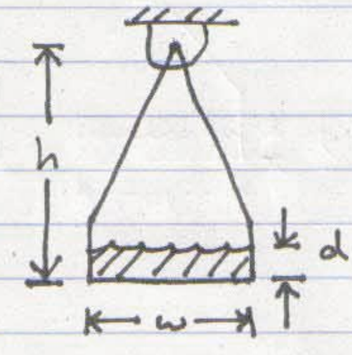
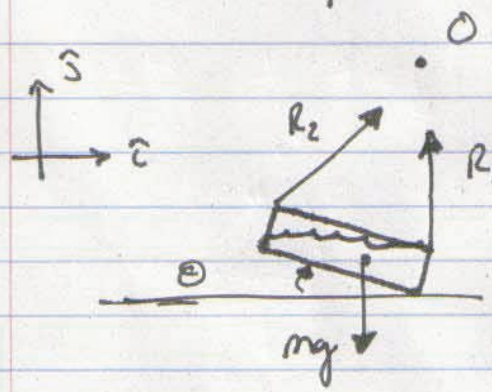
23 OCT 2002

Bucket of Water Stability



What is the minimum "h" for a stable system?

F.B.D. of tray



Moment balance about pt. O

$$\circlearrowleft + \ \Sigma \vec{M}_{/O} = \vec{r}_{cm/O} \times -mg\,\hat{j}$$

$$= (r_x \hat{i} + r_y \hat{j}) \times -mg\,\hat{j}$$

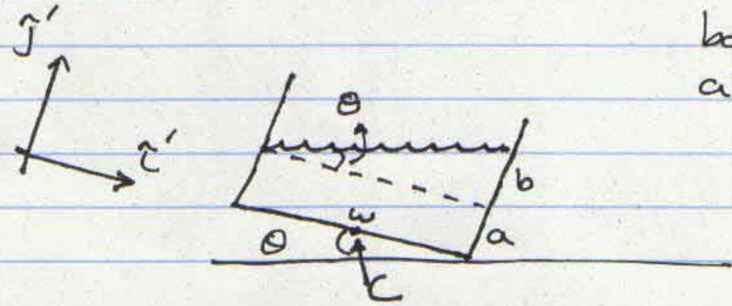
$$= -r_x mg\,\hat{k} + 0$$

(note R_1 and R_2 produce no moment because they are directed through pt. O)

The system is stable if after a small angular displacement, θ, the system returns to its initial horizontal position.

Thus, if θ is positive (as drawn above) the net moment must be positive for rotation to occur in the stable direction. Otherwise, the tray will continue to tip in the direction of initial displacement and the water will spill.

We need an expression for the center of mass of the water in terms of known variables (ω, d, h, and θ) to determine stability requirements for h.



Point "C" lies on the bottom of the tray at the center.

We assume no water spills initially, so the area of the water is a constant.

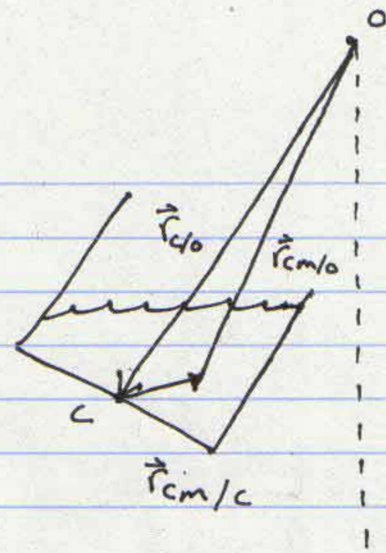
$$A = \omega d$$

Using this fact, we can solve for a and b in terms of d and θ and ω.

$$A = \omega d = \omega a + \tfrac{1}{2}\omega b$$
$$\Rightarrow \quad d = a + \tfrac{1}{2} b$$
$$b = \omega \tan\theta$$
$$a = d - \tfrac{1}{2} b = d - \tfrac{1}{2}\omega \tan\theta$$

Now we can find the center of mass of water with respect to point "C".

Area of rectangle $= \omega a$
Area of triangle $= \frac{1}{2}\omega b$
C.M. of rectangle $= \frac{1}{2} a \hat{j}'$
C.M. of triangle $= \frac{1}{6}\omega \hat{i}' + (a + \frac{1}{3} b)\hat{j}'$



$$\vec{r}_{cm/o} = \vec{r}_{cm/c} + \vec{r}_{c/o}$$

$$\vec{r}_{c/o} = -h\sin\theta\,\hat{\imath} - h\cos\theta\,\hat{\jmath}$$

$$\vec{r}_{cm/c} = \frac{1}{A_{tot}}\left[A_{rect}\,CM_{rect} + A_{tri}\,CM_{tri}\right]$$

$$= \frac{1}{wd}\left[wa\tfrac{1}{2}a\,\hat{\jmath}' + \tfrac{1}{2}wb\left(\tfrac{1}{6}w\,\hat{\imath}' + \left(a+\tfrac{1}{3}b\right)\hat{\jmath}'\right)\right]$$

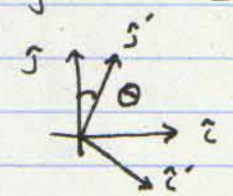
$$= \frac{1}{d}\left[\frac{1}{2}\left(a^2 + ab + \frac{b^2}{3}\right)\hat{\jmath}' + \frac{wb}{12}\,\hat{\imath}'\right]$$

$$= \frac{1}{d}\left[\frac{1}{2}\left(\left(d-\tfrac{1}{2}w\tan\theta\right)^2 + w\tan\theta\left(d-\tfrac{1}{2}w\tan\theta\right) + \tfrac{1}{3}w^2\tan^2\theta\right)\hat{\jmath}' + \frac{w^2\tan\theta}{12}\,\hat{\imath}'\right]$$

$$= \frac{1}{d}\left[\left(\frac{d^2}{2} + \frac{1}{24}w^2\tan^2\theta\right)\hat{\jmath}' + \frac{w^2\tan\theta}{12}\,\hat{\imath}'\right]$$

Note: $\hat{\imath}' = \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}$

$\hat{\jmath}' = \sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$



$$\vec{r}_{cm/c} = \frac{1}{d}\left[\left(\frac{d^2}{2} + \frac{1}{24}w^2\tan^2\theta\right)(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) + \frac{w^2\tan\theta}{12}(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath})\right]$$

Recall that we only care about the $\hat{\imath}$ component of $\vec{r}_{cm/o}$ $\left(\Sigma\vec{M}_{/o} = -r_x mg\,\hat{k}\right)$

So, we can write:

$$\vec{r}_{cm/o} \cdot \hat{\imath} = (\vec{r}_{cm/c} + \vec{r}_{c/o}) \cdot \hat{\imath} = r_x$$

$$r_x = -h\sin\theta + \frac{1}{d}\left[\left(\frac{d^2}{2} + \frac{1}{24}\omega^2\tan^2\theta\right)\sin\theta + \frac{\omega^2\tan\theta\cos\theta}{12}\right]$$

Recall that for stability, $r_x \leq 0$ because the net moment must be positive.

$$\Sigma\vec{M}_{/o} = -r_x mg\,\hat{k}$$

Thus, $-h\sin\theta + \frac{1}{d}\left[\left(\frac{d^2}{2} + \frac{1}{24}\omega^2\tan^2\theta\right)\sin\theta + \frac{\omega^2\sin\theta}{12}\right] \leq 0$

$$h \geq \frac{1}{d}\left[\frac{d^2}{2} + \frac{1}{24}\omega^2\tan^2\theta + \frac{\omega^2}{12}\right]$$

Because θ is small, $\tan\theta \approx \theta$

we notice that $\frac{1}{24}\omega^2\tan^2\theta \approx \frac{1}{24}\omega^2\theta^2$

Compared to $\frac{\omega^2}{12}$, this term is negligible.

$$h \geq \frac{1}{d}\left[\frac{d^2}{2} + \frac{\omega^2}{12}\right]$$

$$\boxed{h \geq \left[\frac{d}{2} + \frac{\omega^2}{12d}\right]}$$