BICYCLE STABILITY: A MATHEMATICAL AND NUMERICAL ANALYSIS

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May 13, 1979

To Mom -
Thanks for putting up with me

love
Mark

Bicycle Stability: A Mathematical
and Numerical Analysis

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Submitted in partial fulfillment of the requirements for a Bachelor of Arts degree in physics from Princeton University.

April 27, 1979

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MURPHY'S LAW:

If anything can go wrong, it will.

Corollary:

Everything takes longer than you think.

ACKNOWLEDGMENTS

Special thanks go to Professor Shoemaker for his sustained enthusiasm and willingness to freely listen and give advice at a moment's notice. Thanks go to Professor Callan for steering me clear of the incorrect derivation noted on page 26 of this text.

I would also like to express my appreciation for the help my parents gave by sending me to Princeton in the first place.

And thanks to the Lord God for all of the above.

INTRODUCTION

Classical mechanics has been well understood for quite some time, yet there remain many unsolved physical systems for which this oldest branch of physics would provide a totally adequate description. In many cases such problems have proven far too complicated in terms of number of degree of freedom and/or form of the equations of motion. An analytic solution is just not feasible. With the present availability of sophisticated computing devices, however, at least some of these physical systems may be studied using a numerical approach.

The steady motion of a bicycle is one such problem. There have been a number of attempts at the problem using a purely analytic approach such as the one appearing in Advanced Dynamics by Timoshenko and Young.[1] These studies have been far too simplistic, i.e. the angular momentum of the wheels is not taken into account or the geometry is not as complex as that of actual machines. Some of the more recent studies have managed to go much further. Robert N. Collins did a particularly detailed study of the steady motion of a riderless two-wheeled vehicle traveling in a straight line. His only assumptions were rigidity of the frame, fork and wheels, infinitely think disks for wheels, and zero slippage of the tires.[2] D.V. Singh went on to add tire slippage to his model a year later and did a comparison with actual machines.[3]

An interesting experimental study was conducted by David E.H. Jones. He actually built various odd bicycle configurations including one with a counter-rotating extra front wheel to nullify the effect of angular momentum. While not actually developing a comprehensive theory of his own, he did demonstrate the "ridability" of many supposedly unstable configurations.[4]

The two most recent mathematical works mentioned used a vector analysis approach to obtain equations of motion. To facilitate this, they made some of their linearizing assumptions at the start. Because the following analysis will not deal with dissipative forces of any nature, it can make use of a Lagrange undetermined multipliers derivation of the equations of motion. This method will yield the fully general equations for a bicycle. These can then be linearized around any steady turn as well as along a straight line as in the previous analyses. Due to the difficulty of the problem, the rider inputs will not be considered here. This rules out any verification of the results of Jones' study.

ASSUMPTIONS

In the interest of developing a more tractable model, a number of assumptions have been made. While at least some of the following suppositions could have been replaced by approximations to make the model more realistic, the resulting analysis would have been far too complex for the scope of this project and far less elegant than what was done. Also, for most of the cases to be studied, the deviations from reality will not be very significant. The following describe the idealizations of this model:

1. The system has no dissipative forces whatsoever: no friction of the bearings in the steering column or the wheels, no tire drag, no air resistance, and specifically, no slippage of the tires on the ground.

2. There is no driving force. The bicycle coasts.

3. The wheels are infinitely thin disks in that each intersects the ground at a single point, not at a contact area of deformed tire.

4. The system consists of four rigid bodies: the frame, the fork and the two wheels. The frame and the fork each have right-left symmetry so that each has an inertia tensor with only four independent elements. The wheels as disks have only two independent elements of their respective inertia tensors.

5. There is no rider input in that the front wheel fork system is free to rotate with respect to the frame-rear wheel system about the steering axis. Also, there is no shift of rider position with respect to the frame. The rider is treated as part of the frame.

With these assumptions the position of the bike with respect to an inertial coordinate system fixed on the ground can be completely specified in terms of seven degrees of freedom. It is then possible to express the kinetic energy, the potential energy and the constraints of no slippage of the tires in terms of these seven degrees of freedom, their time derivatives, and the parameters of the bicyle.

GLOSSARY OF NOTATION

Term	Explanation
r	radius of each of the wheels
d_1	perpendicular distance from the steering axis to the center of the rear wheel
d_3	perpendicular distance from the center of the front wheel to the steering axis
d_2	distance along the steering axis between the perpendiculars to each wheel
(x, y, z)	coordinate system fixed on the frame. Its origin is at the intersection of the steering axis with the perpendicular to the rear wheel. The x axis is perpendicular to the plane of the frame. The y axis is in the plane of symmetry and perpendicular to the steering axis. The z axis is parallel to the steering axis.
m, n, h	coefficients defining the plane of the ground in the (x, y, z) coordinate system
ψ	the angle of rotation of the front wheel fork system with respect to the rear wheel-frame system about the steering axis
θ	the angle between the plane of the rear wheel and the plane of the ground

Term	Explanation
$(\bar{X}, \bar{Y}, \bar{Z})$	the inertial coordinate system fixed on the ground
(X, Y)	the point of contact between the rear wheel and the ground as measured in the $(\bar{X}, \bar{Y}, \bar{Z})$ system
(X_F, Y_F)	the point of contact between the front wheel and the ground as measured in the $(\bar{X}, \bar{Y}, \bar{Z})$ system
φ	the angle between the $\bar{X}$ axis and the line tangent to the rear wheel in the plane of the ground
α_F	the angel of rotation of the front wheel with respect to the fork
α_R	the angle of rotation of the rear wheel with respect to the frame
ν	the angel between the steering axis and the tangent to the rear wheel in the ground plane
γ	geometric quantity depending on ψ and θ
λ	geometric quantity depending on ψ and θ
y_F	the y coordinate of the point of contact of the front wheel as measured in the bike fixed system
L	the distance between the point of contact of the rear wheel and the point of contact of the front wheel
Δ	the angle between the line tangent to the rear wheel in the ground plane and the line tangent to the front wheel in the ground plane
σ	the angle between the line tangent to the rear wheel and the line connecting the two points of contact
ρ	the angle between the line connecting the center of the front wheel with its point of contact and the line of the steering axis
q_k	the generalized expression for a degree of freedom with the following correspondence between the q_k and the previously defined degrees of freedom

Term	Explanation
	$\psi \longleftrightarrow q_1$
	$\theta \longleftrightarrow q_2$
	$\varphi \longleftrightarrow q_3$
	$\alpha_R \longleftrightarrow q_4$
	$\alpha_F \longleftrightarrow q_5$
	$X \longleftrightarrow q_6$
	$Y \longleftrightarrow q_7$
n_i	the partial derivative of n with respect to q_i
L_i	the partial derivative of L with respect to q_i
σ_i	the partial derivative of σ with respect to q_i
ρ_i	the partial derivative of ρ with respect to q_i
Fr_2	the perpendicular distance from the steering axis to the center of mass of the rear wheel-frame system
Fo_2	the perpendicular distance from the steering axis to the center of mass of the front wheel-fork system
Fr_3	the z coordinate of the center of mass of the frame-rear wheel system
Fo_3	the z coordinate of the center of mass of the front wheel-fork system
$(X_{cmb}, Y_{cmb}, Z_{cmb})$	the coordinates of the center of mass of the b th body expressed in the ground fixed system, $(\bar{X}, \bar{Y}, \bar{Z})$ Note: for the remainder of the analysis $b = 1$ refers to the rear wheel-frame system and $b = 2$ refers to the front wheel-fork system
F_{bi}	the i th coordinate of the center of mass of the b th body expressed in the (x, y, z) system
A_b, B_b, C_b	terms involved in the expression for the center of mass of the b th body in the ground fixed system

Term	Explanation
A_{bi}, B_{bi}, C_{bi}	partial derivatives of A_b , B_b and C_b with respect to the q_i degree of freedom
A_{bij}	the partial derivative of A_{bi} with respect to q_j 5
B_{bij}	the partial derivative of B_{bi} with respect to q_j 6
m_b	mass of the b th system
$I_{Fr\,ij}$	inertia tensor of the frame
$I_{Fo\,ij}$	inertia tensor of the fork
$I_{FW\,ij}$	inertia tensor of the front wheel
$I_{RW\,ij}$	inertia tensor of the rear wheel
I_{bij}	modified inertia tensor of the b th system
ω_{bi}	the angular velocity of the b th body ($b=1$ for the frame and $b=2$ for the fork) along its i th body-fixed axis
ω_{bij}	partial derivative of ω_{bi} with respect to the time derivative of the j th degree of freedom, $\dot{q}_j$
E_{ij}	coefficients in the quadratic expression of the total kinetic energy
E_{ijk}	the partial derivative of E_{ij} with respect to q_k
V	the total potential energy
T	the total kinetic energy
$\dot{\gamma}$	the ground speed of the geometric point of contact of the rear wheel with the ground
$\dot{\gamma}_F$	the ground speed of the geometric point of contact of the front wheel with the ground
γ_i	the partial derivative of $\dot{\gamma}$ with respect to $\dot{q}_i$

Term	Explanation
X_i	the partial derivative of $\dot{\mathrm{X}}$ with respect to $\dot{q}_i$
Y_i	the partial derivative of $\dot{\mathrm{Y}}$ with respect to $\dot{q}_i$
φ_i	the partial derivative of $\dot{\varphi}$ with respect to $\dot{q}_i$
α_{Fi}	the partial derivative of $\dot{\alpha}_F$ with respect to $\dot{q}_i$
φ_{ij}	the partial derivative of φ_i with respect to q_j
γ_{ij}	the partial derivative of γ_i with respect to q_j
α_{Fij}	the partial derivative of α_{Fi} with respect to q_j
$\mathcal{L}$	the Lagrangian, equal to $T - V$
Q_i	the generalized force of constraint of no slippage corresponding to the coordinate q_i
T_{ij} , U_{ijk}	coefficients in the final three general equations of motion
P_{ijk}	coefficients in the final two linearized equations of motion

GEOMETRY

In this model, the bike is contrained to have both wheels tangent to the ground. In the (x, y, z) coordinate system the ground is defined by the plane (Figure 1):

$$z = mx + ny - h$$

The point of contact of the rear wheel is a solution to the following system of equations:

$$\text{(1)} \quad \left.\begin{aligned} z &= mx + ny - h \\ x &= 0 \\ r^2 &= x^2 + (y + d_1)^2 + z^2 \end{aligned}\right\} \quad \text{equations defining the rear wheel}$$

This system reduced to one quadratic equation, and the requirement that the wheel intersect the ground in just one point is equivalent to requiring the quadratic to have a double root. The resultant equation of constraint is:

$$0 = (d_1 - nh)^2 - (n^2+1)(h^2 + d_1^2 - r^2)$$

$$\text{or} \quad h = -nd_1 + r\sqrt{n^2+1}$$

Although there are two solutions for h here, the one corresponding to the ground being above the wheel is ruled out.

From analytic geometry, the cosine of the angle, between the plane of the rear wheel, $x = 0$, and the ground plane is:

$$\cos\theta = \frac{m}{\sqrt{m^2+n^2+1}}$$

$$\text{or} \quad m = \sqrt{n^2+1}\,\cot\theta$$

Because of the sign convention chosen here, the time derivative of θ will be in the direction opposite to that of the forward motion of the bicycle (Figure 1.), and $\theta = \frac{\pi}{2}$ will correspond to the bike being upright. Now, using the equation constraining the front wheel to be tangent to the ground, one can solve for h in terms of ψ and θ .

The point of contact of the front wheel is a solution to the following set of equations:

$$z = mx + ny - h$$

$$(2) \quad \left.\begin{aligned} x &= -y\tan\psi \\ r^2 &= (x + d_3\sin\psi)^2 + (y - d_3\cos\psi)^2 \\ &\quad + (z + d_2)^2 \end{aligned}\right\} \text{equations defining the front wheel}$$

Again requiring that the resultant quadratic have a double root, one gets:

$$
\begin{aligned}
0 = {} & n^2\left[(d_3 \tan\psi \cot\theta + r\sec\psi)^2 + (d_3 + d_1\sec\psi)^2 - r^2(\tan^2\psi\cot^2\theta + 1)\right] \\
& + n\left[2(d_3 + d_1\sec\psi)\, d_2\sec\psi\right] \\
& + \left[(d_3\tan\psi\cot\theta + r\sec\psi)^2 + d_2^2\sec^2\psi - r^2(\tan^2\psi\cot^2\theta + \sec^2\psi)\right] \\
& + n\sqrt{n^2+1}\left[-2(d_3\tan\psi\cot\theta + r\sec\psi)(d_3 + d_1\sec\psi) + 2r^2\tan\psi\cot\theta\right] \\
& + \sqrt{n^2+1}\left[-2(d_3\tan\psi\cot\theta + r\sec\psi)\, d_2\sec\psi\right]
\end{aligned}
$$

While a closed form solution of this equation exists, numerical methods prove to be far more practical. In any case, this is as far as the derivation need be pursued at this point. It should be noted that there are two solutions to this equation. The correct one corresponds to the plane of the ground being below the front wheel. From analytic geometry one sees that n is the cotangent of the angle, ν , between the steering axis, $x = 0$ and $y = 0$,and the line tangent to the rear wheel in the ground plane, $z = ny - h$ and $x = 0$. For $\psi = 0$ and $\theta = \frac{\pi}{2}$ one then has $n = -d_2/(d_1 + d_3)$.

The seven degrees of freedom of the bicycle are then: ψ , the angle of steer; θ , the incline of the rear wheel and frame with respect to the ground; φ , the direction of the tangent to the rear wheel measured in the ground plane; X and Y , the cartesian coordinates of the point of contact of the rear wheel expressed in the ($\bar{X}$, $\bar{Y}$, $\bar{Z}$) system; and α_F and α_R the angles of rotation of the front and rear wheels respectively as measured from systems fixed on the fork and the frame respectively.

By requiring the following correspondences between the bike coordinate system and the inertial system one can find the linear transformation between the two systems.

Bike System		Inertial System	
$z = mx + ny - h$	$\longleftrightarrow$	$\bar{Z} = 0$	plane
$x = 0,\ z = ny - h$	$\longleftrightarrow$	$\bar{Z} = 0,$ $\bar{Y} = \tan\varphi\,\bar{X} - (\tan\varphi\,\mathbb{X} - \mathbb{Y})$	line tangent
$\left(0,\ \frac{-d_1 + nh}{n^2+1},\ \frac{-nd_1 - h}{n^2+1}\right)$	$\longleftrightarrow$	$(\mathbb{X}, \mathbb{Y}, 0)$	point of contact

The first relation is the requirement that the ground plane be the same in both systems. The second relates the two expressions for the line tangent to the rear wheel in the ground. The third set of expressions represents the point of contact of the rear wheel in each system (here equations (1) have been solved for the point as expressed in the bike-fixed system).

For convenience the quantity $\gamma = \frac{1}{\sqrt{n^2+1}}$ is defined. The transformation between the two coordinate systems is then:

$$\begin{pmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{pmatrix} = \begin{pmatrix} \sin\theta\sin\varphi & \gamma(\cos\varphi - n\cos\theta\sin\varphi) & \gamma(n\cos\varphi + \cos\theta\sin\varphi) \\ -\sin\theta\cos\varphi & \gamma(\sin\varphi + n\cos\theta\cos\varphi) & \gamma(n\sin\varphi - \cos\theta\cos\varphi) \\ -\cos\theta & -n\gamma\sin\theta & \gamma\sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$+ \begin{pmatrix} \mathbb{X} + \gamma(d_1\cos\varphi + h\cos\theta\sin\varphi) \\ \mathbb{Y} + \gamma(d_1\sin\varphi - h\cos\theta\cos\varphi) \\ h\gamma\sin\theta \end{pmatrix}$$

Other important geometric quantities in this problem are λ defined as $\lambda = n - \frac{\cot\theta \tan\psi}{\gamma}$ and y_F which is the y coordinate of the solution of the system of equations (2) for the point of contact of the front wheel:

$$y_F = \frac{d_3 \sec\psi + \lambda(h - d_2)}{\lambda^2 + \sec^2\psi}$$

Using analytic geometry in the bike-fixed coordinate system one can then determine the distance between the two points of contact L, the angle between the two lines tangent, Δ, and the angle between the tangent to the back wheel and the line connecting the two points of contact, σ. In the actual process of derivation it was easier to arrive at the following relations which are sufficient to determine L, Δ and σ:

$$L\cos\sigma = y_F\left(1/\gamma - n\cot\theta\tan\psi\right) + d_1/\gamma - rn$$

$$L\sin\sigma = y_F \csc\theta \tan\psi$$

$$\cos\Delta = \frac{\sin\theta - n\gamma\cos\theta\tan\psi}{\sqrt{(\sin\theta - n\gamma\cos\theta\tan\psi)^2 + \gamma^2\tan^2\psi}}$$

$$\sin\Delta = \frac{\gamma\tan\psi}{\sqrt{(\sin\theta - n\gamma\cos\theta\tan\psi)^2 + \gamma^2\tan^2\psi}}$$

One final purely geometric quantity which is necessary to the constraints is ρ, the angle between the steering axis and the radius vector of the front wheel to its point of contact with the ground. Doing the necessary geometry in the (x, y, z) system yields:

$$\cos\rho = \frac{-\lambda}{\sqrt{\lambda^2 + \sec^2\psi}}$$

$$\sin\rho = \frac{-\sec\psi}{\sqrt{\lambda^2 + \sec^2\psi}}$$

This along with the three previously mentioned parameters becomes essential to the writing of the constraints of no tire slippage. Noteworthy is the fact that all four of these quantities depend only upon the ψ and θ degrees of freedom.

KINETIC & POTENTIAL ENERGIES

In order to formulate the equations of motion using Lagrange undetermined multipliers, it is necessary to express the kinetic and potential energies in terms of the seven degrees of freedom. Because the bicycle is assumed to consist of four rigid bodies, the kinetic energy can be expressed as the sum of the translational energies of the centers of mass and the rotational energies about the centers of mass.

The positions of the centers of mass in the ($\bar{X}, \bar{Y}, \bar{Z}$) system can be derived from the linear transformation between the two systems. The coordinates of the frame center of mass and the fork center of mass in the bike-fixed system are:

Position of frame C.M. $(0, -Fr_2, Fr_3)$

Position of fork C.M. $(-Fo_2 \sin\psi, Fo_2 \cos\psi, Fo_3)$

In the ground system these positions become:

$$X_{cmb} = X + \cos\varphi\, A_b + \sin\varphi\, B_b$$

$$Y_{cmb} = Y + \sin\varphi\, A_b - \cos\varphi\, B_b$$

$$Z_{cmb} = C_b$$

Where b is a subscript referring to the body. If F_{bi} is the i th coordinate of the center of mass of the b th body as expressed in the bike system, then the following define A_b, B_b and C_b:

$$A_b = \gamma(d_1 + F_{b2}) + n\gamma F_{b3}$$

$$B_b = \cos\theta\left[\gamma(h + F_{b3}) - n\gamma F_{b2}\right] + \sin\theta\, F_{b1}$$

$$C_b = \sin\theta\left[\gamma(h + F_{b3}) - n\gamma F_{b2}\right] - \cos\theta\, F_{b1}$$

As before, these three quantities depend only upon ψ and θ.

Because the center of mass of each of the wheels is on its respective axle, the center of mass of the rear wheel is fixed with respect to the frame as is the center of mass of the front wheel with respect to the fork. The translational energies of the wheels, therefore, do not have to be treated independently. It is only necessary to correct for the positions of the centers of mass of the frame and fork in the usual manner.

The translational kinetic energy can then be written down as the sum of the rear wheel-frame translational kinetic energy and the front wheel-fork translational kinetic energy.

$$\text{Trans. K.E.} = \frac{1}{2}\sum_{b=1}^{2} m_b \Big\{ \big[\dot{X} + \cos\varphi(\dot{A}_b + B_b\dot{\varphi}) + \sin\varphi(\dot{B}_b - A_b\dot{\varphi})\big]^2 + \big[\dot{Y} + \sin\varphi(\dot{A}_b + B_b\dot{\varphi}) - \cos\varphi(\dot{B}_b - A_b\dot{\varphi})\big]^2 + \dot{C}_b^2 \Big\}$$

Here, and in the remainder of the analysis, the convention of using dots above a variable to indicate its time derivative is followed.

The next step is to find the rotational kinetic energies in terms of the degrees of freedom, but first it is convenient to write down the angular velocity of the bike-fixed axis with respect to the ground fixed axis. The components of this angular velocity should be expressed in terms of the bike-fixed axis.

The three angles which are sufficient to determine the orientation of the bike system are θ, φ and ν, the angle between the steering axis and the tangent to the rear wheel in the ground. As previously mentioned, n is the cotangent of this angle, so $\gamma = \sin\nu$ and $n\gamma = \cos\nu$. The time derivative of the angle is $\dot{\nu} = -\gamma^2(n_1\dot{\psi} + n_2\dot{\theta})$. The direction of this time derivative is along the positive X axis fixed on the bike. Referring to Figures 1 and 2 the components of the angular velocity along the bike are:

$$\omega_{11} = \dot{\psi}(-\gamma^2 n_1) + \dot{\theta}(-\gamma^2 n_2) + \dot{\varphi}(-\cos\theta)$$

$$\omega_{12} = \dot{\theta}(-\gamma) + \dot{\varphi}(-n\gamma\sin\theta)$$

$$\omega_{13} = \dot{\theta}(-n\gamma) + \dot{\varphi}(\gamma\sin\theta)$$

Given the tensor of inertia, $I_{Fr\,ij}$, for the frame about the axis passing through its center of mass and parallel to the (x, y, z) axis, the rotational kinetic energy of the frame about its center of mass is then:

$$\text{Rot. K.E. of frame} = \frac{1}{2} \sum_{i,j=1}^{2} I_{Fr\,ij}\, \omega_{1i}\, \omega_{1j}$$

To find the rotational kinetic energy of the fork it is necessary to find its angular velocity with respect to the ground system in terms of components along its body-fixed axis. The only difference between the orientation of the frame-fixed system and the fork-fixed system is the angle of steer, ψ. This implies that the angular velocity of the fork differs from that of the frame only by $\dot{\psi}$. Choosing fork-fixed axis (x', y', z') with x' parallel to the axle of the front wheel, y' in the plane of the front wheel and perpendicular to the steering axis and z' parallel to the steering axis, one gets the following components for the angular velocity of the fork:

$$\omega_{21} = \cos\psi\, \omega_{11} + \sin\psi\, \omega_{12}$$

$$\omega_{22} = -\sin\psi\, \omega_{11} + \cos\psi\, \omega_{12}$$

$$\omega_{23} = \omega_{13} + \dot{\psi}$$

If this fork-fixed system is chosen with origin at its center of mass and the tensor of inertia is $I_{Fo\,ij}$, then the rotational kinetic energy of the fork about its center of mass is:

$$\text{Rot. K.E. of fork} = \frac{1}{2} \sum_{i,j=1}^{3} I_{Fo\,ij}\, \omega_{2i}\, \omega_{2j}$$

The angular velocity of the rear wheel differs from that of the frame only by $\dot{\alpha}_R$ while the front wheel angular velocity equals the fork angular velocity plus $\dot{\alpha}_F$. In terms of body-fixed axis for each of the wheels which for the rear wheel are parallel to the frame-fixed axis when $\alpha_R = 0$ and for the front wheel are parallel to the fork-fixed axis when $\alpha_F = 0$, the wheel angular velocities are:

$$\omega_{FW1} = \omega_{21} + \dot{\alpha}_F$$
$$\omega_{FW2} = \cos\alpha_F\,\omega_{22} - \sin\alpha_F\,\omega_{23}$$
$$\omega_{FW3} = \sin\alpha_F\,\omega_{22} + \cos\alpha_F\,\omega_{23}$$
$$\omega_{RW1} = \omega_{11} + \dot{\alpha}_R$$
$$\omega_{RW2} = \cos\alpha_R\,\omega_{12} - \sin\alpha_R\,\omega_{13}$$
$$\omega_{RW3} = \sin\alpha_R\,\omega_{12} + \cos\alpha_R\,\omega_{23}$$

with 1,2 and 3 referring to the respective body-fixed axis. Because of the symmetry, each wheel has only two independent elements in its inertia tensor and this tensor is diagonal. The rotational kinetic energies are then:

$$\text{F.W. Rot. K.E.} = \tfrac{1}{2}\left[I_{FW11}\,\omega_{FW1}^2 + I_{FW22}\left(\omega_{FW2}^2 + \omega_{FW3}^2\right)\right]$$
$$\text{R.W. Rot. K.E.} = \tfrac{1}{2}\left[I_{RW11}\,\omega_{RW1}^2 + I_{RW22}\left(\omega_{RW2}^2 + \omega_{RW3}^2\right)\right]$$

Substitution for the angular velocities yields:

$$\text{F.W. Rot. K.E.} = \tfrac{1}{2}\left[I_{FW11}\left(\omega_{21} + \dot{\alpha}_F\right)^2 + I_{FW22}\left(\omega_{22}^2 + \omega_{23}^2\right)\right]$$
$$\text{R.W. Rot. K.E.} = \tfrac{1}{2}\left[I_{RW11}\left(\omega_{11} + \dot{\alpha}_R\right)^2 + I_{RW22}\left(\omega_{12}^2 + \omega_{13}^2\right)\right]$$

The total kinetic energy of the bicycle can now be written down, but first it is convenient to define a new inertia tensor, $I_{b\,ij}$. $I_{1\,ij}$ is the same as $I_{F,ij}$

and I_{2ij} is the same as I_{Foij} with the following exceptions:

$$I_{111} = I_{Fr11} + I_{RW11}$$
$$I_{122} = I_{Fr22} + I_{RW22}$$
$$I_{133} = I_{Fr33} + I_{RW22}$$
$$I_{211} = I_{Fo11} + I_{FW11}$$
$$I_{222} = I_{Fo22} + I_{FW22}$$
$$I_{233} = I_{Fo33} + I_{FW33}$$

This notation convention allows for the concise expression of the total rotational kinetic energy:

$$\text{Total Rot. K.E.} = \frac{1}{2}\Big[\sum_{b=1}^{2}\Big\{ \sum_{i,j=1}^{3} I_{bij}\,\omega_{bi}\,\omega_{bj} \Big\} + I_{FW11}\left(2\dot{\alpha}_F\,\omega_{21} + \dot{\alpha}_F^2\right) + I_{RW11}\left(2\dot{\alpha}_R\,\omega_{11} + \dot{\alpha}_R^2\right)\Big]$$

One is now prepared to write down the total kinetic energy in a form which is most useful when later deriving the equations of motion. For convenience, the generalized degrees of freedom, q_K ,have been used in part:

$$T = \sum_{i,j=1}^{5} E_{ij}\,\dot{q}_i\,\dot{q}_j + \frac{1}{2}\sum_{b=1}^{2} m_b \Big\{ \left[\dot{X} + \cos\varphi(\dot{A}_b + B_b\dot{\varphi}) + \sin\varphi(\dot{B}_b - A_b\dot{\varphi})\right]^2 + \left[\dot{Y} + \sin\varphi(\dot{A}_b + B_b\dot{\varphi}) - \cos\varphi(\dot{B}_b - A_b\dot{\varphi})\right]^2 \Big\}$$

The glossary contains an explanation of the q_K .
From the preceding analysis the E_{ij} are defined as follows:

$$E_{ij} = \frac{1}{2}\sum_{b=1}^{2}\Big\{ m_b\, C_{bi}\, C_{bj} + \sum_{K,\ell=1}^{3} I_{bK\ell}\,\omega_{bKi}\,\omega_{b\ell j} \Big\} \qquad i,j = 1,2,3$$

$$E_{i4} = E_{4i} = \frac{1}{2} I_{RW11}\,\omega_{11i} \qquad i = 1,2,3$$

$$E_{i5} = E_{5i} = \frac{1}{2} I_{FW11}\,\omega_{21i} \qquad i = 1,2,3$$

$$E_{44} = \frac{1}{2} I_{RW11}$$

$$E_{55} = \frac{1}{2} I_{FW11}$$

$$E_{45} = E_{54} = 0$$

where

$$\omega_{bi} = \sum_{j=1}^{3} \omega_{bij} \dot{q}_i \qquad \text{and} \qquad C_{bi} = \frac{\partial C_b}{\partial q_i}$$

Recalling that the ω_{bij}, A_b, B_b and C_b depend only upon the degrees of freedom ψ and θ, the kinetic energy is seen to depend only on ψ, θ and φ as coordinates with φ appearing in a cyclic manner. This fact will take on importance when it comes time to reduce the number of equations of motion by eliminating the dependent ones.

Knowing the $\bar{Z}$ coordinate of the center of mass for both the rear wheel-frame system and the front wheel-fork system, it is straightforward to derive the potential energy. It is simply the sum over body systems of the system mass times its height ($\bar{Z}$ coordinate) times g, the gravitational constant:

$$V = g \sum_{b=1}^{2} m_b C_b$$

Again, it should be remembered that the potential energy depends only upon ψ and θ because C_b depends only upon these two degrees of freedom.

Although the kinetic energy, the potential energy and therefore the Lagrangian are already derived, it is not yet possible to formulate the equations of motion. It is first necessary to determine the kinematical equations which express the constraints of zero slippage of the tires.

The condition for zero slippage requires the velocity to be zero for that point of each wheel which is in contact with the ground at any given instant. An equivalent condition for no slippage which is more easily applied to the geometry of the problem is the following: the velocity of the geometric point of contact((X, Y) for the rear wheel, (X_F, Y_F) for the front wheel) of each wheel is in the direction of the tangent to that wheel in the ground, and the speed of the geometric point measured in the ground system is equal to the speed of the geometric point expressed in a system fixed on the wheel.

By performing a matrix transformation from the ground system to a system fixed on a wheel, one can write down the coordinates of the point of contact in terms of the wheel-fixed system. By taking time derivatives the speeds are obtained:

$$\dot{\gamma} = -r(\omega_{11}\dot{\psi} + \omega_{12}\dot{\theta} + \dot{\alpha}_R)$$

$$\dot{\gamma}_F = -r(\rho_1\dot{\psi} + \rho_2\dot{\theta} + \dot{\alpha}_F)$$

The tangent to the rear wheel makes an angle of φ with respect to the $\bar{X}$ axis, and the tangent to the front wheel makes an angle of $\varphi + \Delta$ with respect to the $\bar{X}$

axis. The kinematical equations of constraint are then:

$$\dot{X} = \cos\varphi\,\dot{\gamma}$$
$$\dot{Y} = \sin\varphi\,\dot{\gamma}$$
$$\dot{X}_F = \cos(\varphi + \Delta)\,\dot{\gamma}_F$$
$$\dot{Y}_F = \sin(\varphi + \Delta)\,\dot{\gamma}_F$$

The negative signs in the expressions for $\dot{\gamma}$ and $\dot{\gamma}_F$ are necessary to make a negative $\dot{\alpha}_R$ and $\dot{\alpha}_F$ correspond to the bike traveling forward. X_F and Y_F and their derivatives can be expressed in terms of the previously defined geometric quantities:

$$X_F = X + L\cos(\varphi + \sigma)$$
$$Y_F = Y + L\sin(\varphi + \sigma)$$

Taking derivatives, one can solve the last two equations of constraint for $\dot{\varphi}$ and $\dot{\alpha}_F$:

$$\dot{\varphi} = \varphi_1\dot{\psi} + \varphi_2\dot{\theta} + \varphi_4\dot{\alpha}_R$$
$$\dot{\alpha}_F = \alpha_{F1}\dot{\psi} + \alpha_{F2}\dot{\theta} + \alpha_{F4}\dot{\alpha}_R$$

It is also convenient to rewrite the first two constraint relations:

$$\dot{X} = X_1\dot{\psi} + X_2\dot{\theta} + X_4\dot{\alpha}_R$$
$$\dot{Y} = Y_1\dot{\psi} + Y_2\dot{\theta} + Y_4\dot{\alpha}_R$$

The coefficients in the above four equations are defined as follows:

$$\varphi_i = \frac{L_i\tan(\Delta - \sigma)}{L} - \sigma_i - \frac{r\,\omega_{1i}\sin\Delta}{L\cos(\Delta - \sigma)} \qquad i = 1,2$$

$$\varphi_4 = \frac{-r\sin\Delta}{L\cos(\Delta - \sigma)}$$

$$\alpha_{Fi} = -\frac{L_i}{r\cos(\Delta-\sigma)} + \frac{\omega_{1i}\cos\sigma}{\cos(\Delta-\sigma)} - \rho_i \qquad i=1,2$$

$$\alpha_{F4} = \frac{\cos\sigma}{\cos(\Delta-\sigma)}$$

$$X_i = \cos\varphi\,\gamma_i \qquad i=1,2,4$$

$$Y_i = \sin\varphi\,\gamma_i \qquad i=1,2,4$$

$$\gamma_i = -r\omega_{1i} \qquad i=1,2$$

$$\gamma_4 = -r$$

It is now possible to formulate the equations of motion. Again note that neither γ_i, φ_i nor α_{Fi} depend upon any of the degrees of freedom besides ψ and θ .

EQUATIONS OF MOTION

If there were no constraints on the tires, that is, if they were perfectly slippery, then the equations of motion would simply be the usual unconstrained Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \qquad i=1\ldots7$$

In the case of constrained motion, however, there are forces of constraint, Q_i , which act to cause the motion of the system to satisfy the equations of constraint.

The generalized equations of motion then become:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i \qquad i = 1 \ldots 7$$

Knowing that the forces of constraint dissipate no energy, the tires have zero slippage so there is no distance over which the constraint forces act, the following energy conservation relation must be satisfied:

$$0 = \sum_{i=1}^{7} Q_i \dot{q}_i$$

Substitution of the kinematical constraints into this relation gives:

$$\begin{aligned} 0 = \dot{\psi}&\left[Q_1 + Q_3 \varphi_1 + Q_5 \alpha_{F1} + Q_6 X_1 + Q_7 Y_1 \right] \\ +\dot{\theta}&\left[Q_2 + Q_3 \varphi_2 + Q_5 \alpha_{F2} + Q_6 X_2 + Q_7 Y_2 \right] \\ +\dot{\alpha}_R&\left[Q_4 + Q_3 \varphi_4 + Q_5 \alpha_{F4} + Q_6 X_4 + Q_7 Y_4 \right] \end{aligned}$$

$\dot{\psi}$, $\dot{\theta}$ and $\dot{\alpha}_R$ are now independent so their coefficients in the above relation must vanish. Substitution for the Q_i yields the following equations of motion:

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) - \frac{\partial \mathcal{L}}{\partial \psi} + \varphi_1\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi}\right] + \alpha_{F1}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}_F}\right)\right] + X_1\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right)\right] + Y_1\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Y}}\right)\right]$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} + \varphi_2\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi}\right] + \alpha_{F2}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}_F}\right)\right] + X_2\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right)\right] + Y_2\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Y}}\right)\right]$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}_R}\right) + \varphi_4\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi}\right] + \alpha_{F4}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}_F}\right)\right] + X_4\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right)\right] + Y_4\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Y}}\right)\right]$$

remembering that $\frac{\partial \mathcal{L}}{\partial \alpha_R}$, $\frac{\partial \mathcal{L}}{\partial \alpha_F}$, $\frac{\partial \mathcal{L}}{\partial X}$ and $\frac{\partial \mathcal{L}}{\partial Y}$ are all equal to zero. The other four necessary equations to determine the time evolution of the seven degrees of freedom are simply the kinematic equations of constraint.[7]

At this point, bicycle dynamics has been described by seven equations of motion relating seven degrees of freedom and their first and second derivatives with respect to time. The problem can now be further reduced to three independent equations with three degrees of freedom.

As was already noted and should now be recalled, only ψ, θ and φ appear as coordinates in the seven equations. By substituting for $\dot{X}$, $\dot{Y}$, $\ddot{X}$ and $\ddot{Y}$ in the above three equations using their expressions from the first two equations of constraint, one can immediately eliminate these two degrees of freedom from the equations of motion leaving five equations in five degrees of freedom. Furthermore, the cyclic nature of φ, the coordinate, as it enters into these two constraint relations and in the above three equations causes dependence upon it to vanish from the remaining five equations. It is now a simple matter to eliminate $\dot{\varphi}$, $\dot{\alpha}_F$, $\ddot{\varphi}$ and $\ddot{\alpha}_F$ from the above three equations by direct substitution of the last two constraint relations and their derivatives.

The resultant three equations of motion involve only the ψ, θ and α_R degrees of freedom. The equations take the following form:

$$0 = \sum_{j=1,2,4} T_{ij}\ddot{q}_j + \sum_{j,k=1,2,4} U_{ijk}\dot{q}_i\dot{q}_k + \frac{\partial V}{\partial q_i} \qquad i = 1,2,4$$

where the coefficients are defined as follows:

$$T_{ij} = 2\Big[E_{ij} + E_{i3}\varphi_j + E_{j3}\varphi_i + E_{i5}\alpha_{Fj} + E_{j5}\alpha_{Fi} + E_{33}\varphi_i\varphi_j + E_{55}\alpha_{Fj}\alpha_{Fi} + E_{35}(\alpha_{Fi}\varphi_j + \alpha_{Fj}\varphi_i)\Big] + \sum_{b=1}^{2} m_b\Big\{(A_{bi} + B_b\varphi_i + \gamma_i)(A_{bj} + B_b\varphi_j + \gamma_j) + (B_{bi} - A_b\varphi_i)(B_{bj} - A_b\varphi_j)\Big\}$$

$$\begin{aligned}
U_{ijk} = {} & \Big[E_{ijk} + E_{ikj} + E_{i3j}\varphi_k + E_{i3k}\varphi_j + E_{i3}(\varphi_{jk} + \varphi_{kj}) + E_{i5j}\alpha_{Fk} \\
& + E_{i5k}\alpha_{Fj} + E_{i5}(\alpha_{Fjk} + \alpha_{Fkj})\Big] - \Big[E_{jki} + E_{33i}\varphi_j\varphi_k + E_{55i}\alpha_{Fj}\alpha_{Fk} \\
& + E_{j3i}\varphi_k + E_{k3i}\varphi_j + E_{j5i}\alpha_{Fk} + E_{k5i}\alpha_{Fj} + E_{35i}(\varphi_j\alpha_{Fk} + \varphi_k\alpha_{Fj})\Big] \\
& + \varphi_i\Big[E_{j3k} + E_{k3j} + E_{33j}\varphi_k + E_{33k}\varphi_j + E_{33}(\varphi_{jk} + \varphi_{kj}) + E_{35j}\alpha_{Fk} \\
& + E_{35k}\alpha_{Fj} + E_{35}(\alpha_{Fjk} + \alpha_{Fkj})\Big] + \alpha_{Fi}\Big[E_{j5k} + E_{k5j} \\
& + E_{55j}\alpha_{Fk} + E_{55k}\alpha_{Fj} + E_{55}(\alpha_{Fjk} + \alpha_{Fkj}) + E_{35j}\varphi_k + E_{35k}\varphi_j \\
& + E_{35}(\varphi_{jk} + \varphi_{kj})\Big] + \frac{1}{2}\sum_{b=1}^{2} m_b\Big\{\big[A_{bi} + B_b\varphi_i + \gamma_i\big]\big[\gamma_{ki} + \gamma_{ik} \\
& + 2A_{bjk} + 2B_{bj}\varphi_k + 2B_{bk}\varphi_j + B_b(\varphi_{jk} + \varphi_{kj})\big] \\
& + \big[B_{bi} - A_b\varphi_i\big]\big[2B_{bjk} - 2A_{bj}\varphi_k - 2A_{bk}\varphi_j - A_b(\varphi_{jk} + \varphi_{kj}) - \varphi_j\gamma_k \\
& - \varphi_k\gamma_j\big] - 2\varphi_j\varphi_k\big[B_b B_{bi} + A_b A_{bi} + A_b\gamma_i\big]\Big\}
\end{aligned}$$

[8]

By inspection it is seen that $T_{ij} = T_{ji}$ and $U_{ijk} = U_{ikj}$. Also, none of the coefficients depend upon the coordinate α_R which is precisely as things should be: this degree of freedom defines the angle of rotation of the rear wheel with respect to the frame and is obviously cyclic because the wheel is a symmetric disk. Normally this fact would allow the integration of the third equation of motion, $i = 4$, but one sees by inspection that:

$$\frac{\partial T_{4j}}{\partial q_k} + \frac{\partial T_{4k}}{\partial q_j} \neq u_{4jk} + u_{4kj}$$

When dealing with non-holonomic constraints, cyclic coordinates do not become ignorable.

An interesting sidelight to the analysis seems appropriate here. In writing down an expression for the kinetic energy, T , it is possible to substitute the constraints directly into the kinetic energy. The resulting expression is a function only of the ψ , θ and α_R degrees of freedom. V depends only upon ψ and θ . It might seem reasonable to formulate the Lagrangian in this manner as a function only of these three degrees of freedom and to write down three uncontrained Lagrange equations as the correct equations of motion. This method does not, however, yield the same equations as those already derived. It is therefore, incorrect. Holonomic and non-holonomic constraints are physically different. Non-holonomic forces of constraint cannot be eliminated from the equations of motion.

LINEARIZED EQUATIONS

The complexity of the general equations of motion for the bicycle make an analytic integration virtually impossible. One much more practical approach is to study the steady motion of a bicycle and small disturbances therefrom.

Steady motion for a bicycle is when $\psi = \psi_o$, $\theta = \theta_o$, $\dot{\alpha}_R = \dot{\alpha}_{Ro}$ $\dot{\psi} = 0$, $\dot{\theta} = 0$ and $\ddot{\alpha}_R = 0$. ψ_o , θ_o and $\dot{\alpha}_{Ro}$ must, of course, satisfy the equations of motion:

$$(8) \qquad \dot{\alpha}_{Ro}^2 \, U_{144}\Big|_{\psi_o,\theta_o} + \frac{\partial V}{\partial \psi}\Big|_{\psi_o,\theta_o} = 0$$

$$\dot{\alpha}_{Ro}^2 \, U_{244}\Big|_{\psi_o,\theta_o} + \frac{\partial V}{\partial \theta}\Big|_{\psi_o,\theta_o} = 0$$

Because α_R is cyclic, $U_{444} = 0$ and $\partial V/\partial \alpha_R = 0$ so the third equation is trivial. This steady motion corresponds to the bike traveling around a constant turn the radius of which is determined by ψ_o and θ_o . One of these constant turns is with infinite radius: $\psi_o = 0$, $\theta_o = \frac{\pi}{2}$. In this case $U_{144} = U_{244} = \frac{\partial V}{\partial \psi} = \frac{\partial V}{\partial \theta} = 0$. $\dot{\alpha}_{Ro}$ can take on any value. The fact that motion is steady does not imply that it is stable, i.e. if it is disturbed by a small amount does the motion remain bounded about the steady state or is it unbounded? This question can be answered in a quantitative way by studying the linearized equations of motion about the steady state.

Small disturbances from steady motion can be defined as follows:

$$\psi = \psi_0 + \psi_1 \quad , \quad \dot{\psi}_0 = 0 \quad , \quad \psi_1 << 1 \quad , \quad \dot{\psi}_1 << 1$$

$$\theta = \theta_0 + \theta_1 \quad , \quad \dot{\theta}_0 = 0 \quad , \quad \theta_1 << 1 \quad , \quad \dot{\theta}_1 << 1$$

$$\dot{\alpha}_R = \dot{\alpha}_{R0} + \dot{\alpha}_{R1} \quad , \quad \ddot{\alpha}_{R0} = 0 \quad , \quad \dot{\alpha}_{R1} << 1$$

Rewriting the equations of motion and neglecting all terms of order higher than 1 in the infintesmal quantities, the linearized equations become

$$(9) \qquad 0 = \sum_{j=1,2,4} \left\{ T_{ij} \ddot{q}_j + 2 U_{ij4} \dot{\alpha}_{R0} \dot{q}_j + \left(\frac{\partial^2 V}{\partial q_i \partial q_j} + \dot{\alpha}_{R0}^2 \frac{\partial U_{i44}}{\partial q_j} \right) q_j \right\} \qquad i = 1,2,4$$

where the new correspondence between the remaining degrees of freedom and the generalized degrees of freedom is $\psi_1 = q_1$, $\theta_1 = q_2$ and $\alpha_{R1} = q_4$ and all of the coefficients are evaluated at ψ_0 and θ_0 . In this linearized case, taking into account the fact that $U_{444} = 0$ and $\frac{\partial^2 V}{\partial q_i \partial \alpha_R} = 0$, the third equation of motion can be integrated:

$$\text{Constant} = T_{41} \dot{\psi}_1 + T_{42} \dot{\theta}_1 + T_{44} \dot{\alpha}_{R1} + 2 U_{414} \dot{\alpha}_{R0} \psi_1 + 2 U_{424} \dot{\alpha}_{R0} \theta_1$$

This constant is chosen to be zero because the combination $\psi_1 = 0$, $\theta_1 = 0$ and $\dot{\alpha}_{R1} = 0$ must be a solution to the linearized equations. One can now substitute for $\dot{\alpha}_{R1}$ and $\ddot{\alpha}_{R1}$ in the first two relations. The resulting equations are:

$$(10) \qquad 0 = \sum_{j=1}^{2} \left(P_{2ij} D^2 + P_{1ij} D + P_{0ij} \right) q_j \qquad i = 1,2$$

with the usual convention: $D^K q_j = \frac{d^K q_j}{dt^K}$

and the coefficients defined as:

$$P_{2ij} = T_{ij} - \frac{T_{i4}\,T_{4j}}{T_{44}}$$

$$P_{1ij} = 2\dot{\alpha}_{R0}\left\{U_{ij4} - \frac{(T_{i4}\,U_{4j4} + T_{4j}\,U_{i44})}{T_{44}}\right\}$$

$$P_{0ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} + \dot{\alpha}_{R0}^2\left\{\frac{\partial U_{i44}}{\partial q_j} - \frac{4\,U_{i44}\,U_{4j4}}{T_{44}}\right\}$$

Solutions for ψ_1 and θ_1 take on the form $e^{\lambda t}$ times a constant where the constant depends upon the initial conditions and λ is a solution of the characteristic quartic of the following fourth order differential equation in one unknown:

$$0 = \left[(P_{211}D^2 + P_{111}D + P_{011})(P_{222}D^2 + P_{122}D + P_{022}) - (P_{212}D^2 + P_{112}D + P_{012})(P_{221}D^2 + P_{121}D + P_{021})\right]q_i \qquad i = 1 \text{ or } 2$$

which was derived in the usual manner from the two preceding linearized equations in two unknowns. The coefficients of this polynomial are real, so the roots occur as real numbers or pairs of complex conjugates. The condition for stability of the bike in steady motion is reduced to the requirement that the real part of each of the four roots be less than or equal to zero. If the system is stable, the stability can be quantified by the magnitude of the imaginary part of the root. This imaginary part is equal to 2π times the frequency of oscillation about the steady state. As in the case of a simple harmonic oscillator, the higher the frequency of oscillation the more stable the state.

In the case of linearized equations of motion about an equilibrium configuration (as opposed to steady motion) or the case of steady motion with only holonomic constraints, the absence of dissipative forces and driving forces implies that the roots of the characteristic polynomial are either pure real or pure imaginary. In the case of the bicycle, however, the linearlized equations of motion take a form which admits the possibility of mixed complex roots to the characteristic polynomial. When the bicycle is disturbed from steady motion, it oscillates about that steady motion, but the oscillations eventually die out, and the system returns to its steady state configuration. This happens in the absence of damping forces.

It is not useful to attempt to further the purely analytic development of the model beyond this point, given the assumptions involved. More valuable insight can be gained through a numerical study of the dependence of the roots of the characteristic quartic upon the various design parameters of the bike and upon varying configurations of steady turning.

NUMERICAL ANALYSIS

A brief description of the computer programming used in the numerical study is in order here. Two programs were actually written. Both of them appear in the appendix. The first studies the effects upon the stability in a straight line of the various bicycle design parameters as were included in the preceding formulation and of the bicycle speed. The other finds values for ψ_0 and $\dot{\alpha}_{R0}$ given θ_0 corresponding to a steady turn for a given bicycle design and solves the characteristic equation for that steady turn. Although the first program is merely a special case of the second, it was written as a separate program for the sake of computational efficiency. The first was also one fifth the length of the second, and therefore, far less complicated to check out.

The straight line program only involved computing the coefficients of equations (10) for the values $\psi = 0$, $\theta = \frac{\pi}{2}$. The equation for n could be solved by inspection in this case: $n = \dfrac{-d_2}{d_1 + d_3}$. Also, the third of equations (9) is trivial: $\dot{\alpha}_{R1} = 0$. Otherwise the programming was straightforward.

The second of the two programs was far more involved. Given θ_0, it was necessary to eliminate $\dot{\alpha}_{R0}^2$ from equations (8) and solve numerically for ψ_0. This was done using a Newton-Rapheson approximation with a first guess of $\psi_0 = \dfrac{(\theta_0 - \pi/2)}{10}$. The equation for n was also solved in this manner with the straight ahead value $n = \dfrac{-d_2}{d_1 + d_3}$, used as a first guess for n. A wrong first guess would have yielded n corresponding to the ground being above the front wheel instead of below it.

Once a correct configuration of ψ_0, θ_0 and $\dot{\alpha}_{RO}$ was determined, the program computed the coefficients of equations (10) and solved the characteristic quartic. Also, in both programs, the phase difference between ψ_1 and θ_1 and the magnitude ratio of these two variables were also computed for each root.

The first program was checked out by comparing it to Collins' program in the case of zero friction and zero driving force. The results of the two were close for two of the four roots but not identical. Further inspection revealed that the difference was in the mathematical analysis and not in program operation. Collins made an error in his geometric analysis. The discrepancy, in effect, involves the dependence of n on ψ near the upright position. When this difference is eliminated from the programs, their results fall within a reasonable margin of error for the computer accuracy.

The only way to check out the second program is to run it for values of θ_0 approaching $\frac{\pi}{2}$. It should then approach the first program in results, given the same $\dot{\alpha}_{RO}$ and bicycle design parameters for both. The program did check out in this manner. Although it was not possible to verify that the solutions to equations (8) were indeed correct, they did behave in a proper manner in that if (ψ_0, θ_0, $\dot{\alpha}_{R0}$) was a solution, then ($-\psi_0$, $\pi - \theta_0$, $\dot{\alpha}_{R0}$) was also a solution with both of these solutions generating identical coefficients for the linearized equations of motion.

The programs were written in Fortran IV and run on the IBM 360 batched processor at Princeton University. The results were graphed by hand.

METHODS OF ANALYSIS

The method used to study the variations of the stability with the different design parameters was to first find a stable configuration, then to vary one parameter at a time. A somewhat different set of geometric parameters was used in this study. F_{r_2}, F_{r_3}, d_1, d_2 and d_3 are replaced by:

Rake angle $= \frac{\pi}{2} - \arcsin\left(\frac{d_1 + d_3}{\sqrt{d_2^2 + (d_1+d_3)^2}}\right)$

Trail $= \frac{r d_2 - d_3 \sqrt{d_2^2 + (d_1+d_3)^2}}{d_1 + d_3}$

Wheel base $= \sqrt{d_2^2 + (d_1+d_3)^2}$

Frame center of mass height $= r + \frac{(d_1 - F_{r_2}) d_2 + F_{r_3}(d_1+d_3)}{\sqrt{d_2^2 + (d_1+d_3)^2}}$

Horizontal distance from frame c.m. to rear wheel center $= \frac{(d_1 - F_{r_2})(d_1+d_3) - d_2 F_{r_3}}{\sqrt{d_2^2 + (d_1+d_3)^2}}$

The rake angle is the angle between the steering axis and the vertical when the bike is upright. The trail is the distance by which the point of contact of the front wheel trails the point where the line extended from the steering axis intersects the ground when the bike is upright.

The stable configuration from which variations were made is the following:

rake angle = 15	degrees	wheel radius = 0.35	meters
trail = 0.10	meters	wheel base = 1.5	meters
frame c.m. height = 1.0	meters	frame c.m. horiz. offset from rear wheel = 0.3	meters
F_{o2} = 0.09	meters	m_1 = 60.0	kg
F_{o3} = -0.10	meters	I_{Fr11} = 15.0	kg-meter^2
m_2 = 4.0	kg	I_{Fr33} = 6.0	kg-meter^2
I_{Fr22} = 5.0	kg-meter^2	I_{Fo11} = 1.0	kg-meter^2
I_{Fr23} = 0.15	kg-meter^2	I_{Fo33} = 0.5	kg-meter^2
I_{Fo22} = 0.5	kg-meter^2	I_{FW11} = 0.10	kg-meter^2
I_{Fo23} = -0.04	kg-meter^2	I_{FW22} = 0.05	kg-meter^2
I_{RW11} = 0.10	kg-meter^2	$\dot{\alpha}_{R0}$ = -30.0	radians/sec.
I_{RW22} = 0.05	kg-meter^2		

The parameters which were then varied were the trail, the rake angle, the wheel base, the frame center of mass height, the distance of the fork-front wheel c.m. from the steering axis, F_{o2}, the frame-rear wheel mass, m_1, the fork-front wheel mass, m_2, the moment of inertia of the front wheel about its axle, I_{FW11}, the moment of inertia of the fork about the axis through its center of mass and parallel to the steering axis, I_{Fo33}, and the speed of the bicycle, $\dot{\alpha}_{R0}$.

Also, for the above listed set of bicycle design parameters, the variation of the stability with the angle of lean, θ_0 , for an equilibrium turn was studied.

RESULTS AND DISCUSSION

A qualitative description of how a bicycle maintains stability is as follows: If the bike is perturbed slightly from its steady state, it begins to fall, that is, its lean, θ , increases. This, in turn, causes the steering angle, ψ , to respond in the direction of the lean sending the bike into a turn. Once in this turn, the centrifugal force acts to restore the bike to zero lean. It is with this in mind that one must interpret the variation of the stability of a bicycle.

In most cases the solutions of the characteristic quartic consisted of one pair of complex conjugates, a small real root, and a large negative real root. In such extreme cases as low bike speed the roots varied from this to become four real roots, two positive and two negative. In these cases, the bicycle was obviously unstable.

The condition for stability requires that the four real parts of the roots to the characteristic quartic be less than or equal to zero. As the last root always takes on a large negative value, it need not be considered in studying the stability. The other real root and the real part of the complex conjugates, however, take on both positive and negative values depending upon the design configuration. These are the two roots which determine whether or not a given design is stable. As will be seen, each corresponds to a different way in which the bike goes unstable.

For the solution to the equations of motion corresponding to the small pure real root, the lean, θ, and the steer, ψ, are always in phase, with the magniture of lean about 3-5 times as large as the magnitude of the steer. When this root is positive, the bike simply falls over to one side or the other. The bike never goes into enough of a turn to counteract the force of gravity on the lean.

The variation of the real root with the bicycle speed illustrates this point well. With increased speed the root goes unstable. This seems strange because it is obvious that less of a turn is needed to counteract the force of gravity at higher speeds: the centrifugal acceleration goes as speed^2/radius of turn. Why then does this root go unstable at higher speeds? The answer is that the magnitude ratio of the steer to the lean for this root is decreasing more rapidly than the steer necessary to counteract the force of gravity for a given lean (Graph 1). The reason for this decrease in the magnitude ratio is as follows: The constraint forces, which keep the front wheel from slipping sideways and which increase with speed, act to restore the steer to zero when the trail of the front wheel point of contact behind the steering axis is positive. This effect is known as caster action. An illustration of this is a trailer falling in behind a car after the car has changed direction.

The instability that arises when the real part of the complex conjugates goes positive is much different. Here the bicycle over-reacts to a fall. When the lean increases, the resultant turn is such that it not only restores the lean, but causes a fall in the other direction which is larger than the original fall. The oscillation blows up instead of damping out. As can be seen from graphs 2,3, and 4, the magnitude of this real part depends largely upon the phase difference between the lean and the steer for this root. When the lean is too far ahead of the steer the root goes unstable. This makes sense because the lean will be experiencing its maximum restoring force (the steer is at its maximum) past the time of its maximum displacement. Also, the lower the ratio of the magnitude of steer to the magnitude of lean, the lower the phase lead of lean over steer that is still stable. Although only shown for these three cases, this relationship between the real part of the complex conjugates and the phase difference between ψ and θ for this root was found to be the same for all variations of design parameters and for variation of the steady turn.

The way in which the various parameters affected the stability are as follows:

1. The distance of the center of mass of the front wheel-fork system from the steering axis, F_{o2}, was very important to the stability (Graph 5). If it was too close to the axis or behind it, then the steering moment due to gravity was too small and the bike fell over (meaning the real root was positive). This low steering moment also caused a large enough phase lead of the lean for the bicycle to exhibit oscillatory instability in this region. If the center of mass was too far ahead of the steering axis, then the moment of inertia of the front wheel-fork system was too large and the bike developed sufficient phase lead of lean over steer to go unstable in the real part of the complex conjugates. This is exactly the same as what happens when the moment of inertia of the fork about the axis parallel to the steering axis and passing through its center of mass is increased beyond a certain point or when the mass of this system is increased beyond a certain point. In all three cases the increased moment of inertia about the steering axis causes a decreased frequency of vibration of the solution corresponding to the complex root.

2. When the trail of the front wheel point of contact behind the steering axis is too small (Graph 6) there is only a small gravitational moment causing the steer to respond in the direction of the lean. This causes the real root to be unstable in this region. The complex root is also unstable here because the restoring moment on the steer due to the caster forces of constraint is very small. When the trail is increased it eventually reaches a point where the torque due to the caster forces acts to delay the phase of the steer enough to cause instability of the complex root as was previously described. The frequency of the complex root decreases with increased trail because the magnitude of the reaction of the steer to the lean is decreasing thus decreasing the restoring forces.

3. The moment of inertia of the front wheel about its axle proved to be of great importance to the stability. If it was too low, the steer was too far behind the lean in phase and the solution corresponding to the complex root became unstable (Graph 7). This is due to the fact that the response of the steer is caused in part by the torque of gravity acting upon the angular momentum of the front wheel. For reasons which are not clear, the bike exhibits instability in the real root when the moment of inertia of the front wheel is increased beyond a certain point.

4. The same lack of sufficient steering response due to low angular momentum of the front wheel is exhibited at low speeds of the bicycle causing instability in the complex root (Graph 8).

5. The dependence of the stability upon the height of the frame center of mass seems perfectly reasonable as regards the real root. As the height increases, the real root decreases until the bike goes unstable in the falling-over mode (Graph 9). The reason for its going unstable in the oscillatory mode in the small region between 0.4 and 0.7 meters, on the other hand, is not at all clear.

6. For too small a wheel base, the bike exhibited instability in the real root or falling-over mode (Graph 10), whereas it goes unstable in the oscillatory mode at large values of this parameter. The reason for each of these responses are at best unclear.

7. Rake angle had a great effect upon the stability (Graph 11). Steeper designs were more stable in every sense excepting the frequency of vibration corresponding to the complex root. This effect is due to the fact that for a given steer, the actual turn is sharper (of smaller radius) for the bike design with the steeper rake angle. The restoring centrifugal forces are thus greater for the bike with the steeper steering axis.

8. A rather surprising result is that for values of the frame mass within a well-defined region the bike

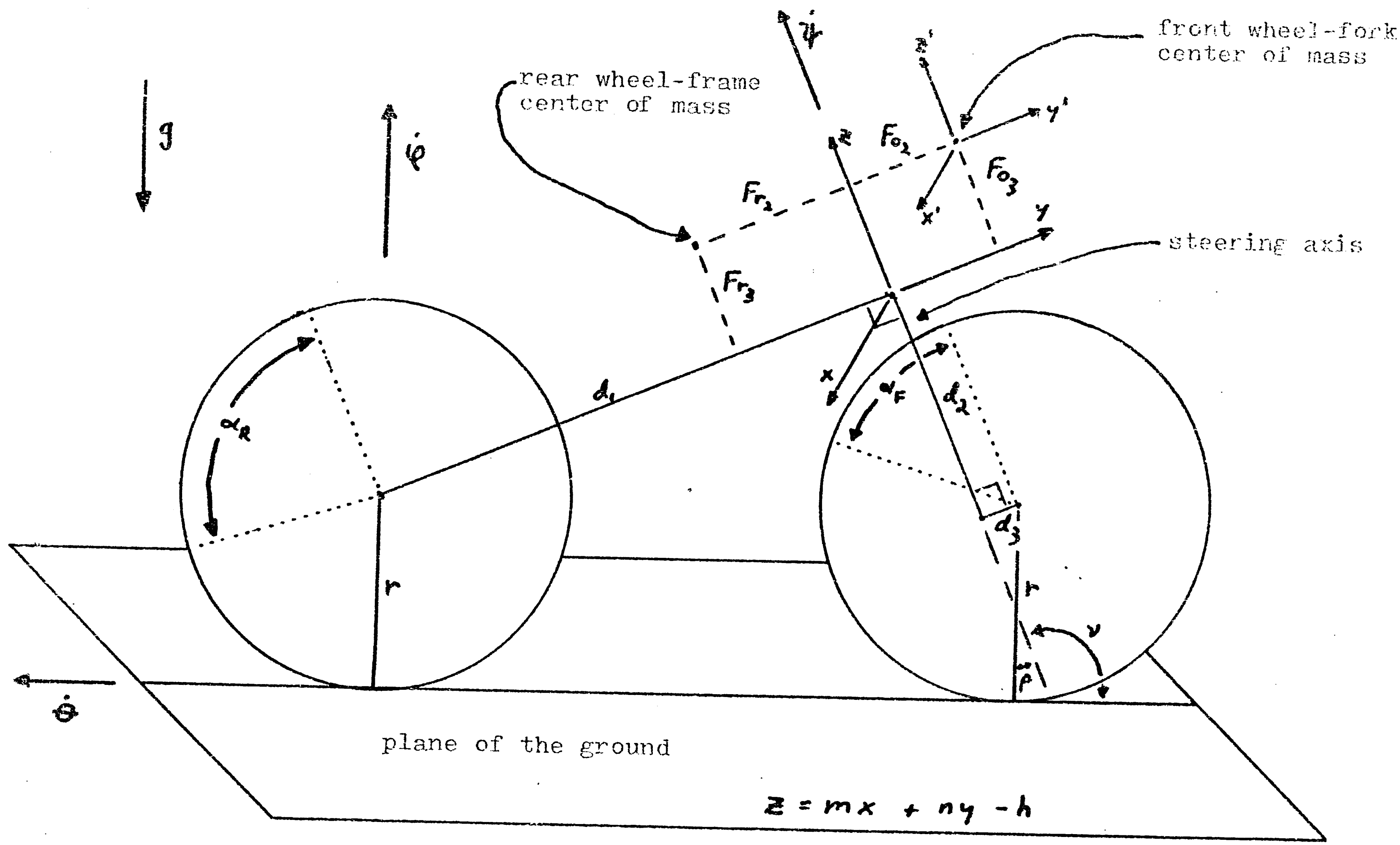
is unstable in the real root (Graph 12). No explanation of this phenomenon will be attempted here.

9. A study of the equilibrium turns for the above mentioned bicycle design revealed that the equilibrium steer varies almost linearly with the equilibrium lean over the region of stability (Graph 13). The speed, $\dot{\alpha}_{R0}$, decreases only slightly with increased lean. There is only a certain amount of lean which is allowable before the turn goes unstable in the oscillatory mode (Graph 14). At even more drastic leans, not only does stability disappear, but there is also no equilibrium.

In the majority of the cases studied, the complex root would go unstable while its frequency was low. This suggests a correspondence between larger restoring forces and stability, but the fact that this root also goes unstable at higher frequencies suggests the presence of other important effects such as phase difference between the lean and the steer.

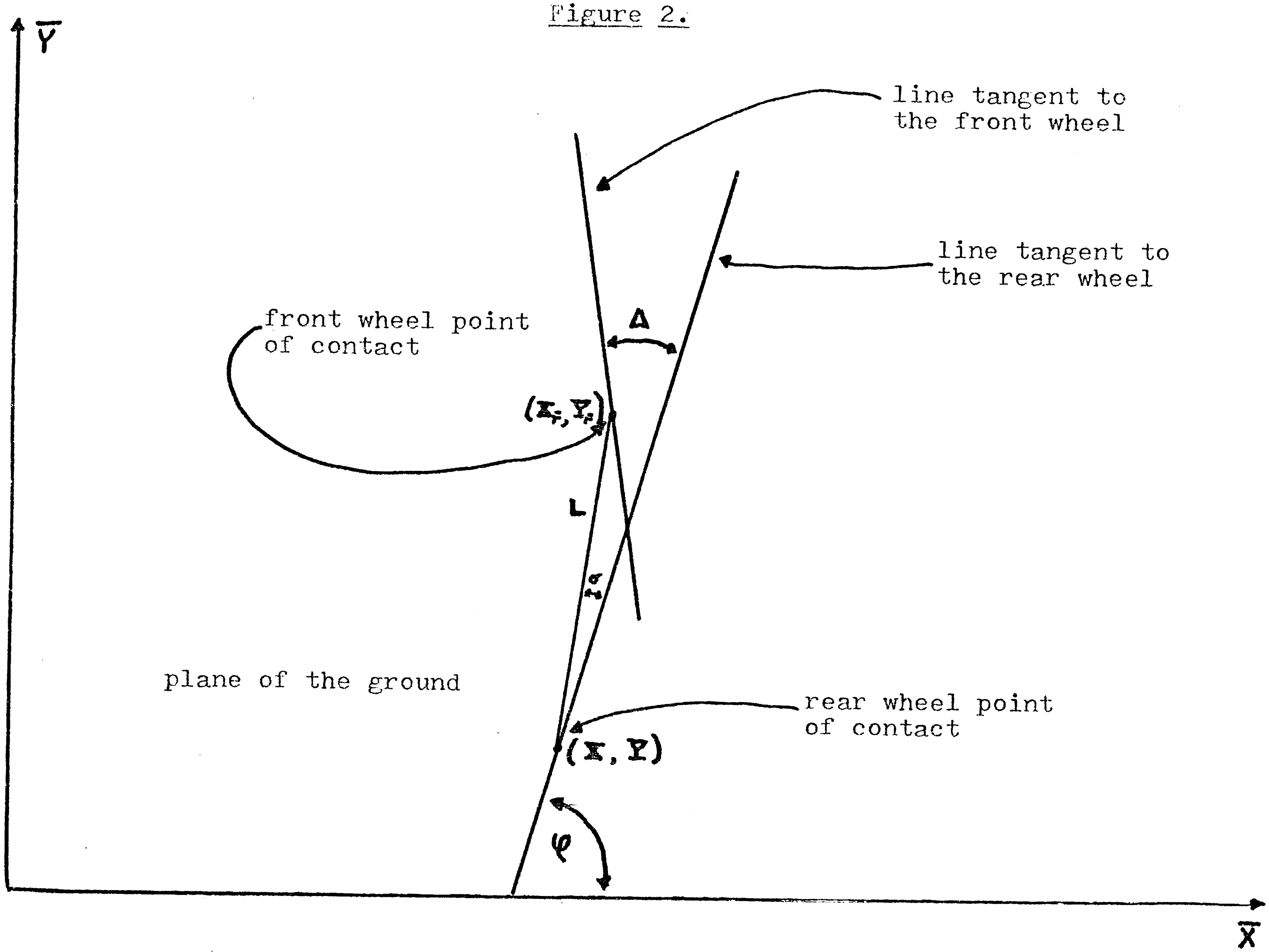
The instability of the bike with respect to the real root in regions which are known to represent ridable configurations, i.e. at higher speeds, suggests the relative unimportance of this root to considerations of rideability. This root is generally small and proper inputs of the rider should be enough to overcome its effects.

Figure 1.



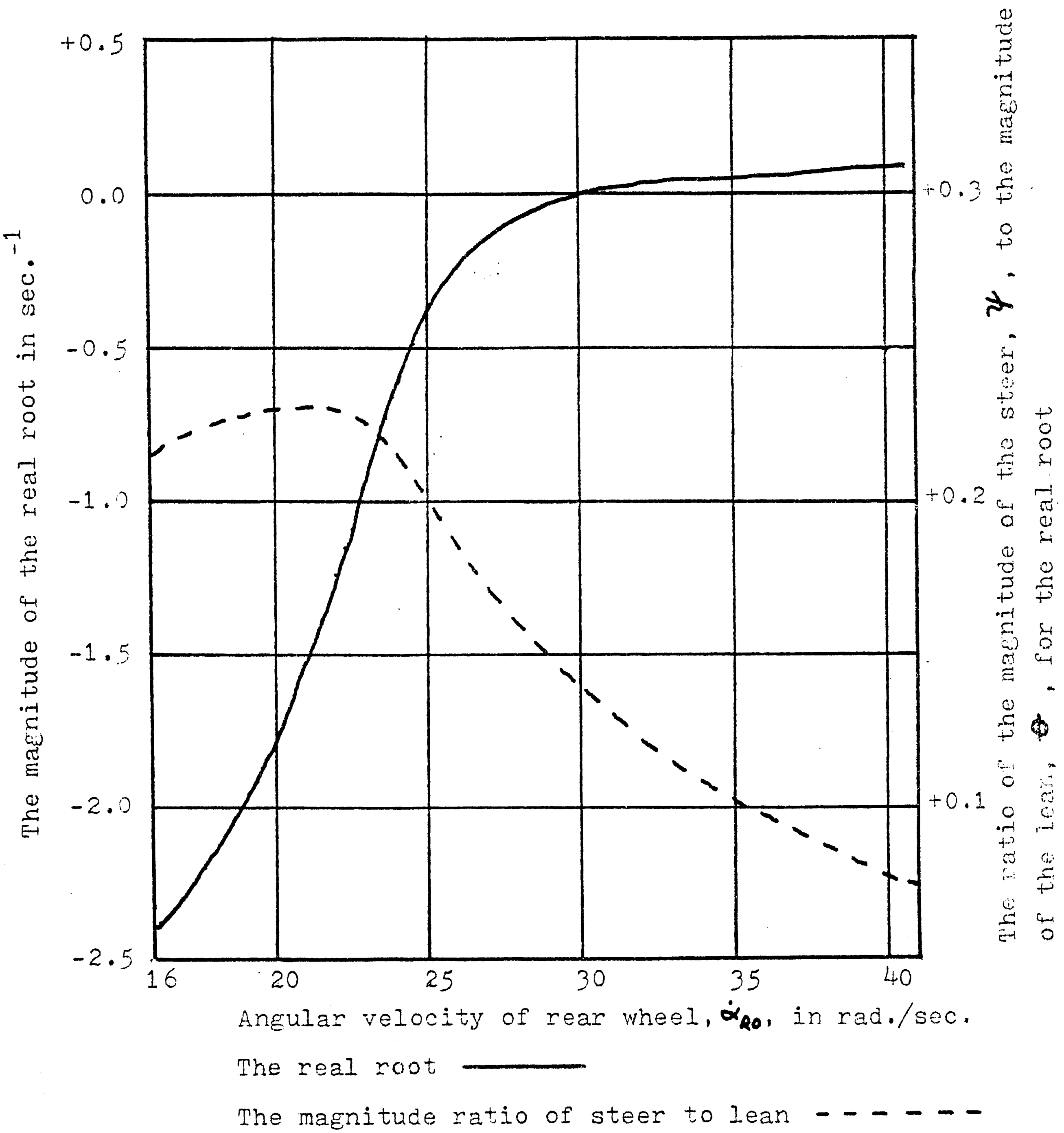
rear wheel-frame center of mass
front wheel-fork center of mass
steering axis
plane of the ground
z = mx + ny - h

Figure 2.

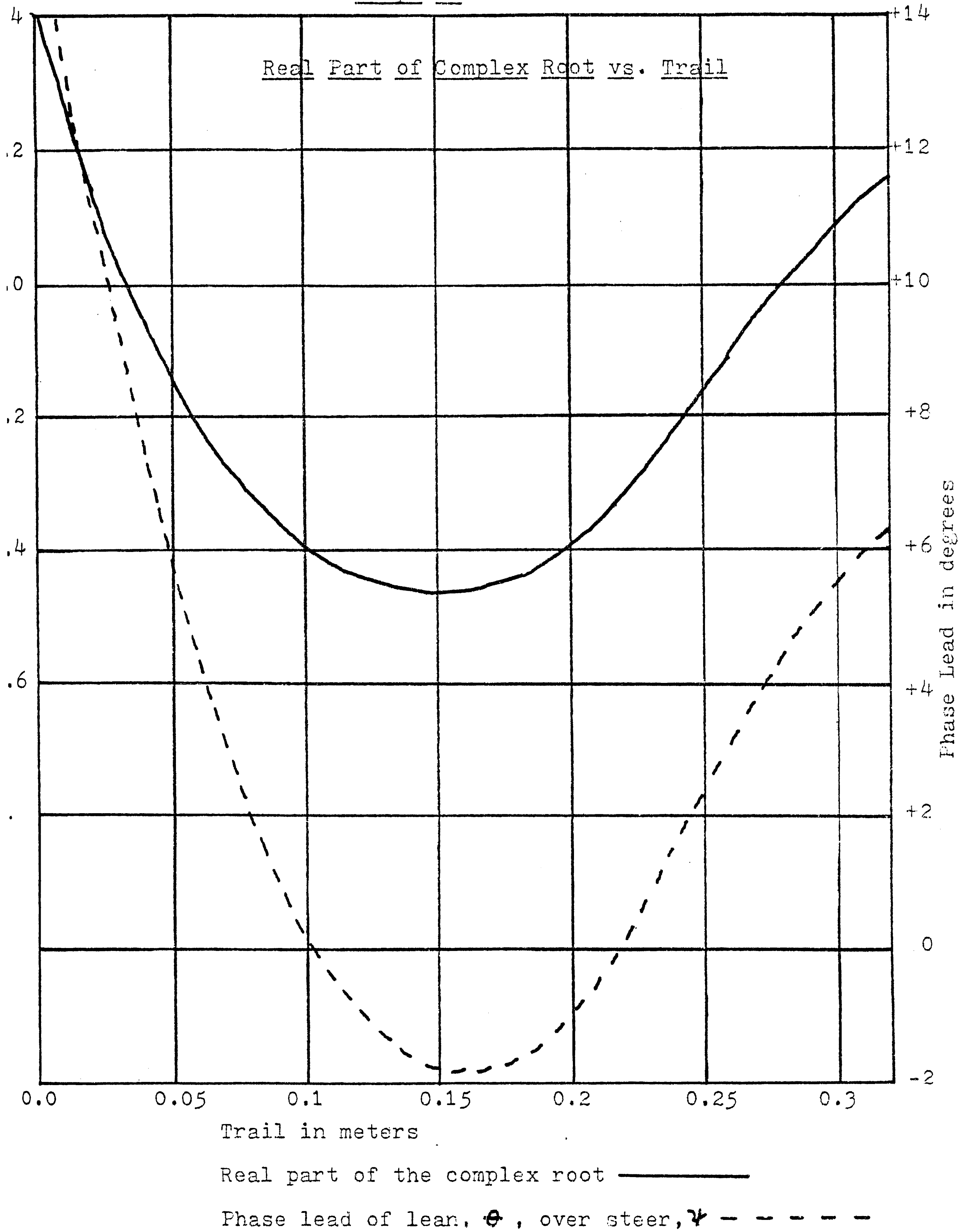

$\overline{Y}$
line tangent to
the front wheel
line tangent to
the rear wheel
Δ
front wheel point
of contact
(X_f, Y_f)
L
σ
plane of the ground
rear wheel point
of contact
(X, Y)
φ
$\overline{X}$

Graph 1.

The Real Root vs. Bicycle Speed



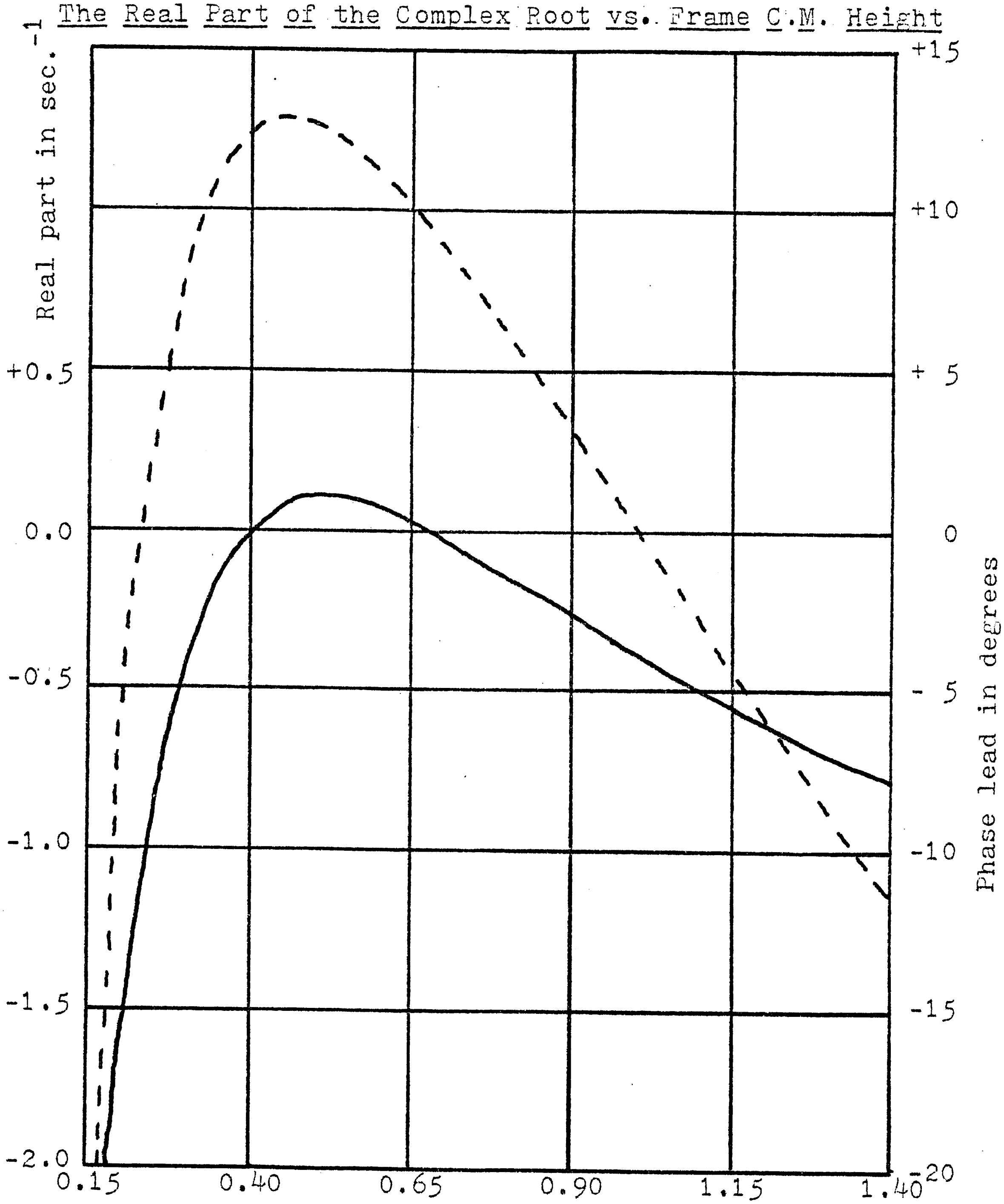
Graph 2.



Real Part of Complex Root vs. Trail
4
.2
.0
.2
.4
.6
+14
+12
+10
+8
+6
+4
+2
0
-2
Phase Lead in degrees
0.0
0.05
0.1
0.15
0.2
0.25
0.3
Trail in meters
Real part of the complex root
Phase lead of lean, θ, over steer, ψ

Graph 3.

The Real Part of the Complex Root vs. Frame C.M. Height



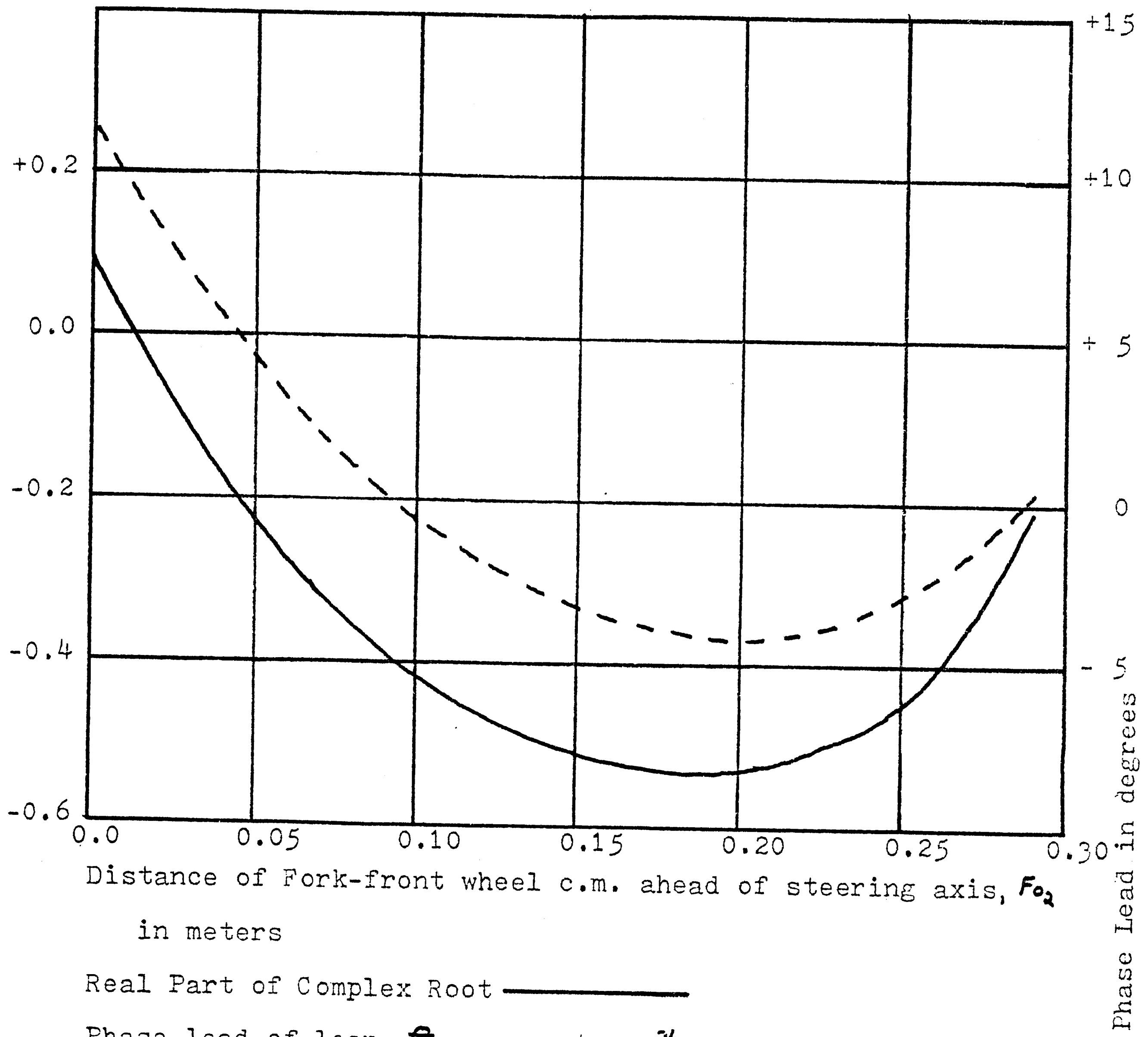
Frame center of mass height in meters

Real Part of Complex Root ————

Phase lead of lean, θ, over steer, ψ - - - - -

Graph 4.

Real Part of the Complex Root vs. Offset of Fork C.M. Ahead of Steering Axis



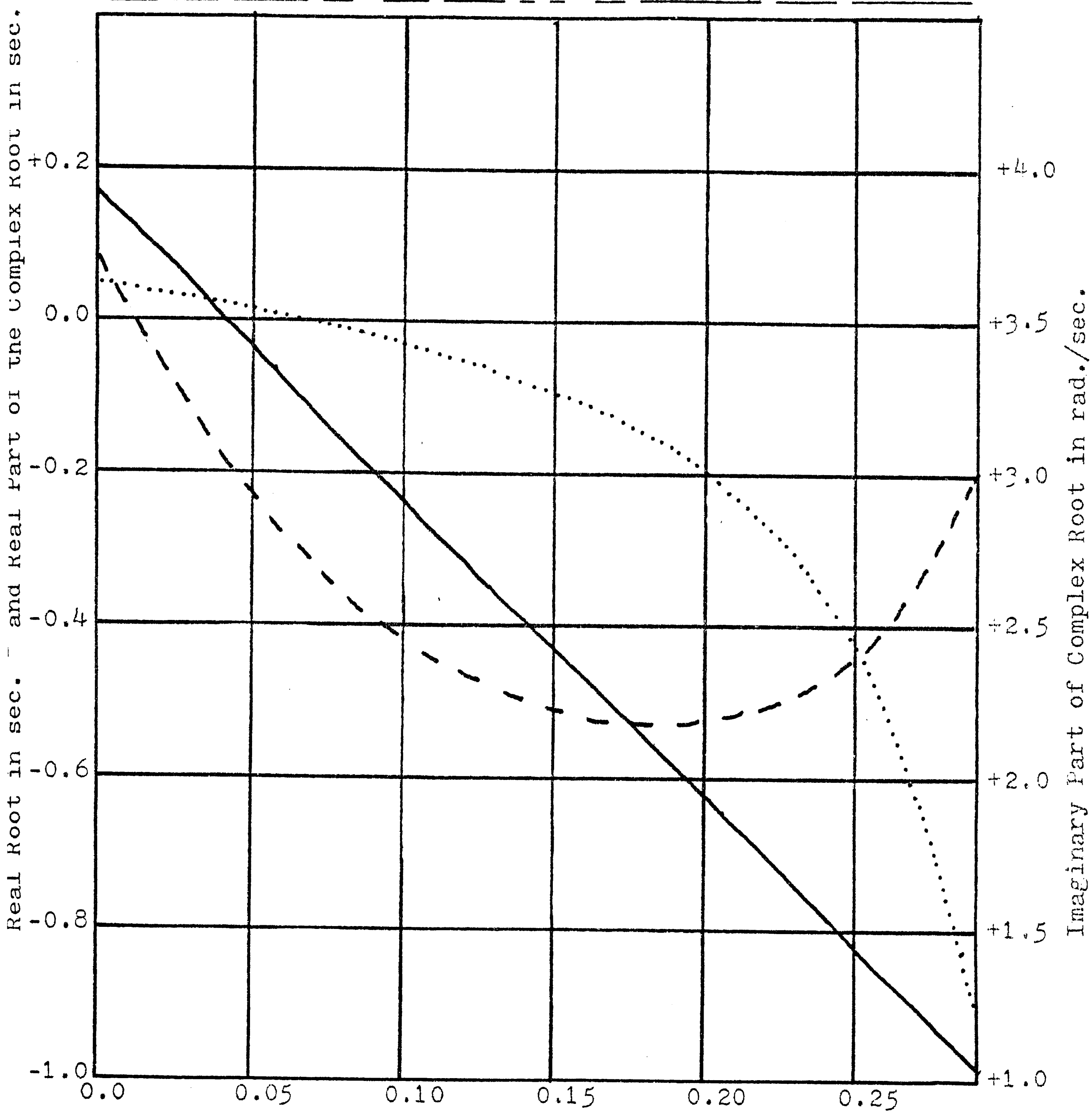
Distance of Fork-front wheel c.m. ahead of steering axis, F_{o_2} in meters

Real Part of Complex Root ————

Phase lead of lean, θ, over steer, ψ — — — — —

Graph 5.

The Two Roots vs. the Fork C.M. to Steering Axis Distance



Distance from fork-front wheel c.m. to steering axis, F_{o_2}

in meters

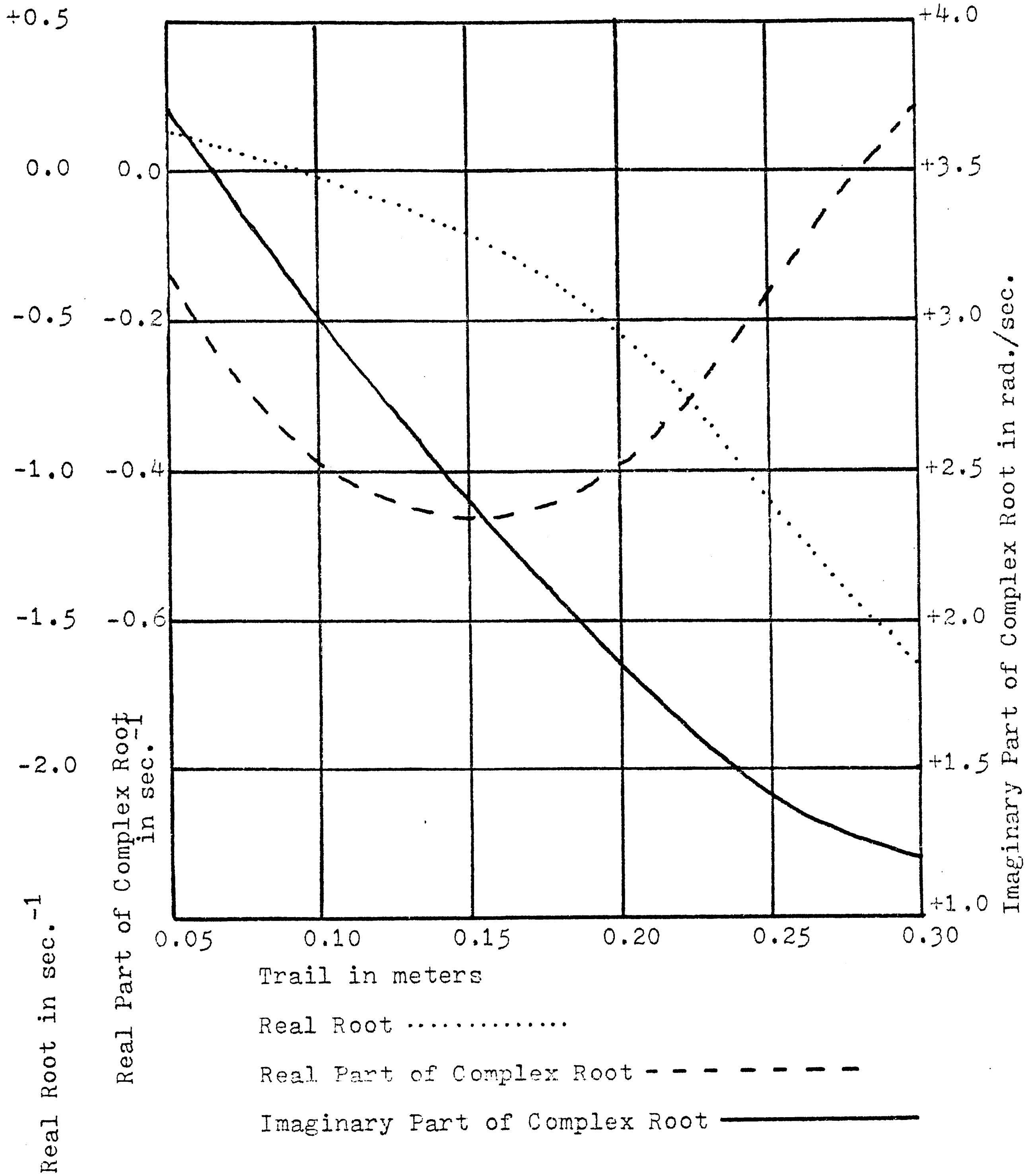
Real Root ....

Real Part of Complex Root - - - - -

Imaginary Part of Complex Root ————

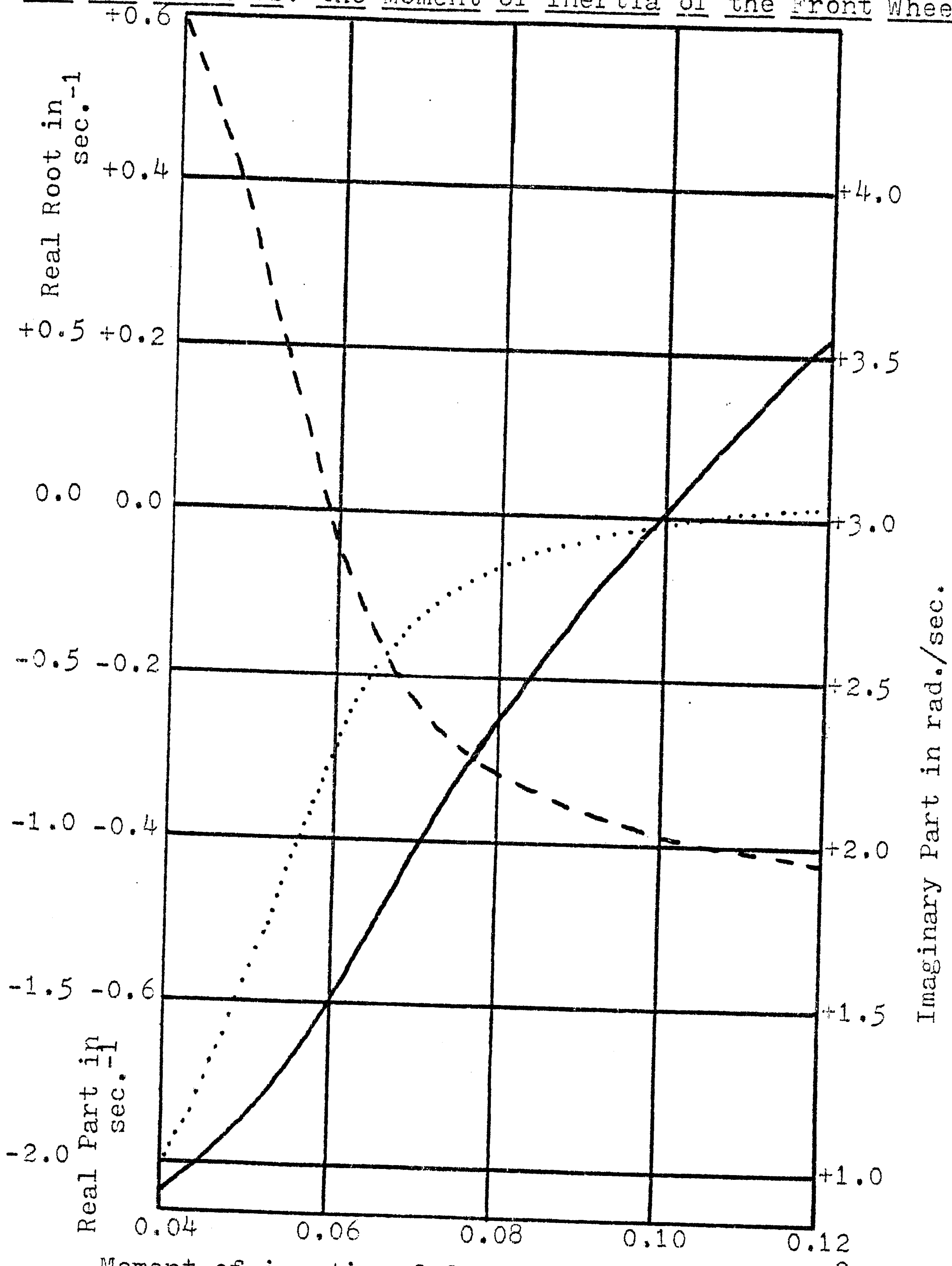
Graph 6.

The Two Roots vs. the Trail of the Front Wheel Contact Point



Graph 7.

The Two Roots vs. the Moment of Inertia of the Front Wheel



Moment of inertia of front wheel in kg.-meter^2

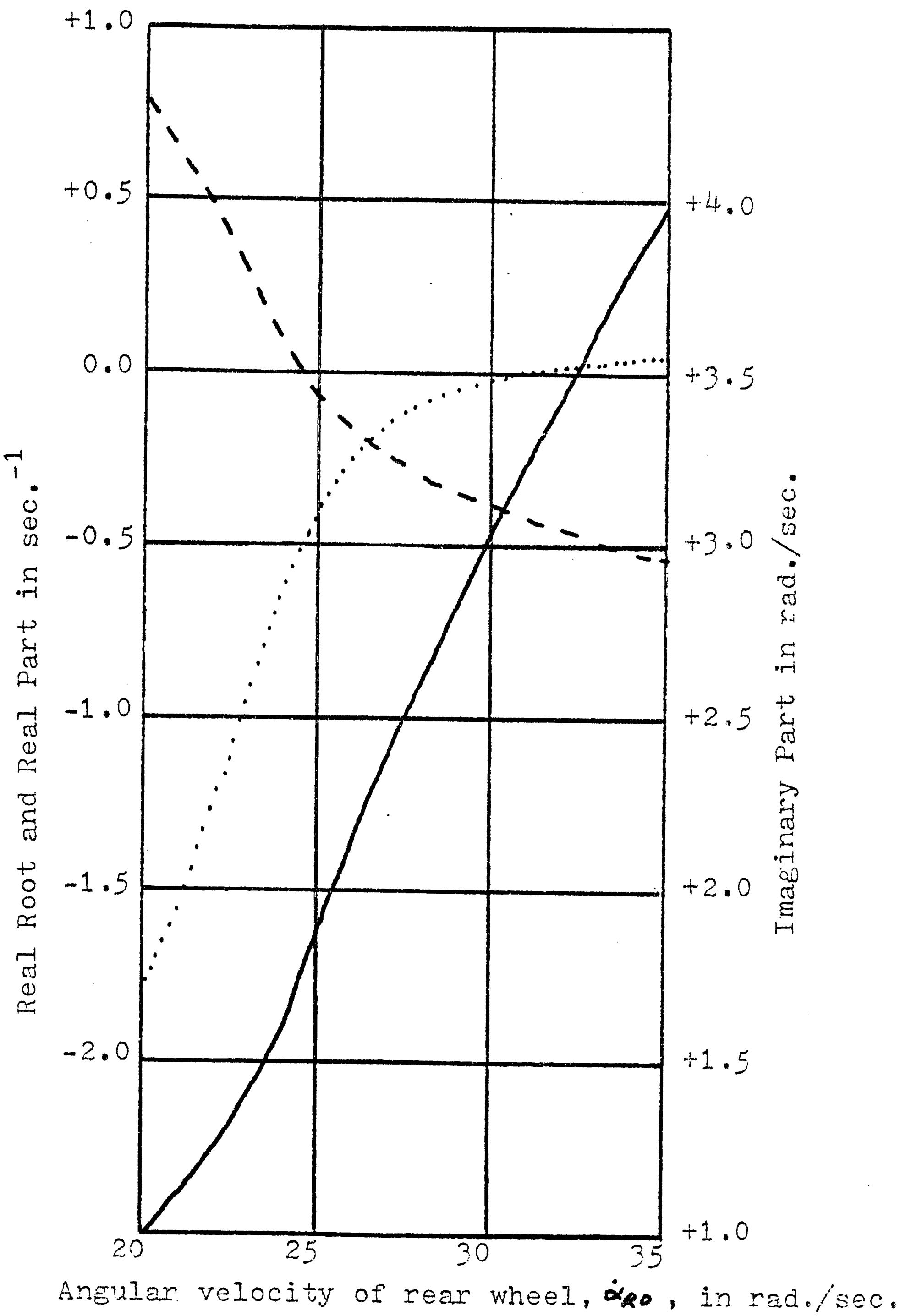
Real Root ....

Real Part of Complex Root - - - - - - -

Imaginary Part of Complex Root ————————

Graph 8.

The Two Roots vs. Bicycle Speed



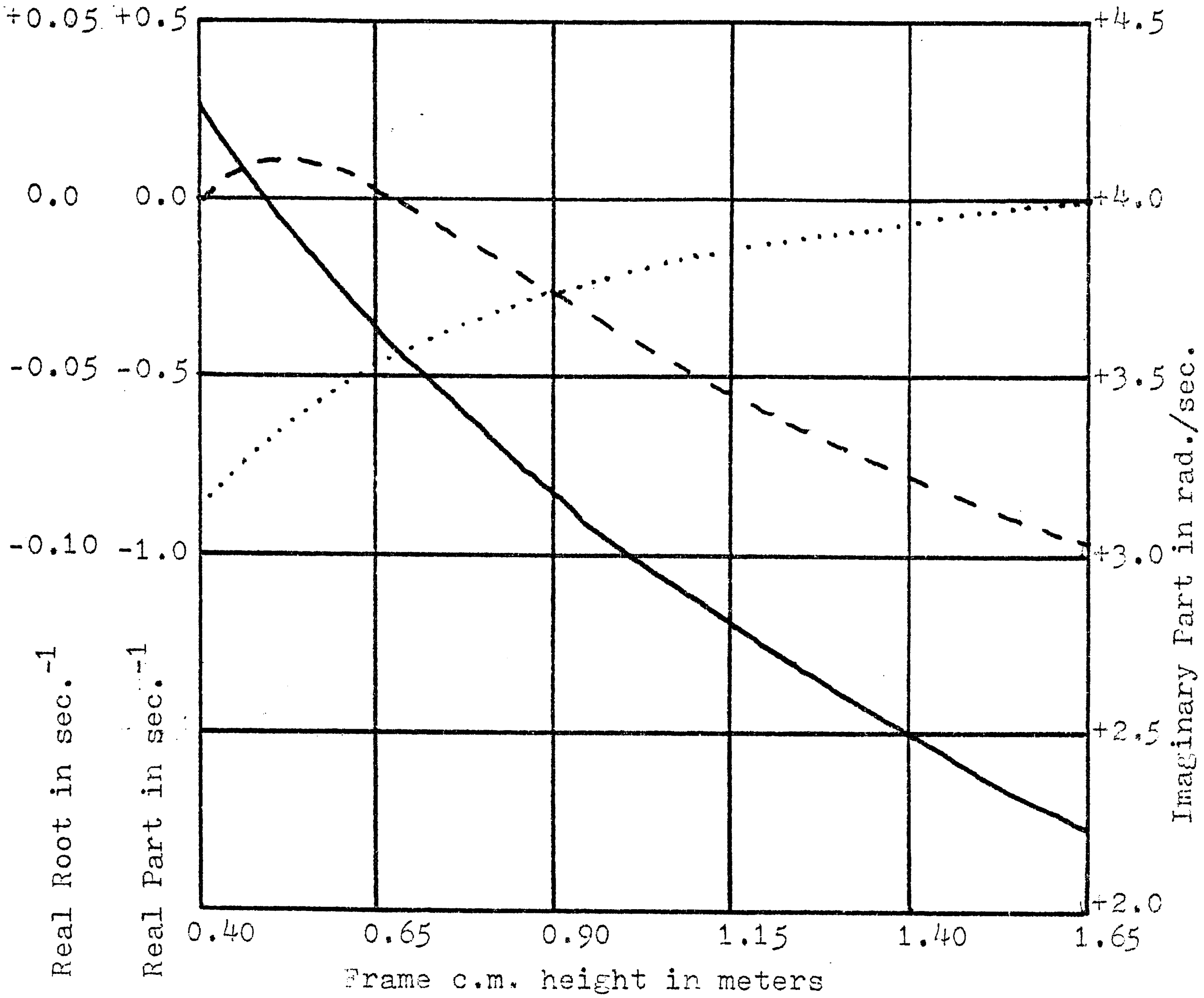
Real Root . . . .

Real Part of Complex Root – – – – – –

Imaginary Part of Complex Root ————————

Graph 9.

The Two Roots vs. Frame C.M. Height



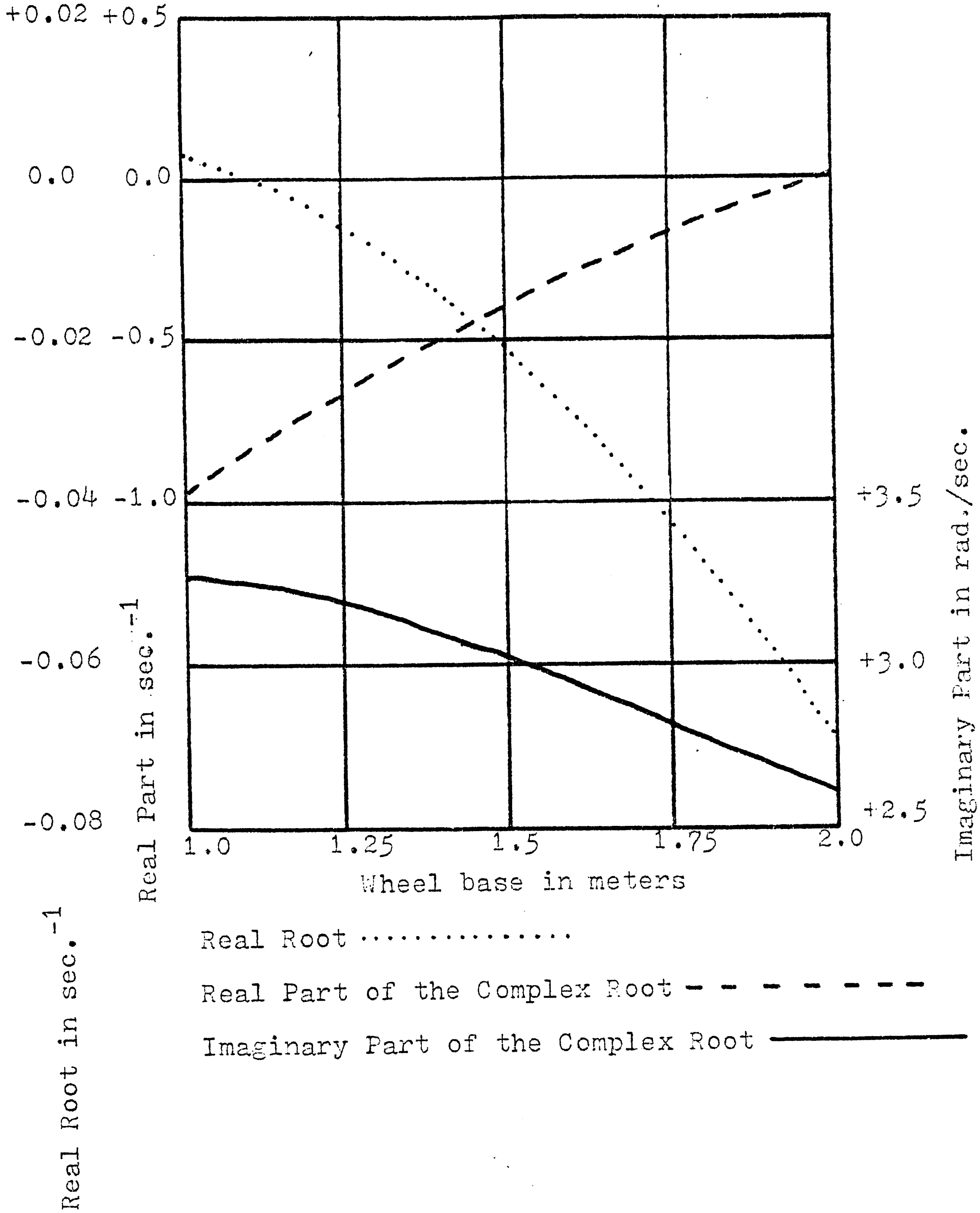
Real Root ....

Real Part of Complex Root – – – – – – –

Imaginary Part of Complex Root ————————

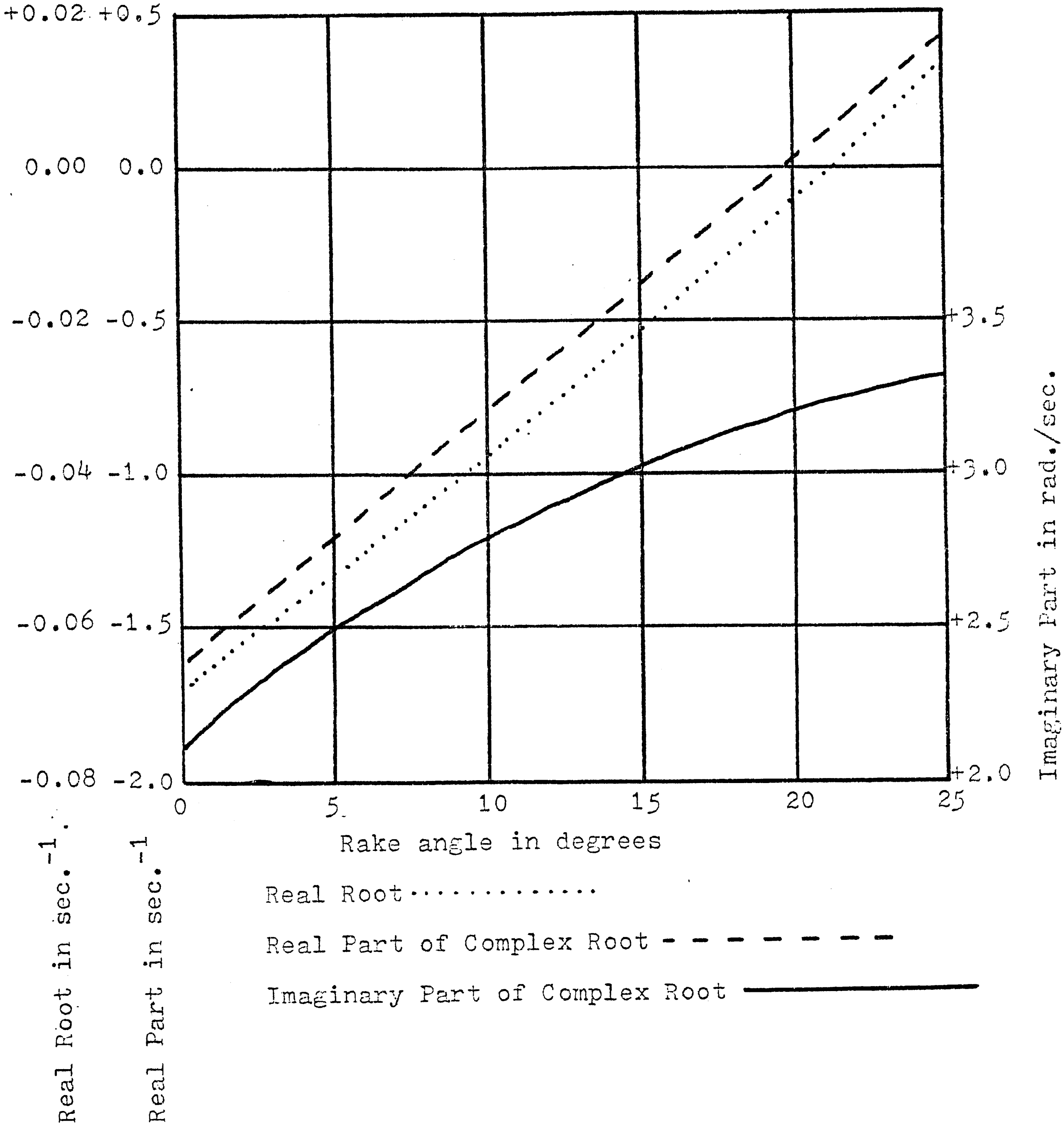
Graph 10.

The Two Roots vs. the Wheel Base



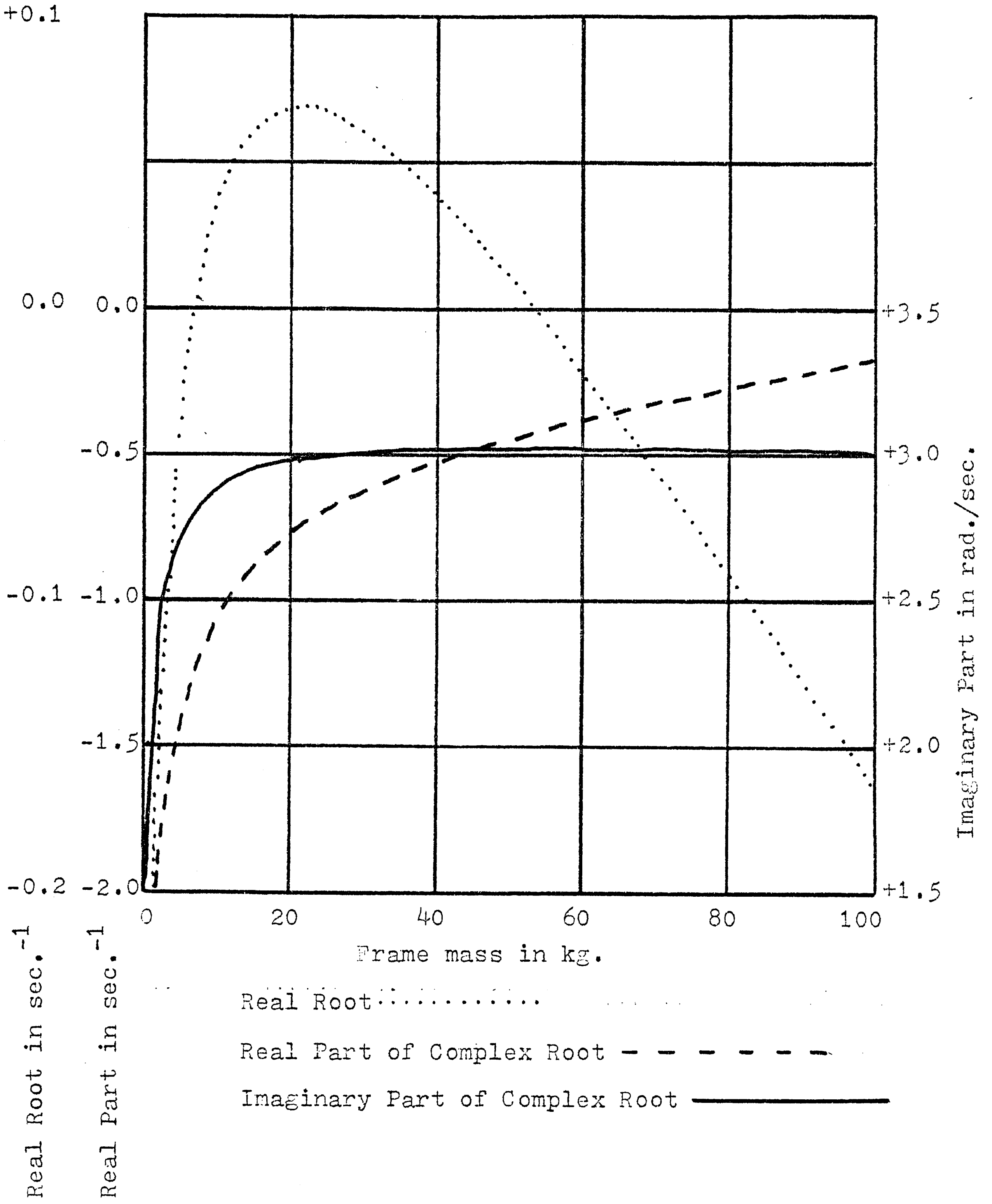
Graph 11.

The Two Roots vs. Rake Angle



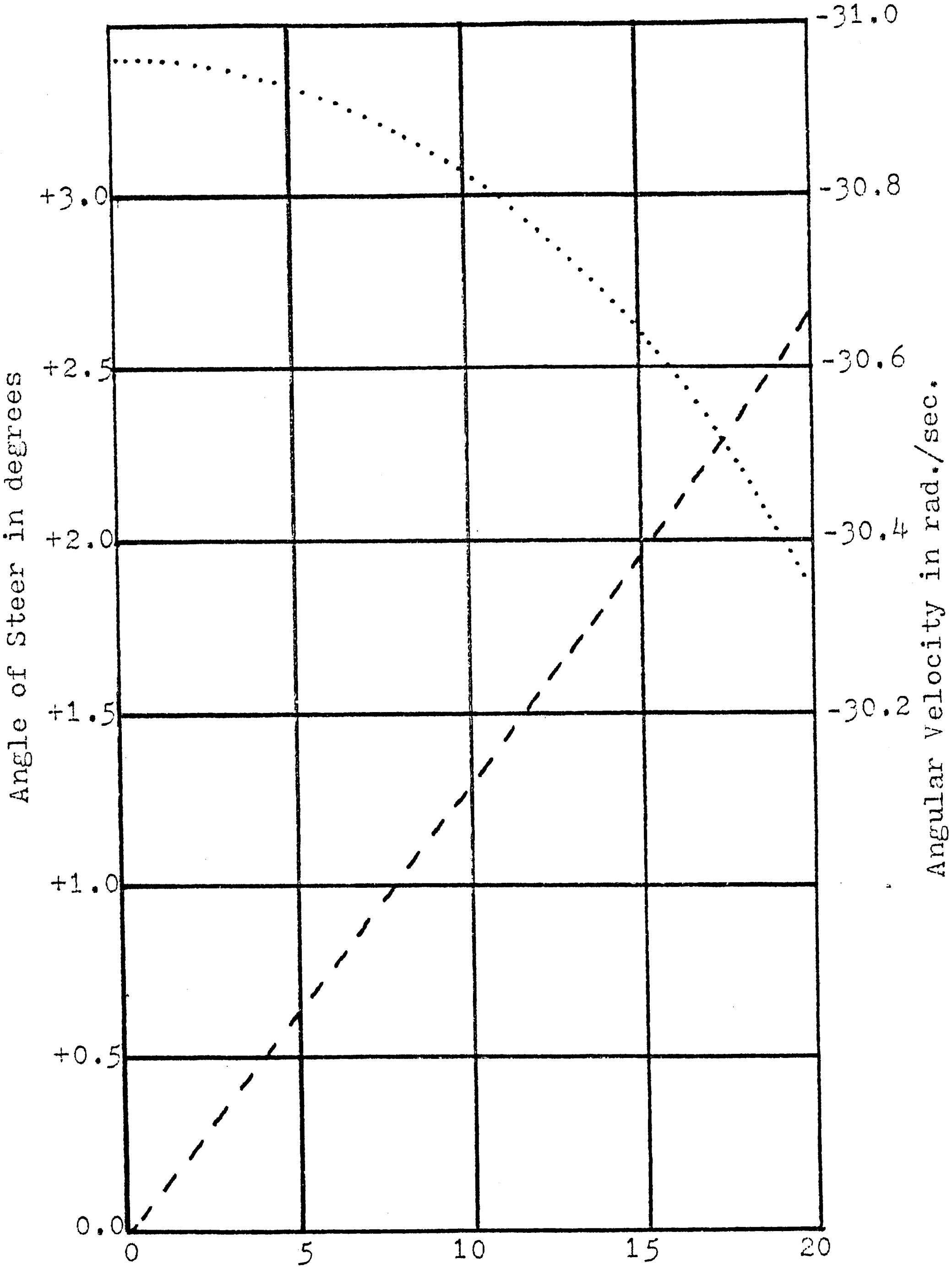
Graph 12.

The Two Roots vs. the Frame Mass



Graph 13.

Equilibrium Steer and Speed vs. Equilibrium Lean



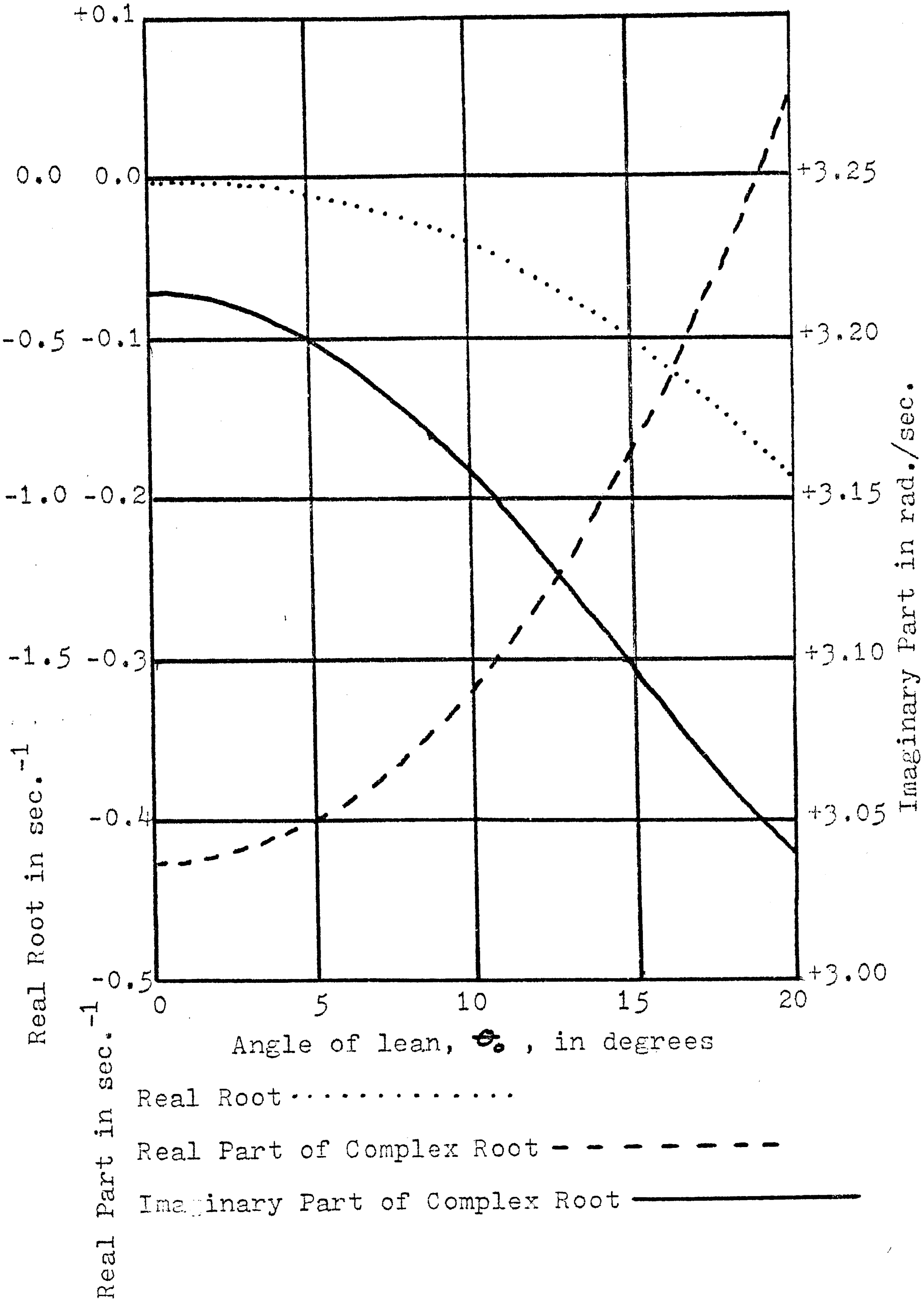
Angle of lean, θ_0, in degrees

Angle of steer, ψ_0, - - - - - - - -

Angular velocity of rear wheel, $\dot{\alpha}_{R0}$, ...

Graph 14.

The Two Roots vs. the Angle of Equilibrium Lean, θ_0



FOOTNOTES

1. Timoshenko, S. and Young, D.H., *Advanced Dynamics*, (New York: McGraw-Hill, 1948), p. 239.

2. Collins, Robert Niel, " A Mathematical Analysis of the Stability of Two Wheeled Vehicles," Ph.D. thesis, University of Wisconsin, 1963.

3. Singh, Digvijai," Advanced Concepts of the Stability of Two Wheeled Vehicles, Application of Mathematical Analysis to Actual Vehicles," Ph.D. thesis, University of Wisconsin, 1964.

4. Jones, D.E.H., " The Stability of the Bicycle," Physics Today, April 1970, pp. 34-40.

5. Note that $A_{bij} = A_{bji}$

6. Note that $\beta_{bij} = \beta_{bji}$

7. A similar development of the general Lagrange-undetermined multipliers solution to a problem with similar non-holonomic constraints can be found in: Whittaker, E.T., *Analytical Dynamics*, (New York: MacMillan Co., 1904), pp. 210-211.

8. Note that $\varphi_{ij} \neq \varphi_{ji}$, $\gamma_{ij} \neq \gamma_{ji}$, $\alpha_{Fij} \neq \alpha_{Fji}$

ADDITIONAL REFERENCE

Whitt, F.R., Wilson, D.G., *Bicycling Science*, (Cambridge, Mass.: The MIT Press, 1974), pp. 171-184.

Appendix

Listed in this appendix are the main program for the stability in a straight line, a subroutine for the solution of a general quartic equation with real coefficients, and the main program for the determination of equilibrium turning configurations and their stability. Included after the main for turns are the subroutines used by this main. A glossary of the terminology used in the straight-ahead program is as follows:

Program symbol	Explanation or equivalent symbol from analysis
M(b)	m_b
IijFR	I_{FRij}
IijFO	I_{Foij}
I11FW	I_{FW22}
I33FW	I_{FW11}
I11BW	I_{BW22}
I33BW	I_{BW11}
I(b,i,j)	I_{bij}
ALPHAO	$\dot{\alpha}_{Ro}$
BASE	The wheel base
R	r
HRCM	Height of the frame center of mass
DCMW	Horizontal distance between frame c.m. and center of rear wheel
F02	F_{o2}
F03	F_{o3}
ALP	The rake angle in degrees
D1	d_1

Symbol	Explanation
D2	d_2
D3	d_3
N	n
GAM	γ
GAMN	γ_n
H	h
DDN(i,j)	The second partial derivative of h with q_i and q_j
DDGAM(i,j)	The second partial derivative of γ
DDGAMN(i,j)	The second partial derivative of γ_n
DDH(i,j)	The second partial derivative of h
YF	y_F
L	L
PHI(i)	φ_i
ALPHA(i)	α_{Fi}
Z(i)	γ_i
DPHI(i,j)	φ_{ij}
FR2	F_{r_2}
FR3	F_{r_3}
A(b)	A_b
B(b)	B_b
DA(b,i)	$\partial A_b / \partial q_i$
DB(b,i)	$\partial B_b / \partial q_i$
DDC(b,i,j)	$\partial^2 C_b / \partial q_i \partial q_j$
W(b,i,j)	ω_{bij}
DW(b,i,j,k)	$\partial \omega_{bij} / \partial q_k$

Symbol	Explanation
E(i,j)	E_{ij}
DE(i,j,k)	$\partial E_{ij} / \partial q_k$
DDV(j,k)	$\partial^2 V / \partial q_j \partial q_k$
DVT44(j,k)	$\partial u_{j44} / \partial q_k$
VT4(j,k)	u_{jk4}
T(j,k)	T_{jk}
P2(j,k)	P_{2jk}
P1(j,k)	P_{1jk}
PO(j,k)	P_{0jk}
Qi	Coefficient of the i th degree term in the characteristic quartic
AR(j,k)	Term used in calculating the phase difference and magnitude ratio of the steer to the lean for the j th root
BR(j,k)	Another term in the phase and magnitude calculation for the j th root
PHASE(j)	The phase lead of θ over ψ in radians for the j th root
ANG(j)	PHASE(j) expressed in degrees
RATi(j)	Terms used in determining the magnitude ratio of steer to lean for the j th root

Many of the symbols in the analysis have no equivalent in the program for straight line motion because their values are zero in this case.

```
 MAIN PROGRAM FOR SOLVING THE CHARACTERISTIC POLYNOMIAL FOR THE
 LINEARIZED EQUATIONS OF MOTION OF A BICYCLE TRAVELING IN A
 STRAIGHT LINE
  IMPLICIT REAL*8 (A,B,C,D,E,F,G,H,I,L,M,N,P,O,R,S,T,V,W,X,Y,Z)
  DIMENSION XREAL(4), XIMAG(4)
  DIMENSION DDN(2,2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDLAM(1,2)
 1    , DDYF(1,2), DDLCSS(1,2), DDL(1,2), DDROH(1,2), PHI(4),
 2    ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4), F(2,3), A(2),
 3    BC(2), B(2), DF(2,2,2), DA(2,4), DB(2,4), DC(2,3), DDF(2,2,2,2)
 4    , DDBC(2,2,2), DDC(2,3,2), W(2,3,3), DW(2,3,3,2), E(5,5),
 5    DE(5,5,4), DDV(2,2), DVT44(2,2), VT4(4,4), T(4,4), P2(2,2),
 6    P1(2,2),P0(2,2)
  DIMENSION AR(4,2), BR(4,2), PHASE(4), RAT1(4), RAT2(4),ANG(4)
  DIMENSION I(2,3,3), M(2)
  M(1) = 60.0
  M(2) = 4.0
  I11FR = 15.0
  I22FR = 5.0
  I33FR = 5.0
  I23FR = 0.15
  I11FO = 1.0
  I22FO = 0.5
  I33FO = 0.5
  I23FO = -0.04
  I11FW = 0.05
  I33FW = 0.1
  I11BW = 0.05
  I33BW = 0.1
  DO 4000  J = 1,2
     DO 3990  J1 = 1,3
        DO 3980  J2 = 1,3
           I(J,J1,J2) = 0.0
80      CONTINUE
90     CONTINUE
00  CONTINUE
  I(1,1,1) = I11FR +I33BW
  I(1,2,2) = I22FR +I11BW
  I(1,3,3) = I33FR +I11BW
  I(1,2,3) = I23FR
  I(1,3,2) = I23FR
  I(2,1,1) = I11FO +I33FW
  I(2,2,2) = I22FO +I11FW
  I(2,3,3) = I33FO +I11FW
  I(2,2,3) = I23FO
  I(2,3,2) = I23FO
  ALPHA0 = -30.0
  BASE = 1.5
  R = 0.35
  HRCM = 1.0
  DCMW = 0.3
  CAST = -0.10
  FO2 = 0.09
  FO3 = -0.10
  ALP = 15.0000000
  COSALP = DCOS( ALP*3.141592/180.0 )
  SINALP = DSQRT( 1.000000000 -COSALP**2 )
  D2 = SINALP*BASE
  D3 = CAST*COSALP +R*SINALP
  D1 = COSALP*BASE -D3
```

```
      N = -D2/( D1 +D3 )
      GAM = 1.0/DSQRT( N**2 +1.0 )
      GAMN = GAM*N
      H = -N*D1 +R/GAM
      DDN(1,1) = ( R*N**2*GAM -D2 -N*D1 )/( D1 +D3 )
      DDN(1,2) = -( D3/GAM +R*N )/( D1 +D3 )
      DDN(2,1) = DDN(1,2)
      DDN(2,2) = 0.0
      DO 600  J = 1,2
         DO 590   J1 = 1,2
            DDGAM(J,J1) = -N*GAM**3*DDN(J,J1)
            DDGAMN(J,J1) = GAM**3*DDN(J,J1)
            DDH(J,J1) = DDN(J,J1)*( R*GAMN -D1 )
590      CONTINUE
600   CONTINUE
      YF = ( N*( H -D2 ) +D3 )/( N**2 +1.0 )
      L = DSQRT( D2**2 +( D1 +D3 )**2 )
      DO  650  J = 1,4
         PHI(J) = 0.0
         ALPHA(J) = 0.0
         Z(J) = 0.0
         DO 640  K = 1,4
            DPHI(J,K) = 0.0
640      CONTINUE
650   CONTINUE
      PHI(1) = -YF/L
      DPHI(4,1) = -R*GAM/L
      ALPHA(4) = 1.0
      Z(4) = -R
      FR2 = D1 -SINALP*( HRCM -R ) -COSALP*DCMW
      FR3 = COSALP*( HRCM -R ) -SINALP*DCMW
      A(1) = GAM*( D1 -FR2 ) +GAMN*FR3
      A(2) = GAM*( D1 +FO2 ) +GAMN*FO3
      B(1) = 0.0
      B(2) = 0.0
      DA(1,1) = 0.0
      DA(1,2) = 0.0
      DA(2,1) = 0.0
      DA(2,2) = 0.0
      DB(1,1) = 0.0
      DB(1,2) = -GAM*( H +FR3 ) -GAMN*FR2
      DB(2,1) = -FO2
      DB(2,2) = -GAM*( H +FO3 ) +GAMN*FO2
      DDC(1,1,1) = DDGAM(1,1)*( H +FR3 ) +GAM*DDH(1,1) +DDGAMN(1,1)*FR2
      DDC(1,1,2) = DDGAM(1,2)*( H +FR3 ) +GAM*DDH(1,2) +DDGAMN(1,2)*FR2
      DDC(1,2,1) = DDC(1,1,2)
      DDC(1,2,2) = DB(1,2)
      DDC(2,1,1) = FO2*( GAMN -DDGAMN(1,1) ) +DDGAM(1,1)*( H +FO3 )
     1  +GAM*DDH(1,1)
      DDC(2,1,2) = -FO2*( 1.0 +DDGAMN(1,2) ) +DDGAM(1,2)*( H +FO3 )
     1  +GAM*DDH(1,2)
      DDC(2,2,1) = DDC(2,1,2)
      DDC(2,2,2) = DB(2,2)
      DO 1300 J = 1,3
         DO 1290  J1 = 1,3
            W(1,J,J1) = 0.0
            DO 1280  J2 = 1,2
               DW(1,J,J1,J2) = 0.0
1280        CONTINUE
1290     CONTINUE
```

```
1300    CONTINUE
        W(1,2,2)  = -GAM
        W(1,2,3)  = -GAMN
        W(1,3,2)  = -GAMN
        W(1,3,3)  = GAM
        DW(1,1,1,1)  = -GAM**2*DDN(1,1)
        DW(1,1,1,2)  = -GAM**2*DDN(1,2)
        DW(1,1,2,1)  = -GAM**2*DDN(2,1)
        DW(1,1,3,2)  = 1.0
        DO 1350  J = 1,3
           DO 1340  J1 = 1,3
              W(2,J,J1)  = W(1,J,J1)
1340       CONTINUE
1350    CONTINUE
        W(2,3,1)  = 1.0
        DO 1370  J = 1,2
           DIS1 = 0.0
           IF( J . EQ . 1 ) DIS1 = 1.0
           DO 1360  J1 = 1,3
              DW(2,1,J1,J)  = DW(1,1,J1,J)  +W(1,2,J1)*DIS1
              DW(2,2,J1,J)  = -W(1,1,J1)*DIS1  +DW(1,2,J1,J)
              DW(2,3,J1,J)  = DW(1,3,J1,J)
1360       CONTINUE
1370    CONTINUE
        DO 1450  J = 1,3
           E(J,4)  = I33BW*W(1,1,J)/2.0
           E(J,5)  = I33FW*W(2,1,J)/2.0
           E(4,J)  = E(J,4)
           E(5,J)  = E(J,5)
           DO 1440  K = 1,3
              E(J,K)  = 0.0
              DO 1430  J1 = 1,2
                 DO 1420  J2 = 1,3
                    DO 1410  J3 = 1,3
                       E(J,K)  = E(J,K)  +I(J1,J2,J3)*W(J1,J2,J)*W(J1,J3,K)
     1                  /2.0
1410                CONTINUE
1420             CONTINUE
1430          CONTINUE
1440       CONTINUE
1450    CONTINUE
        E(4,4)  = I33BW/2.0
        E(5,5)  = I33FW/2.0
        E(4,5)  = 0.0
        E(5,4)  = 0.0
        DO 1500 JD = 1,2
           DO 1490  J = 1,3
              DE(J,4,JD)  = I33BW*DW(1,1,J,JD)/2.0
              DE(J,5,JD)  = I33FW*DW(2,1,J,JD)/2.0
              DE(4,J,JD)  = DE(J,4,JD)
              DE(5,J,JD)  = DE(J,5,JD)
              DO 1480  K = 1,3
                 DE(J,K,JD)  = 0.0
                 DO 1470  J1 = 1,2
                    DO 1460  J2 = 1,3
                       DO 1455  J3 = 1,3
                          DE(J,K,JD)  = DE(J,K,JD)  +I(J1,J2,J3)*( DW(J1,
     1                     J2,J,JD)*W(J1,J3,K)  +W(J1,J2,J)*DW(J1,J3,K,
     2                     JD) )/2.0
1455                   CONTINUE
```

```
 1460               CONTINUE
 1470             CONTINUE
 1480           CONTINUE
 1490         CONTINUE
              DE(4,4,JD) = 0.0
              DE(4,5,JD) = 0.0
              DE(5,4,JD) = 0.0
              DE(5,5,JD) = 0.0
 1500       CONTINUE
            DO 2000  J = 1,5
              DO 1990  K = 1,5
                DE(J,K,4) = 0.0
 1990       CONTINUE
 2000       CONTINUE
            DO 1000  J = 1,2
              DO 990  K = 1,2
                DDV(J,K) = 0.0
                DVT44(J,K) = -2.0*( DE(3,4,J)*DPHI(4,K) +DE(3,5,J)*
     1            DPHI(4,K)*ALPHA(4) )
                DO 980  J1 = 1,2
                  DDV(J,K) = DDV(J,K) +9.80*M(J1)*DDC(J1,J,K)
                  DVT44(J,K) = DVT44(J,K) +M(J1)*( -Z(4)*DPHI(4,K)*
     1              ( DB(J1,J) -A(J1)*PHI(J) ) )
  980         CONTINUE
  990         CONTINUE
 1000       CONTINUE
            DO 1050  J = 1,4
              IF( J .EQ. 3 ) GO TO 1050
              DO 1040  K = 1,4
                IF( K .EQ. 3 ) GO TO 1040
                VT4(J,K) = DE(J,4,K) +E(J,3)*DPHI(4,K) +DE(J,5,K)*ALPHA(4)
     1            -( DE(K,4,J) +DE(4,3,J)*PHI(K) +DE(K,5,J)*ALPHA(4)
     2            +DE(3,5,J)*PHI(K)*ALPHA(4) ) +PHI(J)*( E(3,3)*DPHI(4,K)
     3            +DE(4,3,K) +DE(3,5,K)*ALPHA(4) ) +ALPHA(J)*E(3,5)*
     4            DPHI(4,K)
                T(J,K) = 2.0*( E(J,K) +E(J,3)*PHI(K) +E(K,3)*PHI(J) +E(J,5)*
     1            ALPHA(K) +E(K,5)*ALPHA(J) +E(3,5)*( PHI(J)*ALPHA(K)
     2            +PHI(K)*ALPHA(J) ) +E(3,3)*PHI(J)*PHI(K) +E(5,5)*
     3            ALPHA(J)*ALPHA(K) )
                DO 1030  J1 = 1,2
                  DA(J1,4) = 0.0
                  DB(J1,4) = 0.0
                  VT4(J,K) = VT4(J,K) +M(J1)*( ( DA(J1,J) +B(J1)*PHI(J)
     1              +Z(J) )*( B(J1)*DPHI(4,K) ) +( DB(J1,J) -A(J1)*PHI(J)
     2              )*( -A(J1)*DPHI(4,K) -PHI(K)*Z(4) ) )/2.0
                  T(J,K) = T(J,K) +M(J1)*( ( DA(J1,J) +B(J1)*PHI(J) +Z(J)
     1              )*( DA(J1,K) +B(J1)*PHI(K) +Z(K) ) +( DB(J1,J) -A(J1)*
     2              PHI(J) )*( DB(J1,K) -A(J1)*PHI(K) ) )
 1030         CONTINUE
 1040       CONTINUE
 1050     CONTINUE
          DO 1150  J = 1,2
            DO 1140  J1 = 1,2
              P2(J,J1) = T(J,J1) -T(J,4)*T(J1,4)/T(4,4)
              P1(J,J1) = 2.0*ALPHA0*( VT4(J,J1) -( T(J,4)*VT4(4,J1)
     1          +T(J1,4)*VT4(J,4) )/T(4,4) )
              P0(J,J1) = DDV(J,J1) +ALPHA0**2*( DVT44(J,J1) -4.0*VT4(J,4)
     1          *VT4(4,J1)/T(4,4) )
 1140       CONTINUE
 1150     CONTINUE
```

```
      Q4 = P2(1,1)*P2(2,2) -P2(2,1)*P2(1,2)
      Q3 = P2(1,1)*P1(2,2) +P2(2,2)*P1(1,1) -P2(2,1)*P1(1,2) -P2(1,2)*
     1  P1(2,1)
      Q2 = P2(1,1)*P0(2,2) +P2(2,2)*P0(1,1) +P1(1,1)*P1(2,2)
     1  -P2(2,1)*P0(1,2) -P2(1,2)*P0(2,1) -P1(1,2)*P1(2,1)
      Q1 = P1(1,1)*P0(2,2) +P1(2,2)*P0(1,1) -P1(1,2)*P0(2,1) -P1(2,1)*
     1  P0(1,2)
      Q0 = P0(1,1)*P0(2,2) -P0(1,2)*P0(2,1)
      CALL QUART( Q3/Q4, Q2/Q4, Q1/Q4, Q0/Q4, XREAL, XIMAG)
      DO 6020  J = 1,4
      DO 5090  K = 1,2
        AR(J,K) = P2(1,K)*( XREAL(J)**2 -XIMAG(J)**2 ) +P1(1,K)*
     1  XREAL(J) +P0(1,K)
      BR(J,K) = 2.0*XREAL(J)*XIMAG(J)*P2(1,K) +XIMAG(J)*P1(1,K)
 5090 CONTINUE
      PHASE(J) = DATAN2( BR(J,1)*AR(J,2) -AR(J,1)*BR(J,2) ,
     1  BR(J,1)*BR(J,2) +AR(J,1)*AR(J,2) )
      RAT1(J) = -AR(J,2)*DCOS( PHASE(J) ) +BR(J,2)*DSIN( PHASE(J) )
      RAT2(J) = AR(J,1)
      ANG(J) = PHASE(J)*180.0/3.141592
      WRITE ( 6, 5095 ) J
 5095 FORMAT('  FOR ROOT ',I2 /)
      WRITE(6,6000 ) ANG(J)
 6000 FORMAT( '  PHASE THETA = PHASE PSI + ',1PD30.15,'  DEGREES'/)
      WRITE(6,6010 )  RAT1(J), RAT2 (J)
 6010 FORMAT('  MAG PSI : MAG THETA AS ',1PD30.15,'  : ',D30.15 // )
 6020 CONTINUE
      WRITE(6,987 )
  987 FORMAT( ////)
      END
```

```
C       SUBROUTINE FOR SOLVING A GENERAL QUARTIC EQUATION WIHT REAL COEFF.
        SUBROUTINE  QUART(A,B,C,D,XREAL,XIMAG)
        REAL*8 F,P,Q,R,Y1,Y2,Y3,RTPQ,FPRATY,Y1NEW,T
        REAL*8 A,B,C,D,Y, RSQ, DE1, R1, DE2,DSQ,ESQ,D1,E,XREAL,XIMAG,
     1   DE2SQ, ROH, DEROH,THETA,DREAL,DIMAG,TREAL,TIMAG
        DIMENSION TREAL(4), TIMAG(4)
        DIMENSION XREAL(4), XIMAG(4)
        F(T) = T**3 +P*T**2 +Q*T +R
        P = -B
        Q = A*C -4.0*D
        R = -A**2*D +4.0*B*D -C**2
        WRITE(6, 25 )    P, Q, R
   25   FORMAT('  P = ',1PD20.9,'  Q = ',D20.9,'  R = ',D20.9//)
        Y1 = -P/3.0
        RTPQ = DSQRT( DABS( P**2 -3.0*Q ) )
        Y2 = Y1 +RTPQ/3.0
        Y3 = Y1 -RTPQ/3.0
        IF( F(Y2). GT . 0.0 . AND . F(Y3) . GT . 0.0 ) Y1 = Y1 -2.0*RTPQ
     1   /3.0
        IF( F(Y2) . LT . 0.0 . AND . F(Y3) . LT . 0.0 ) Y1 = Y1 +2.0*RTPQ
     1   /3.0
        IF( F(Y2) . EQ.0.0 ) GO TO 50
        IF( F(Y3) . EQ . 0.0 ) GO TO 45
        IF( DABS(R) . LT . 1.0D-20 ) GO TO 40
        DO 30  J = 1, 30
           FPRATY = 3.0*Y1**2 +2.0*P*Y1 +Q
        Y1NEW = Y1 -F(Y1)/FPRATY
           IF( DABS( ( Y1NEW -Y1)/Y1 ). LT . 1.0D-10 ) GO TO 60
        Y1 = Y1NEW
   30   CONTINUE
        WRITE ( 6, 35)
   35   FORMAT( '  Y1 CALC. TOOK MORE THAN 20 STEPS ' )
        RETURN
   40   CONTINUE
        Y1 = 0.0
        GO TO 70
   45   CONTINUE
        Y1 = Y3
        GO TO 70
   50   CONTINUE
        Y1 = Y2
        GO TO 70
   60   CONTINUE
        Y1 = Y1NEW
   70   CONTINUE
        FATY = Y1**3 +P*Y1**2 +Q*Y1 +R
        WRITE( 6, 80 ) FATY
  80    FORMAT ( ' THE TEST FOR THE CUBIC = ' ,1PE15.6 )
        Y = Y1
        RSQ = A**2/4.0 -B +Y
        DE1 = 3.0*A**2/4.0 -RSQ -2.0*B
        R1 = DSQRT( DABS( RSQ) )
        IF (RSQ) 200, 150, 127
  127   CONTINUE
        DE2 = ( 4.0*A*B -8.0*C -A**3 )/4.0/R1
  126   DSQ = DE1 +DE2
        ESQ = DE1 -DE2
        D1 = DSQRT( DABS( DSQ) )
        E = DSQRT( DABS ( ESQ ) )
```

```
      XREAL(1)  = -A/4.0 +R1/2.0
      XREAL(2)  = XREAL(1)
      XIMAG(1)  = D1/2.0
      XIMAG(2)  = -D1/2.0
      IF( DSQ . LT . 0.0 ) GO TO 129
      XREAL(1)  = XREAL(1)  +XIMAG(1)
      XREAL(2)  = XREAL(2)  +XIMAG(2)
      XIMAG(1)  = 0.0
      XIMAG(2)  = 0.0
129   CONTINUE
      XREAL(3)  = -A/4.0 -R1/2.0
      XREAL(4)  = XREAL(3)
      XIMAG(3)  = E/2.0
      XIMAG(4)  = -E/2.0
      IF( ESQ . LT . 0.0 ) GO TO 128
      XREAL(3)  = XREAL(3)  +XIMAG(3)
      XREAL(4)  = XREAL(4)  +XIMAG(4)
      XIMAG(3)  = 0.0
      XIMAG(4)  = 0.0
128   CONTINUE
      GO TO 500
150   CONTINUE
      DE2SQ = Y**2 -4.0*D
      DE2 = 2.0*DSQRT( DABS( DE2SQ ) )
      IF( DE2SQ ) 202, 126, 126
200   CONTINUE
      DE2 = -( 4.0*A*B -8.0*C -A**3 )/4.0/R1
202   CONTINUE
      ROH = DSQRT( DE1**2 +DE2**2 )
      DEROH = DSQRT( ROH)
      THETA = DATAN2( DE2, DE1 )
      DREAL = DEROH*DCOS( THETA/2.0 )
      DIMAG = DEROH*DSIN( THETA/2.0 )
      XREAL(1)  = -A/4.0 +DREAL/2.0
      XIMAG(1)  = R1/2.0 +DIMAG/2.0
      XREAL(2)  = -A/4.0 -DREAL/2.0
      XIMAG(2)  = R1/2.0 -DIMAG/2.0
      XREAL(3)  = -A/4.0 +DREAL/2.0
      XIMAG(3)  = -R1/2.0 -DIMAG/2.0
      XREAL(4)  = -A/4.0 -DREAL/2.0
      XIMAG(4)  = -R1/2.0 +DIMAG/2.0
500   CONTINUE
      DO 120  J = 1,4
      WRITE ( 6, 110 ) J, XREAL(J), XIMAG(J)
110   FORMAT('     X',I1,'    =    ',1PD20.9,'  + ',D20.9,' *SQRT(-1)'//)
      TREAL(J)  = ( XREAL(J)**2 -XIMAG(J)**2)**2 -4.0*XREAL(J)**2*
     1   XIMAG(J)**2 +A*( XREAL(J)*( XREAL(J)**2 -XIMAG(J)**2 ) -2.0*
     2   XREAL(J)*XIMAG(J)**2 ) +B*( XREAL(J)**2 -XIMAG(J)**2 ) +C*
     3   XREAL(J) +D
      TIMAG(J)  = 4.0*XREAL(J)*XIMAG(J)*( XREAL(J)**2 -XIMAG(J)**2 )
     1   +A*( XIMAG(J)*( 3.0*XREAL(J)**2 -XIMAG(J)**2 ) )+B*2.0*XREAL(J)
     2   *XIMAG(J) +C*XIMAG(J)
      WRITE ( 6, 115 ) J, TREAL(J), TIMAG(J)
115   FORMAT(' TEST',I1,'    =    ',1PD20.9,'  + ',D20.9,' *SQRT(-1)'//)
120   CONTINUE
      RETURN
      END
```

```
C     MAIN PROGRAM FOR THE CALCULATION OF THE EQUILIBRIUM TURNS AND
C     THE STABILITY IN THOSE TURNS FOR ANY GIVEN BICYCLE
      IMPLICIT REAL*8 (A,B,C,D,E,F,G,H,I,L,M,N,P,Q,R,S,T,V,W,X,Y,Z)
      DIMENSION XREAL(4), XIMAG(4)
      DIMENSION P2(2,2), P1(2,2), P0(2,2)
      DIMENSION AR(4,2), BR(4,2),  PHASE(4), ANG(4), RAT1(4), RAT2(4)
      COMMON /JE/ JERROR
      COMMON /POTE/ DV(2), DDV(2,2)
      COMMON /ERG/ E(5,5), T(4,4), DE(5,5,4), VT4(4,4), DVT44(2,2),
     1   DDE33(2,2), DDE34(2,2), DDE35(2,2)
      COMMON /OMG/ W(2,3,3), DW(2,3,3,2)
      COMMON /BAC/ FO2, FO3, FR2, FR3, A(2), B(2), BC(2), F(2,3),
     1   DF(2,2,2), DA(2,4), DBC(2,2), DB(2,4), DC(2,3), DDA(2,2,2),
     2   DDB(2,2,2), DDC(2,3,2)
      COMMON /MA/ I(2,3,3), IW(2), M(2)
      COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1   DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2   DDCOTT(2,2)
      COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1   DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
      COMMON /BGEO/ LAM, YFN, YFD, LSIG, SCDHL,NDNM,DLAM(2), DYFN(2),
     1   DYFD(2),DYF(2),DLSIG(2)
      M(1)  = 60.0
      M(2)  = 4.0
      I11FR  =  15.0
      I22FR  =  5.0
      I33FR  =  6.0
      I23FR  =  0.15
      I11BW  =  0.05
      I33BW  =  0.1
      I11FO  =  1.0
      I22FO  =  0.5
      I33FO  =  0.5
      I23FO  =  -0.04
      I11FW  =  0.05
      I33FW  =  0.1
      BASE  =  1.5
      R  =  0.35
      HRCM  =  1.0
      DCMW  =  0.3
      FO2  =  0.09
      FO3  =  -0.10
      CAST  =  -0.10
      ALP  =  15.0
      COSALP  =  DCOS( ALP*3.141592/180.0 )
      SINALP  =  DSQRT( 1.0000000000 -COSALP**2 )
      D2  =  SINALP*BASE
      D3  =  CAST*COSALP  +R*SINALP
      D1  =  COSALP*BASE  -D3
      FR2  =  D1  -SINALP*( HRCM  -R )  -COSALP*DCMW
      FR3  =  COSALP*( HRCM  -R )  -SINALP*DCMW
      DO 80   J1  =  1,2
          DO 70   J2  =  1,3
              DO 60   J3  =  1,3
                  I(J1,J2,J3)  =  0.0
60            CONTINUE
70        CONTINUE
80    CONTINUE
```

```
      I(1,1,1)  = I11FR +I33BW
      I(1,2,2)  = I22FR +I11BW
      I(1,3,3)  = I33FR +I11BW
      I(1,2,3)  = I23FR
      I(1,3,2)  = I23FR
      I(2,1,1)  = I11FO +I33FW
      I(2,2,2)  = I22FO +I11FW
      I(2,3,3)  = I33FO +I11FW
      I(2,2,3)  = I23FO
      I(2,3,2)  = I23FO
      IW(1)  = I33BW
      IW(2)  = I33FW
      PIE = 3.141592653590
 85   READ ( 5, 90 ) THETA
 90   FORMAT ( F15.0 )
      WRITE ( 6, 100 ) THETA
100   FORMAT ( '    THETA = ', 1PD30.15,'  DEGREES '// )
        THETA = THETA*PIE/180.000000 +PIE/2.0000
      IF( THETA . LE . 0.0 . OR . THETA . GE . PIE ) STOP
      CALL DTHETA( THETA )
      PSI = ( THETA  -PIE/2.00000 )/10.0
      JCOUNT = 1
110   CONTINUE
      CALL DPSI( PSI )
      CALL NCOMP
      IF ( JERROR . EQ . 1 ) GO TO 85
      DO 120 J = 1,2
         CALL SECDER(J,1)
120   CONTINUE
      W(1,1,3)  = -COSTHE
      W(1,2,3)  = -GAMN*SINTHE
      W(1,3,3)  = GAM*SINTHE
      CALL WFORK(3)
      CALL ABC
      CALL PHICON(4)
      CALL ALPCON(4)
      CALL ZCON(4)
      DO 130  J = 1,2
         DW(1,1,3,J)  = -DCOSTH(J)
         DW(1,2,3,J)  = -DGAMN(J)*SINTHE -GAMN*DSINTH(J)
         DW(1,3,3,J)  = DGAM(J)*SINTHE +GAM*DSINTH(J)
         CALL DWFORK(3,J)
         CALL  DEIJK(3,3,J)
         CALL DEIJK(3,4,J)
         CALL DEIJK(3,5,J)
         CALL DABC(J)
         CALL PHICON(J)
         CALL ZCON(J)
         VT4(J,4)  = -( DE(3,3,J)*PHI(4)**2 +2.0*DE(3,4,J)*PHI(J) +2.0*
     1     DE(3,5,J)*PHI(4)*ALPHA(4) )
         DO 125  J1 = 1,2
            VT4(J,4)  = VT4(J,4) +M(J1)*( -Z(4)*PHI(4)*( DB(J1,J) -A(J1)
     1        *PHI(J) ) -PHI(4)**2*( B(J1)*DB(J1,J) +A(J1)*( DA(J1,J)
     2     +Z(J) ) ) )
125      CONTINUE
         DV(J)  = 9.80*( M(1)*DC(1,J) +M(2)*DC(2,J) )
130   CONTINUE
      CALL PHIDER(4,1)
      CALL ALPDER(4,1)
      DO 140  J = 1,2
```

```
         CALL PHIDER(J,1)
         CALL ZDER(J,1)
         CALL VTDER(J,1)
140   CONTINUE
      FATPSI = DV(1)*VT4(2,4)  -DV(2)*VT4(1,4)
      FPRATP = DDV(1,1)*VT4(2,4)  +DV(1)*DVT44(2,1)  -DDV(2,1)*VT4(1,4)
     1   -DV(2)*DVT44(1,1)
      IF ( FPRATP . EQ . 0.0 ) GO TO 300
      PSINEW = PSI - FATPSI/FPRATP
      IF( DABS ( ( PSINEW -PSI )/PSINEW ) . LT . 1.0D-8 ) GO TO 500
      PSI = PSINEW
      JCOUNT = JCOUNT + 1
      IF ( JCOUNT . GE . 31 ) GO TO 400
      GO TO 110
300   WRITE ( 6, 310 )
310   FORMAT ( ' F PRIME AT PSI IS ZERO. PSI CALC ABORTED ' /// )
      GO TO 85
400   WRITE ( 6, 410 )
410   FORMAT( ' THE PSI CALC. FAILED TO CONV. IN 20 ITERATIONS ' /// )
      GO TO 85
450   WRITE ( 6, 460 )
460   FORMAT ( ' ALPHA SQUARED IS LESS THAN ZERO ' /// )
      GO TO 85
500   CONTINUE
      ALPSQ = -DV(1)/VT4(1,4)
      IF ( ALPSQ . LT . 0.0 ) GO TO 450
      ALPHA0 = -DSQRT( ALPSQ )
      PSIDEG = PSINEW*180.000/PIE
      WRITE( 6, 510 ) JCOUNT, PSIDEG, ALPHA0
510   FORMAT ( '  JCOUNT = ',I3,/'       PSI = ',1PD30.15,/'   ALPHA0 =
     1',D30.15 // )
      CALL DPSI( PSINEW )
      CALL NCOMP
      IF ( JERROR . EQ . 1 ) GO TO 85
      YF = YFN/YFD
      WRITE (6, 245 ) YF
245   FORMAT( '  YF = ', 1PD30.15// )
      CALL ABC
      W(1,1,1)  = -GAM**2*DN(1)
      W(1,1,2)  = -GAM**2*DN(2)
      W(1,1,3)  = -COSTHE
      W(1,2,1)  = 0.0
      W(1,2,2)  = -GAM
      W(1,2,3)  = -GAMN*SINTHE
      W(1,3,1)  = 0.0
      W(1,3,2)  = -GAMN
      W(1,3,3)  = GAM*SINTHE
      CALL WFORK(1)
      CALL WFORK(2)
      CALL WFORK(3)
      CALL SECDER(1,1)
      CALL SECDER(1,2)
      CALLSECDER(2,1)
      CALL SECDER(2,2)
      DO 550  J = 1,2
      DW(1,1,1,J)  = -2.0*GAM*DGAM(J)*DN(1)  -GAM**2*DDN(1,J)
      DW(1,1,2,J)  = -2.0*GAM*DGAM(J)*DN(2)  -GAM**2*DDN(2,J)
      DW(1,1,3,J)  = -DCOSTH(J)
      DW(1,2,1,J)  = 0.0
      DW(1,2,2,J)  = -DGAM(J)
```

```
      DW(1,2,3,J)  =  -DGAMN(J)*SINTHE -GAMN*DSINTH(J)
      DW(1,3,1,J)  =  0.0
      DW(1,3,2,J)  =  -DGAMN(J)
      DW(1,3,3,J)  =  DGAM(J)*SINTHE +GAM*DSINTH(J)
      CALL DWFORK(1,J)
      CALL DWFORK(2,J)
      CALL DWFORK(3,J)
      CALL DABC(J)
 550  CONTINUE
      DO 600  J = 1,5
         DO 590  J1 = 1,5
            CALL EIJ(J,J1)
 590     CONTINUE
 600  CONTINUE
      DO 650  J = 1,4
     IF ( J . EQ . 3 ) GO TO 650
         CALL PHICON(J)
         CALL ZCON(J)
         CALL ALPCON(J)
         CALL ALPDER(4,J)
         CALL PHIDER(4,J)
 650  CONTINUE
      DO 750  J = 1,5
         DO 740   J1 = 1,4
            IF ( J1 . EQ . 3 ) GO TO 740
            DO 730  J2 = 3,5
               CALL DEIJK(J,J2,J1)
 730        CONTINUE
 740     CONTINUE
 750  CONTINUE
      DO 800  J = 1,4
         IF ( J . EQ . 3 ) GO TO 800
         DO 790  J1 = 1,4
            IF ( J1 . EQ . 3 ) GO TO 790
            CALL VTIJ4(J,J1)
 790     CONTINUE
 800  CONTINUE
      DO 850  J = 1,2
         DO 840  J1 = 1,2
            CALL ZDER(J,J1)
            CALL PHIDER(J,J1)
            CALL VTDER(J,J1)
 840     CONTINUE
 850  CONTINUE
      DO 900  J = 1,2
         DO 890  J1 = 1,2
            P2(J,J1)  =  T(J,J1)  -T(J,4)*T(J1,4)/T(4,4)
      P1(J,J1)  =  2.0*ALPHA0*( VT4(J,J1)  -( T(J,4)*VT4(4,J1)  +T(J1,4)*
     1      VT4(J,4)  )/T(4,4)  )
            P0(J,J1)  =  DDV(J,J1)  +ALPHA0**2*( DVT44(J,J1)  -4.0*(
     1      VT4(J,4)*VT4(4,J1)  )/T(4,4)  )
 890     CONTINUE
 900  CONTINUE
      Q4 = P2(1,1)*P2(2,2)  -P2(2,1)*P2(1,2)
      Q3 = P2(1,1)*P1(2,2)  +P2(2,2)*P1(1,1)  -P2(2,1)*P1(1,2)  -P2(1,2)*
     1  P1(2,1)
      Q2 = P2(1,1)*P0(2,2)  +P2(2,2)*P0(1,1)  +P1(1,1)*P1(2,2)  -P2(2,1)*
     1  P0(1,2)  -P2(1,2)*P0(2,1)  -P1(2,1)*P1(1,2)
      Q1 = P1(1,1)*P0(2,2)  +P1(2,2)*P0(1,1)  -P1(2,1)*P0(1,2)  -P1(1,2)*
     1  P0(2,1)
```

```
      Q0 = P0(1,1)*P0(2,2) -P0(2,1)*P0(1,2)
      CALL  QUART( Q3/Q4, Q2/Q4, Q1/Q4, Q0/Q4, XREAL, XIMAG)
      DO 6020  J = 1,4
         DO 5090  K = 1,2
            AR(J,K)  = P2(1,K)*( XREAL(J)**2 -XIMAG(J)**2 ) +P1(1,K)*
     1        XREAL(J)  +P0(1,K)
            BR(J,K)  = 2.0*XREAL(J)*XIMAG(J)*P2(1,K)  +XIMAG(J)*P1(1,K)
 5090    CONTINUE
         PHASE(J)  = DATAN2( BR(J,1)*AR(J,2)  -AR(J,1)*BR(J,2),
     1     BR(J,1)*BR(J,2)  +AR(J,1)*AR(J,2)  )
         RAT1(J)  = -AR(J,2)*DCOS( PHASE(J)  ) +BR(J,2)*DSIN( PHASE(J)  )
      RAT2(J)  = AR(J,1)
      ANG(J)  = PHASE(J)*180.0000000/PIE
      WRITE( 6, 5095 ) J
 5095 FORMAT( '    FOR ROOT ',I2 /)
      WRITE(6,6000) ANG(J)
 6000 FORMAT('  PHASE THETA = PHASE PSI + ',1PD30.15,'  DEGREES'/)
      WRITE(6,6010 ) RAT1(J), RAT2(J)
 6010 FORMAT('  MAG PSI : MAG THETA AS ',1PD30.15,'  : ',D30.15//)
 6020 CONTINUE
      GO TO 85
      END
C
C     SUBROUTINE FOR THE CALCULATION OF THE TRIG THETA FUNCTIONS

      SUBROUTINE  DTHETA(THETA)
      IMPLICIT REAL*8  (C,S,D,T)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1  DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2  DDCOTT(2,2)
      COSTHE = DCOS( THETA )
      SINTHE = DSIN( THETA )
      COTTHE = COSTHE/SINTHE
      CSCTHE = 1.0/SINTHE
      DO 4000  J = 1,2
         DIS = 0.0
         IF ( J . EQ . 2 )  DIS = 1.0
         DCOSTH(J)  = -SINTHE*DIS
         DSINTH(J)  = COSTHE*DIS
         DCSCTH(J)  = -CSCTHE*COTTHE*DIS
         DCOTTH(J)  = -CSCTHE**2*DIS
 4000 CONTINUE
      DO 4200  J = 1,2
         DO 4100  K = 1,2
            DIS = 0.0
            IF( J . EQ . 2 . AND . K . EQ . 2 )  DIS = 1.0
            DDCOST(J,K)  = -COSTHE*DIS
            DDSINT(J,K)  = -SINTHE*DIS
            DDCSCT(J,K)  = ( CSCTHE**3 +CSCTHE*COTTHE**2 )*DIS
            DDCOTT(J,K)  = 2.0*COTTHE*CSCTHE**2*DIS
 4100    CONTINUE
 4200 CONTINUE
      RETURN
      END
C
C     SUBROUTINE FOR THE CALCULATION OF THE TRIG PSI FUNCTIONS

      SUBROUTINE  DPSI( PSI )
      IMPLICIT REAL*8  (C,S,D,T,P)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
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     1   DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2   DDCOTT(2,2)
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1   DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2   CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COSPSI = DCOS( PSI )
      SINPSI = DSIN( PSI )
      TANPSI = SINPSI/COSPSI
      SECPSI = 1.0/COSPSI
      CTTP = COTTHE*TANPSI
      DO 4300  J = 1,2
         DIS = 0.0
         IF ( J . EQ . 1 ) DIS = 1.0
         DCOSPS(J)  = -SINPSI*DIS
         DSINPS(J)  = COSPSI*DIS
         DSECPS(J)  = TANPSI*SECPSI*DIS
         DTANPS(J)  = SECPSI**2*DIS
      DCTTP(J)  = DCOTTH(J)*TANPSI  +COTTHE*DTANPS(J)
 4300 CONTINUE
      DO 4500  J = 1,2
         DO 4400  K = 1,2
            DIS = 0.0
            IF ( J . EQ . 1 . AND . K . EQ . 1 ) DIS = 1.0
            DDCOSP(J,K)  = -COSPSI*DIS
            DDSINP(J,K)  = -SINPSI*DIS
            DDSECP(J,K)  = ( SECPSI**3 +TANPSI**2*SECPSI )*DIS
            DDTANP(J,K)  = 2.0*TANPSI*SECPSI**2*DIS
            DDCTTP(J,K)  = DDCOTT(J,K)*TANPSI  +DCOTTH(J)*DTANPS(K)
     1        +DCOTTH(K)*DTANPS(J)  +COTTHE*DDTANP(J,K)
 4400    CONTINUE
 4500 CONTINUE
      RETURN
      END
C
C     SUBROUTINE FOR THE CALCULATION OF GEOMETRIC QUANTITIES AND THEIR
C     FIRST DERIVATIVES WITH RESPECT TO PSI AND THETA

      SUBROUTINE  NCOMP
      IMPLICIT REAL*8 (C,S,T,D,R,G,H,N,L,Y,F)
      DIMENSION DLCSS(2), DLSIS(2), DDELCO(2), DDELSI(2)
      COMMON /JE/ JERROR
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1   DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2   CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1   DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2   DDCOTT(2,2)
      COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1   DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
      COMMON /BGEO/ LAM, YFN, YFD, LSIG, SCDHL, NDNM, DLAM(2), DYFN(2),
     1   DYFD(2), DYF(2), DLSIG(2)
      COMMON /LGEO/ L, SINDEL, COSDEL, SINSIG, COSSIG, TANDMS, SECDMS,
     1   DL(2), DDEL(2), DSIG(2), DDMS(2), DROH(2), DDL(2,2), DDSIG(2,2)
      JERROR = 0
      CNSQ = ( D3*TANPSI*COTTHE +R*SECPSI )**2 +( D3 +D1*SECPSI )**2
     1   -( R*TANPSI*COTTHE )**2 -R**2
      CN = 2.0*( D3 +D1*SECPSI )*D2*SECPSI
      CO = ( D3*TANPSI*COTTHE +R*SECPSI )**2 +( D2*SECPSI )**2
     1   -( R*TANPSI*COTTHE )**2 -( R*SECPSI )  **2
      CNSQRF = -2.0*( D3*TANPSI*COTTHE +R*SECPSI )*( D3 +D1*SECPSI )
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     1   +2.0*R**2*TANPSI*COTTHE
         CSORT = -2.0*( D3*TANPSI*COTTHE +R*SECPSI )*D2*SECPSI
         N = -D2/( D1 +D3 )
         DO 36  JCOUNT = 1,30
         RTN2P1 = DSQRT( N**2 +1.0 )
         FATN = CNSQ*N**2 +CN*N +C0 +CNSQRT*N*RTN2P1 +CSQRT*RTN2P1
         FPRATN = 2.0*CNSQ*N +CN +CNSQRT*RTN2P1 +CNSQRT*N**2/RTN2P1
     1   +CSQRT*N/RTN2P1
         IF ( FPRATN . EQ . 0.0 ) GO TO 1500
         N = N -FATN/FPRATN
         IF ( DABS ( FATN/FPRATN/N) . LT . 1.0D-10 ) GO TO 39
 36      CONTINUE
         WRITE ( 6, 37 )
 37      FORMAT ( '  THE N CALCULATION TOOK MORE THAN, 20 STEPS ' )
         JERROR = 1
         RETURN
 1500    WRITE ( 6, 1600 )
 1600    FORMAT ( '  F PRIME AT N = 0.0 ' )
         JERROR = 1
         RETURN
  39     CONTINUE
         GAM = 1.0/DSQRT( N**2 +1.0 )
         GAMN = GAM*N
         H = -N*D1 +R/GAM
         LAM = N -CTTP/GAM
         YFN = LAM*( H -D2 ) +D3*SECPSI
         YFD = LAM**2 +SECPSI**2
         YF = YFN/YFD
         LSIG = YF*( 1.0 +N*LAM ) -H*N +D1
         LCSS = GAM*LSIG
         LSIS = YF*CSCTHE*TANPSI
         L = DSQRT( LCSS**2 +LSIS**2 )
         COSSIG = LCSS/L
         SINSIG = LSIS/L
         DELCOS = SINTHE -GAMN*COSTHE*TANPSI
         DELSIN = GAM*TANPSI
         DELDNM = DELCOS**2 +DELSIN**2
         RTDELD = DSQRT( DELDNM )
         COSDEL = DELCOS/RTDELD
         SINDEL = DELSIN/RTDELD
         SECDMS = 1.0/( COSDEL*COSSIG +SINDEL*SINSIG )
         TANDMS = ( SINDEL*COSSIG -COSDEL*SINSIG )*SECDMS
         SCDHL = SECPSI*( D2 -H ) +D3*LAM
         NDNM = R**2*LAM*( 1.0 -GAMN*CTTP ) -SCDHL*( SECPSI*( D1 -R*GAMN )
     1   +D3*( 1.0 -GAMN*CTTP ) )
         DO 1000 J = 1,2
            DN(J) = ( SCDHL*( DSECPS(J)*( D2 -H ) -D3*DCTTP(J)/GAM ) +R**2
     1   *LAM*DCTTP(J)/GAM -R**2*SECPSI*DSECPS(J) )/NDNM
         DGAM(J) = -N*GAM**3*DN(J)
         DGAMN(J) = GAM**3*DN(J)
         DH(J) = DN(J)*( R*GAMN -D1 )
         DLAM(J) = DN(J)*( 1.0 -GAMN*CTTP) -DCTTP(J)/GAM
         DYFN(J) = DLAM(J)*( H -D2 ) +LAM*DH(J) +D3*DSECPS(J)
         DYFD(J) = 2.0*( LAM*DLAM(J) +SECPSI*DSECPS(J) )
         DYF(J) = ( YFD*DYFN(J) -YFN*DYFD(J) )/YFD**2
         DLSIG(J) = DYF(J)*( 1.0 +N*LAM ) +YF*( DN(J)*LAM +N*DLAM(J) )
     1   -DH(J)*N -H*DN(J)
         DLCSS(J) = DGAM(J)*LSIG +GAM*DLSIG(J)
         DLSIS(J) = DYF(J)*CSCTHE*TANPSI +YF*( DCSCTH(J)*TANPSI +CSCTHE*
     1   DTANPS(J) )
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      DL(J) = COSSIG*DLCSS(J) +SINSIG*DLSIS(J)
      DSIG(J) = ( -SINSIG*DLCSS(J) +COSSIG*DLSIS(J) )/L
      DROH(J) = ( -DLAM(J)*SECPSI +LAM*DSECPS(J) )/YFD
      DDELCO(J) = DSINTH(J)*( 1.0 -GAMN*CTTP ) +SINTHE*( -DGAMN(J)*
     1  CTTP -GAMN*DCTTP(J) )
      DDELSI(J) = DGAM(J)*TANPSI +GAM*DTANPS(J)
      DDEL(J) = ( DELCOS*DDELSI(J) -DELSIN*DDELCO(J) )/DELDNM
      DDMS(J) = DDEL(J) -DSIG(J)
 1000 CONTINUE
      RETURN
      END
C
C     SUBROUTINE FOR THE CALCULATION OF THE GEOMETRIC SECOND DERIVATIVES

      SUBROUTINE  SECDER(I,J)
      IMPLICIT REAL*8 (D,L,Y,S,N,C,T,R,G,H)
      DIMENSION DSCDHL(2), DNUM(2,2), DNDNM(2), DDLAM(2,2), DDYFN(2,2),
     1  DDYFD(2,2), DDLSIG(2,2), DDLCSS(2,2), DDLSIS(2,2) , DDYF(2,2)
      COMMON /BGEO/ LAM, YFN, YFD, LSIG, SCDHL, NDNM, DLAM(2), DYFN(2),
     1  DYFD(2), DYF(2), DLSIG(2)
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1  DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2  CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1  DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2  DDCOTT(2,2)
      COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1  DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
      COMMON /LGEO/ L, SINDEL, COSDEL, SINSIG, COSSIG, TANDMS, SECDMS,
     1  DL(2), DDEL(2), DSIG(2), DDMS(2), DROH(2), DDL(2,2), DDSIG(2,2)
      DSCDHL(J) = DSECPS(J)*( D2 -H ) -SECPSI*DH(J) +D3*DLAM(J)
      DNUM(I,J) = DSCDHL(J)*( DSECPS(I)*( D2 -H ) -D3*DCTTP(I)/GAM)
     1  +SCDHL*( DDSECP(I,J)*( D2 -H ) -DSECPS(I)*DH(J) -D3*GAMN*DN(J)
     2  *DCTTP(I) -D3*DDCTTP(I,J)/GAM ) +R**2*( DCTTP(I)*( DLAM(J)/GAM
     3  +GAMN*DN(J)*LAM ) +LAM*DDCTTP(I,J)/GAM -DSECPS(I)*DSECPS(J)
     4  -SECPSI*DDSECP(I,J) )
      DNDNM(J) = R**2*DLAM(J)*( 1.0 -GAMN*CTTP ) +R**2*LAM*( -DGAMN(J)
     1  *CTTP -GAMN*DCTTP(J) ) -DSCDHL(J)*( SECPSI*( D1 -R*GAMN ) +D3*
     2  ( 1.0 -GAMN*CTTP ) ) -SCDHL*( DSECPS(J)*( D1 -R*GAMN ) -R*
     3  SECPSI*DGAMN(J) -D3*( DGAMN(J)*CTTP +GAMN*DCTTP(J) ) )
      DDN(I,J) = DNUM(I,J)/NDNM -DN(I)*DNDNM(J)/NDNM
      DDGAM(I,J) = DN(I)*DN(J)*GAM**3*( 3.0*N**2*GAM**2 -1.0 ) -N*GAM
     1  **3*DDN(I,J)
      DDGAMN(I,J) = GAM**3*DDN(I,J) +3.0*GAM**2*DGAM(J)*DN(I)
      DDH(I,J) = DDN(I,J)*( R*GAMN -D1 ) +R*DGAMN(J)*DN(I)
      DDLAM(I,J) = DDN(I,J)*( 1.0 -GAMN*CTTP ) -DN(I)*( DGAMN(J)*CTTP
     1  +GAMN*DCTTP(J) ) -GAMN*DN(J)*DCTTP(I) -DDCTTP(I,J)/GAM
      DDYFN(I,J) = DDLAM(I,J)*( H -D2 ) +DLAM(I)*DH(J) +DLAM(J)*DH(I)
     1  +LAM*DDH(I,J) +D3*DDSECP(I,J)
      DDYFD(I,J) = 2.0*( DLAM(I)*DLAM(J) +LAM*DDLAM(I,J) +DSECPS(I)*
     1  DSECPS(J) +SECPSI*DDSECP(I,J) )
      DDYF(I,J) = ( DYFD(J)*DYFN(I) +YFD*DDYFN(I,J) -DYFN(J)*DYFD(I)
     1  -YFN*DDYFD(I,J) )/YFD**2 -2.0*DYF(I)*DYFD(J)/YFD
      DDLSIG(I,J) = DDYF(I,J)*( 1.0 +N*LAM ) +DYF(I)*( DN(J)*LAM +N*
     1  DLAM(J) ) +DYF(J)*( DN(I)*LAM +N*DLAM(I) ) +YF*( DDN(I,J)*LAM
     2  +DN(I)*DLAM(J) +DN(J)*DLAM(I) +N*DDLAM(I,J) ) -DDH(I,J)*N
     3  -DH(I)*DN(J) -DH(J)*DN(I) -H*DDN(I,J)
      DDLCSS(I,J) = DDGAM(I,J)*LSIG +DGAM(I)*DLSIG(J) +DGAM(J)*DLSIG(I)
     1  +GAM*DDLSIG(I,J)
      DDLSIS(I,J) = DDYF(I,J)*CSCTHE*TANPSI +DYF(I)*( DCSCTH(J)*TANPSI
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1     +CSCTHE*DTANPS(J)  )  +DYF(J)*(  DCSCTH(I)*TANPSI  +CSCTHE*DTANPS
2     (I)  )  +YF*(  DDCSCT(I,J)*TANPSI  +DCSCTH(I)*DTANPS(J)  +DCSCTH(J)*
3     DTANPS(I)  +CSCTHE*DDTANP(I,J)  )
   DDL(I,J)  =  L*DSIG(I)*DSIG(J)  +COSSIG*DDLCSS(I,J)  +SINSIG*DDLSIS
1     (I,J)
   DDSIG(I,J)  =  (  -DL(I)*DSIG(J)  -DL(J)*DSIG(I)  -SINSIG*DDLCSS(I,J)
1     +COSSIG*DDLSIS(I,J)  )/L
   RETURN
   END
```

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C
C      SUBROUTINE FOR CALCULATION OF THE CONSTRAINTS ON PHI
        SUBROUTINE  PHICON(J)
        IMPLICIT REAL*8 (A,Z,D,R,G,H,N,L,S,C,T,P)
        COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
        COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1    DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
        COMMON /LGEO/ L, SINDEL, COSDEL, SINSIG, COSSIG, TANDMS, SECDMS,
     1    DL(2), DDEL(2), DSIG(2), DDMS(2), DROH(2), DDL(2,2), DDSIG(2,2)
        PHI(J) = -R*SINDEL*SECDMS/L
        IF ( J . NE . 4 ) PHI(J) = DL(J)*TANDMS/L -DSIG(J) +R*SINDEL*
     1    SECDMS*GAM**2*DN(J)/L
        RETURN
        END
C
C      SUBROUTINE FOR CALCULATING THE CONSTRAINTS ON ALPHA

        SUBROUTINE  ALPCON(J)
        IMPLICIT REAL*8 (P,A,Z,D,R,G,H,N,L,S,C,T)
        COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
        COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1    DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
        COMMON /LGEO/ L, SINDEL, COSDEL, SINSIG, COSSIG, TANDMS, SECDMS,
     1    DL(2), DDEL(2), DSIG(2), DDMS(2), DROH(2), DDL(2,2), DDSIG(2,2)
        ALPHA(J) = COSSIG*SECDMS
        IF ( J . NE . 4 ) ALPHA(J) = -DL(J)*SECDMS/R -COSSIG*SECDMS*
     1    GAM**2*DN(J) -DROH(J)
        RETURN
        END
C
C      SUBROUTINE FOR THE CALCULATION OF THE CONSTRAINTS ON Z

        SUBROUTINE  ZCON(I)
        IMPLICIT REAL*8 (P,A,Z,D,R,G,H,N)
        COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
        COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1    DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
        IF ( I . EQ . 4 ) GO TO 100
        Z(I) = R*GAM**2*DN(I)
        RETURN
  100   Z(I) = -R
        RETURN
        END
C
C      SUBROUTINE FOR CALCULATION OF THE DERIVATIVES OF THE CONSTRAINTS
C      ON PHI

        SUBROUTINE  PHIDER(I,J)
        IMPLICIT REAL*8 (L,S,C,T,D,R,G,H,N,P,A,Z)
        COMMON /LGEO/ L, SINDEL, COSDEL, SINSIG, COSSIG, TANDMS, SECDMS,
     1    DL(2), DDEL(2), DSIG(2), DDMS(2), DROH(2), DDL(2,2), DDSIG(2,2)
        COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1    DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
        COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
        DPHI(I,J) = 0.0
        IF( I . EQ . 4 . AND . J . NE . 4 ) DPHI(I,J) = -R*SECDMS*(
     1    -DL(J)*SINDEL/L +COSDEL*DDEL(J) +SINDEL*TANDMS*DDMS(J) )/L
        IF( I . NE . 4 . AND . J . NE . 4 ) DPHI(I,J) = TANDMS*( L*
     1    DDL(I,J) -DL(I)*DL(J) )/L +DL(I)*SECDMS**2*DDMS(J)/L -DDSIG(I,
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     2   J) +R*SECDMS*GAM**2*( -DL(J)*SINDEL*DN(I)/L +COSDEL*DDEL(J)*
     3   DN(I) +SINDEL*TANDMS*DDMS(J)*DN(I) -2.0*SINDEL*N*GAM**2*DN(J)*
     4   DN(I) +SINDEL*DDN(I,J) )/L
      RETURN
      END
C
C    SUBROUTINE FOR CALCULATION OF THE DERIVATIVES OF THE CONSTRAINTS
C    ON ALPHA

      SUBROUTINE  ALPDER(I,J)
      IMPLICIT REAL*8 (L,S,C,T,D,P,A,Z)
      COMMON /LGEO/ L, SINDEL, COSDEL, SINSIG, COSSIG, TANDMS, SECDMS,
     1   DL(2), DDEL(2), DSIG(2), DDMS(2), DROH(2), DDL(2,2), DDSIG(2,2)
      COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
      DALPHA(I,J) = 0.0
      IF( I . EQ . 4 . AND . J . NE . 4 ) DALPHA(I,J) = SECDMS*( TANDMS
     1   *DDMS(J)*COSSIG -SINSIG*DSIG(J) )
      RETURN
      END
C
C    SUBROUTINE FOR CALCULATING THE DERIVATIVES OF THE CONSTRAINTS ON Z

      SUBROUTINE  ZDER(I,J)
      IMPLICIT REAL*8 (P,A,Z,D,R,G,H,N)
      COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
      COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1   DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
      DZ(I,J) = 0.0
      IF( I . NE . 4 . AND . J . NE . 4 ) DZ(I,J) = R*( 2.0*GAM*
     1   DGAM(J)*DN(I) +GAM**2*DDN(I,J) )
      RETURN
      END
C
C    SUBROUTINE FOR THE CALCULATION OF THE VELOCITY TERMS, A, B, AND C

      SUBROUTINE  ABC
      IMPLICIT REAL*8 (F,A,B,D,C,S,T,R,G,H,N)
      COMMON /BAC/ FO2, FO3, FR2, FR3, A(2), B(2), BC(2), F(2,3),
     1   DF(2,2,2), DA(2,4), DBC(2,2), DB(2,4), DC(2,3), DDA(2,2,2),
     2   DDB(2,2,2), DDC(2,3,2)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1   DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2   DDCOTT(2,2)
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1   DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2   CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1   DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
      F(1,1) = 0.0
      F(2,1) = -FO2*SINPSI
      F(1,2) = -FR2
      F(2,2) = FO2*COSPSI
      F(1,3) = FR3
      F(2,3) = FO3
      DO 100 J = 1,2
      A(J) = GAM*( D1 +F(J,2) ) +GAMN*F(J,3)
      BC(J) = GAM*( H +F(J,3) ) -GAMN*F(J,2)
      B(J) = COSTHE*BC(J) +SINTHE*F(J,1)
  100 CONTINUE
      RETURN
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      END
C
C     SUBROUTINE FOR CALCULATING THE DERIVATIVES OF A, B, AND C
C
      SUBROUTINE  DABC(I)
      IMPLICIT REAL*8 (F,A,B,D,C,S,T,R,G,H,N)
      COMMON /BAC/ FO2, FO3, FR2, FR3, A(2), B(2), BC(2), F(2,3),
     1  DF(2,2,2), DA(2,4), DBC(2,2), DB(2,4), DC(2,3), DDA(2,2,2),
     2  DDB(2,2,2), DDC(2,3,2)
      COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1  DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2  DDCOTT(2,2)
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1  DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2  CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1  DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
      DF(2,1,I) = -FO2*DSINPS(I)
      DF(2,2,I) = FO2*DCOSPS(I)
      DF(1,1,I) = 0.0
      DF(1,2,I) = 0.0
      DO 100 J = 1,2
      DA(J,I) = DGAM(I)*( D1 +F(J,2) ) +GAM*DF(J,2,I) +DGAMN(I)*F(J,3)
      DBC(J,I) = DGAM(I)*( H +F(J,3) ) +GAM*DH(I) -DGAMN(I)*F(J,2)
     1  -GAMN*DF(J,2,I)
      DB(J,I) = DCOSTH(I)*BC(J) +COSTHE*DBC(J,I) +DSINTH(I)*F(J,1)
     1  +SINTHE*DF(J,1,I)
      DC(J,I) = DSINTH(I)*BC(J) +SINTHE*DBC(J,I) -DCOSTH(I)*F(J,1)
     1  -COSTHE*DF(J,1,I)
  100 CONTINUE
      RETURN
      END
C
C     SUBROUTINE FOR CALCULATION OF THE ANGULAR VELOCIYIES OF THE FORK
C     GIVEN THE ANGULAR VELOCITIES OF THE FRAME
C
      SUBROUTINE WFORK(I)
      IMPLICIT REAL*8 (C,S,T,D,W)
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1  DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2  CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COMMON /OMG/ W(2,3,3), DW(2,3,3,2)
      W(2,1,I) = COSPSI*W(1,1,I) +SINPSI*W(1,2,I)
      W(2,2,I) = -SINPSI*W(1,1,I) +COSPSI*W(1,2,I)
      W(2,3,I) = W(1,3,I)
      W(2,3,1) = 1.0
      RETURN
      END
C
C     SUBROUTINE FOR THE CALCULATION OF THE DERIVATIVES OF THE FORK
C     ANGULAR VELOCITIES GIVEN THE DERIVATIVES OF THE FRAME ANGULAR
C     VELOCITIES
C
      SUBROUTINE  DWFORK(I,J)
      IMPLICIT REAL*8 ( C,S,T,D,W)
      COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1  DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2  CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
      COMMON /OMG/ W(2,3,3), DW(2,3,3,2)
      DW(2,1,I,J) = DCOSPS(J)*W(1,1,I) +COSPSI*DW(1,1,I,J) +DSINPS(J)*
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     1     W(1,2,I)  +SINPSI*DW(1,2,I,J)
         DW(2,2,I,J)  = -DSINPS(J)*W(1,1,I)  -SINPSI*DW(1,1,I,J)  +DCOSPS(J)*
     1     W(1,2,I)  +COSPSI*DW(1,2,I,J)
         DW(2,3,I,J)  = DW(1,3,I,J)
         RETURN
         END
C
C       SUBROUTINE FOR CALCULATING THE EIJ COEFFICIENTS

         SUBROUTINE  EIJ(J,K)
         IMPLICIT REAL*8  ( E,T,D,V,W,F,A,B,I,M)
         COMMON /ERG/ E(5,5), T(4,4), DE(5,5,4), VT4(4,4), DVT44(2,2),
     1     DDE33(2,2), DDE34(2,2), DDE35(2,2)
         COMMON /OMG/ W(2,3,3), DW(2,3,3,2)
         COMMON /BAC/ FO2, FO3, FR2, FR3, A(2), B(2), BC(2), F(2,3),
     1     DF(2,2,2), DA(2,4), DBC(2,2), DB(2,4), DC(2,3), DDA(2,2,2),
     2     DDB(2,2,2), DDC(2,3,2)
         COMMON /MA/ I(2,3,3), IW(2), M(2)
         IF( J . GT . 3 . OR . K . GT . 3 ) GO TO 500
         E(J,K)  = 0.0
         DO 400 L = 1,2
            DC(L,3)  = 0.0
            E(J,K)  = E(J,K)  +M(L)*DC(L,J)*DC(L,K)/2.0
            DO 300 J1 = 1,3
               DO 200 K1 = 1,3
                  E(J,K)  = E(J,K)  +I(L,J1,K1)*W(L,J1,J)*W(L,K1,K)/2.0
  200          CONTINUE
  300       CONTINUE
  400    CONTINUE
         RETURN
  500    CONTINUE
         IF( J . GT . 3 . AND . K . GT . 3 ) GO TO 600
         IF( J . GT . 3 ) E(J,K)  = IW(J-3)*W(J-3,1,K)/2.0
         IF( K . GT . 3 ) E(J,K)  = IW(K-3)*W(K-3,1,J)/2.0
         RETURN
  600    CONTINUE
         E(J,K)  = 0.0
         IF( J . EQ . K ) E(J,K)  = IW(J-3)/2.0
         RETURN
         END
C
C       SUBROUTINE FOR CALCULATING THE DE(J,K,L)  WITH K RESTRICTED TO 3-5

         SUBROUTINE  DEIJK(J,K,L)
         IMPLICIT REAL*8  (W,D,I,M,E,T,V)
         COMMON /OMG/ W(2,3,3), DW(2,3,3,2)
         COMMON /MA/ I(2,3,3), IW(2), M(2)
         COMMON /ERG/ E(5,5), T(4,4), DE(5,5,4), VT4(4,4), DVT44(2,2),
     1     DDE33(2,2), DDE34(2,2), DDE35(2,2)
         DE(J,K,L)  = 0.0
         IF  ( L . EQ . 4 ) RETURN
         IF( J . GT . 3 . AND . K . GT . 3 ) RETURN
         IF( J . GT . 3 . OR . K . GT . 3 ) GO TO 500
         DO 400  L1 = 1,2
            DO 300  J1 = 1,3
               DO 200  K1 = 1,3
                  DE(J,K,L)  = DE(J,K,L)  +I(L1,J1,K1)*( DW(L1,J1,J,L)*
     1              W(L1,K1,K)  +W(L1,J1,J)*DW(L1,K1,K,L)  )/2.0
  200          CONTINUE
  300       CONTINUE
```

```
  400  CONTINUE
       RETURN
  500  CONTINUE
       IF( J . GT . 3 ) DE(J,K,L) = IW(J-3)*DW(J-3,1,K,L)/2.0
       IF( K . GT . 3 ) DE(J,K,L) = IW(K-3)*DW(K-3,1,J,L)/2.0
       RETURN
       END
C
C       SUBROUTINE FOR THE CALCULATION OF THE TIJ COEFFICIENTS
C      AND THE VT4(I,J) COEFFICIENTS

       SUBROUTINE  VTIJ4(I,J)
       IMPLICIT REAL*8 (Q,M,E,T,D,V,P,A,Z,F,B)
       COMMON /MA/ Q(2,3,3), Q1(2), M(2)
       COMMON /ERG/ E(5,5), T(4,4), DE(5,5,4), VT4(4,4), DVT44(2,2),
     1   DDE33(2,2), DDE34(2,2), DDE35(2,2)
       COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
       COMMON /BAC/ FO2, FO3, FR2, FR3, A(2), B(2), BC(2), F(2,3),
     1   DF(2,2,2), DA(2,4), DBC(2,2), DB(2,4), DC(2,3), DDA(2,2,2),
     2   DDB(2,2,2), DDC(2,3,2)
       T(I,J) = 2.0*( E(I,J) +E(I,3)*PHI(J) +E(J,3)*PHI(I) +E(I,5)*
     1   ALPHA(J) +E(J,5)*ALPHA(I) +E(3,5)*( PHI(I)*ALPHA(J) +PHI(J)*
     2   ALPHA(I) ) +E(3,3)*PHI(I)*PHI(J) +E(5,5)*ALPHA(I)*ALPHA(J) )
       VT4(I,J) = DE(I,4,J) +DE(I,3,J)*PHI(4) +E(I,3)*DPHI(4,J)
     1   +DE(I,5,J)*ALPHA(4) +E(I,5)*DALPHA(4,J) -( DE(J,4,I)
     2   +DE(3,3,I)*PHI(J)*PHI(4) +DE(5,5,I)*ALPHA(J)*ALPHA(4)
     3   +DE(J,3,I)*PHI(4) +DE(4,3,I)*PHI(J) +DE(J,5,I)*ALPHA(4)
     4   +DE(3,5,I)*( PHI(J)*ALPHA(4) +PHI(4)*ALPHA(J) ) )  +PHI(I)*
     5   ( DE(3,3,J)*PHI(4) +E(3,3)*DPHI(4,J) +DE(4,3,J) +DE(3,5,J)*
     6   ALPHA(4) +E(3,5)*DALPHA(4,J) ) +ALPHA(I)*( DE(5,5,J)*ALPHA(4)
     7   +E(5,5)*DALPHA(4,J) +DE(3,5,J)*PHI(4) +E(3,5)*DPHI(4,J) )
       DO 100  L = 1,2
       DA(L,4) = 0.0
       DB(L,4) = 0.0
          T(I,J) = T(I,J) +M(L)*( ( DA(L,I) +B(L)*PHI(I) +Z(I) )*( DA
     1      (L,J) +B(L)*PHI(J) +Z(J) ) +(DB(L,I) -A(L)*PHI(I) )*( DB(L,
     2      J) -A(L)*PHI(J) ) )
          VT4(I,J) = VT4(I,J) +M(L)*( ( DA(L,I) +B(L)*PHI(I) +Z(I) ) *
     1      ( 2.0*DB(L,J)*PHI(4) +B(L)*DPHI(4,J) ) +( DB(L,I) -A(L)*
     2      PHI(I) )*( -2.0*DA(L,J)*PHI(4) -A(L)*DPHI(4,J) -PHI(4)*Z(J)
     3      -PHI(J)*Z(4) ) -2.0*PHI(J)*PHI(4)*( B(L)*DB(L,I) +A(L)*
     4      ( DA(L,I) +Z(I) ) ) ) /2.0
  100  CONTINUE
       RETURN
       END
C
C      SUBROUTINE FOR CALCULATING THE DERIVATIVE OF VT4(J,4)
C      AND THE SECOND DERIVATIVE OF THE POTENTIAL

       SUBROUTINE  VTDER(I,J)
       IMPLICIT REAL*8 (D,M,P,E,T,V,A,Z,F,B,C,S,R,G,H,N,W)
       DIMENSION DDW(2,3,3,2,2)
       DIMENSION DDF(2,2,2,2), DDBC(2,2,2)
       COMMON /MA/ PI(2,3,3), PIW(2),M(2)
       COMMON /POTE/ DV(2), DDV(2,2)
       COMMON /ERG/ E(5,5), T(4,4), DE(5,5,4), VT4(4,4), DVT44(2,2),
     1   DDE33(2,2), DDE34(2,2), DDE35(2,2)
       COMMON /C/PHI(4), ALPHA(4), Z(4), DPHI(4,4), DALPHA(4,4), DZ(4,4)
       COMMON /BAC/ FO2, FO3, FR2, FR3, A(2), B(2), BC(2), F(2,3),
     1   DF(2,2,2), DA(2,4), DBC(2,2), DB(2,4), DC(2,3), DDA(2,2,2),
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     2    DDB(2,2,2), DDC(2,3,2)
          COMMON /PS/ COSPSI, SINPSI, SECPSI, TANPSI, DCOSPS(2), DSINPS(2),
     1      DSECPS(2), DTANPS(2), DDCOSP(2,2), DDSINP(2,2), DDSECP(2,2),
     2      CTTP, DCTTP(2), DDCTTP(2,2), DDTANP(2,2)
          COMMON /TH/ COSTHE, SINTHE, COTTHE, CSCTHE, DCOSTH(2), DSINTH(2),
     1      DCSCTH(2), DCOTTH(2), DDCOST(2,2), DDSINT(2,2), DDCSCT(2,2),
     2      DDCOTT(2,2)
          COMMON /GEO/ R, D1, D2, D3, GAM, GAMN, H, N, DGAM(2), DGAMN(2),
     1      DH(2), DN(2), DDGAM(2,2), DDGAMN(2,2), DDH(2,2), DDN(2,2)
          COMMON /OMG/ W(2,3,3), DW(2,3,3,2)
          DDW(1,1,3,I,J) = -DDCOST(I,J)
          DDW(1,2,3,I,J) = -DDGAMN(I,J)*SINTHE -DGAMN(I)*DSINTH(J) -DGAMN
     1      (J)*DSINTH(I) -GAMN*DDSINT(I,J)
          DDW(1,3,3,I,J) = DDGAM(I,J)*SINTHE +DGAM(I)*DSINTH(J) +DGAM(J)*
     1      DSINTH(I) +GAM*DDSINT(I,J)
          DDW(2,1,3,I,J) = DDCOSP(I,J)*W(1,1,3) +DCOSPS(I)*DW(1,1,3,J)
     1      +DCOSPS(J)*DW(1,1,3,I) +COSPSI*DDW(1,1,3,I,J) +DDSINP(I,J)*
     2      W(1,2,3) +DSINPS(I)*DW(1,2,3,J) +DSINPS(J)*DW(1,2,3,I) +SINPSI*
     3      DDW(1,2,3,I,J)
          DDW(2,2,3,I,J) = -DDSINP(I,J)*W(1,1,3) -DSINPS(I)*DW(1,1,3,J)
     1      -DSINPS(J)*DW(1,1,3,I) -SINPSI*DDW(1,1,3,I,J) +DDCOSP(I,J)*
     2      W(1,2,3) +DCOSPS(I)*DW(1,2,3,J) +DCOSPS(J)*DW(1,2,3,I) +COSPSI*
     3      DDW(1,2,3,I,J)
          DDW(2,3,3,I,J) = DDW(1,3,3,I,J)
          DDE33(I,J) = 0.0
          DO 50  L = 1,2
              DO 40  L1 = 1,3
                  DO 30  L2 = 1,3
                      DDE33(I,J) = DDE33(I,J) +PI(L,L1,L2)*( DDW(L,L1,3,I,J)*
     1                  W(L,L2,3) +DW(L,L1,3,I)*DW(L,L2,3,J) +DW(L,L1,3,J)*
     2                  DW(L,L2,3,I) +W(L,L1,3)*DDW(L,L2,3,I,J) )/2.0
   30             CONTINUE
   40         CONTINUE
   50     CONTINUE
          DDE34(I,J) = PIW(1)*DDW(1,1,3,I,J)/2.0
          DDE35(I,J) = PIW(2)*DDW(2,1,3,I,J)/2.0
          DDF(1,1,I,J) = 0.0
          DDF(1,2,I,J) = 0.0
          DDF(2,1,I,J) = -FO2*DDSINP(I,J)
          DDF(2,2,I,J) = FO2*DDCOSP(I,J)
          DVT44(I,J) = -( DDE33(I,J)*PHI(4)**2 +2.0*( DE(3,3,I)*PHI(4)*
     1      DPHI(4,J) +DDE34(I,J)*PHI(4) +DE(3,4,I)*DPHI(4,J) +DDE35(I,J)*
     2      PHI(4)*ALPHA(4) +DE(3,5,I)*( DPHI(4,J)*ALPHA(4) +PHI(4)*
     3      DALPHA(4,J) ) ) )
          DO 100  L = 1,2
              DDA(L,I,J) = DDGAM(I,J)*( D1 +F(L,2) ) +DGAM(I)*DF(L,2,J)
     1          +DGAM(J)*DF(L,2,I) +GAM*DDF(L,2,I,J) +DDGAMN(I,J)*F(L,3)
              DDBC(L,I,J) = DDGAM(I,J)*( H +F(L,3) ) +DGAM(I)*DH(J) +DGAM(J)
     1          *DH(I) +GAM*DDH(I,J) -DDGAMN(I,J)*F(L,2) -DGAMN(I)*
     2          DF(L,2,J) -DGAMN(J)*DF(L,2,I) -GAMN*DDF(L,2,I,J)
              DDB(L,I,J) = DDCOST(I,J)*BC(L) +DCOSTH(I)*DBC(L,J) +DCOSTH(J)*
     1          DBC(L,I) +COSTHE*DDBC(L,I,J) +DDSINT(I,J)*F(L,1) +DSINTH(I)*
     2          DF(L,1,J) +DSINTH(J)*DF(L,1,I) +SINTHE*DDF(L,1,I,J)
              DDC(L,I,J) = DDSINT(I,J)*BC(L) +DSINTH(I)*DBC(L,J) +DSINTH(J)*
     1          DBC(L,I) +SINTHE*DDBC(L,I,J) -DDCOST(I,J)*F(L,1) -DCOSTH(I)*
     2          DF(L,1,J) -DCOSTH(J)*DF(L,1,I) -COSTHE*DDF(L,1,I,J)
              DVT44(I,J) = DVT44(I,J) +M(L)*( -Z(4)*DPHI(4,J)*( DB(L,I)
     1          -A(L)*PHI(I) ) -Z(4)*PHI(4)*( DDB(L,I,J) -DA(L,J)*PHI(I)
     2      -A(L)*DPHI(I,J) ) -2.0*PHI(4)*DPHI(4,J)*( B(L)*DB(L,I) +A(L)*
     3      ( DA(L,I) +Z(I) ) ) -PHI(4)**2*( DB(L,I)*DB(L,J) +B(L)*
```

```
     4    DDB(L,I,J)  +DA(L,J)*( DA(L,I)  +Z(I)  )  +A(L)*( DDA(L,I,J)
     5    +DZ(I,J) ) ) )
 100      CONTINUE
          DDV(I,J)  = 9.80*( M(1)*DDC(1,I,J)  +M(2)*DDC(2,I,J) )
          RETURN
          END
```