EFFECT OF INCLINATION OF STEERING AXIS AND OF STAGGER OF THE FRONT WHEEL ON STABILITY OF MOTION OF A BICYCLE

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Paper [1] derived the equations of motion of a controlled and uncontrolled bicycle and investigated the stability of motion of simplified models. Simplification of the model involved disregarding the angle of inclination λ of the steering axis and the stagger σ of the front wheel. Since, for actual bicycles, λ and σ are usually nonzero, it is of interest to investigate the stability of motion of an uncontrolled bicycle in relation to these parameters.

In this paper we refine the equations of motion of [1] for an uncontrolled bicycle. As a result of investigating these equations we identify the region of stable motion in the space of parameters σ, λ, V, where V is the velocity of the bicycle.

As regards rolling dynamics, bicycles and motorcycles are machines of the same type, and therefore everything in the paper is applicable to motorcycles as well.

1. Consider a model of a bicycle with rigid wheels. In accordance with the notation in Figs. 1 and 2, the coordinates of the centers of mass M_1 and M_2 of the rear and front parts of the bicycle respectively are expressed in terms of generalized coordinates x, θ, χ, ψ to within second-order small quantities:

$$
\begin{gathered}
x_1=x+h_1\chi-l_1\theta \\
y_1=y+h_1\chi\theta+\left(1-\frac{\theta^2}{2}\right)l_1+(h_1-R)\left(\frac{\sin\lambda}{2}\psi^2-\chi\psi\right)\frac{c_1}{c} \\
z_1=h_1\left(1-\frac{\chi^2}{2}\right)-l_1\frac{c_1}{c}\left(\frac{\sin\lambda}{2}\psi^2-\chi\psi\right) \\
x_2=x+h_2\chi-l_2\theta-d\psi \\
y_2=y+h_2\chi\theta+l_2\left(1-\frac{\theta^2}{2}\right)-d\psi\,\theta+\left[(h_2-R)\frac{c_1}{c}\sin\lambda-d\cos\lambda\right]\frac{\psi^2}{2}- \\
-(h_2-R)\frac{c_1}{c}\chi\psi \\
z_2=h_2\left(1-\frac{\chi^2}{2}\right)-\left(l_2\frac{c_1}{c}+d\right)\left(\frac{\sin\lambda}{2}\psi^2-\chi\psi\right)
\end{gathered}
\tag{1.1}
$$

Here we have employed the following notation: R is the wheel radius (m); c is the base of the bicycle (m); d is the distance from the center of mass of the front part to the steering axis (m); b is the distance from the center of the front wheel to the steering axis (m); δ is the coefficient of viscous friction in the steering column ($kg\cdot m^2\cdot sec^{-1}$); λ is the angle of inclination of the steering axis relative to the vertical (rad); V is the velocity of the bicycle ($m\cdot sec^{-1}$); c_2 is the stagger of the front wheel, $c_1=c_2\cos\lambda$ (m); $\sigma=c_2 c^{-1}$ is the dimensionless stagger of the front wheel; g is the acceleration due to gravity ($m\cdot sec^{-2}$); F is the Rayleigh dissipation function; U is the potential energy of the bicycle and rider; m_1 is the mass of the rear part of the bicycle with rider (kg); l_1 and h_1 are the coordinates of the center of mass M_1 of the rear part (m); A_1 and B_1 are the central moments of inertia relative to the horizontal axis and the axis that is perpendicular to it, which lie in the plane of the rear wheel ($kg\cdot m^2$); D_1 is the centrifugal moment of inertia relative to these axes ($kg\cdot m^2$); C_1 is the moment of inertia of the rear wheel relative to its axis of intrinsic rotation ($kg \cdot m^2$); and T_1 is the kinetic energy of the rear part of the bicycle with rider.

The parameters of the front part of the bicycle are denoted by the same letters, with a subscript 2:

$$l_1 = \alpha \cos \varphi_0 + \beta \sin \varphi_0, \quad h_1 = R - \alpha \sin \varphi_0 + \beta \cos \varphi_0$$
$$l_2 = a \cos \varphi_0 + d \cos \lambda - d_1 \sin \lambda, \quad h_2 = R - a \sin \varphi_0 + d \sin \lambda + d_1 \cos \lambda$$
$$d = h_2 \sin \lambda + (l_2 - c) \cos \lambda - c_1$$

The refinement of the equations of motion of a bicycle and motorcycle as obtained in [1] involves the fact that in the expression for the angle φ in terms of ψ, χ we should take account of terms of the power series expansion in ψ, χ up to second-order small quantities. In this approximation, the angle φ is expressed by the relationship

$$\varphi = \varphi_0 + \frac{c_1}{2c}(\psi^2 \sin \lambda - 2\psi\chi) \tag{1.2}$$

Note that in [1, 2] the expression for φ was obtained only to within first-order small terms inclusively. It follows from (1.2), however, that in this approximation the angle φ is constant.

Taking account of (1.2), the kinetic energies T_1 and T_2 of the rear and front parts respectively have the form

$$T_1 = \frac{1}{2} m_1 \Big[(x^{\cdot} - l_1\theta^{\cdot} + h_1\chi^{\cdot})^2 - 2V(l_1\theta - h_1\chi)\theta^{\cdot} + 2Vh_1\chi^{\cdot}\theta +$$
$$+ 2V\frac{c_1}{c}(h_1 - R)(\sin\lambda\psi\psi^{\cdot} - \psi\chi^{\cdot} - \chi\psi^{\cdot}) \Big] + \frac{1}{2}(A_1\chi^{\cdot 2} - 2D_1\chi^{\cdot}\theta^{\cdot} + B_1\theta^{\cdot 2}) +$$
$$+ \frac{VC_1}{R}\chi\theta^{\cdot}$$

$$T_2 = \frac{1}{2} m_2(x^{\cdot} + h_2\chi^{\cdot} - l_2\theta^{\cdot} - d\psi^{\cdot})^2 + m_2 V \Big\{ (h_2\chi - l_2\theta - \tfrac{}{2}d\psi)\theta^{\cdot} +$$
$$+ \Big[\Big(\frac{c_1(h_2 - R)}{c} \sin\lambda - d\cos\lambda \Big)\psi - d\theta - \frac{c_1(h_2 - R)}{c}\chi \Big]\psi^{\cdot} +$$
$$+ \Big[h_2\theta - \frac{c_1(h_2 - R)}{c}\psi \Big]\chi^{\cdot} \Big\} + \frac{1}{2}[A_2(\chi^{\cdot} - \psi^{\cdot}\sin\lambda)^2 +$$
$$+ 2D_2(\chi^{\cdot} - \psi^{\cdot}\sin\lambda)(\theta^{\cdot} + \psi^{\cdot}\cos\lambda) + B_2(\theta^{\cdot} + \psi^{\cdot}\cos\lambda)^2] -$$
$$- \frac{C_2 V}{R}[\psi\chi^{\cdot}\cos\lambda + \theta^{\cdot}(\psi\sin\lambda - \chi)]$$

With the same degree of approximation, we can define the potential energy of the bicycle by the expression

$$U = -\frac{g}{2}\Big[h\chi^2 + \Big(m_2 d + \frac{lc_1}{c} \Big)(\psi^2 \sin\lambda - 2\psi\chi) \Big]$$

The Lagrange function $L = T_1 + T_2 - U$. We introduce the operator $L_q = (d/dt)(\partial L/\partial q^{\cdot}) - (\partial L/\partial q)$ and compute $L_x, L_\chi, L_\theta, L_\psi$:

$$L_x = mx^{\cdot\cdot} - l\theta^{\cdot\cdot} + h\chi^{\cdot\cdot} - m_2 d\psi^{\cdot\cdot}, \quad L_\chi = hx^{\cdot\cdot} - J_{12}\theta^{\cdot\cdot} - V(C_1^{\circ} + C_2^{\circ})\theta^{\cdot} + J_1\chi^{\cdot\cdot} -$$
$$- gh\chi - J_3\psi^{\cdot\cdot} - VC_2^{\circ}\cos\lambda\psi^{\cdot} + g\Big(m_2 d - \frac{lc_1}{c} \Big)\psi$$
$$L_\theta = -lx^{\cdot\cdot} + J_2\theta^{\cdot\cdot} - J_{12}\chi^{\cdot\cdot} + V(C_1^{\circ} + C_2^{\circ})\chi^{\cdot} + J_4\psi^{\cdot\cdot} - \psi^{\cdot}C_2^{\circ}V\sin\lambda \tag{1.3}$$
$$L_\psi = -m_2 dx^{\cdot\cdot} + J_4\theta^{\cdot\cdot} + VC_2^{\circ}\sin\lambda\theta^{\cdot} - J_3\chi^{\cdot\cdot} + C_2^{\circ}V\cos\lambda\chi^{\cdot} +$$
$$+ g\Big(m_2 d + \frac{lc_1}{c} \Big)\chi + J\psi^{\cdot\cdot} - g\Big(m_2 d + \frac{lc_1}{c} \Big)\sin\lambda\psi$$

Here we have introduced the following notation: $m = m_1 + m_2$ is the mass of the entire sys-

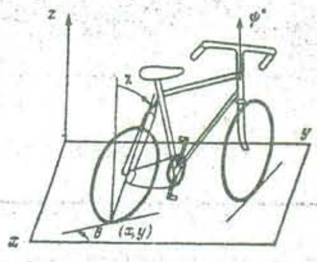

Fig. 1

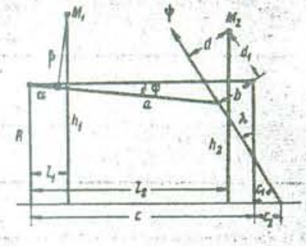

Fig. 2

tem (bicycle and rider),

$$\begin{gathered} l=m_1l_1+m_2l_2, \quad h=m_1h_1+m_2h_2 \\ J=m_2d^2+A_2\sin^2\lambda-D_2\sin 2\lambda+B_2\cos^2\lambda \\ J_1=m_1h_1^2+A_1+m_2h_2^2+A_2, \quad J_2=m_1l_1^2+B_1+m_2l_2^2+B_2 \\ J_3=m_2h_2d+A_2\sin\lambda-D_2\cos\lambda, \quad J_4=m_2dl_2-D_2\sin\lambda+B_2\cos\lambda \\ J_{12}=m_1h_1l_1+D_1+m_2h_2l_2-D_2, \quad C_1^\circ=C_1R^{-1}, \quad C_2^\circ=C_2R^{-1} \end{gathered}$$

In accordance with the equations of the kinematic relations that express rolling of the wheels without slip,

$$x^{\cdot}+V\theta=0, \quad c\theta^{\cdot}=c_1\psi^{\cdot}+\psi V\cos\lambda \tag{1.4}$$

the dynamic equations of the bicycle model under consideration can be written as follows:

$$L_\chi=0, \quad c\left(L_\psi+\frac{\partial F}{\partial\psi^{\cdot}}\right)+c_1L_\theta=0$$

Using (1.3), we can eliminate parameters x and θ from them using nonholonomic linkage equations (1.4), after which the equations of motion become

$$\begin{gathered} a_0\chi^{\cdot\cdot}-a_1\chi-a_2\psi^{\cdot\cdot}-a_3V\psi^{\cdot}+(a_4-a_5V^2)\psi=0 \\ b_0\psi^{\cdot\cdot}+b_1V\psi^{\cdot}+(b_2V^2-b_3)\psi-b_4\chi^{\cdot\cdot}+b_5V\chi^{\cdot}+b_6\chi=0 \end{gathered} \tag{1.5}$$

Here the expressions for the coefficients a_i and b_i are the same as the corresponding expressions in [1]. The only difference involves the coefficients a_4, b_3, b_6, in which the cofactor m_{2d} is replaced by m_2d+lc_1/c. This constitutes the refinement of the equations of motion of the bicycle.

2. Let us investigate the stability of rectilinear motion of an uncontrolled bicycle. Equations (1.5) describe the motion of the representative point in configuration space χ,ψ in which the equilibrium state at the coordinate origin $\chi=\psi=0$ corresponds to rectilinear motion of an uncontrolled bicycle. The stability of this state is determined by the roots of a fourth-order characteristic equation:

$$\begin{gathered} \alpha_0p^4+\alpha_1p^3+(\alpha_2-\varepsilon\alpha_3)p^2+(\alpha_4-\varepsilon\alpha_5)p+\varepsilon(\varepsilon\alpha_6-\alpha_7)=0 \\ \alpha_0=a_0b_0-a_2b_4, \quad \alpha_1=a_0b_1+a_2b_5-a_3b_4 \\ \alpha_2=a_0b_2+a_3b_5-a_5b_4, \quad \alpha_3=a_1b_0+a_0b_3-a_2b_6-a_4b_4 \\ \alpha_4=a_5b_5, \quad \alpha_5=a_1b_1+a_4b_5-a_3b_6, \quad \alpha_6=a_1b_3-a_4b_6 \\ \alpha_7=a_1b_2-a_5b_6, \quad \varepsilon=1/V \end{gathered} \tag{2.1}$$

The quantities a_i and b_i are given by the expressions

$$a_0=cJ_1, \quad a_1=cgh, \quad a_2=cJ_3+c_1J_{12}$$

$$a_3 = (J_{13}+cC_2^\circ)\cos\lambda + c_1(h+C_1^\circ+C_2^\circ), \quad a_4 = cg\left(m_2 d + \frac{c_1}{c}l\right)$$

$$a_5 = (h+C_1^\circ+C_2^\circ)\cos\lambda, \quad b_0 = cJ + 2c_1J_4 + \frac{c_1^2}{c}J_2, \quad b_1 = J_4\cos\lambda +$$

$$+ c_1\left(m_2 d + \frac{1}{c}J_2\cos\lambda\right) + \frac{lc_1}{c} + c\delta_1, \quad b_2 = \left(m_2 d + C_2^\circ\sin\lambda + \frac{lc_1}{c}\right)\cos\lambda$$

$$b_3 = cg(m_2 d + lc_1/c)\sin\lambda, \quad b_4 = cJ_3 + c_1J_{12}$$

$$b_5 = cC_2^\circ\cos\lambda + c_1(C_1^\circ+C_2^\circ), \quad b_6 = cg\left(m_2 d + \frac{lc_1}{c}\right), \quad \delta_1 = \frac{\delta}{V}$$

We fix all the parameters of the system under consideration except for the angle of inclination λ of the steering axis, the stagger σ of the front wheel, and the velocity V of the bicycle, which will vary in the following ranges: $0<\lambda<0{,}3$ rad, $0<\sigma<0{,}15$, $0<V<15$ m/sec.

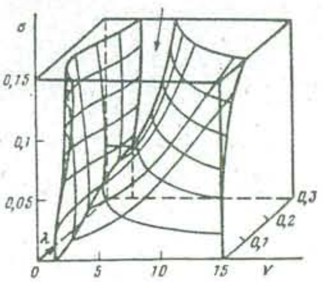

Fig. 3

Assume that the remaining parameters have the following values: $R = 0.35$ m, $l_1 = 0.4$ m, $c = 1.04$ m, $l_2 = 0.9$ m, $g = 9.8$ m/sec^2, $A_1 = 3.3$, kg$\cdot$m^2, $m_1 = 80$ kg, $A_2 = 0.036$ kg$\cdot$m^2, $m_2 = 2.7$ kg, $B_1 = 6.7$ kg$\cdot$m^2, $h_1 = 0.9$ m, $B_2 = 0.22$ kg$\cdot$m^2, $h_2 = 0.5$ m, $C_1 = C_2 = 0.5$ kg$\cdot$m^2.

Using the Routh-Hurwitz criterion, and an M-220 computer, we plotted the boundaries of the stability region in the space of parameters λ, σ, V; they are shown in Fig. 3, where the arrow indicates the stability region.

In plotting the boundaries we obtained values of V, σ, λ, satisfying the equations

$$\alpha_0=0,\ \alpha_1=0,\ \alpha_2-\varepsilon\alpha_3=0$$
$$\alpha_4-\varepsilon\alpha_5=0,\ \varepsilon\alpha_6-\alpha_7=0 \tag{2.2}$$
$$(\alpha_4-\varepsilon\alpha_5)[\alpha_1(\alpha_2-\varepsilon\alpha_3)-\alpha_0(\alpha_4-\varepsilon\alpha_5)]-\varepsilon(\varepsilon\alpha_6-\alpha_7)\alpha_1^2=0$$

which are obtained when the Routh-Hurwitz conditions are turned into equalities.

The above results yield the following conclusions.

1. Rectilinear motion of an uncontrolled bicycle is stable for any physical parameter relationships only on a finite velocity interval $V_1<V<V_2$, where the minimum and maximum velocities V_1 and V_2 are determined on Fig. 3 as the points of intersection of the straight line $\lambda=\mathrm{const}, \sigma=\mathrm{const}$ and the boundaries of the stability region.

2. To obtain motion that is stable on the largest interval of velocities, we should try to reduce the ange of inclination λ of the steering axis, while maintaining a sufficiently large stagger σ of the front wheel.

3. The resultant boundaries of the stability region enable the designer to choose, for a given bicycle model, the inclination λ of the steering axis and stagger σ of the front wheel in such a way as to ensure the greatest stability margin. A bicycle with these parameters will evidently possess optimum controllability as well.

4. Construction of the boundaries of the region of stable motion for various values of the coefficient of viscous friction δ in the steering column revealed that this coefficient does not markedly affect the behavior of these boundaries.

REFERENCES

1. Yu. I. Neimark and N. A. Fufaev, "Stability of an uncontrolled and controlled bicycle and motorcycle," Izv. AN SSSR. MTT [Mechanics of Solids], no. 2, p. 12, 1967.
2. Yu. I. Neimark and N. A. Fufaev, Dynamics of Nonholonomic Systems [in Russian], Nauka Press, Moscow, 1967.

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