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ON THE MOTION OF A BICYCLE.

By *G. R. R. Routh.*

THE four problems we here propose to discuss are as follows:

(1) The motion of a bicycle when the front wheel is locked.

(2) The oscillations about a state of steady motion in a straight line.

(3) Steady motion in a circle.

(4) Oscillations about a state of steady motion in a circle.

In the first of these the machine is constrained to move in a straight line, and the problem is that of three rigid bodies connected together. We shall, using D'Alembert's principle, reverse the effective forces of the wheels and apply them to the system.

In the second we shall neglect the effective forces due to the wheels, for the mass of a wheel is very small compared with the mass of the framework and rider. The mass of the steering pillar and handles will also be neglected.

In the third and fourth propositions we shall assume the radius to be large compared with the length and height of the machine, but we shall not in the first instance assume the angular velocity about the centre of the circle, or the inclination of the plane of the frame-work to the vertical, to be small.

I. We have here first to find the effective forces due to the motion of a wheel.

Let G be the centre of the wheel, GC normal to its plane, GA the radius through its point of contact with the ground, and GB at right angles to GA and GC.

Let GM be the perpendicular on the horizontal plane.

Let θ be the angle GC makes with the vertical, and ψ the angle MA makes with any fixed straight line in the horizontal plane.

Let ω_1, ω_2, ω_3 be the angular velocities of the wheel about GA, GB, GC, and let u, v be the velocities of G resolved parallel to MA and GB, then $\omega_1 = -\sin\theta\psi'$ and $\omega_2 = \theta'$.

Since the point A is at rest, we have

$$u = a\theta' \sin\theta, \quad v = -a\omega_3, \quad z = a\sin\theta,$$

where a is the radius of the wheel.

Suppose the wheel to be of mass unity, then the forces are resolved parallel to MA, GB, and vertically

$$\frac{du}{dt} - v\frac{d\psi}{dt} = a\sin\theta\theta'' + a\cos\theta\theta'^2 + a\omega_3\psi',$$

$$\frac{dv}{dt} + u\frac{d\psi}{dt} = -a\omega_3' + a\sin\theta\theta'\psi',$$

$$\frac{d^2z}{dt^2} = a\cos\theta\theta'' - a\sin\theta\theta'^2.$$

The forces $\frac{du}{dt}$, $\frac{dv}{dt}$ can be replaced by $a\theta'^2$ along GA and $a\theta''$ along GC.

Let θ_1, θ_2, θ_3 be the angular velocities of GA, GB, GC about their instantaneous position, then $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, for GC is fixed on the body and $\theta_3 = \psi' \cos\theta$.

Hence the couples round GA, GB, GC are

$$(1)\quad -A\frac{d}{dt}(\sin\theta\psi') - A\theta'\psi'\cos\theta + C\theta'\omega_3$$

$$= -A\sin\theta\psi'' - 2A\theta'\psi'\cos\theta + C\theta'\omega_2,$$

$$(2)\quad A\theta'' + C\omega_3\sin\theta\psi' - A\sin\theta\cos\theta\psi'^2,$$

$$(3)\quad C\omega_3',$$

where A and C are the moments of inertia of the wheel about GA and GC.

We have now to apply these results to the case of a bicycle with the front wheel locked.

Since the bicycle is moving in a straight line, it follows that ψ', ψ'' are zero, and each wheel is constrained to move along a parallel to GB through A. Hence the moment of the effective forces of the wheel about its trace on the ground is $A\theta'' + A^2\theta''$.

Let m_1, m_2 be the masses of the two wheels and a_1, a_2 their radii, and let A_1, A_2 be their moments of inertia about a radius. Let MK^2 be the moment of inertia of the rider and framework about the path of the bicycle, and let h be the length of the perpendicular from the centre of gravity on to the path.

Hence, taking moments for the whole system

$$m_1(A_1 + a_1^2)\theta'' + m_2(A_2 + A_2^2)\theta'' + MK^2\theta''$$

$$= -m_1ga_1\cos\theta - m_2ga_2\cos\theta - Mgh\cos\theta.$$

This is an equation of the form $\theta'' = -n^2\cos\theta$, which becomes $\phi'' = n^2\sin\phi$ if we write $\frac{1}{2}\pi - \phi$ for θ.

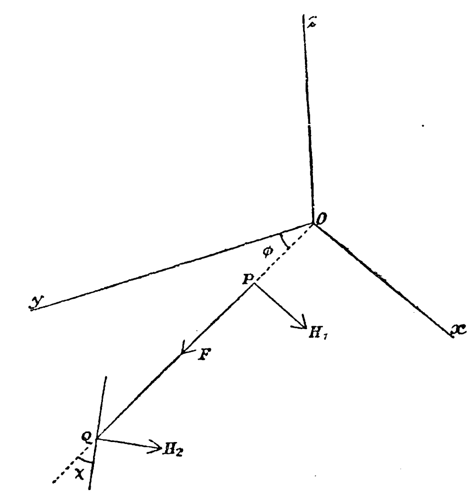
When ϕ is small we have therefore $\phi'' = n^2\phi$ and $\phi = L_1e^{nt} + L_2e^{-nt}$. Hence the state of steady motion is unstable, and the machine will fall to the ground.

The equation of motion does not involve the velocity of the bicycle. Heuce the time it takes to fall to the ground is independent of the velocity with which it is started.

We see then that the motion of two wheels rigidly connected together is very different from that of a single wheel. In the case of a single wheel oscillations are possible about a vertical axis, that is to say ψ' and ψ'' are not zero, and it is found that the terms involving ψ' and ψ'' make the motions stable.

II. In this proposition we neglect the effective forces due to the motion of a wheel; for the mass of a wheel is very small compared with that of the rider and framework.

Let P, Q be the points of contact of the hind and fore



wheels with the ground, and let G be the centre of gravity of the rider and frame.

Let Oy be the mean direction of motion and Ox be horizontal and at right angles to Oy, and let Ox be measured to the left as we look in the direction of motion.

The position of the machine is fixed by the coordinates x, y of the projection of G, the angle ϕ which PQ makes with Oy, and the angle p the plane GPQ makes with the vertical plane through PQ. We suppose p measured to the right and ϕ to the left, as we look in the direction of motion. Let χ be the angle the tangent to the front wheel at Q makes with PQ.

The rider works on the pedals, and the friction between the hind wheel and ground along the trace of the hind wheel is the impressed force which causes motion. Let F be this friction. The friction along the trace of the front wheel merely turns that wheel, and as its mass is here neglected this friction may also be neglected.

The rider works with an arbitrary force, and turns the handles at his pleasure. We shall treat F and χ as given.

When the machine is in steady motion F is small, being only sufficient to balance the resistances. As these are neglected we shall make F zero when the machine goes uniformly.

Let H_1, H_2 be the resolved parts of the frictions at P and Q perpendicular to traces of the wheels, and let R_1, R_2 be the vertical pressures of the ground on the hind and fore wheels.

The equations of motion are therefore

$$\left.\begin{aligned} Mx'' &= F\sin\phi + H_1\cos\phi + H_2\cos(\phi+\chi) \\ My'' &= F\cos\phi - H_1\sin\phi - H_2\sin(\phi+\chi) \\ Mz'' &= R_1 + R_2 - Mg \end{aligned}\right\}.$$

When the machine is moving nearly on Oy we regard ϕ, χ, H_1, H_2 as small quantities, and z'' is of the second order. Hence the above equations reduce to

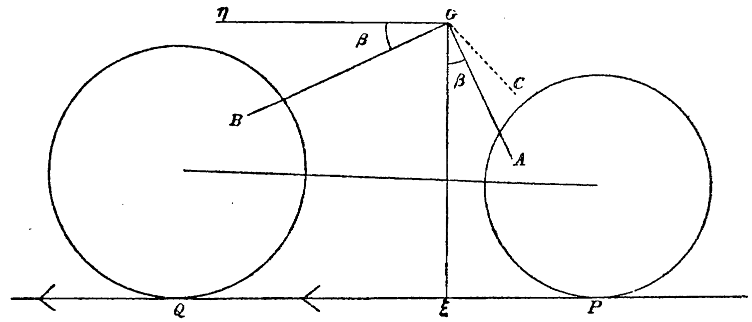
$$Mx'' = F\phi + H_1 + H_2 \quad \dots\dots\dots\dots\dots\dots (1),$$

$$My'' = F \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2),$$

$$0 = R_1 + R_2 - Mg \quad \dots\dots\dots\dots\dots\dots (3).$$

We have now to find the angular motion.

Let GA, GB, GC be the principal axes at G, then GC is perpendicular to the plane GPQ, and we suppose it measured to left as we look in the direction of motion.



Let $G\xi$, $G\eta$ be perpendicular and parallel to PQ.

Let Ω_1, Ω_2, ω_3 be the angular velocities about $G\xi$, $G\eta$, GC, and ω_3, ω_2, ω_3 about GA, GB, GC.

ω_3 is given by the vertical displacement of the front wheel and is of the second order.

We clearly have $\Omega_2 = -\dfrac{dp}{dt}$, $\quad \Omega_1 = \dfrac{d\phi}{dt}$.

Let β be the angle GA makes with $G\xi$.

Then by resolution about GA, GB,

$$\omega_1 = -\Omega_2\sin\beta + \Omega_1\cos\beta = \quad p'\sin\beta + \phi'\cos\beta,$$

$$\omega_2 = \quad \Omega_2\cos\beta + \Omega_1\sin\beta = -p'\cos\beta + \phi'\sin\beta.$$

Let us take Euler's dynamical equations and resolve along $G\xi$, $G\eta$, GC, then we get

$$\{A\omega_1' - (B-C)\,\omega_2\omega_3\}\cos\beta + \{B\omega_2' - (C-A)\,\omega_3\omega_1\}\sin\beta$$
$$= \text{moment about } G\xi.$$

Since $\omega_3 = 0$, this reduces to

$$A\omega_1'\cos\beta + B\omega_2'\sin\beta = \text{moment about } G\xi.$$

Similarly

$$-A\omega_1'\sin\beta + B\omega_2'\cos\beta = \text{moment about } G\eta,$$

and the moment about GC is zero.

We have now to find the moments of the forces.

The components of R_2 are $-R_2 \cos p$ along $G\xi$ and $R_2 \sin p$ along GC. The components of H_2 are $H_2 \cos\chi \sin p$ along $G\xi$, $H_2 \cos\chi \cos p$ along GC, and $-H_2 \sin\chi$ along $G\eta$. The coordinates of Q referred to $G\xi$, $G\eta$, GC are h, l_2, 0.

Therefore moments of R_2, H_2, and F about $G\xi$, $G\eta$, GC are

$$l_2 (R_2 \sin p + H_2 \cos\chi \cos p) \text{ about } G\xi,$$

$$-h (R_2 \sin p + H_2 \cos\chi \cos p) \text{ about } G\eta,$$

$$h (F - H_2 \sin\chi) - l_2 (-R_2 \cos p + H_2 \cos\chi \sin p) \text{ about } GC.$$

The moments of R_1, H_1 are found from those for R_2, H_2 by writing $-l_1$ for l_2 and putting χ zero. They must be added to the above.

Hence, substituting for ω_1, ω_2, and remembering that χ and p are small, we have

$$A \cos\beta (p'' \sin\beta + \phi'' \cos\beta) + B \sin\beta (-p'' \cos\beta + \phi'' \sin\beta)$$
$$= (A - B) p'' \sin\beta \cos\beta + (A \cos^2\beta + B \sin^2\beta) \phi''$$
$$= l_2 (R_2 p + H_2) - l_1 (R_1 p + H_1) \text{ .... } (4),$$

$$-A \sin\beta (p'' \sin\beta + \phi'' \cos\beta) + B \cos\beta (-p'' \cos\beta + \phi'' \sin\beta)$$
$$= -p'' (A \sin^2\beta + B \cos^2\beta) - (A - B) \phi'' \sin\beta \cos\beta$$
$$= -h (R_2 p + H_2) - h (R_1 p + H_1) \text{ .... } (5),$$

$$0 = hF + l_2 R_2 - l_1 R_1 \text{ .... } (6).$$

We have now to find two geometrical equations.

The velocity of the point of contact of each wheel perpendicular to the wheel is zero.

The velocities of P due to rotation are zero parallel to PQ and $(-l_1\Omega_1 - h\Omega_2) \cos p$ perpendicular to PQ. Therefore velocity of Q due to rotation is $(l_2\Omega_1 - h\Omega_2) \cos p$ perpendicular to PQ.

The velocities of G parallel and perpendicular to PQ are $x' \sin\phi + y' \cos\phi$ and $x' \cos\phi - y' \sin\phi$. Therefore the conditions are

$$x' \cos\phi - y' \sin\phi - (l_1\Omega_1 + h\Omega_2) \cos p = 0,$$

and

$$-(x' \sin\phi + y' \cos\phi) \sin\chi + (x' \cos\phi - y' \sin\phi) \cos\chi$$
$$+ l_2\Omega_1 - h\Omega_2) \cos p \cos\chi = 0.$$

Reject the squares of small quantities and subtract the first of these equations from the second.

Therefore $\quad x' - \phi y' - (l_1\Omega_1 + h\Omega_2) = 0 \text{ .... } (7),$

and $\quad -y'\chi + l\Omega_1 = 0, \text{ where } l = l_1 + l_2 \text{ .... } (8).$

We have now to solve these equations.

In uniform motion $F=0$, and we write V for y' in (7) and (8), for ϕ and χ are small.

Substituting in (5) from (1) and (3), we have

$$-p'' (A \sin^2\beta + B \cos^2\beta) - (A - B) \phi'' \sin\beta \cos\beta$$
$$= -hp (Mg) - h (Mx'').$$

From (8) we have $l\phi' = V\chi$, and from (7)

$$x'' - V\phi' - (l_1\phi'' - hp'') = 0.$$

Therefore

$$-p'' (A \sin^2\beta + B \cos^2\beta) - (A - B) \phi'' \sin\beta \cos\beta$$
$$= -hpMg - hMV\phi' - hM (l_1\phi'' - hp'') \ldots (9).$$

In practice a rider, when he finds his machine inclined to the vertical, turns the handles a little more than is sufficient to rectify it, and turns them slowly back as perpendicularity is restored, so that the machine becomes vertical and the front wheel straight at the same time. We are here concerned with first powers of p and χ only, and we put $\chi = -\mu p$. Hence $l\phi' = -\mu V p'$ and therefore $\phi' = -\frac{\mu V}{l} p$.

Equation (9) becomes

$$p'' (A' + Mh^2) + \frac{\mu V}{l} p' (Mhl_1 - B') + Mhp \left(\frac{\mu V^2}{l} - g\right) = 0,$$

where

$$A' = A \sin^2\beta + B \cos^2\beta,$$
$$B' = (A - B) \sin\beta \cos\beta.$$

To solve this let $p = e^{\kappa t}$, and we get a quadratic for κ.

If the last term be negative the roots will be real and one is positive. In this case p will increase indefinitely with t and the motion is unstable.

Hence $\frac{\mu V^2}{l} - g$ is positive or $\mu > \frac{gl}{V^2}$.

Hence μ decreases rapidly as V increases, and μ is smaller for bicycles with a short base than for those with a long base such as tandems, &c.

For an ordinary single bicycle $l = 3\frac{1}{2}$ feet. Hence at 11 miles an hour $\mu > \frac{7}{16}$, and at $5\frac{1}{2}$ miles an hour $\mu > \frac{7}{4}$.

Let $\mu = \frac{\lambda g l}{V^2}$, so that λ is a little greater than unity.

The quadratic for κ now becomes

$$\kappa^2 (A' + Mh^2) + \kappa \frac{\lambda g}{V} (Mhl_1 - B') + Mgh (\lambda - 1) = 0.$$

The roots of this equation may be real or complex. If real they must both be negative, and if complex the real part of each must be negative. Either of these conditions is satisfied if $Mhl_1 - B'$ is positive. In a single bicycle l_1 is about 8 inches and h is about $4\frac{1}{2}$ feet, so Mhl is not large. Hence B' must be negative or a small positive quantity. If the rider sit nearly erect $(A - B)$ is considerable, but β is very small, while if the rider bends forward β may be considerable, but A will nearly equal B. In either case then B' is small and the condition is satisfied.

The roots of the quadratic are real if

$$\lambda^2 - \frac{4MhV^2 (A' + Mh^2)}{g (Mhl_1 - B')^2} (\lambda - 1)$$

is positive, λ is greater than unity, but if the rider be skilful λ will only exceed unity by a small quantity, so we can write $1 + \delta$ for λ and reject δ^2. We thus get

$$1 - \left\{\frac{4MhV^2 (A' + Mh^2)}{g (Mhl_1 - B')^2} - 2\right\} \delta$$

is to be positive.

In a single bicycle $l_1 = 8$ inches and $h = 4\frac{1}{2}$ feet approximately. Let us regard the rider as a rod 6 feet long, then $A' = 3M$, $B' = 0$. Let $V = 16$ feet per second, then the above condition reduces to $\delta < \frac{1}{370}$. Hence λ must lie between 1 and $\frac{371}{370}$. This will be impossible in practice, so that in general there will be a periodic term in the solution giving an oscillation whose period is

$$\frac{4\pi (A' + Mh^2)}{\left\{4 (A' + Mh^2) Mgh (\lambda - 1) - \frac{\lambda^2 g^2}{V^2} (Mhl_1 - B')^2\right\}^{\frac{1}{2}}},$$

Taking A', h, and V, as before, and supposing $\lambda = \frac{4}{3}$, we obtain 4·4 seconds for the period of a complete oscillation.

Alterations in the velocity have very little effect on this period. At 8 feet a second the period is 4·5 secs., while at 32 feet a second the period is 4·37 secs.

The real part of each root is $-\frac{\lambda g (Mhl_1 - B')}{V (A' + Mh^2)}$.

Taking V to be 16 feet per second and giving the other constants their previous values, we find this expression to be $\frac{2}{93}$. Thus the amplitude of the oscillations will be reduced to half its original value in 7 9 seconds and to one-tenth its original value in 12·6 seconds.

III. In this proposition we shall use polar coordinates, taking the centre of the circular path for origin.

The equations of motion are then

$$M (r'' - r\theta'^2) = F \sin\phi + H_1 \cos\phi + H_2 \cos(\phi + \chi) \ldots\ldots(1),$$

$$M \frac{1}{r} \frac{d}{dt} (r^2\theta') = F \cos\phi - H_1 \sin\phi - H_2 \sin(\phi + \chi) \ldots\ldots(2),$$

$$M \frac{d^2}{dt^2} (h \cos p) = R_1 + R_2 - Mg \ldots\ldots\ldots\ldots(3).$$

Taking axes $G\xi$, $G\eta$, GC as in the last proposition, we have for the angular velocities of the system

$$\Omega_1 = (\phi' - \theta') \cos p, \quad \Omega_2 = - p', \quad \omega_3 = - (\phi' - \theta') \sin p.$$

From these we get by resolution

$$\omega_1 = - \Omega_2 \sin\beta + \Omega_1 \cos\beta = p' \sin\beta + (\phi' - \theta') \cos p \cos\beta,$$

$$\omega_2 = \Omega_2 \cos\beta + \Omega_1 \sin\beta = - p' \cos\beta + (\phi' - \theta') \cos p \sin\beta,$$

$$\omega_3 = - (\phi' - \theta') \sin p.$$

We are going, as in the last proposition, to use Euler's dynamical equations for motion about GA, GB, GC. These involve ω_1', ω_2, ω_3, &c. ..., when differentiating the above expressions for ω_1, ω_2, ω_3, we remember that p' and ϕ' are small and θ' is nearly equal to the mean angular velocity of the bicycle in its path which we shall call ω.

We thus get

$$\omega_1 = p'' \sin\beta + (\phi'' - \theta'') \cos p \cos\beta + \omega \sin p \cos\beta p',$$

$$\omega_2' = -p'' \cos\beta + (\phi'' - \theta'') \cos p \sin\beta + \omega \sin p \sin\beta p',$$

$$\omega_3 = -(\phi'' - \theta'') \sin\beta + \omega p' \cos p,$$

$$\omega_2\omega_3 = -p'\omega \cos\beta \sin p + 2\omega\phi' \sin p \cos p \sin\beta - \theta'^2 \sin p \cos\beta \sin\beta,$$

$$\omega_3\omega_1 = p'\omega \sin\beta \sin p + 2\omega\phi' \sin p \cos p \cos\beta - \theta'^2 \sin p \cos p \cos\beta,$$

$$\omega_1\omega_2 = p'\omega \cos p \cos 2\beta - \omega\phi' \cos^2 p \sin 2\beta - \theta'^2 \cos^2 p \sin\beta \cos\beta.$$

The moments of the forces R_1, H_1, F, &c. about $G\xi$, $G\eta$, GC will be the same as in the last proposition.

We thus get the three equations of angular motion about $G\xi$, $G\eta$, GC, viz.

$$L_1 = l_2 (R_2 \sin p + H_2 \cos\chi \cos p) - l_1 (R_1 \sin p + H_1 \cos p) \ldots (4),$$

$$L_2 = -h (R_2 \sin p + H_2 \cos\chi \cos p) - h (R_1 \sin p + H_1 \cos p) \ldots (5),$$

$$L_3 = h (F - H_3 \sin\chi) - l_2 (-R_2 \cos p + H_2 \cos\chi \cos p) + l_1 (-R_1 \cos p + H_1 \sin p) \ldots (6),$$

where L_1, L_2, L_3 are Euler's dynamical equations resolved about $G\xi$, $G\eta$, GC, and can be written down as we require them from the values of ω_1', ω_2, ω_3, &c. just found.

We have now to find two geometrical relations.

The coordinates of P, referred to $G\xi$, $G\eta$, GC, are $h, -l_1, 0$.

Therefore the velocities of P due to rotations are $l_1\Omega_3$ parallel to $G\xi$, $(-l_1\Omega_1 - h\Omega_2)$ parallel to GC, and $h\Omega_3$ parallel to $G\eta$.

The velocities of G are $r' \sin\phi + r\theta' \cos\phi$ parallel to PQ, and $r' \cos\phi - r\theta' \sin\phi$ perpendicular to PQ.

Now the horizontal velocity of P resolved perpendicular to the trace of the wheel is zero.

Hence we have

$$r' \cos\phi - r\theta' \sin\phi - (l_1\Omega_1 + h\Omega_2) \cos p + l_1\Omega_3 \sin p = 0,$$

or since $\qquad \Omega_1 \cos p - \Omega_3 \sin p = \phi' - \theta',$

$$r' \cos\phi - r\theta' \sin\phi - l_1 (\phi' - \theta') - h\Omega_2 \cos p = 0 \ldots (7).$$

Again the velocity of Q resolved perpendicular to the trace of the front wheel is zero.

To find the velocities of Q from those of P we write $-l_2$ for l_1, and we get

$$-(r' \sin\phi + r\theta' \cos\phi) \sin\chi + (r' \cos\phi - r\theta' \sin\phi) \cos\chi$$
$$+ l_2 (\phi' - \theta') \cos\chi - h\Omega_2 \cos p \cos\chi - h\Omega_3 \sin\chi = 0.$$

Multiply (7) by $\cos\chi$ and subtract from this equation. Therefore

$$-(r' \sin\phi + r\theta' \cos\phi) \sin\chi + l (\phi' - \theta') \cos\chi\, h\Omega_3 \sin\chi = 0 \ldots (8),$$

where $l = l_1 + l_2$ as before.

In steady motion r', ϕ', p' are zero and

$$\theta' = \omega, \quad \Omega_1 = -\omega \cos p, \quad \Omega_2 = 0, \quad \omega_3 = \omega \sin p.$$

Hence (7) becomes

$$-a\omega \sin\phi + l_1\omega = 0.$$

Therefore $\qquad \sin\phi = \dfrac{l_1}{a}.$

Hence ϕ is small. It does not depend on the velocity and is smaller for those bicycles in which l_1 is small.

Equation (8) becomes

$$-a\omega \cos\phi \sin\chi - l\omega \cos\chi - h\omega \sin p \sin\chi = 0.$$

Therefore $\qquad -\cot\chi = \dfrac{a \cos\phi + h \sin p}{l},$

or, if we reject $h \sin p$ as compared with l, $\chi = -\dfrac{l}{a}$.

Hence χ is small also and if we reject $h \sin p$ does not depend on the velocity.

We shall in future reject squares of ϕ and χ.

Hence in steady motion the first three equations become

$$-Ma\omega^2 = H_1 + H_2,$$

$$0 = -(H_1 + H_2)\phi - H_2\chi,$$

$$0 = R_1 + R_2 - Mg.$$

Hence, substituting in (5), we get

$$L_2 = -h \sin p\,(Mg) + Ma\omega^2 h \cos p.$$

Now

$$L_2 = -\{A\omega_1' - (B-C)\,\omega_2\omega_3\}\sin\beta + \{B\omega_3' - (C-A)\,\omega_3\omega_1\}\cos\beta$$

$$= -(B-C)\,\omega^2 \sin p \cos p \sin^2\beta + (C-A)\,\omega^2 \sin p \cos p \cos^2\beta$$

$$= (C - A\cos^2\beta - B\sin^2\beta)\,\omega^2 \sin p \cos p.$$

Therefore

$$(C - A\cos^2\beta - B\sin^2\beta)\,\omega^2 \sin p \cos p = -Mgh \sin p + Ma\omega^2 h \cos p\,;$$

or, writing $\frac{V}{a}$ for ω,

$$(C - A\cos^2\beta - B\sin^2\beta)\frac{V^2}{a^2}\sin p \cos p = -Mgh \sin p + M\frac{V^2}{a}h\cos p.$$

This is the equation for steady motion.

$A\cos^2\beta + B\sin^2\beta$ is the moment of inertia about $G\xi$, and it seems probable from the position of the rider that this moment of inertia is less than C, that is the left-hand side of the equation is positive.

If we regard the rider as a rod as before, then

$$A\cos^2\beta + B\sin^2\beta$$

is zero and $C = 3M$. Now $\frac{V}{a}$ cannot be very large, for even if V be 30 miles an hour and a only 10 yards, $\frac{V^2}{a^2}$ will be < 4. Hence the left-hand side of the equation will be very small compared with Mgh.

If we neglect the left-hand side we obtain $\tan p = \frac{V^2}{ga}$, which is approximately correct. Hence the inclination is very nearly independent of h. The following table approximately gives the inclination of the machine to the vertical for different values of V and a.

The possibility of the machine skidding is not here considered, which would make some of the larger inclinations impossible.

The first row indicates the miles per hour and the first column the radius in yards.

	5	10	15	20	25
10	3°13′	12°38′	26°44′	41°50′	54°23′
20	1°36′	6°24′	14°9′	24°7′	34°58′
30	1°4′	4°16′	9°32′	16°37′	25°0′
40	0°48′	3°13′	7°11′	12°38′	19°17′
100	0°19′	1°17′	2°53′	5°7′	7°58′

To take into account the possibility of either wheel skidding, we require the values of H_1, H_2, R_1, R_2.

Substituting for ϕ and χ in (2), we have

$$0 = -H_1 l_1 + H_2 l_2.$$

Hence (4) becomes

$$\frac{L_1}{\sin p} = l_2 R_2 - l_1 R_1.$$

We also have

$$-Ma\omega^2 = H_1 + H_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$Mg = R_1 + R_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Solving these equations and writing $\frac{V}{a}$ for ω, we obtain

$$-\frac{H_1}{R_1} = \frac{Ml_2V^2}{a\,(Mgl_3 - L_1\,\mathrm{cosec}\,p)}, \quad -\frac{H_2}{R_1} = \frac{Ml_1V^2}{a\,(Mgl_1 + L_1\,\mathrm{cosec}\,p)},$$

The back or front wheel will skid first according as $-\frac{H_1}{R_1}$ is greater or less than $-\frac{H_2}{R_2}$.

In steady motion L_1 is found to be

$$-(A-B)\sin\beta\cos\beta\sin p\cos p.$$

Hence, in order that there may be no skidding, the coefficient of friction between a wheel and the ground must be greater than the greatest of the fractions

$$\frac{Ml_2V^2}{a\{Mgl_2+(A-B)\sin\beta\cos\beta\cos p\}}$$

and

$$\frac{Ml_1V^2}{a\{Mgl_1-(A-B)\sin\beta\cos\beta\cos p\}}.$$

Generally A will be greater than B, but β, though small, will be negative, so the back wheel will skid first.

Each of these fractions is nearly equal to $\frac{V^2}{ag}$ that is to $\tan p$. Hence the approximate condition that there shall be no skidding is that the angle of friction shall be greater than the inclination of the machine to the vertical.

IV. We have now to find the small oscillations about steady motion. Let

$$r=a+R,\quad \theta'=\omega+\Omega,\quad \phi=\phi_0+\Phi,\quad p=p_0+P,\quad \chi=\chi_0+X,$$

where ϕ_0, p_0, χ_0 are the values of ϕ, p, χ in steady motion.

Let us write H_1+K_1, H_2+K_2, R_1+S_1, R_2+S_2 for H_1, H_2, R_1, R_2 so that H_1, H_2, R_1, R_2 represent the values of the reactions in steady motion. We shall reject terms involving the squares of small quantities, and shall retain only those which involve the first powers of R, Ω, H, K, &c. Equations (1), (2), (3), and (5) become

$$M(R''-R\omega^2-2a\omega\Omega)=K_1+K_2 \quad\text{.......................(1)},$$

$$M(2R'\omega+a\Omega')=-H_1\Phi-H_2\Phi-H_2X\quad\text{.................(2)},$$

$$-Mh\sin p_0P''=S_1+S_2\quad\text{...............................(3)},$$

$$L_2=-h(R_2P\cos p_0+S_2\sin p_0-H_2\sin p_0P+K_2\cos p_0)\ (5),$$

$$-h(R_1P\cos p_0+S_1\sin p_0-H_1\sin p_0P+K_1\cos p_0),$$

Equations (7) and (8) become

$$R'-a\omega\Phi-l_1(\Phi'-\Omega)+hP'\cos p_0=0\quad\text{...........(7)},$$

$$-a\omega X+l(\Phi'-\Omega)-h\omega\sin p_0X=0\quad\text{...........(8)}.$$

Now $\qquad H_1+H_2=-Ma\omega^2,$

and $\qquad 0=-(H_1+H_2)\phi_0-H_2\chi_0,$

therefore $\qquad H_2\chi_0=Ma\omega^2\phi_0=Ml_1\omega^2,$

therefore $\qquad H_2=\frac{Ml_1\omega^2}{\chi_0},$

therefore, substituting in (2) and dividing by M,

$$2R'\omega+a\Omega'=a\omega^2\Phi-\frac{l_1\omega^2}{\chi_0}X\quad\text{............(9)},$$

and substituting in (5)

$$L_2=-hMgP\cos p_0+Mh^2\sin^2p_0P''-Mah\omega^2\sin p_0P$$
$$-Mh\cos p_0(R''-R\omega^2-2a\omega\Omega).$$

Next, to find L_2,

$$\omega_1'=\ \ P''\sin\beta+(\Phi''-\Omega')\cos p_0\cos\beta+\omega\sin p_0\cos\beta P',$$

$$\omega_2'=-P''\cos\beta+(\Phi''-\Omega')\cos p_0\sin\beta+\omega\sin p_0\sin\beta P',$$

$$\omega_2\omega_3=-P'\omega\cos\beta\sin p_0+\omega\Phi'\sin 2p_0\sin\beta$$
$$-\omega^2P\cos 2p_0-\omega\Omega\sin 2p_0\sin\beta,$$

$$\omega_3\omega_1=P'\omega\sin\beta\sin p_0+\omega\Phi'\sin 2p_0\cos\beta$$
$$-\omega^2P\cos 2p_0\cos\beta-\omega\Omega\sin 2p_0\cos\beta,$$

and $L_2=-\{A\omega_1'-(B-C)\omega_2\omega_3\}\sin\beta$
$$+\{B\omega_2'-(C-A)\omega_3\omega_1\}\cos\beta.$$

From these we obtain

$$L_2=-P''(A\sin^2\beta+B\cos^2\beta)-(\Phi''-\Omega')(A-B)\sin\beta\cos\beta$$
$$-(\Phi'-\Omega)\omega\sin 2p_0(C-A\cos^2\beta-B\sin^2\beta)$$
$$+\omega^2P\cos 2p_0(C-A\cos^2\beta-B\sin^2\beta).$$

Hence we have four equations, viz., (7), (8), (9), and

$$-P''(A\sin^2\beta+B\cos^2\beta)-(\Phi''-\Omega')(A-B)\sin\beta\cos\beta$$
$$-(\Phi'-\Omega)\,\omega\sin 2p_0(C-A\cos^2\beta-B\sin^2\beta)$$
$$+\omega^2P\cos 2p_0(C-A\cos^2\beta-B\sin^2\beta)$$
$$=-MghP\cos p_0+Mh^2P''\sin^2p_0-Mah\omega^2P\sin p_0$$
$$-Mh\cos p_0(R''-R\omega^2-2a\omega\Omega)\ \ldots\ldots(10).$$

To solve these let $R=M_1e^{\kappa t}$, $\Omega=M_2\kappa e^{\kappa t}$, $P=M_3e^{\kappa t}$, $\Phi=M_4e^{\kappa t}$, $X=M_5e^{\kappa t}$.

Then the four equations become

$$M_1\kappa-a\omega M_4-l_1\kappa(M_4-M_2)+h\kappa M_3\cos p_0=0\ldots(11),$$
$$-a\omega M_5+l\kappa(M_4-M_2)-h\omega\sin p_0M_5=0\ldots(12),$$
$$2M_1\kappa\omega+a\kappa^2M_2=a\omega^2M_4-\frac{l_1}{\chi_0}\omega^2M_5\ldots(13),$$
$$-M_3\kappa^2(A\sin^2\beta+B\cos^2\beta)-\kappa^2(M_4-M_2)(A-B)\beta\sin\cos\beta$$
$$-\omega\kappa(M_4-M_2)(C-A\cos^2\beta-B\sin^2\beta)\sin 2p_0$$
$$+\omega^2M_3\cos 2p_0(C-A\cos^2\beta-B\sin^2\beta)$$
$$=-MghM_3\cos p_0+Mh^2\sin^2p_0\kappa^2M_3-Mah\omega^2\sin p_0M_3$$
$$-Mh\cos p_0(M_1\kappa^2-M_1\omega^2-2a\omega M_2\kappa)\ \ldots\ldots(14).$$

From (12) we obtain

$$\kappa(M_4-M_2)=\frac{a+h\sin p_0}{l}M_5\omega,$$
$$=-\frac{M_5\omega}{\chi_0}.$$

Multiply (11) by ω, subtract from (13) and divide by κ, and we get

$$M_1\omega+a\kappa M_2=h\omega M_3\cos p_0\ \ldots\ldots(15).$$

Again, $M_4=M_2-\dfrac{M_5\omega}{\kappa\chi_0}$; hence, substituting in (11), we get

$$M_1\kappa-a\omega M_2=-\frac{M_5a\omega^2}{\kappa\chi_0}-\frac{l_1M_5\omega}{\chi_0}-h\kappa M_3\cos p_0\ldots(16).$$

Multiply (15) by $-\omega$ and (16) by κ, and add. Therefore

$$M_1\kappa^2-M_1\omega^2-2M_2a\omega\kappa=-\frac{M_5a\omega^2}{\chi_0}-\frac{l_1M_5\kappa\omega}{\chi_0}$$
$$-hM_3\cos p_0(\kappa^2+\omega^2).$$

Substitute in (14), and put $M_5=-\mu M_3$, and we get

$$-A'\kappa^2-\frac{\mu\omega\kappa}{\chi_0}B'-\frac{\omega^2\mu}{\chi_0}(C-A\cos^2\beta-B\sin^2\beta)\sin 2p_0$$
$$+\omega^2\cos 2p_0(C-A\cos^2\beta-B\sin^2\beta),$$
$$=-Mgh\cos p_0+Mh^2\kappa^2\sin^2p_0-Mah\omega^2\sin p_0$$
$$-Mh\cos p_0\left(\frac{\mu a\omega^2}{\chi_0}+\frac{\mu l_1\kappa\omega}{\chi_0}\right)+Mh^2\cos^2p_0(\kappa^2+\omega^2),$$

or
$$-\kappa^2(A'+Mh^2)+\frac{\mu\omega\kappa}{\chi_0}(Mhl_1-B')$$
$$-\frac{\omega^2\mu}{\chi_0}(C-A\cos^2\beta-B\sin^2\beta)\sin 2p_0$$
$$+\omega^2\cos 2p_0(C-A\cos^2\beta-B\sin^2\beta)$$
$$+Mgh\cos p_0+Mah\omega^2\sin p_0+Mh\cos p_0\frac{\mu a\omega^2}{\chi_0}-Mh^2\omega^2\cos^2p_0=0.$$

If we substitute for $(C-A\cos^2\beta-B\sin^2\beta)$ from the equation of steady motion, we obtain for the term independent of κ

$$\frac{2\mu Mgh}{\chi_0}\sin p_0-\frac{\mu Mah\omega^2\cos p_0}{\chi_0}+Mgh\frac{\sin^2p_0}{\cos p_0}$$
$$+Mah\omega^2\frac{\cos^2p_0}{\sin p_0}-Mh^2\omega^2\cos^2p_0,$$

or

$$Mgh\sin p_0\left(\frac{2\mu}{\chi_0}+\tan p_0\right)+Mah\omega^2\cos p_0\left(-\frac{\mu}{\chi_0}+\cot p_0\right)$$
$$-Mh^2\omega^2\cos^2p_0.$$

This must be negative or there will be a positive value of κ. This term would increase indefinitely with t, and would destroy equilibrium.

Let us reject terms involving $\frac{V^2}{a^2}$; then by the equation of steady motion $Mgh \sin p_0 = Mah\omega^2 \cos p_0$.

Hence $-\frac{\mu}{\chi_0}$ must be greater than $\tan p_0 + \cot p_0$, which is satisfied by

$$\frac{\mu a}{l} > \frac{1}{\sin p_0 \cos p_0},$$

or

$$\mu > \frac{l}{a \sin p_0 \cos p_0}.$$

Hence μ is smaller for bicycles with a short base than for those with a long base.

This can be put in another form, viz.

$$\frac{\mu a}{l} > \frac{V^2}{ag} + \frac{ag}{V^2},$$

or

$$\mu > \frac{V^2 l}{a^2 g^2} + \frac{gl}{V^2}.$$

Since we are rejecting $\frac{V^2}{a^2}$ this reduces to $\mu > \frac{gl}{V^2}$, which is the same as for rectilinear motion.

Supposing this condition satisfied, the roots may be real or complex. If real they will be negative, or if complex the real part will be negative if

$$\frac{\mu\omega}{\chi_0} \frac{Mhl_1 - B'}{Mh^2 + A'} \text{ is negative,}$$

i.e. if

$$\frac{\mu V}{l} \frac{Mhl_1 - B'}{Mh^2 + A'} \text{ is positive.}$$

Hence, just as in rectilinear motion, Mhl_1 must be greater than B'.

Thus we see that, if a be so large or V be so small that $\frac{V^2}{a^2}$ may be rejected, the equations and periods for small oscillations are identical with those for rectilinear motion.

We have throughout the work neglected the fact that when the front wheel is turned its point of contact no longer lies in the plane of the framework.

This deviation is extremely small, but makes it possible to ride the machine without using the handles.

In the problems above considered we have supposed the rider to turn the handles, and this deviation merely makes it easier for him to do so.