‘Singularity’ functions

(Notes from office hours 12/1/2010 . . . and then some. Someone in office hours asked me to write this up and send it out, so here you go.)

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Motivation. For loads $w(x)$ that are *not* smooth, singularity functions make it easier to find the shear force V, bending moment M, slope u' and displacement u. To see singularity functions in use, see the solutions to Problem 15.27.

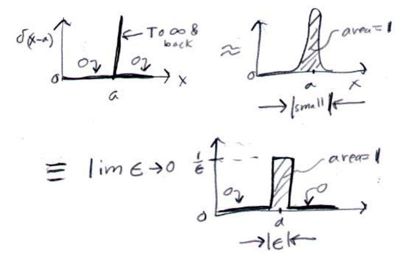

Figure 0.1: **The Dirac delta function** $\delta(x-a)$. By definition, the Dirac delta function is the bracket function with a superscript -1: $< x-a >^{-1} \equiv \delta(x-a)$. You can think of $\delta(x-a)$ as very tall and very narrow with a total area underneath of 1. In the third case above you have a box with width ϵ and height $1/\epsilon$.

Context

In general if you know the downwards load per unit length $w(x)$ on a beam you can integrate 4 times to find

$$\begin{aligned} V(x) &= -\int w(x)\,dx + C_1 \\ EIu'' = M(x) &= \int V(x)\,dx + C_2, \\ u' &= \int u''\,dx + C_3 \qquad \text{and} \\ u &= \int u'\,dx + C_4. \end{aligned} \tag{1}$$

These apply for any distributed loading $w(x)$.

But what kind of function $w(x)$ do you use for a concentrated load P; or a reaction force F; or for a load that is constant in some regions and zero in others; or when there is an applied couple? What are the functions $w(x)$ for such loads and how do you integrate them? The answers are: *singularity functions*.

Without singularity functions you have to find separate expressions for V, M,*etc.* for the regions to the left and to the right of such discontinuities. And then, as described in most books about beams, you have to pick integration constants in the two regions so that there is an appropriate jump, or not of the V, M, u' or u. Using singularity functions you can skip all this matching. You just follow the integration rules and the right quantities jump up and down the right ways.

Delta function and step function

Dirac delta function. The most famous[1] singularity function is called the Dirac delta function, or the ‘impulse’ function. If it applies at a it is written:

$$\delta(x-a) = \text{‘delta of x minus a’}.$$

What function is it? In the classical mathematics sense the delta function $\delta(x-a)$ isn’t a function. Mathematicians were upset about this for a while.

[1] Actually the most famous singularity function, the delta function, isn’t the most singular one, at least not in the mathematical sense of the word ‘singular’.

There are lots of ways to think about this function (See fig. 0.1). One way (that any real mathematician would hate) is is to think of $\delta(x-a)$ as a function of x that is zero except for very close to $x = a$. There, it is as tall as it is narrow. So the area underneath is one. Actually $\delta(x-a)$ is infinitely tall and infinitely narrow, but the area underneath is still one.

Concentrated load P. You can replace a downwards force P with a very large force per unit length $w(x)$ acting on a small region near $x = a$ but that has total force P:

$$\int_{\text{left of } a}^{\text{right of } a} w(x)dx = P.$$

But if the concentrated load is very high in value and very narrow in spatial extent, its just like the delta function. Thus for a concentrated load P at a the associated distributed load w is

$$w(x) = P\delta(x-a).$$

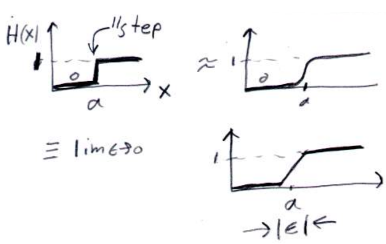

Figure 0.2: **Heaviside step function.** Compare these threecurves for the Heaviside step function with the three ways of thinking about the delta function.

Heaviside step function. There are two primary ways to think of the Heaviside step function:

1. The step function is the integral of the delta function, informally:

$$\int \delta(x-a)\, dx = H(x-a)$$

or more formally

$$\int_{-\infty}^{x} \delta(x'-a)\, dx' = H(x-a).$$

That is, the Heaviside step function is the cumulative area under the delta function curve.

2. The Heaviside step function is that function of x that is zero to the left of a and one to the right of a,

$$H(x-a) \equiv \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x \geq a. \end{cases}$$

This is pictured in the first of fig. 0.2.

Shear V for a concentrated load. If there is various loading on a beam, part of which is a concentrated load P at a then the whole $w(x)$ function is several terms of which we only write out the key term $P\delta(x-a)$.

$$w(x) = \ldots P\delta(x-a)\ldots.$$

So, integrating,

$$\begin{aligned} V(x) &= -\int w(x)\,dx \\ &= -\int \ldots \qquad \underbrace{P\delta(x-a)}_{P \text{ times the delta function}} \qquad \ldots\,dx \\ &= \ldots \qquad \underbrace{-PH(x-a)}_{-P \text{ times the step function}} \qquad \ldots. \end{aligned}$$

If there is a concentrated load P at a then the shear force function $V(x)$ has a step down of size P at a.

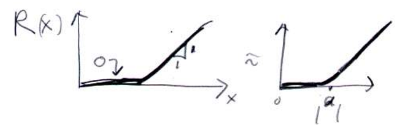

Figure 0.3: **The unit ramp function.**

The ramp function. See fig. 0.3. This one is not so famous②. Think of the ramp function as $(x-a)$ but for that its set to zero if $x < a$. So we define

$$\underbrace{R(x-a)}_{\text{Unit ramp function}} \qquad \equiv < x-a >^1 \equiv \begin{cases} (x-a) & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$

②The ramp function is actually the Macauley ramp function, but Macauley isn't famous like Dirac or Heaviside.

The brackets <> mean that you should think of the whole expression as being zero if $x < a$. For $x > a$ the brackets are like ordinary parentheses (at least for $n \geq 0$). Its easy to see that

$$\int H(x-a)\,dx = R(x-a)$$

$$\text{or more formally,} \quad \int_{-\infty}^{x} H(x'-a)\,dx' = R(x-a)$$

The switched on parabola, *etc*. We can keep defining new functions this way with higher and higher powers, *e.g.,*

$$< x-a >^2 \equiv \begin{cases} (x-a)^2 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$

and

$$< x-a >^3 \equiv \begin{cases} (x-a)^3 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$

In the same way that the step function integrates to the ramp we have

$$\int < x-a >^1\,dx = < x-a >^2 /2, \qquad \text{and}$$

$$\int < x-a >^2\,dx = < x-a >^3 /3.$$

Moment, slope and deflection due to a concentrated load. A concentrated load P at a contributes $P\delta(x-a)$ to $w(x)$. We already integrated to get shear, now lets keep going,

$$
\begin{aligned}
& w(x) = \ldots P\delta(x-a) + \ldots \\
V = -\int w \quad \Rightarrow \quad & V(x) = \cdots - PH(x-a) + \ldots \\
M = \int V \quad \Rightarrow \quad EIu''(x) = & M(x) = \cdots - PR(x-a) + \ldots \\
u' = \int u' \quad \Rightarrow \quad & EIu'(x) = \cdots - P < x-a >^2 /2 + \ldots \\
u = \int u' \quad \Rightarrow \quad & EIu(x) = \cdots - P < x-a >^3 /6 + \ldots
\end{aligned} \tag{2}
$$

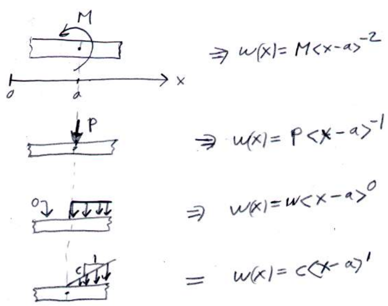

Figure 0.4: **How to represent loads with singularity functions.** From top to bottom: an applied couple at $x = a$, a concentrated load at $x = a$, a step in the distributed load at $x = a$, and a ramp function starting at $x = a$. The most important case is the concentrated load P.

Every concentrated load leads to a displacement term that is cubic in x.

More brackets. So that we don't have to remember all those names (Dirac, Heaviside and Macauley) nor the names of their functions (δ, H, R) we invent a notation that covers all cases, the last of which we already defined.

$$
\begin{aligned}
< x-a >^{-1} &\equiv \delta(x-a), \\
< x-a >^{0} &\equiv H(x-a), \qquad \text{and} \\
< x-a >^{1} &\equiv R(x-a).
\end{aligned}
$$

We can now write eqn. (2) with this notation.

$$
\begin{aligned}
\text{Given that} \quad w(x) &= \ldots \quad P < x-a >^{-1} + \ldots \\
\Rightarrow \quad V(x) &= \cdots - P < x-a >^{-0} + \ldots \\
\Rightarrow \quad \underbrace{M(x)}_{EIu''} &= \cdots - P < x-a >^{1} + \ldots \\
\Rightarrow \quad EIu'(x) &= \cdots - P < x-a >^{2} /2 + \ldots \\
\Rightarrow \quad EIu(x) &= \cdots - P < x-a >^{3} /6 + \ldots
\end{aligned} \tag{3}
$$

Note, if the superscript n is positive, think of it as an exponent. It it is negative it is *not* and exponent. Its just a label marking the degree of singularity: -1 is singular, -2 is more so, etc.

Applied couples. If a couple is applied to a beam it is like a two big equal and opposite forces next to each other. You can see this as the derivative of the delta function. For an applied counterclockwise couple M applied at a we write

$$
w(x) = M < x-a >^{-2}
$$

and use the integration rule $\int < x-a >^{-2}\, dx =< x-a >^{-1}$. Remember, for negative n you don't think of the superscipt n as a power, but just as a label.

The general case

Here is the full definition of the singularity functions and the general formulas for their integration.

$$
\text{If} \quad n < 0 : < x-a >^n \equiv \begin{cases} 0 & \text{if } x < a, \\ \text{undefined} & \text{if } x = a, \\ 0 & \text{if } x > a, \end{cases}
$$

$$
\text{If} \quad n \geq 0 : < x-a >^n \equiv \begin{cases} 0 & \text{if } x < a, \\ (x-a)^n & \text{if } x \geq a, \end{cases} \tag{4}
$$

$$
\text{If} \quad n \leq 0 : \quad \int < x-a >^n \; dx \equiv < x-a >^{n+1}
$$

$$
\text{If} \quad n \geq 0 : \quad \int < x-a >^n \; dx \equiv < x-a >^{n+1} / (n+1) \tag{5}
$$

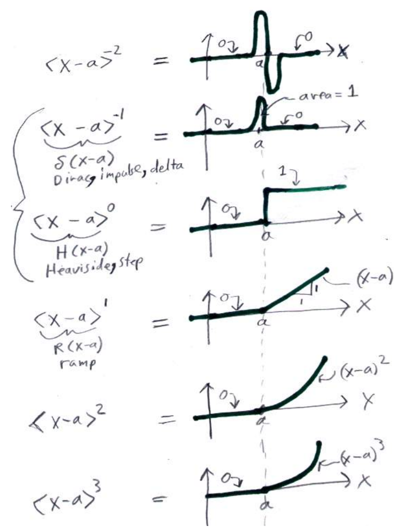

Figure 0.5: Graphical representation of the singularity functions $< x-a >^n$ for $-2 \leq n \leq 3$. The most famous and most used of these, in all manner of scientific and engineering disciplines, are **the Dirac delta function** $\delta(x-a) =< x-a >^{-1}$ and **the Heaviside step function** $H(x-a) =< x-a >^0$. The ramp, quadratic functions and so on are useful for beam problems, but not so useful in the rest of science and engineering. The derivative of the delta function, $< x-a >^{-2}$ is mathematically even more singular than the delta function. It has some uses outside of beam deflection problems, but is much more rarely seen than the delta impulse function or the Heaviside step function.