}$. By subtracting the rear-wheel constraint from the front-wheel constraint, we obtain

$$-c_w\dot{\theta} + c_f\dot{\psi} = V\psi\cos\lambda \ ,$$

which may be solved for $\dot{\theta}$ to give

$$\dot{\theta} = \frac{c_f}{c_w}\dot{\psi} + V\frac{\cos\lambda}{c_w}\psi \ .$$

This may be differentiated once to give $\ddot{\theta}$:

$$\ddot{\theta} = \frac{c_f}{c_w}\ddot{\psi} + V\frac{\cos\lambda}{c_w}\dot{\psi} \ .$$

Finally, the rear constraint relation may be differentiated once, and $\dot{\theta}$ may be replaced:

$$\ddot{X} = -V\dot{\theta} = -V\frac{c_f}{c_w}\dot{\psi} - V^2\frac{\cos\lambda}{c_w}\psi$$

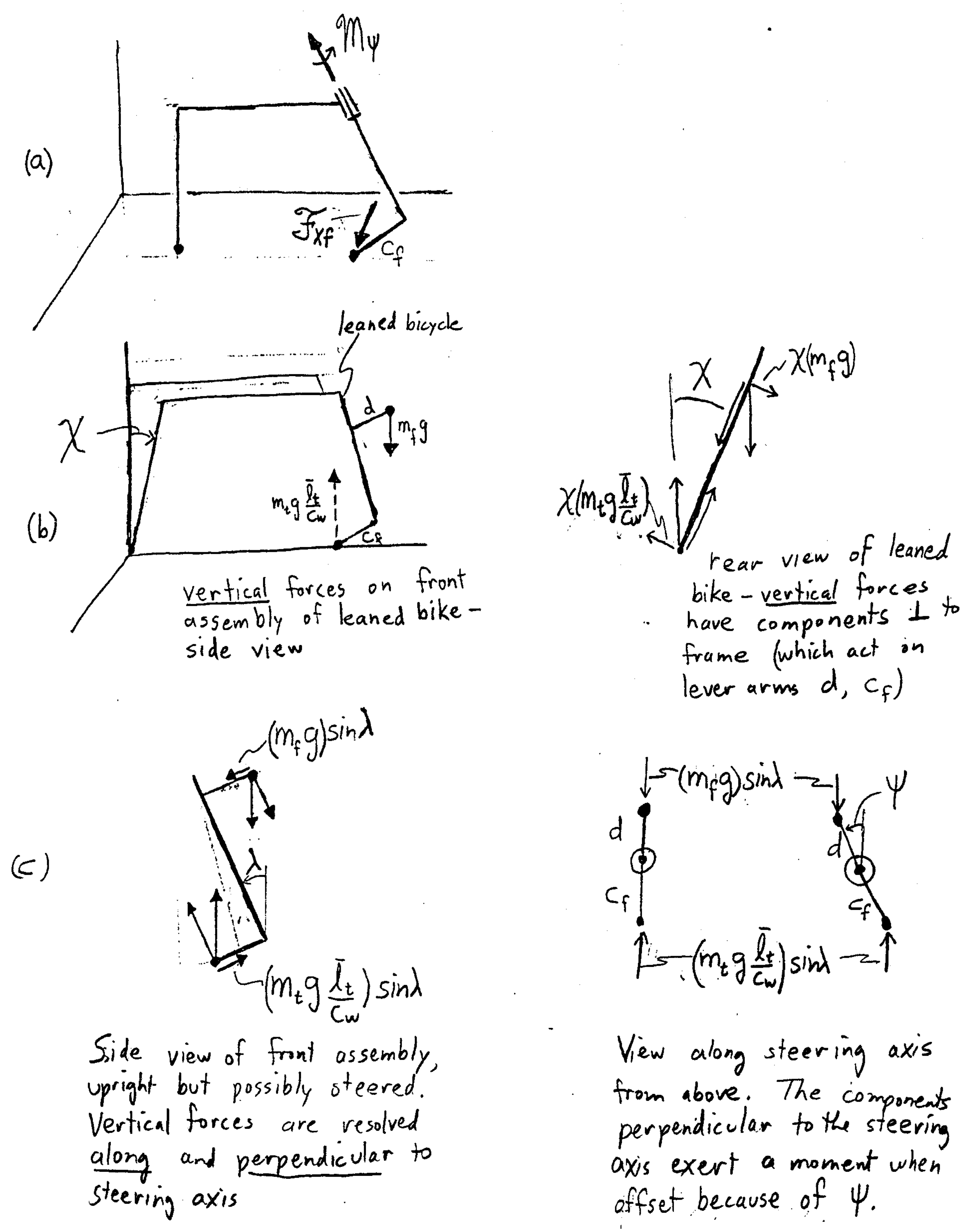
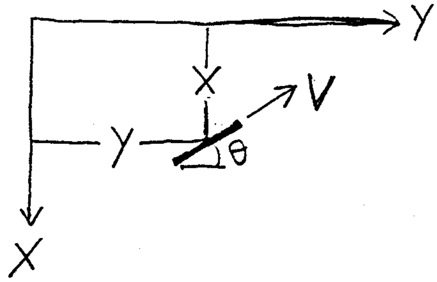


Fig.A9: Moments acting on the front assembly about the steering axis (none transmitted by the steering bearing):
(a) Steer moment $\mathcal{M}_\psi$, and horizontal force $\mathcal{F}_{Xf}$ at front contact P_f
(b) Moment of vertical forces when bicycle is leaned only
(c) Moment of vertical forces when bicycle is steered only (which arise if $\lambda = 0$)

rear wheel:

(a)

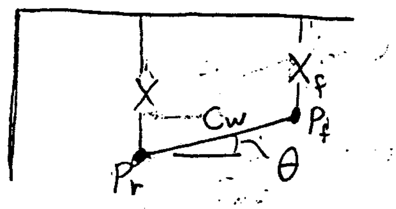


$$\frac{\Delta X}{\Delta Y} = -\tan\theta, \quad \text{or} \quad \dot{X} \approx -V\theta$$

(for small θ)

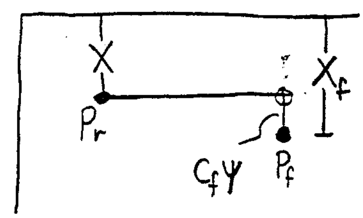
Similarly for X_f, θ_f at front contact

(b)



For bike with $\psi = 0$,

$X_f \approx X - c_w\theta$

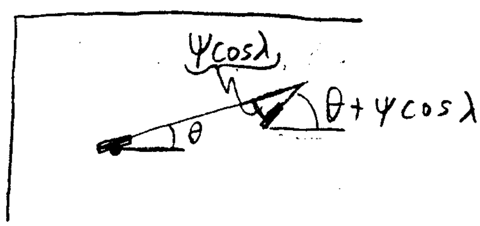


For bike with $\theta = 0$,

$X_f \approx X + c_f\psi$ due to offset of front contact

Combined : $X_f \approx X - c_w\theta + c_f\psi$

(c)



Heading of front wheel θ_f depends on heading of rear wheel plus steer of front relative to rear.

$$\theta_f \approx \theta + \psi\cos\lambda$$

Fig. A10 (a) Illustration of rolling constraint due to rear wheel
(b) Defining X_f in terms of X, θ, ψ
(c) Defining θ_f in terms of θ, ψ

After substituting these relations into Eqs. 1,2,3, to eliminate $\dot{\theta}, \ddot{\theta}$ and X, we have four equations but apparently only two variables. However, demanding that the horizontal contact forces should prevent the wheel sideslip means that these forces are no longer freely selectable, but must be exactly the right magnitude at every instant. In essence, they are the two remaining variables.

Since we are concerned at present only with the *motion*, the most convenient thing is to rearrange the equations so that the unkown forces do not appear in two of them; these two will allow us to solve for the unknowns χ, ψ.

Equation (2a) (with X and θ eliminated) is already in that form; because it dealt with moments about a line in the ground it will be called the lean equation. For the other equation we simply eliminate $\mathcal{F}_{Xf}$ from (2b) and (3), and leave $\mathcal{M}_\psi$ on the right hand side; this is called the steer equation. (Evidently equation (1) is not needed, unless we wish to find $\mathcal{F}_{Xr}$.)

We write these two equations in the form:

$$M_{\chi\chi}\ddot{\chi} + M_{\chi\psi}\ddot{\psi} + C_{\chi\psi}\dot{\psi} + K_{\chi\chi}\chi + K_{\chi\psi}\psi = 0 \text{ , the lean equation*}$$

(note that there is no $C_{\chi\chi}\dot{\chi}$ term); and

$$M_{\psi\chi}\ddot{\chi} + M_{\psi\psi}\ddot{\psi} + C_{\psi\chi}\dot{\chi} + C_{\psi\psi}\dot{\psi} + K_{\psi\chi}\chi + K_{\psi\psi}\psi = \mathcal{M}_\psi \text{ , the steer equation.}$$

The coefficients to the lean equation are

$$\begin{aligned}
M_{\chi\chi} &= T_{yy} \\
M_{\chi\psi} &= F'_{\lambda y} + \frac{c_f}{c_w}T_{yz} \\
C_{\chi\chi} &= 0 \\
C_{\chi\psi} &= -\left(H_f\cos\lambda + \frac{c_f}{c_w}H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w}m_t\overline{h}_t\right) \\
K_{\chi\chi} &= -g m_t\overline{h}_t \\
K_{\chi\psi} &= g\nu - H_tV\frac{\cos\lambda}{c_w} - V^2\frac{\cos\lambda}{c_w}m_t\overline{h}_t
\end{aligned}$$

and the coefficients to the steer equation are:

$$\begin{aligned}
M_{\psi\chi} &= F'_{\lambda y} + \frac{c_f}{c_w}T_{yz} \\
M_{\psi\psi} &= F'_{\lambda\lambda} + 2\frac{c_f}{c_w}F''_{\lambda z} + \frac{c_f^2}{c_w^2}T_{zz} \\
C_{\psi\chi} &= H_f\cos\lambda + \frac{c_f}{c_w}H_t \\
C_{\psi\psi} &= V\left(\frac{\cos\lambda}{c_w}F''_{\lambda z} + \frac{c_f}{c_w}\left(\frac{\cos\lambda}{c_w}T_{zz} + \nu\right)\right) \\
K_{\psi\chi} &= g\nu \\
K_{\psi\psi} &= -g\nu\sin\lambda + VH_f\frac{\sin\lambda\cos\lambda}{c_w} + V^2\frac{\cos\lambda}{c_w}\nu
\end{aligned}$$

Note that most coefficients are functions of velocity. In fact the angular momentum H_f for the front wheel typically could be written as $H_f = V(J_f/a_f)$, where a_f is front wheel radius and J_f is front wheel polar moment of inertia; and similarly for the rear. However there is also the possibility of adding independent high-speed gyros to the bicycle, in which case H_f and/or H_r might be constant, or a negative multiple of speed, etc.

When developed in this form, the equations display a degree of symmetry.

* If we had allowed flexible training wheels (say) to help support the rear assembly against leaning, the lean equation would have to have the supporting moment $\mathcal{M}_\chi$ on the right hand side.

APPENDIX B

APPLYING AND EXTENDING THE EQUATIONS OF MOTION

[Not yet completed.]

From APPENDIX A the equations of bicycle motion, which govern rear-assembly lean-angle χ and steer angle ψ, are:

$$M_{\chi\chi}\ddot{\chi} + K_{\chi\chi}\chi + M_{\chi\psi}\ddot{\psi} + C_{\chi\psi}\dot{\psi} + K_{\chi\psi}\psi = \mathcal{M}_{\chi} \quad \text{(the lean equation)},$$

and

$$M_{\psi\chi}\ddot{\chi} + C_{\psi\chi}\dot{\chi} + K_{\psi\chi}\chi + M_{\psi\psi}\ddot{\psi} + C_{\psi\psi}\dot{\psi} + K_{\psi\psi}\psi = \mathcal{M}_{\psi} \quad \text{(the steer equation)}.$$

$\mathcal{M}_{\psi}$ is the steering moment exerted by the rider (or by some device attached to the *rear assembly*), and the tipping (or supporting) moment $\mathcal{M}_{\chi}$ is normally zero.

By solving, or studying the stability of, these equations, a variety of topics may be studied. Here, several are described very briefly, to give the reader an idea of the equations' potential usefulness.

Steering Moment Given as Function of Motion

If the steering moment $\mathcal{M}_{\psi}$ is given as a function of bicycle motion (for example, by a 'balance — controller' which applies a steering moment depending on the bicycle's lean), this may be incorporated in the analysis by modifying the steer equation. Then the lean equation and the modified steer equation can be used simultaneously to determine the behaviour of χ and ψ. This approach can be used to evaluate balancing strategies or controller designs.

Some potentially interesting ways of improving bicycle stability include low-mass passive components such as steer dampers and springs, or a fast front-assembly gyro. With careful design, these devices could be made to adapt their properties to the bicycle's velocity, potentially improving handling at all speeds.

Steer Angle Given as Function of Motion

On the other hand, we may want to study the effect of controlling the steer angle as a function of bicycle lean. If ψ is given in terms of χ, the lean equation must be used to solve for χ. If desired, the solution may be inserted into the steer equation to find the resulting steer moment.

No-Hands Stability

After setting the steering moment to zero, we may use both equations simultaneously to study no-hands motion and stability (the primary focus of this report).

Steer Moment Given as Function of Time

The heading of a stable bicycle may be affected (and thus in some sense controlled) by applying steering moments. The steering produced by instantaneous or ramped application of a steer moment can be deduced by solving the equations of motion with the given $\mathcal{M}_{\psi}(t)$ on the right-hand-side, under the assumption that $\psi, \dot{\psi}, \chi, \dot{\chi} = 0$ initially. Similarly, if the wind tends to tip the bicycle, the time-dependent tipping moment $M_{\chi}(t)$ must be used in the lean equation.

Skateboards

Skateboards have axles designed so that the wheels steer by an amount proportional to the lean angle. Since both the front and the rear wheels of a skateboard steer, to apply the bicycle analysis it is necessary to define an imaginary 'non-steering rear-wheel contact point', P_r^{im}, as the correct origin for defining moments of inertia, etc. If the front and rear axles steer exactly opposite amounts, P_r^{im} is half-way between the front and rear contact points (any trail is ignored for simplicity). More generally, for a given lean angle imagine that there is a series of wheels along the length of the skateboard, whose steer angles vary linearly between the actual values at the front and back. Then P_r^{im} is at the imaginary wheel whose steer angle is zero (this might be off the skateboard altogether). Then the wheelbase (i.e. the distance to the front contact) and other important quantities are calculated with respect to P_r^{im}, not with respect to the actual rear contact.

Once the appropriate equation coefficients have been provided, the analysis is performed on the lean equation after setting the steer angle ψ proportional to the lean angle χ (cf. *"Steer Angle Given"*, above).

Tricycles

We can study the behaviour of tricycles by ignoring the lean equation (which should have the supporting moment $\mathcal{M}_\chi$ on the right hand side), and employing the steer equation with $\chi = 0$. If $\psi(t)$ is given, this equation delivers the steering torque $\mathcal{M}_\psi$ required; and if we set $\mathcal{M}_\psi = 0$, it determines hands-off stability, and allows us to find $\psi(t)$ if desired. Once $\psi(t)$ is either specified or found, the lean equation gives the supporting moment $\mathcal{M}_\chi$ supplied by the rear wheels.

This tricycle analysis can also be used to investigate whether a shopping cart will tend to travel straight, when rolling freely either forwards or backwards.

Lateral Forces on Wheels

Bicycle wheels, especially in the traditional large size, are rather weak laterally — side loads cause high spoke stresses and may even lead to collapse. The equations of motion (including those in APPENDIX A) can be used to find these forces in any given motion; perhaps the most interesting case is when the rider varies the steer angle rapidly, or applies a large steer torque. [There is reason to believe that very large front–wheel side forces sometimes occur, perhaps due to the $\dot{\psi} V \cos\lambda / c_w$ contribution to $\ddot{\theta}$.]

In this case we first solve for the motion as above so that <u>both</u> $\chi(t)$ and $\psi(t)$ are known. Then we use equations (1) and (2b) to solve for the horizontal forces $\mathcal{F}_{Xr}$ and $\mathcal{F}_{Xf}$, in terms of χ and ψ and their derivatives. The lateral force at each wheel is not just the horizontal force, but because of lean also includes a component of the vertical force. So at the rear the lateral force is $\mathcal{F}_{Xr} - \chi(g m_t \frac{c_w - \bar{l}_t}{c_w})$, and at the front it is $\mathcal{F}_{Xf} - (\chi - \psi \sin\lambda)(g m_t \frac{\bar{l}_t}{c_w})$.

* * * * * * * * * * * * * *

The above topics can be treated in a straightforward way with essentially no modifications to the equations of motion. To study subjects such as the following requires more thought, and often a significant alteration of the equations (here discussed with reference to equations (1), (2a), (2b), (3) of APPENDIX A). The greatest complication arises when new *degrees of freedom* are introduced: additional independent ways mass can move in the problem.

Most of these problems have been treated by one or more investigators, but we are not yet in a position to evaluate much of that work.

Finite Fore and Aft Forces

This concerns the effects of finite forces acting mostly in the forward or backwards direction. In a small steering and leaning motion, these may slightly shift their direction or point of application in such a way as to enter the lean and steer equations. The forces under consideration include: braking or driving forces (applied to front and/or rear wheel contacts); and forces due to aerodynamic drag, towing, or gravity on a hill; these forces may balance each other, or cause acceleration / deceleration. Such phenomena can evidently be important for motorcycles, which have been known to behave well while accelerating, but to oscillate violently when decelerating.

It is easy to work out such things as the drive force or net acceleration, but harder to understand their contributions to the equations of motion, which depend at least on (1) the altered vertical force at the front contact, and (2) the approximate lateral forces and yaw or roll moments applied to the front and rear assemblies for small steering and leaning motions.

Steady motion down a hill of angle α with the brakes on may be one of the easier cases to analyse. The equations are altered for four reasons: (a)'downwards' gravity is modified to $g\cos\alpha$; (b) horizontal $mg\sin\alpha$ forces act on front and rear masses in the Y direction; (c) the front 'vertical' contact force is increased; (d) the front and/or rear wheels have braking forces. The <u>right-hand sides</u> of equations (1),(2a),(2b),(3) must be modified in the following way:

(1) X-components of the braking forces act on each wheel that is not parallel to the Y-axis.

(2a) g is modified to $g\cos\alpha$ in the first two terms. The front-contact 'vertical' force in the third term is modified. The horizontal force $m_t g \sin\alpha$ exerts a tipping force proportional to θ.

(2b) The front braking force exerts a moment about the rear contact proportional to ψ. $mg \sin\alpha$ forces at front and rear mass centers exert moments due to θ (rear) and θ, ψ (front).

(3) In each gravity term, $m_f d$ has $g \cos\alpha$, and the other part has the modified front contact force. $m_f g \sin\alpha$ exerts a steering moment due to the heading $\theta + \psi \cos\lambda$ of the front wheel.

Acceleration and deceleration appear much more difficult to study. It seems appropriate to make the approximation that instability "at a given speed" is sufficient information, even though the velocity, and hence the equation coefficients, are changing continuously (such 'parametric variation' can sometimes completely change the stability picture). Even so, it is not easy to work out the necessary modifications to the left-hand sides of the four equations.

Aerodynamic 'Lift' Forces in Still Air

When a 'flat' object moves relative to the air, often the largest aerodynamic forces are 'lift' forces perpendicular to its relative motion (a horizontal force is still called lift). There is some concern that a covered (disk) wheel or covered bicycle might generate such forces in small steering motions, which would destroy stability. The idea is perhaps that if the front wheel turns a little, the lift forces will cause it to turn even more (and also apply a tipping moment to the bicycle).

In still air, the effect of lift forces is perhaps not as bad as imagined. Any point on a body supported by a single rolling wheel moves exactly in the direction the wheel is pointed, except for lateral motions due to tipping and yawing (and tire 'slip' angle; see below). The angle of attack is proportional to the tipping and yawing rates, and the lift force resists the tipping and yawing — more or less like a viscous damper. In a typical bicycle motion the front and rear assemblies each have tipping and yawing rates (recall that a steady turn causes yawing), leading to net moments and forces* which are conveniently referred to the front and rear contacts.

While we have not worked out whether these effects destroy self-stability, at least it seems likely that any instability arising from this still–air 'viscous damping' will occur slowly, and may not be too important for a rider-controlled bicycle. However sidewinds — steady, or especially sudden — will produce large disturbing forces. (This liability is probably not offset by a streamlined bicycle's potential for 'sailing', i.e. extracting energy from a crosswind!) Small wheels have a clear advantage here: less side force, lower tiping moment.

Tire Phenomena

[Not completed; mostly need to condense a larger essay which is already drafted. Outline:

— Explain how tire deformation under maneuvering forces gives rise to 'sideslip', 'camber thrust', etc. (raising the pressure reduces the effect).

— Argue that ***bicycle*** tire behaviour may be well modelled by a series of springy fingers. (Note: a tire's vertical stiffness apparently is not affected by its 'fatness'.)

— Cite derived results (any confirmation?) for tire side force and scrub torque as linear functions of (steady) slip angle, lean, and path curvature. Display dependence of coefficients on tire pressure, wheel radius, and load carried.

— Point out difficulty of including lateral tire flexibility, since for consistency's sake it should add several kinds of 'unsteadiness' to the steady tire equations. However lateral wheel flexibility may legitimately be added, since it is usually considerably greater than tire flexibility.

— Is tire 'sideslip' an important factor in bicycle dynamics? Analytical estimates of tire 'sideslip' behaviour, and of the forces which cause it, suggest that tire deformation phenomena may be much more important for motorcycles in their typical riding conditions than for bicycles in theirs (and also that sideslip may be insignificant in steady turns, because the force of the ground is nearly in the plane of the wheel). On the other hand, wheel (and bicycle) flexibility appears to be an important factor in bicycle shimmy. The question is whether tire behaviour is important in bicycle shimmy conditions.

— Brief instructions for incorporating tire behaviour and wheel flexibility in the equations of motion — and the order of equation which results from each choice. For best understanding, perhaps it is preferable to retain simplicity by modelling the front tire only.]

* [Forces and moments arising from yawing motions appear to be impossible to measure on a steady basis in a traditional wind tunnel: either the tunnel should be curved, or the object should be whirled at the end of an arm in still air.]

Rider Body Articulation

[Not yet written. Riders may both bend and shift sideways at the waist; are both important separately? Number of equations; sketch how to derive them.]

Frame Flexibility

[Not yet written. 'Torsional' bicycle flexibility may be important for the study of shimmy — in contrast to wheel flex, it involves the gyroscopic effects of wheel tilt. How to derive equations. Possibility of developing simplified equations which represent high-frequency oscillations alone?]

APPENDIX C

GENERAL STABILITY OBSERVATIONS FROM EQUATIONS

VELOCITY DEPENDENCE OF COEFFICIENTS

The coefficients A–E (see *Determination of No–Hands Stability* in the text) are functions of the velocity V, with the forms

$$A = a_0,\ B = b_1 V,\ C = c_o + c_2 V^2,\ D = d_1 V + d_3 V^3,\ E = e_0 + e_2 V^2$$

where the magnitudes and signs of $a_0 ... e_2$ depend on the parameters of the bicycle (plus rider). a_0 is positive for ***any*** conceivable design, while for a ***conventional design***, b_1, c_2, d_3, e_0 are positive, and c_0, d_1, e_2 are negative (these signs will not always hold). In the conventional case the velocity must be large enough to make C and D positive (in fact somewhat greater, to make $C > AD/B + EB/D$), yet when it is too great E will be negative. So if a design is stable at one speed it need not be at another (e.g. at rest). In general, self–stability occurs in a single range of speeds if at all;* for an approximate model of a conventional bicycle this range is from about 12 mph to about 16 mph.

LIMITS OF STABLE VELOCITY RANGE

It turns out that for ***any*** design, IF there is a range of stable speeds, the limiting speeds occur when

$$C = A\left(\frac{D}{B}\right) + E\left(\frac{B}{D}\right),$$

and usually (if not always) when

$$E = 0 .$$

The first equation is actually the condition for the existence of <u>purely imaginary</u> eigenvalues,** while the second is the condition for a <u>zero</u> eigenvalue.

In the first case, which typically is the low-speed limit, the bicycle executes a weaving motion which never dies away. At a slightly slower speed the weave grows with each oscillation, while at a slightly faster speed the weave amplitude decreases with each oscillation.

In the second case (which can provide at most *one* of the limiting velocities, typically the high-speed one) the bicycle tends to remain in any turn. Slightly slower, and it slowly straightens up; slightly faster, and it slowly falls over without any weaving. It can be shown that $-E$ is proportional to the steering moment exerted by the rider when the bicycle is balanced in any steady turn. For a ***stable*** bike, therefore, the handlebars must be ***restrained*** (gently) from turning further — very different from a car! The wheel ***tries*** to turn into the lean, so when the handlebar is released the front contact point is steered back under the bicycle. On the other hand, if the steering is trying to straighten out ($E < 0$, as could be arranged with a steer-centering spring), then when the handlebars are released the steering returns towards straight ahead, and the bicycle falls over.

As noted below in "STUDY OF COEFFICIENTS", even when E is somewhat negative most bicycles and motorcycles can easily be stabilised by subtle rider steering or leaning behaviour. So while an <u>uncontrolled</u> bicycle absolutely requires $E > 0$ for stability, one might consider ignoring the sign of E for a <u>controlled</u> bicycle (as long as the magnitude of E/D is sufficiently small — see (2) below).

APPROXIMATE SOLUTIONS FOR LARGE VELOCITIES

* Theoretically it appears that <u>two</u> stable speed ranges might be possible. We have never observed this, and would like to prove that it can't happen.

** Actually, imaginary eigenvalues may also occur if B and D go to zero while thir ratio remains finite, as for example when velocity goes to zero. An exceptional limiting case would occur if $a_0, b_1, c_0, d_1, e_0 > 0$ and $c_0^2 > 4a_0e_0$, in which case the velocity would go to zero (and the bicycle would lose stability) without violating the first criterion. Fortunately, it is impossible for c_0 and e_0 both to be positive (unless the bicycle has constant-speed gyroscopes attached), otherwise the bicycle at rest would oscillate without falling!

For our fourth-degree polynomial we may recognise some special velocity magnitudes (use absolute values where appropriate):

(1) First, there are the velocities above which the lower-power contributions to C, D, E may be ignored: $V >> V_C,\ V_D,\ V_E$, where $V_C^2 = (c_0/c_2)$, $V_D^2 = (d_1/d_3)$, $V_E^2 = (e_0/e_2)$. The three quotients are each proportional to gravity g times some measure of bicycle size.

(2) Second, there is the velocity V_{sr} above which a *small real* eigenvalue is given approximately by $s_{sr} \approx -E/D$, that is when $E/D << D/C,\ \sqrt[2]{D/B},\ \sqrt[3]{D/A}$. For large velocities E/D will generally be proportional to V^{-1}, while the other terms will be proportional to V, so that $V_{sr}^2 >> (c_2e_2/d_3^2)$, $(e_2^2b_1/d_3^3)^{1/2}$, $(e_2^3a_0/d_3^4)^{1/3}$. These quantities are also proportional to g and size. (The approximation also holds near V_E if e_0 and e_2 have opposite signs, though if $V_D >> V_E$ it may fail for some range of greater velocities.) This eigenvalue s_{sr}, which produces slow falling if positive, may be relatively unimportant, as it typically reaches only a small (and hence easily controllable) positive value, and then decreases like V^{-1}.

(3) When $V >> V_C,\ V_D,\ V_E,\ V_{sr}$, then the other three eigenvalues will be found approximately from the cubic equation

$$a_0\left(\frac{s}{V}\right)^3 + b_1\left(\frac{s}{V}\right)^2 + c_2\left(\frac{s}{V}\right)^1 + d_3 = 0\ .$$

a_0, b_1, c_2, d_3 do not involve gravity, so we may say this equation describes the behaviour resulting when gravity forces are of negligible importance compared to inertial forces. This equation implies that if V is large enough to make it valid, then the remaining three eigenvalues are ***all proportional to*** V. Thus oscillations, uprighting etc. will each take place over a fixed distance, which of course is traversed more quickly at higher speeds.* (Another way to see this is to note that the operator s/V actually represents d/dY, so the equation may be written entirely in terms of distance Y.) In this velocity range, stability requires a_0, b_1, c_2, d_3 to be positive, and $c_2 > a_0(d_3/b_1)$ — an indication that it may be good to keep a_0 small.

In special cases where some of these coefficients are particularly large or small, further approximations may be possible.

SCALING RULES (similar to rules for a rolling disk)

Mass Scaling: If all the mass densities of all the parts is doubled, the equations are absolutely unaffected. Thus a motorcycle plus rider is perhaps like a heavy bicycle with a young child riding it.

Speed and Length Scaling: If the dimension of every part on the bicycle (including the rider) is multiplied by a given factor, the stability will be the same as long as the speed is correctly scaled:

$$\frac{V^2}{gL} = \text{constant},$$

where L is a typical length such as wheelbase or c.m. height. A larger size is thus like a smaller V — at a given speed, an otherwise standard bicycle which is too large will weave, and one which is too small will 'capsize' (i.e. lean increasingly to one side without weaving). If V_{min} is a bicycle's lowest stable speed, V_{min}^2/gL is a sort of 'index of merit' for bicycle design — the smaller the better. However as there is no rule for defining L, slight differences are only meaningful if bicycle shape varies little.

Note that a lengthened bicycle such as a tandem is similar to a lowered bicycle.

Time Scaling: When similar bicycles of different sizes have the appropriate velocities defined above, their stability or instability will be similar. (That is, the two fourth-order polynomials will have proportional solutions.) However, everything will happen slower with the larger bicycle: the observed period of any oscillation *when the speed is correctly scaled* is L/V or $\sqrt[2]{L/g}$.

Force Scaling: The horizontal ground–contact forces will scale like $(FL/MV^2) = f(gL/V^2)$, which approaches a constant at sufficiently high velocities (i.e., when gravity may be ignored and an oscillation takes place over a fixed distance).

* Ruina points out that velocity-independent motion is to be expected of systems whose state of motion may somehow be characterised by a single velocity, when this is sufficiently large that inertial forces somehow become dominant.

STUDY OF COEFFICIENTS

It may be worth noting that the spin angular momentum of the wheels does not appear in A or B but only in C, D, E.

Using the fact that the kinetic energy of the bicycle "on ice" is never negative, it can be shown that it is physically impossible for A to be negative. However, A can be made very small if the "mechanical trail" — the perpendicular distance of the front contact from the steering axis — is not too large: A is then proportional to the moment of inertia about the steering axis of the wheel+fork+handlebar. If A is sufficiently small, there will be a large real eigenvalue $s_{lr} \approx -B/A$, which must be negative for stability. The conditions for this approximation to be valid are that $A/B << B/C,\ \sqrt[2]{B/D},\ \sqrt[3]{B/E}$.

It is obvious that b_1 ***must*** be made positive, or the bicycle will never be self-stable. However, we have not made a study of how this limits bicycle design. For a fairly standard bike, the sign of B may be found by 'kicking' the front contact sideways — the front wheel should steer toward the 'kicker' (this will normally happen for positive c_f, d).

c_0 can be made positive if $F'_{\lambda\lambda}$, $F'_{\lambda y}$, $F''_{\lambda z}$ and m_f are small, but it is usually negative. c_2 has not been studied (though it is easy to make it positive if these quantities are small).

d_3 is nearly always positive. It is proportional to $C_{\psi\chi}$, and hence vanishes if there are no gyroscopic effects.

To the best of my knowledge, E is the only coefficient which has so far been studied much (by Carvallo, by Rice, by Weir, and — perhaps in greater detail — by ourselves); this is ironic, since the above claims about the small real eigenvalue, and also Whipple's analysis, suggest that instability due to typical negative E is very easily controlled by the rider.* Motorcycles at highway speed, and bicycles at racing speed, are apparently both unstable in this sense.

e_0 can be found from experiments in which *the bicycle at rest or moving very slowly is steered while kept perfectly balanced,* and the (non-frictional part of the) steering torque is measured as a function of the steer angle. For a bicycle with a *massless front assembly* (wheel, fork, and handlebar), e_0 is the product of "Jones' Parameter" (which for an infinite-wheelbase bicycle is just the mechanical trail c_f), times a more complex geometric quantity which is positive only if

$$\sin\lambda > \frac{c_f}{c_w}\frac{\bar{l}_t}{\bar{h}_t}$$

(Here, λ is the head angle measured as a tilt back from vertical, c_w is the wheelbase, and $\bar{l}_t, \bar{h}_t$ are the horizontal and vertical distance of the overall center of mass relative to the rear contact.) That is, for e_0 to be positive the mechanical trail must not be too great for the head angle.

e_2V^2, which is typically negative, contains spin angular momentum as a factor, and so vanishes for a non-gyroscopic bicycle. It can also be made to vanish by adjusting the geometry.

For a general bicycle it is possible to adjust the parameters so that e_0 is positive even though $c_f < 0$, as long as $\lambda < 0$. It is also possible to make e_2 and e_0 positive together, so that E is ***never negative.*** Then (in Sharp's terminology) the *capsize mode* is always stable. [Actually, capsize is conceivable even if $E > 0$ if some other eigenvalues are unstable — does this case ever occur?]

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* It is easy to make $E < 0$ with a steer-centering spring of stiffness k, which is simply added to $K_{\psi\psi}$. (This increases C by kT_{yy}, and decreases E by $kgm_t\bar{h}_t$.) It would be interesting to learn what values of k a rider will either notice, dislike, or find prevents no-hands riding. [A torsional spring of about 30 ft-lbf/rad makes no-hands riding nearly impossible, and makes large steer angles unpleasant to achieve, but straight-line hands-on riding at moderate speeds seems acceptable.] It appears possible to make E more positive by making k negative (i.e. a de-centering spring). Ideally the magnitude of k would be proportional to rider weight; one way to achieve this is for the front fork to move up and down relative to the head tube as it is turned.

7/24/2010

Explanatory note for sheets numbered 23.1, 23.2

These sheets were informally added to my "summary report", but not to the copy available on Andy's website.

The handwritten information is the symbolic versions of eigenvalue equation coefficients A - E. I arranged them to highlight the dependence on velocity, trail c_f, and spin momentum H.
[Note: Arend also did a symbolic expansion around 2008 and I confirmed agreement]
In some cases I toyed with different representations to see if they might simplify the presentation.
I also included a steering spring, k.

At the time when I was seeking a simple notation and organisation of equations of motion, and also for PRS paper, I was focusing on papers with correct, general equations of motion. I did not pay attention to determinantal expansions available in K&S, or the E expressions in Carvallo or Whipple. So it may be that the full expressions for the equation coefficients (A, B, C, D, E) have been given elsewhere, perhaps even in papers we reviewed, but that I didn't notice them.

Jim Papadopoulos

Supplement: Expansion of Coefficients A, B, C, D, E

[which terms dominate for conventional bike?]

$$A = \underbrace{\overbrace{T_{yy}}^{R_{yy}+F'_{yy}} F'_{\lambda\lambda} - F'^{2}_{\lambda y}}_{R_{yy}F'_{\lambda\lambda} \,+\, \cos^2\lambda \det F'} + 2\frac{c_f}{c_w}\left(T_{yy}F''_{\lambda z} - F'_{\lambda y}T_{yz}\right) + \left(\frac{c_f}{c_w}\right)^2 \det T$$

$$B = V\left\{ \frac{\cos\lambda}{c_w}\left(T_{yy}\underbrace{F''_{\lambda z}}_{\sim \text{approx } m_f d} - F'_{\lambda y}T_{yz}\right) + \frac{c_f}{c_w}\left(\overbrace{\frac{\cos\lambda}{c_w}\det T}^{\text{dominant positive definite}} + T_{yy}m_f d + F'_{\lambda y}m_t\bar{h}_t\right) \right.$$

$$\left. + \left(\frac{c_f}{c_w}\right)^2 \left(\underbrace{T_{yy}m_t\bar{l}_t + T_{yz}m_t\bar{h}_t}_{m_t \mathbf{q}\,\mathbf{I}:\mathbf{t}\mathbf{j}}\right)\right\}.$$

$$C = -g\left\{\left[m_t\bar{h}_t F'_{\lambda\lambda} + m_f d\left(\sin\lambda\, T_{yy} + 2F'_{\lambda y}\right)\right] + \frac{c_f}{c_w}\left[2\overbrace{\left(m_t\bar{h}_t F''_{\lambda z} + m_f d\, T_{yz}\right)}^{J} + m_t\bar{l}_t\left(\sin\lambda\, T_{yy} + 2F'_{\lambda y}\right)\right] \right.$$

$$\left. + \left(\frac{c_f}{c_w}\right)^2\left[\overbrace{2T_{yz}m_t\bar{l}_t + T_{zz}m_t\bar{h}_t}^{m_t \mathbf{q}\,\mathbf{I}:\mathbf{t}\mathbf{k} + \cdots}\right]\right\} \qquad (+\,T_{yy}k) \text{ steer spring}$$

$$+ V^2\frac{\cos\lambda}{c_w}\left\{\left[T_{yy}m_f d + F'_{\lambda y}m_t\bar{h}_t\right] + \frac{c_f}{c_w}\left[\overbrace{T_{yy}m_t\bar{l}_t + T_{yz}m_t\bar{h}_t}^{m_t \mathbf{q}\,\mathbf{I}:\mathbf{t}\mathbf{j}}\right]\right\}$$

$$+ V\left\{\frac{\cos\lambda}{c_w}\left[H_t F'_{\lambda y} - H_f \overset{(\mathbf{I}:\boldsymbol{\lambda}\mathbf{j})}{T_{\lambda y}}\right] + \left(\frac{c_f}{c_w}\right)H_f\cos\lambda\, m_t\bar{h}_t + \left(\frac{c_f}{c_w}\right)^2 H_t m_t\bar{h}_t\right\}$$

$$+ \left\{H_f\cos\lambda + \frac{c_f}{c_w}H_t\right\}^2$$

often $\propto d$

$m_t g \underset{\approx}{T} \underset{=}{t} \underset{\sim}{k}$ (rarely negative)

$$D = \frac{V\cos\lambda}{c_w}\left\{-g\left[\overbrace{\left(F''_{\lambda z} m_t \bar{h}_t + T_{yz} m_f d\right)}^{J} + \frac{c_f}{c_w}\left(T_{zz} m_t \bar{h}_t + T_{yz} m_t \bar{l}_t\right)\right]\right.$$

$$\left. + \left[\left(\underbrace{m_t \bar{h}_t V}_{} + H_t\right)\left(\underbrace{H_f \cos\lambda}_{} + \frac{c_f}{c_w} H_t\right)\right]\right\}$$

almost always positive — dominant

$$E = g^2\left[m_f d\left(m_t \bar{h}_t \sin\lambda - m_f d\right) + \frac{c_f}{c_w} m_t \bar{l}_t\left(m_t \bar{h}_t \sin\lambda - 2m_f d\right) - \left(\frac{c_f}{c_w}\right)^2 \left(m_t \bar{l}_t\right)^2\right]$$

$(-g m_t \bar{h}_t k)$ steer spring

$$+ Vg\frac{\cos\lambda}{c_w}\left[\left(H_t m_f d - H_f m_t \bar{h}_t \sin\lambda\right) + \frac{c_f}{c_w} m_t \bar{l}_t H_t\right]$$

dominant — dominant

$$\ldots \quad E = g^2 \nu\left(\underline{m_t \bar{h}_t \sin\lambda - \nu}\right) - g\frac{V\cos\lambda}{c_w}\left(H_f m_t \bar{h}_t \sin\lambda - H_t \nu\right)$$

$$\left(\text{If } H_r = 0, \quad \ldots \quad E = g\left(g\nu - \frac{V\cos\lambda}{c_w} H_f\right)\left(m_t \bar{h}_t \sin\lambda - \nu\right)\right)$$

RH $\Rightarrow$ $(BC - AD)D > EB^2$, so $BC - AD > 0$ is most important at high speeds when $EB^2 \sim 0$

23.2

def: $\nu = m_f d + m_t \bar{l}_t \left(\frac{c_f}{c_w}\right)$