Multiple choice exam.

How can you learn best about stress and strain?

a) fly on Apollo 13,
b) talk to people whose pulse is two standard deviations faster than normal,
c) go to graduate school,
d) take TAM 663.

Answer: c & d. Go to graduate school at Cornell and relaxedly enjoy TAM 663.

T&AM 663, Fall 1998

Introduction to Solid Mechanics

(Introduction to Stress and Strain)

How do you measure the load on a material? How do you describe the deformation of solids? How does the load affect the deformation? How do you describe the differerence between materials? How does load in one place affect deformation in another? How do you quantify the strength of a solid?

Andy Ruina, draft August 15, 1998

Note day, time and room change.

Tuesday, Thursday 1:25-3:00 PM, Thurston 201. Some meetings in a room TBA Thursdays from 3:00-4:30. First meeting Thursday August 27 at 1:25 PM.

Recommended text: *CONTINUUM MECHANICS for ENGINEERS* by Mase & Mase, CRC Press. We will not cover the fluid mechanics part of this book. We will cover elasticity topics not covered in this book, several books will be on reserve in the library that cover these topics (more or less): Malvern, Fung, Schaum's Outline.

Homeworks, from the text and from handouts, will be assigned at most lectures and due about one week later. The homework will marked for needed improvement. At the end of the semester a complete set of correctly done homework will be handed in. There will be a closed book, closed note final exam with some problems closely modeled on the homework. Labs are described in a separate handout. Lab time will be scheduled in lecture.

Approximate Syllabus

Lectures#	Lecture Content
1-3.	**Introduction. Math Preliminaries (1st lecture of three).** Vectors: components, base vectors. Einstein summation convention. Polar coordinates. Tensors: Linear operators, dyadics, matrices. $\delta_{ij}, \epsilon_{ijk}$. Change of basis. Change of coordinates. Rotation. Diagonalization. Gauss theorem. Divergence theorem. ∇ (direct, cartesion and polar), grad and divergence.
4&5	**Introduction to stress.** True stress (Cauchy stress) only(!). Surface forces, body forces, Cauchy tetrahedron. Linear and Angular momentum balance: integral form, differential form, little cubes. Polar coord using $\nabla \cdot \underline{\sigma}$ and with a little rhombus.
6.	**Introduction to Deformation, keeping track of fibers.** x, X, F. Polar Decomposition: **R**, **U** and **V**. Strain: **E**. Examples: isotropic stretch, one dimensional stretch, simple shear, pure rotation.
7.	**Small Deformations.** Approach I: $\mathbf{R} \approx \mathbf{I} + \underline{\underline{\omega}}$ & $\mathbf{U} \approx \mathbf{I} + \underline{\underline{\epsilon}}$. Approach II: $\epsilon_{ij} = sym\{u_{i,j}\}$ and $\omega_{ij} = antisym\{u_{i,j}\}$. Examples: pure stretch, isotropic stretch, simple shear, pure shear, pure rotation. Bulk strain, engineering strain. Compatability (necessary conditions only). Plane strain and compatability. Anti-plane strain. Average strain (using divergence theorem).
8&9	**<u>P</u>rinciple of <u>V</u>irtual <u>W</u>ork.** Linear momentum balance + compatability $+(\sigma_{ij} = \sigma_{ji}) \Rightarrow$ PVW. PVW $+(\sigma_{ij} = \sigma_{ji}) \Rightarrow$ linear momentum balance. A convenient statement of divergence theorem, whether or not equilibrium is satisfied. Use of PVW to approximate equilibrium in numerical calculations. PVW eqn with the actual stress and actual strain rate and the energy (power, actually) balance equation.
10.	**Introduction to Constitutive laws, uniaxial only.** Elastic: linear, non-linear, path independence. Linear visco-elastic (especially simple models). Plastic (simple models). Rate dependent inelastic laws. General constitutive laws.
11.	**Introduction to LINEAR ELASTICITY.** 3D constitutive laws (81 constants $\Rightarrow$ 36 constants). Use strain-displacement relations, or strain-compatability relations. Momentum Balance. Common types of boundary conditions and boundary value problems. Sample solutions: uniform stress and strain.
12.	**Strain Energy.** Definition. Path independence. Force field analogy and the existence of a potential. Strain energy density and the total strain energy. 36 constants $\Rightarrow$ 21 constants.
13.	**Linearity and superposition.** The fact. Examples.
14.	**Symmetry and the elastic constants.** Orthotropic, cubic, axi-symmetric, isotropic. Isotropic elastic constants (E, ν, G, λ,& κ) and their meanings (simple experiments which isolate them).
15.	**Potential and Complementary Energy.** Definition. Minimum principles. Uniqueness.
16.	**Reciprocal Theorem** The fact. Examples.
17.	**Approximate solutions and numerical methods.** Raleigh-Ritz. Finite Element.
18-19.	**Relation of exact 3-D theory to Strength of materials.** Rod in tension. Round rod in torsion. Beam in pure bending.
20.	**2-D Theories.** Plane strain, plane stress, anti-plane strain. Governing equations. Motivating concepts.
21-24.	**2-D Elasticity Solutions** Pressurized hole, squeezed cylinder. Sine-wave on a half-space. Stress concentration at a hole. Stress function. Shear lag (approximate) models. Approximate numerical solutions.
25.	**Saint Venant Introduction to elastodynamics. Plane waves.**
26&27.	**Saint Venant Theory of Torsion.**
28.	**Unscheduled.**

Solid Mechanics 663. 1991 FALL. page 1. Lecture 1.

other textbooks, Malvern, Fung, Long.

Homework Assignments, lab.

1. Introduction to solids; especically linear elasticity.

2. Course Chronology : see syllabus.

3. Structure of Subjects

a) Geometry of deformation & MOTION : Kinematics, compatibility, position, displacement, velocity, deformation, strain.

b) Laws of Mechanics.

A. Free body diagram. (Force is a measure of interaction).

B. Principle of Action & Reaction.

C. Momentum (Linear) Balance & Angular momentum balance.

$$\Sigma \underset{\sim}{F} = \dot{\underset{\sim}{L}} \qquad \Sigma \underset{\sim}{M} = \dot{\underset{\sim}{H}}$$

c) Material behaviour.

Material imposes restriction on relationship between force & motion of stress & strain.

Constitutive law : elastic, plastic, viscous, visco-elastic, friction, collision, etc.

d) General Results based on A, B & C for a particular materials & possibly using 1st & 2nd law of thermodynamics : Elasticity, energy thms, restriction on const. laws, plasticity theory, etc.

e) Solution of Boundary Value Problems. to predict deformation, motion, stress, failure ....

Given a Geometry, loads, material description : Goal is to find some deformation or stress value of interest.

Many Methods : 1) Analytic solutions of PDE.

2) Numerical Solution

3) "Strength of solids" solns (beam, Rod, shaft, plate, shell)

4) 2.Dim. approximation.

Lecture 1.

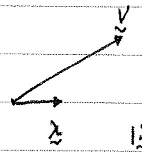
The language of Solid Mechanics uses scalar, vectors & tensors. [Read Chapter 1 & 2].

Scalars : e.g. Work, Energy, energy density, mass density, change of volume / unit volume. principal stresses & components of vectors & tensors.

Vectors : Force, velocity, position, displacement, acceleration, traction, surface normal. base vectors, principal stress directions ....

Tensors ; Stress, deformation gradient, strain, rotation, Elasticity tensor.

Vectors Think of an arrow with length, direction, projection in various direction.



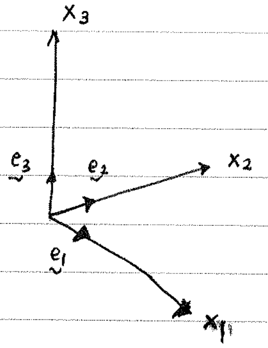
$|\underset{\sim}{\lambda}| = 1$

proj of $\underset{\sim}{v}$ on $\underset{\sim}{\lambda}$ $= \underset{\sim}{v} \cdot \underset{\sim}{\lambda}$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = |\underset{\sim}{v}|\,|\underset{\sim}{w}| \cos\theta_{(\underset{\sim}{v}\underset{\sim}{w})} = |\underset{\sim}{w}|$ proj of $\underset{\sim}{v}$ on $\underset{\sim}{w}$ direction

Look at basis $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ $\quad \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$

$|\underset{\sim}{e}_i| = 1.$



$\underset{\sim}{v} = v_i \underset{\sim}{e}_i$

$v_i = \underset{\sim}{v} \cdot \underset{\sim}{e}_i \qquad i = 1, 2, 3.$

since $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$

Consider a different basis, $\underset{\sim}{e}_1', \underset{\sim}{e}_2', \underset{\sim}{e}_3'.$

$\underset{\sim}{v} = v_i' \underset{\sim}{e}_i' = v_j \underset{\sim}{e}_j$

$$(v_i' \underset{\sim}{e}_i') \cdot \underset{\sim}{e}_K' = v_i' \delta_{iK} = v_K'$$
$$= (v_j \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_K'$$
$$= v_j (\underset{\sim}{e}_j \cdot \underset{\sim}{e}_K')$$

$$\therefore \quad v_K' = (\underset{\sim}{e}_j \cdot \underset{\sim}{e}_K') v_j$$

Notation

Einstein's summation notation: If a given subscript $\overset{i}{\wedge}$ appear twice in a multiplicative expression, then $\sum_{i=1}^{3}$ is assumed.

e.g. $v_i \underset{\sim}{e}_i = \sum_{i=1}^{3} v_i \underset{\sim}{e}_i$ $\qquad$ $v_j w_j = \sum_{j=1}^{3} v_j w_j$

$$a_i b_j c_K d_j = \sum_{j=1}^{3} a_i c_K b_j d_j$$

$$c_{ijKl}\, \epsilon_{Kl} = \sum_{K=1}^{3} \sum_{l=1}^{3} c_{ijKl}\, \varepsilon_{Kl}$$

Repeated subscripts are called Dummy indexes $\quad V_i W_i = V_j W_j = V_K W_K$

Kronecker delta: $\delta_{ij} = 0$ if $i \neq j$

$= 1$ if $i = j$

Ex. $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$

Ex $\underset{\sim}{u} \cdot \underset{\sim}{v} = (u_i \underset{\sim}{e}_i) \cdot (v_j \underset{\sim}{e}_j) = u_i v_j (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j) = u_i v_j \delta_{ij} = u_i v_i$

Puzzle: $\delta_{ii} = 3 \qquad \delta_{11} + \delta_{22} + \delta_{33} = 3$

Alternating symbal ε_{ijK}. (27 nos, $-1, 0, 1$)

$\varepsilon_{ijK} = 0$ if $i=j$ OR $j=K$ OR $i=K$

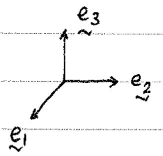
1 if $123, 231, 312 = ijK$

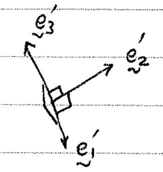
-1 if $ijK = 213, 132, 321$.

Lecture 2 TAM 663. page 1.

Math Preliminaries.

Recall :





$\underset{\sim}{v} = v_i \underset{\sim}{e}_i = v_i' \underset{\sim}{e}_i'$

Question from last class : Given $\underset{\sim}{e}_i'$, $\underset{\sim}{e}_i$, v_i , find v_i'

Ans : $v_i' = (\underset{\sim}{v} \cdot \underset{\sim}{e}_i') = (v_j \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i' = v_j (\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i') = (\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j) v_j$

Define $a_{ij} = \underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j \Rightarrow v_i' = a_{ij} v_j$

$$\underset{\sim}{e}_i' = (\underset{\sim}{e}_i' \cdot \underset{\sim}{e}_j) \underset{\sim}{e}_j = a_{ij} \underset{\sim}{e}_j$$

Also $\underset{\sim}{e}_i = a_{ji} \underset{\sim}{e}_j' = a^T_{ij} \underset{\sim}{e}_j'$

$[A]_\sim$ matrix , Notation.

a_{ij}'s are the direction cosines between $\underset{\sim}{e}_i'$ and $\underset{\sim}{e}_j$

$[A]$ is an orthogonal matrix, i.e., $[A][A]^T = [I] = [A]^T[A]$

To see this, $\underset{\sim}{e}_i = a_{ji} \underset{\sim}{e}_j' = a_{ji} a_{jl} \underset{\sim}{e}_l$

$$\therefore \quad a_{ji} a_{jl} = \delta_{il} \quad \underline{\text{or}} \quad [A]^T[A] = [I]$$

Back to ε_{ijk} :

$\varepsilon_{ijk} = 0$ if $i=j$, $j=k$ or $i=k$

$= 1$ even permutation

$= -1$ odd "

for determinant & vector cross products.

$\underset{\sim}{v} \times \underset{\sim}{w} = |\underset{\sim}{v}||\underset{\sim}{w}| \sin\theta \, \underset{\sim}{\lambda}$ unit vector with direction given by Right hand Rule

$\underset{\sim}{\lambda} \perp \underset{\sim}{v}$ and $\underset{\sim}{\lambda} \perp \underset{\sim}{w}$.

$$= (v_2 w_3 - v_3 w_2)\underset{\sim}{e}_1 + (v_3 w_1 - v_1 w_3)\underset{\sim}{e}_2 + (v_1 w_2 - v_2 w_1)\underset{\sim}{e}_3 = \varepsilon_{ijk} v_i w_j \underset{\sim}{e}_k .$$

Show that $\underset{\sim}{v} = v_i \underset{\sim}{e}_i$ $\underset{\sim}{w} = w_j \underset{\sim}{e}_j$ assume distributive rule $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$ work.

Page 2. Lecture 2.

Determinants (see text). "Useful identity" $\epsilon - \delta$ ~~identical~~. identity.

$$\epsilon_{jkq}\epsilon_{miq} = \delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}$$

Example 2.2.12 in text. $\underset{\sim}{n}$ unit vector.

$$\underset{\sim}{v} = (\underset{\sim}{a}\cdot\underset{\sim}{n})\underset{\sim}{n} + \underset{\sim}{n}\times(\underset{\sim}{a}\times\underset{\sim}{n}) = \underset{\sim}{a}.$$

What is $\underset{\sim}{v}$?

Dyads, Dyadics ; Linear Operator & tensor.

Define "Dyad" as two vectors side by sides e.g. $\underset{\sim}{u}\underset{\sim}{v}$, tensor product. or $\underset{\sim}{u}\otimes\underset{\sim}{v}$.

Define "Dyadics" as a sum of Dyads. $\underset{\sim}{u}\underset{\sim}{v} + \underset{\sim}{s}\underset{\sim}{t} + \underset{\sim}{a}\underset{\sim}{b} \dots$

$$\underset{\sim}{u}\underset{\sim}{v} \neq \underset{\sim}{v}\underset{\sim}{u}$$

Assume the distributive law holds.

so that $\underset{\sim}{u}\underset{\sim}{v} = u_i v_j \underset{\sim}{e}_i \underset{\sim}{e}_j$

Define a dot product of dyads with vectors and with Dyads,

$$\underset{\sim}{u}\underset{\sim}{v}\cdot\underset{\sim}{w} \equiv \underset{\sim}{u}(\underset{\sim}{v}\cdot\underset{\sim}{w}) = (\underset{\sim}{v}\cdot\underset{\sim}{w})\underset{\sim}{u}$$

$$\underset{\sim}{w}\cdot(\underset{\sim}{u}\underset{\sim}{v}) = (\underset{\sim}{w}\cdot\underset{\sim}{u})\underset{\sim}{v}$$

Note $\underset{\sim}{u}\underset{\sim}{v}\cdot\underset{\sim}{w} \neq \underset{\sim}{w}\cdot(\underset{\sim}{u}\underset{\sim}{v})$

$$\underset{\uparrow\ \text{Dyadics}}{\underline{\underline{T}}} = T_{ij}\underset{\uparrow\ \text{sum of 9 Dyads.}}{\underset{\sim}{e}_i\underset{\sim}{e}_j}$$

$$\underline{\underline{T}}\cdot\underset{\sim}{v} = (T_{ij}\underset{\sim}{e}_i\underset{\sim}{e}_j)\cdot(v_K e_K) = T_{ij}\,\underset{\sim}{e}_i\, v_K\delta_{jK} = T_{iK}v_K\underset{\sim}{e}_i$$

page 3 lecture 2.

Linear operators and tensors:

∃ function that have vector input and vector output.

$$\underline{L}(\underset{\sim}{v}) = \text{a vector which depends linearly on } \underset{\sim}{v}$$

Defn. of Linear Operator:

$$\underline{L}(a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2}) = a_1 \underline{L}(\underset{\sim}{v_1}) + a_2 \underline{L}(\underset{\sim}{v_2})$$

for all scalar a_1, a_2 and vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}$.

Look at $\underline{L}(\underset{\sim}{e_i}) = \underset{\sim}{t_i}$, then

$$\underline{L}(\underset{\sim}{v}) = \underline{L}(v_i \underset{\sim}{e_i}) = v_i \underline{L}(\underset{\sim}{e_i}) = v_i \underset{\sim}{t_i} = V_j \underline{t}_j$$

so a linear operator is completely defined by its action on the basis.

Recall Dyadics.

Look at Dyadics. $\underline{\underline{T}} = \underset{\sim}{t_1}\underset{\sim}{e_1} + \underset{\sim}{t_2}\underset{\sim}{e_2} + \underset{\sim}{t_3}\underset{\sim}{e_3} = \underset{\sim}{t_i}\underset{\sim}{e_i}$

$$\underline{\underline{T}} \cdot \underset{\sim}{e_i} = \underset{\sim}{t_i} \qquad \underline{\underline{T}} \cdot \underset{\sim}{v} = v_i \underset{\sim}{t_i}$$

Note $\underset{\sim}{t_i} = \overset{T_{ji}}{\cancel{\sigma_{ij}}} \underset{\sim}{e_j}$ $\qquad \underline{\underline{T}} = \cancel{T_{ij}\underset{\sim}{e_j}\underset{\sim}{e_i}}$ $T_{ji} \underline{e}_j \underline{e}_i = T_{ij} \underline{e}_i \underline{e}_j$ $\qquad \xrightarrow{\quad}$ component of tensor.

~~$\underline{\underline{T}} \cdot \underset{\sim}{v} = v_i \overset{T}{\sigma}_{ij} \underset{\sim}{e_j} = \overset{T}{\sigma}_{ij} v_i e_j$~~

$$\underline{\underline{T}} \cdot \underline{V} = T_{ij}\, \underline{e}_i \underline{e}_j \cdot V_K \underline{e}_K$$

$$= T_{ij}\, \underline{e}_i\, \delta_{Kj}\, V_K$$

$$= T_{ij}\, V_j\, \underline{e}_i$$

Lecture 3. page 1. Sept. 7, 1992.

Matrix representation of a Tensor $\underset{\sim}{T} \equiv [T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$

Change of basis. $\underset{\sim}{e}_i, \underset{\sim}{e}'_j$ $\qquad \underset{\sim}{e}'_i = \underbrace{(\underset{\sim}{e}'_i \cdot \underset{\sim}{e}_j)}_{a_{ij}} \underset{\sim}{e}_j$ $\qquad v'_j = a_{ij} v_j = (\underset{\sim}{e}'_i \cdot \underset{\sim}{e}_j) v_j$

$[V'] = A[V]$

matrix $= [a_{ij}]$

$$T_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j = T'_{Kl} \underset{\sim}{e}'_K \underset{\sim}{e}'_l \qquad T_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j \cdot \underset{\sim}{e}'_r = T'_{Kl} \underset{\sim}{e}'_K \delta_{lr}$$

$$T_{ij} (\underset{\sim}{e}_j \cdot \underset{\sim}{e}'_r) \underset{\sim}{e}_i = T'_{Kr} \underset{\sim}{e}'_K$$

Dotted with $\underset{\sim}{e}'_s$ $\qquad T_{ij} \underbrace{(\underset{\sim}{e}_j \cdot \underset{\sim}{e}'_r)}_{a_{rj}} \underbrace{(\underset{\sim}{e}_i \cdot \underset{\sim}{e}'_s)}_{a_{si}} = T'_{Kr} \delta_{Ks} = T'_{sr}.$

$$T'_{sr} = a_{si} a_{rj} T_{ij}$$

$$[T'] = a_{rj} [AT]_{sj} = [A][T][A]^t$$ ← use as a defn. as a tensor in some books.

Special Tensors. Symmetric tensor $T_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j = T_{ji} \underset{\sim}{e}_i \underset{\sim}{e}_j$ $\qquad \underset{\approx}{T} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\approx}{T}$

or $T_{ij} = T_{ji}$ (for orthogonal basis).

Anti-symmetric tensor $\qquad \underset{\approx}{T} = -\underset{\approx}{T}^t$

Orthogonal tensors : $\underset{\approx}{R} \cdot \underset{\approx}{R}^t = \underset{\approx}{I}$ $\qquad R_{ij} R_{jK} = \delta_{jK}$

$\underset{\approx}{R}^t$ represents a Rotation : $\qquad |\underset{\approx}{R} \cdot \underset{\sim}{v}| = |\underset{\sim}{v}| \qquad (\underset{\approx}{R} \cdot \underset{\sim}{v}) \cdot (\underset{\approx}{R} \cdot \underset{\sim}{w}) = \underset{\sim}{v} \cdot \underset{\sim}{w}$

Key Fact. For any symmetric tensor $\underset{\approx}{T}$, ∃ a orthornormal basis $\underset{\sim}{n}_i$ so that

$$\underset{\approx}{T} = T_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j = \lambda_i \underset{\sim}{n}_i$$

since $\underset{\approx}{T}$ has three orthogonal eigenvectors. $\underset{\approx}{T} \cdot \underset{\sim}{n}_i = \lambda_i \underset{\sim}{n}_i$ (No sum).

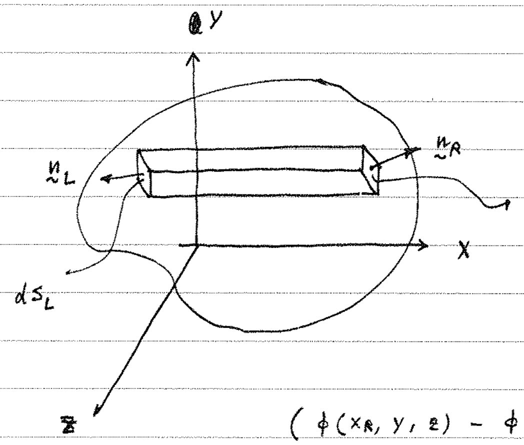
Fundamental thm. of Calculus.

$$\phi_2 = \phi_1 + \int_{\phi_1}^{\phi_2} d\phi = \phi_1 + \int_{x_1}^{x_2} \underset{\substack{\| \\ \phi_{,x}}}{\frac{d\phi}{dx}} dx$$

page 2. Lecture 3.

Now $\phi = \phi(x,y,z)$ think of (y,z) fixed.

Then $\phi(x_R,y,z) - \phi(x_L,y,z) = \int_{x_L}^{x_R} \phi_{,x}\,dx$



ds_R = area of prism top on the Right.

$$dA_x = \Big(ds_R \overbrace{\underset{\sim}{n}_R \cdot \underset{\sim}{i}}^{n_{x_R}}\Big)$$

$$\Big(\underset{\substack{\| \\ \phi_R}}{\phi(x_R,y,z)} - \phi(x_L,y,z)\Big)\overset{dA}{\cancel{dA}} = \left[\int_{x_L}^{x_R} \phi_{,x}\,dx\right] dA$$

$$\underbrace{\left(\phi_R\, ds_R\, n_{x_R} - \phi_L\, ds_L\, n_{x_L}\right)}_{\int_{\text{end caps}} \phi\, n_x\, ds} = \int_{\text{prism}} \phi_{,x}\, dV$$

Add up all prisms in entire volume, ~~the side surfaces cancel each other~~ so that

$$\sum_{\text{prism}} \int_{\text{prism}} \phi_{,x}\,dV = \int_V \phi_{,x}\,dV = \int_S \phi\, n_x\, ds.$$

or $\int_V \phi_{,i}\,dV = \int_S \phi\, n_i\, ds$ Gauss's thm

→ can apply Gauss to anything, components of vectors & tensors.

→ can add Gauss thm applied to various components,

e.g. $\int_S v_i n_i\, ds = \int_V v_{i,i}\, dV \Rightarrow \int_V \nabla\cdot\underset{\sim}{v}\, dV = \int_S \underset{\sim}{v}\cdot\underset{\sim}{n}\, ds$ Divergence theorem.

OR

$$\int_V \phi_{,i}\, dV = \int_S \phi\, n_i\, ds \Rightarrow \underbrace{\int_V \phi_{,i}\, \underset{\sim}{e_i}\, dV}_{} = \int_S \phi\, n_i\, \underset{\sim}{e_i}\, ds$$

$$\int_V \nabla\phi\, dV = \int_S \phi\, \underset{\sim}{n}\, ds.$$

page 3 Lecture 3.

The gradient operator $\nabla\phi \equiv \phi_{,i}\underset{\sim}{e}_i$ in cartesian coordinates.

$$d\phi(x,y,z) = d\phi = \underset{\sim}{\nabla}\phi\cdot d\underset{\sim}{r}$$

$$d\phi(x,y,z) \equiv d\phi \equiv \phi(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - \phi(x,y,z)$$

$$= \phi(\underset{\sim}{r}+d\underset{\sim}{r}) - \phi(\underset{\sim}{r})$$

since $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$= \nabla\phi\cdot d\underset{\sim}{x}$ $\quad d\underset{\sim}{x} = \overset{dx_1}{dx}\,\underset{\underset{\underset{\sim}{e}_1}{\parallel}}{\underset{\sim}{i}} + \overset{dx_2}{dy}\,\underset{\underset{\underset{\sim}{e}_2}{\parallel}}{\underset{\sim}{j}} + \overset{dx_3}{dz}\,\underset{\underset{\underset{\sim}{e}_3}{\parallel}}{\underset{\sim}{k}} = dx_i\underset{\sim}{e}_i$

$= (\phi_{,i}\underset{\sim}{e}_i)(dx_j\underset{\sim}{e}_j)$

$\therefore\ \nabla\phi = \phi_{,i}\underset{\sim}{e}_i$

Define $\nabla\phi$ as the vector so that $d\phi = \underset{\sim}{\nabla}\phi\cdot d\underset{\sim}{r}$

In polar coord: $d\underset{\sim}{r} = dr\,\underset{\sim}{e}_r + (r\,d\theta)\,\underset{\sim}{e}_\theta + dz\,\underset{\sim}{e}_z$

9/9-92 Lecture 4. TAM 663. page 1

Mechanics:

FBD (Free Body Diagram) : can draw a free body diagram of any system or any part of the system. The external forces on the FBD describe the full mechanical effect of the outside world on FBD.

→ Force is the method of mechanical interaction between bodies. (any piece of anything).

All of mechanics is drawing (possibly in your mind) FBD and applying linear and Angular momentum balance.

For any "body", we have linear ~~com~~ momentum balance ⇒

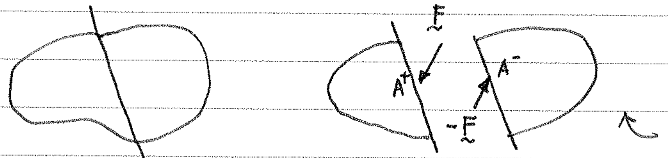
$$\text{(A)} \qquad \sum \underset{\sim}{F} = \sum m\underset{\sim}{a}$$

relative to any pt you like

$$\text{(B)} \qquad \sum \underset{\sim}{r} \times \underset{\sim}{F} = \sum \underset{\sim}{r} \times ma$$

[One needs to be in a "Newtonian" frame for A & B to hold. There is always is a convienient Newton Frame]

Action and Reaction holds for interaction between any pair of systems.



Not a good free Body diagram.

Other momentum balance equations can be used.

e.g. Angular momentum balance about three non-colinear points.

⇒ (A) & (B)

No matter what eqs. we used there are only 6 independent scalar eqs. of momentum balance (lin + Ang.) for any FBD.

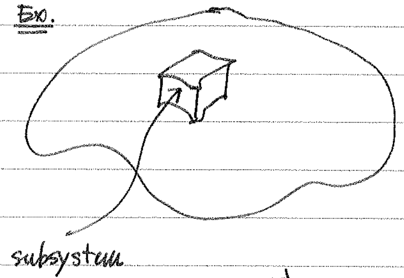
Continuum mechanics. (Extra assumptions)

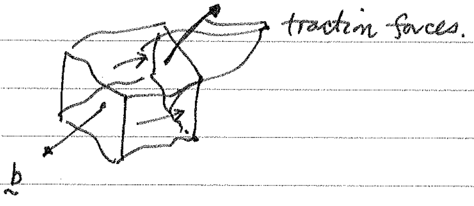
Assume ρ = density, $\underset{\sim}{t}$ = traction, $\underset{\sim}{b}$ body force are smooth fct. of position.

9/9/92 Lec. 4. TAM 663. page 2.

All forces are divided into one of two kinds : surface forces : $\underset{\sim}{t}$ = force per unit area and Body force $\underset{\sim}{b} = \frac{\text{force}}{\text{volume}}$.

Ex.



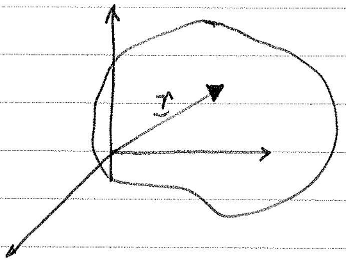


$$\underset{\text{acting on free Body Diagram}}{\sum \underset{\sim}{F}} = \int_S \underset{\sim}{t}\, ds + \int_V \underset{\sim}{b}\, dV$$

Assumption: surface force decays much faster than ~~normal~~ Body forces.

Breaks Down : { composite material, fiber. / Couple stresses.

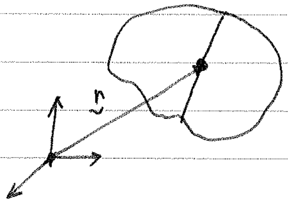
At one point inside a continuum, you can creat imaginary surfaces ~~at~~ various orientations, Let the outward ~~surface~~ normal to that surface be $\underset{\sim}{n}$

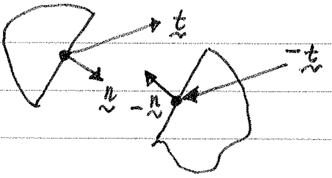


$\underset{\sim}{t}(\underset{\sim}{r}, \underset{\sim}{n})$

what do linear & Angular momentum balance tell us about $\underset{\sim}{t}$

Action & Reaction → $\underset{\sim}{t}(\underset{\sim}{n}) = -\underset{\sim}{t}(-\underset{\sim}{n})$



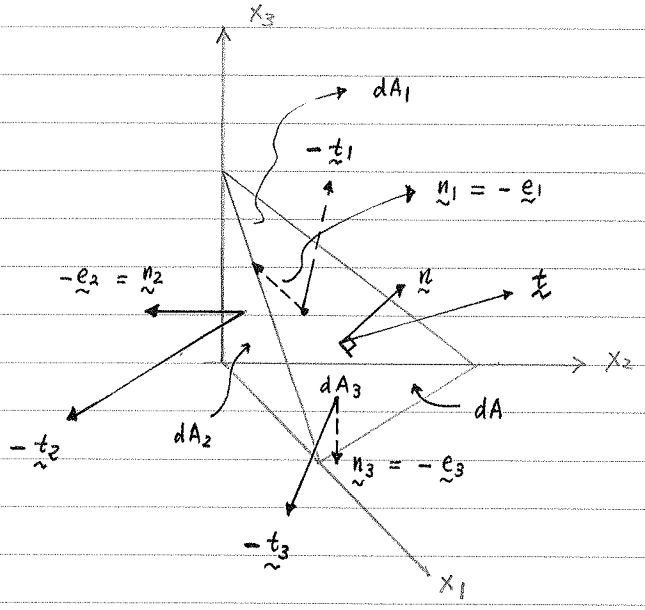


At one point in space, fixed $\underset{\sim}{r}$, what is $\underset{\sim}{t}(\underset{\sim}{n})$?

Is $\underset{\sim}{t}(\underset{\sim}{r}, \underset{\sim}{n})$ dependent on $\underset{\sim}{n}$ for different $\underset{\sim}{n}$.

Draw free body Diagram & apply linear momentum balance:

Cauchy Tetrahedron:



Assume tetrahedron is small:

e.g. $\int_{S_i} \underset{\sim}{t}_i \, ds \sim \underset{\sim}{t}_i \, dA_i$

Linear Momentum balance ⇒

$$\Sigma \underset{\sim}{F} = \Sigma m \underset{\sim}{a} = \underset{\sim}{a} \int dm$$

$$= \underset{\sim}{a} \int \rho \, dV$$

$$dV = c(\underset{\sim}{n}) h^3$$

$$dA = d(\underset{\sim}{n}) h^2$$

Linear momentum balance ⇒

$$\int_S \underset{\sim}{t} \, dS + \int_V \underset{\sim}{b} \, dV = \int_V \underset{\sim}{a} \, dm = a \int_V \rho \, dV$$

$$\Rightarrow \quad \underset{\sim}{t} \, dA - \underset{\sim}{t}_i \, dA_i + \underset{\sim}{b} (c h^3) = \rho a (c h^3)$$

Limit $h \to 0$

$$\underset{\sim}{t} = t_i \frac{dA_i}{dA} = \cancel{t_i(} \; \underset{\sim}{t}_i (\underset{\sim}{e}_i \cdot \underset{\sim}{n})$$

$$= \underset{\sim}{n} \cdot \underbrace{(\underset{\sim}{e}_i \underset{\sim}{t}_i)}$$

Define $\underset{\approx}{\sigma} = \underset{\sim}{e}_i \underset{\sim}{t}_i$

$$\therefore \quad \underset{\sim}{t} = \underset{\sim}{n} \cdot \underset{\approx}{\sigma}$$

OR $\underset{\approx}{\sigma} = \underset{\sim}{e}_1 \underset{\sim}{t}_1 + \underset{\sim}{e}_2 \underset{\sim}{t}_2 + \underset{\sim}{e}_3 \underset{\sim}{t}_3$

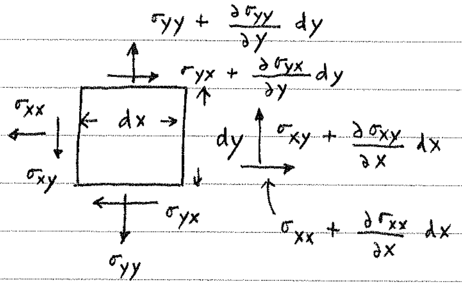
$$= \sigma_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j$$

where $\sigma_{ij} \underset{\sim}{e}_j = \underset{\sim}{t}_i$

9/4. Lecture 5 TAM 663. page 1.

Linear Momentum balance.

1^{st} method. (Sloopy calculus) 2 dim, no body force, statics, FBD of small square.



Linear momentum balance $\sum F = \sum ma = 0 \Rightarrow$ statics

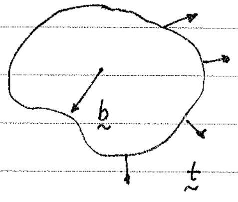
$$\sum (\text{tractions})(\text{areas}) = 0$$

$$\sum F_x = 0 \qquad \left(\frac{\partial \sigma_{xx}}{\partial x} dx\right)(dy) + \left(\frac{\partial \sigma_{yx}}{\partial y} dy\right)(dx) = 0$$

$$\Rightarrow \qquad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

$$\sum F_y = 0 \Rightarrow \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

2^{nd} Method: FBD of an 3D volume [arbitrary]



linear Momentum balance $\sum \underset{\sim}{F} = \sum m \underset{\sim}{a}$

$$\int_S \underset{\sim}{t}\, ds + \int_V \underset{\sim}{b}\, dV = \int_V \overbrace{\rho \underset{\sim}{a}\, dV}^{dm}$$

$$\int_S \sigma_{ji} n_j\, ds + \int_V (b_i - \rho a_i)\, dV = 0$$

$\int \sigma_{ji} n_j ds$
$= \int_V \sigma_{ji,j}\, dV$

$$\Rightarrow \int_V (\sigma_{ji,j} + b_i - \rho a_i)\, dV = 0$$

Valid for all V, $\Rightarrow \sigma_{ji,j} + b_i = \rho a_i$

or $\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \underset{\sim}{b} = \rho \underset{\sim}{a}$

9/14.

Surface force $\underset{\sim}{t}$, Body Force $\underset{\sim}{b}$

L. Momentum balance $\Rightarrow$ $\quad \underset{\sim}{t} = \underset{\sim}{n} \cdot \underset{\approx}{\sigma}$ $\quad$ and $\quad \sigma_{ji,j} + b_i = \rho a_i$

Angular Momentum balance

$$\sum_{\substack{\text{all} \\ \text{ext. forces}}} \underset{\sim}{r} \times \underset{\sim}{F} = \sum_{\substack{\text{all} \\ \text{mass in Body}}} \underset{\sim}{r} \times m\underset{\sim}{a} \qquad \Rightarrow \quad \sum \underset{\sim}{r} \times (\underset{\sim}{F} - m\underset{\sim}{a}) = 0$$

$$\Rightarrow \quad \text{Antisym.}\left[\sum r_i F_j - \sum r_i m a_j \right] = 0$$

$$\text{Antisym}\left[\int_S r_i \overbrace{\sigma_{kj} n_k}^{t_j} \, dS + \underbrace{\int_V r_i b_j \, dV}_{(2)} - \underbrace{\int_V r_i a_j \rho \, dV}_{(3)} \right] = 0$$

$$\int_S r_i \sigma_{kj} n_k \, dS = \int_V (r_i \sigma_{kj})_{,k} \, dV = \int_V r_{i,k} \sigma_{kj} \, dV + \int_V r_i \sigma_{kj,k} \, dV$$

$$= \int_V \delta_{ik} \sigma_{kj} \, dV + \int_V r_i \sigma_{kj,k} \, dV$$

$\uparrow$ cancell the terms (2) & (3)

$$\therefore \quad \text{Antisym}\left[\int_V \sigma_{ij} \, dV \right] = 0 \qquad \forall\, V \quad \Rightarrow \quad \underline{\underline{\sigma_{ij} = \sigma_{ji}}}$$

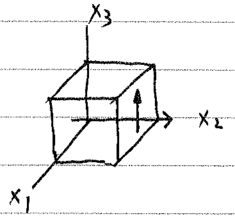
Summary : $\quad \sigma_{ij} = \sigma_{ji} \quad \Rightarrow \underset{\sim}{t} = \underset{\approx}{\sigma} \cdot \underset{\sim}{n} = \underset{\sim}{n} \cdot \underset{\approx}{\sigma}$

$$\sigma_{ij,j} + b_i = \rho a_i$$

Note : These equations are valid even if "Non-linear", finite deformation, plastic, visco-elastic ... etc, so long as $\underset{\approx}{\sigma}$ is the "true stress" $(\quad)_{,j}$ means differentiation with respect to current spatial position.

σ_{ij} = force per unit area on i th surface in j direction

e.g σ_{23} = z component of traction on surface with outer normal z (xz plane)



What can we say about $\underset{\sim}{\sigma}$,

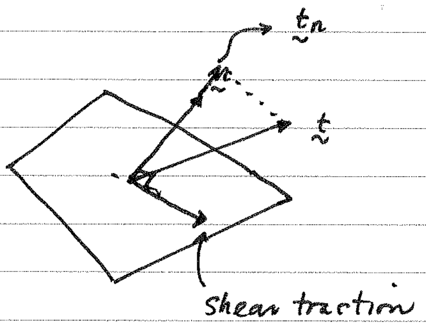
Angular momentum balance $\sigma_{ij} = \sigma_{ji}$
Linear " " $\sigma_{ij,j} + b_i = \rho a_i$

Math Fact : $\underset{\sim}{x} \times \underset{\sim}{F} = 0 \iff \text{Anti sym}\left[\underset{\sim}{x}\,\underset{\sim}{F}\right] = \underset{\approx}{0}$

Today's Topics: 1) Principal stress (intuitive version)
2) Mohr's Circle (Simple Derivation)
3) Introduction to deformation.

Assume $\underset{\sim}{t}(\underset{\sim}{n})$ depends smoothly on $\underset{\sim}{n}$. which is evident since $\underset{\sim}{t} = \underset{\approx}{\sigma}\cdot\underset{\sim}{n}$

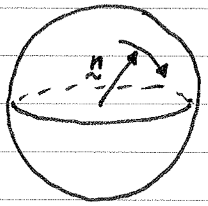
Also $\sigma_{ij} = \sigma_{ji}$



Concept of Normal traction $(\underset{\sim}{t}\cdot\underset{\sim}{n})\underset{\sim}{n} = \underset{\sim}{t}_n$

" " Shear traction $\underset{\sim}{t} - (\underset{\sim}{t}\cdot\underset{\sim}{n})\underset{\sim}{n} = \underset{\sim}{t}_s$

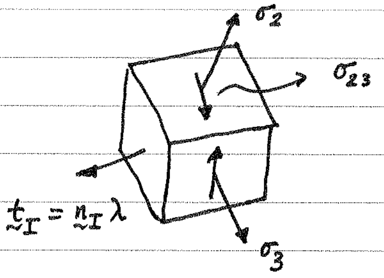
Think of $\underset{\sim}{t}_s$ as function of $\underset{\sim}{n}$. The set of all possible $\underset{\sim}{n}$ are points on a unit circle. $\underset{\sim}{t}_s$ are arrows drawn on sphere.



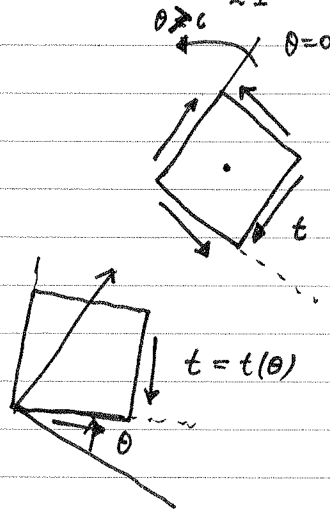
Hairy Ball theorem: there exist a pt on the sphere (actually 2) where $\underset{\sim}{t}^s(\underset{\sim}{n}) = \underset{\sim}{0}$

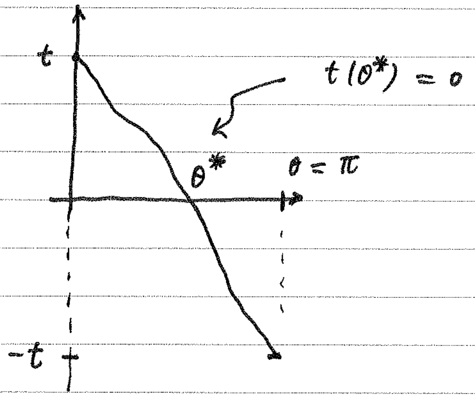
$\Rightarrow$ $\underset{\sim}{t} = \lambda\underset{\sim}{n}$ at some orientation $\underset{\sim}{n}$.

$\Rightarrow$ exist a cube st.



Look down $\underset{\sim}{n}_I$ axis, rotate Box θ about $\underset{\sim}{n}_I$

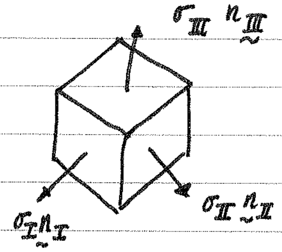




·) $\sigma_{ij} = \sigma_{ji}$ by cube

pg. 2.

∴ DRAW CUBE lined up that way (use θ^*) so that



∴ there ∃ a cube with no shear on any surface

⇒ exist coord. axis st.

$$\underset{\sim}{e}_i = \underset{\sim}{n}_i$$

$$\underset{\sim}{t}_i = \sigma_I \underset{\sim}{e}_i$$

$$\underset{\approx}{\sigma} = \underset{\sim}{t}_i \underset{\sim}{e}_i = \sigma_I e_I e_I + \sigma_{II} \underset{\sim}{e}_{II} \underset{\sim}{e}_{II} + \sigma_3 \underset{\sim}{e}_3 \underset{\sim}{e}_3$$ No sum.

Example of Non-uniqueness

$\sigma = -p\, \delta_{ij} \underset{\sim}{e}_i \underset{\sim}{e}_j$ (isotropic tension)

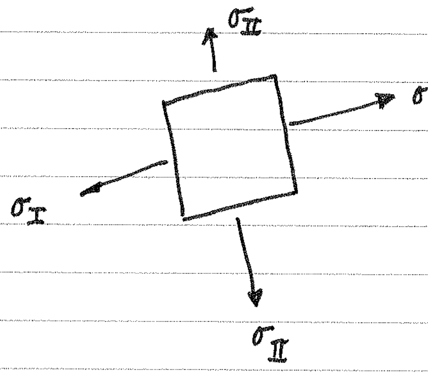
$\underset{\approx}{\sigma} \cdot \underset{\sim}{n} = -p \underset{\sim}{n}$ ∀ $\underset{\sim}{n}$ ⇒ every direction is a principle direction.

Non-uniqueness in principle direction when there are repeated eigenvalues.

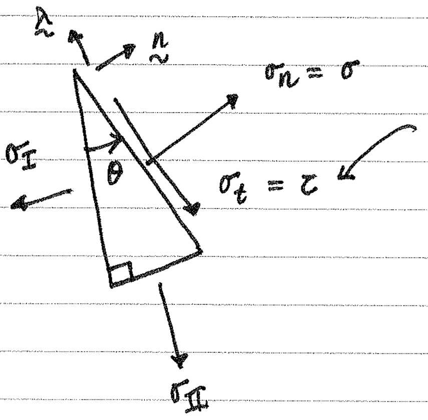
e.g $\underset{\approx}{\sigma} = \sigma_x \underset{\sim}{e}_1 \underset{\sim}{e}_1 + \sigma_x \underset{\sim}{e}_2 \underset{\sim}{e}_2 + 0\, \underset{\sim}{e}_3 \underset{\sim}{e}_3$

2D Mohr's Circle.

Look down $\underset{\sim}{n}_{III}$ axis at specially oriented cube



DRAW FBD



positive if clockwise in picture

pg.3.

$$\Sigma F_n = 0 \quad \Rightarrow$$

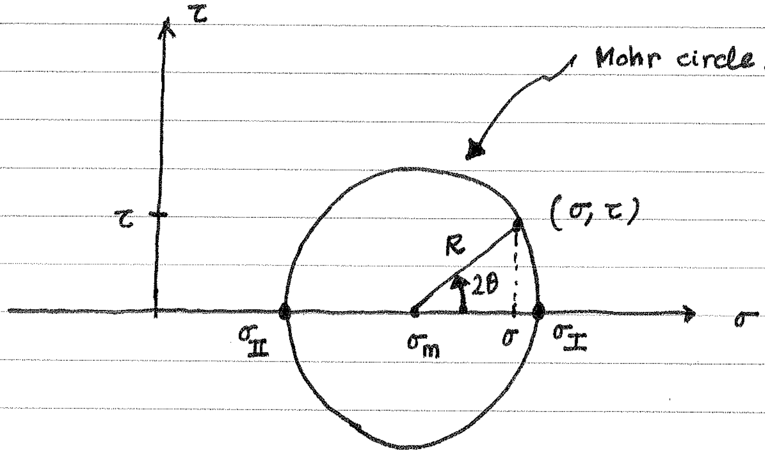
$$\sigma_n dA - (\sigma_I \cos\theta)(\cos\theta dA) - (\sigma_{II} \sin\theta)(\sin\theta dA) = 0 \qquad (1)$$

$$\Sigma F_\lambda = 0 \quad \Rightarrow \quad -\tau dA + \sigma_I \sin\theta \cos\theta \, dA - \sigma_{II} \cos\theta \sin\theta \, dA = 0 \qquad (2)$$

$$(1) \quad \Rightarrow \quad \sigma_n = \frac{(\sigma_I + \sigma_{II})}{2} + \left(\frac{\sigma_I - \sigma_{II}}{2}\right) [\underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta}]$$

$$(2) \quad \Rightarrow \quad \tau = \left(\frac{\sigma_I - \sigma_{II}}{2}\right) \cdot 2 \cos\theta \sin\theta = \left(\frac{\sigma_I - \sigma_{II}}{2}\right) \sin 2\theta.$$

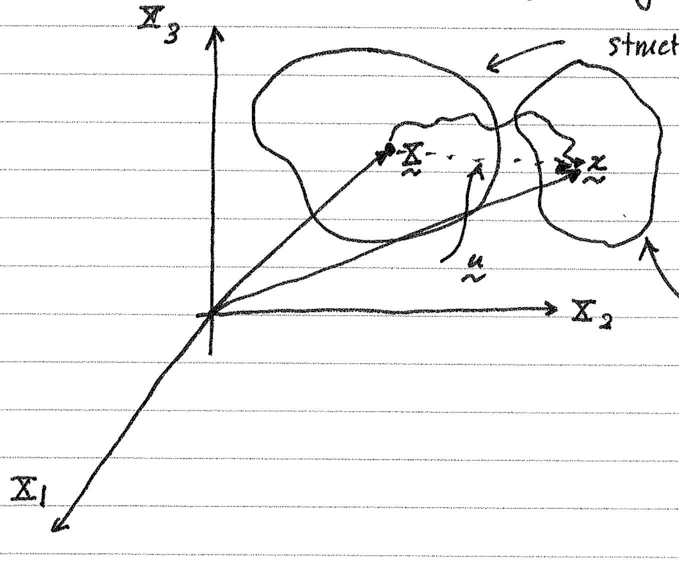
Let $R = \left(\frac{\sigma_I - \sigma_{II}}{2}\right) \qquad \sigma_M = \frac{\sigma_I + \sigma_{II}}{2}$



2θ measured counterclockwise.

Introduction to Deformation

Forget Mechanics (stress, Momentum etc) and just look~~ed~~ at geometry of objects as they deform.



structure of interest before deformation, undeform, reference state.

$\underset{\sim}{X}$ = position of point of interest before deformation

Structure after deformation or deform or in the current state.

$\underset{\sim}{x}$ = ~~deform~~ position of the particle $\underset{\sim}{X}$ after deformation

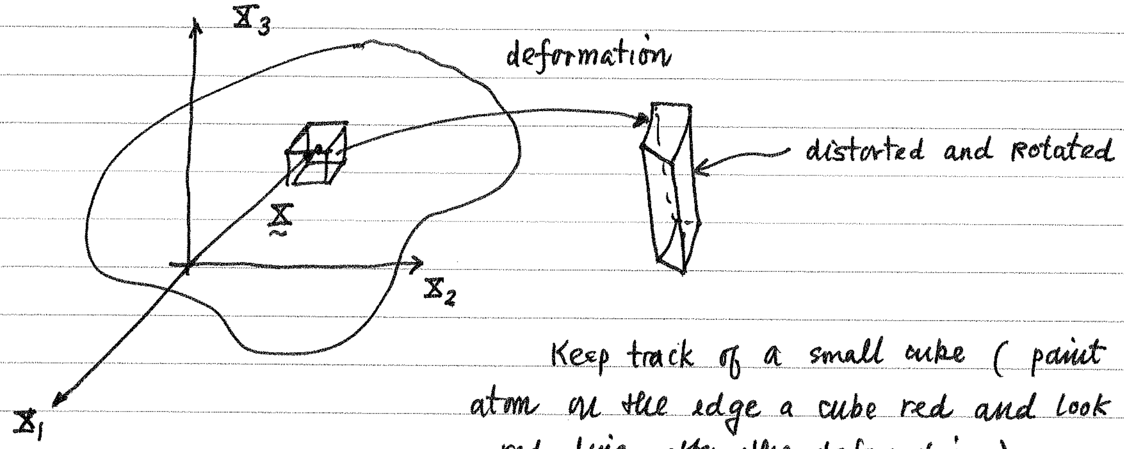
$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{u}$ or $\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X}$ = displacement of $\underset{\sim}{X}$

We labelled particles by their cordinate $\underset{\sim}{x}$ or $\underset{\sim}{X}$. In this course, we will always use $\underset{\sim}{X}$ to label the particle. Description of deformation is the fct

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t) \quad \text{or} \quad x_i = x_i(X_j, t).$$

where $x_i \underset{\sim}{e_i} = \underset{\sim}{x}$ $\quad \underset{\sim}{X} = X_j \underset{\sim}{e_j}$ $\quad \underset{\sim}{u} = u_K \underset{\sim}{e_K}$

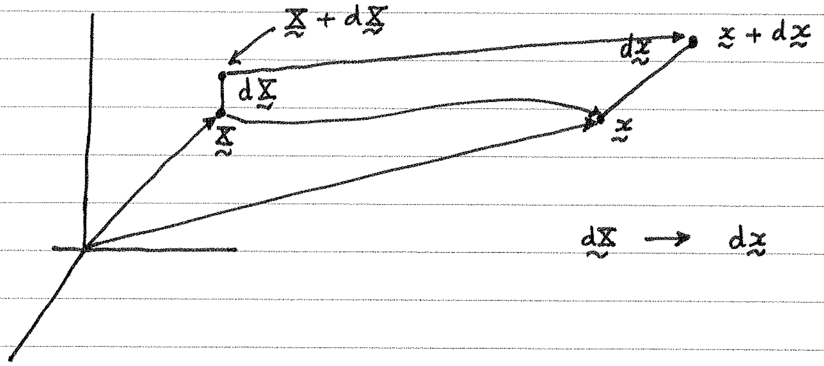
We pay special attention to deformation or distortion of little nbdh at the pt. of interest.



Keep track of a small cube (paint a set of atom on the edge a cube red and look at the red line after the deformation)

Note : Angles are not ~~pre~~ persened.
lengths are 〃 〃
straight lines stay straight
parallel line stay parallel.

Keep track of the little red lines/ material sibers.



$$\underset{\sim}{x} + d\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X} + d\underset{\sim}{X})$$

$$= \underset{\sim}{x}(\underset{\sim}{X}) + \frac{\partial \underset{\sim}{x}}{\partial X_1} dX_1 + \frac{\partial \underset{\sim}{x}}{\partial X_2} dX_2 + \frac{\partial \underset{\sim}{x}}{\partial X_3} dX_3$$

$$\therefore \quad d\underset{\sim}{x} = \frac{\partial \underset{\sim}{x}}{\partial X_j} dX_j$$

$$= \frac{\partial \underset{\sim}{x}}{\partial X_K} e_K \cdot (dX_j \underset{\sim}{e}_j)$$

Let $\underset{\sim}{x} = x_i \underset{\sim}{e_i}$

$$= \left(\frac{\partial x_i}{\partial X_K} \underset{\sim}{e_i} \underset{\sim}{e_K} \right) \cdot \underbrace{(dX_j \underset{\sim}{e_j})}_{d\underset{\sim}{X}}$$

$$= \left(\frac{\partial x_i}{\partial X_j} \underset{\sim}{e_i} \underset{\sim}{e_j} \right) \cdot (dX_K \underset{\sim}{e_K})$$

$$\therefore \quad d\underset{\sim}{x} = \underset{\approx}{F} \cdot d\underset{\sim}{X} = \nabla_{\underset{\sim}{X}} \underset{\sim}{x} \cdot d\underset{\sim}{X}$$

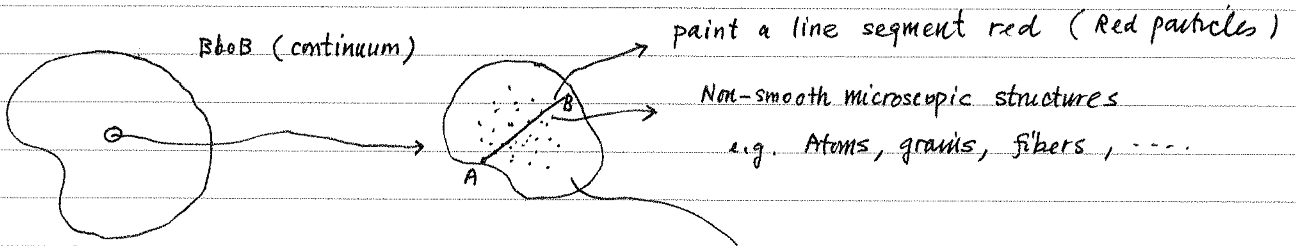
Characterize the motions of points near reference pt. relative to ref. pt.

$\underset{\approx}{F}$ = deformation gradient. = $F_{ij}\, \underset{\sim}{e}_i \underset{\sim}{e}_j$ $\quad F_{ij} = \frac{\partial x_i}{\partial x_j}$

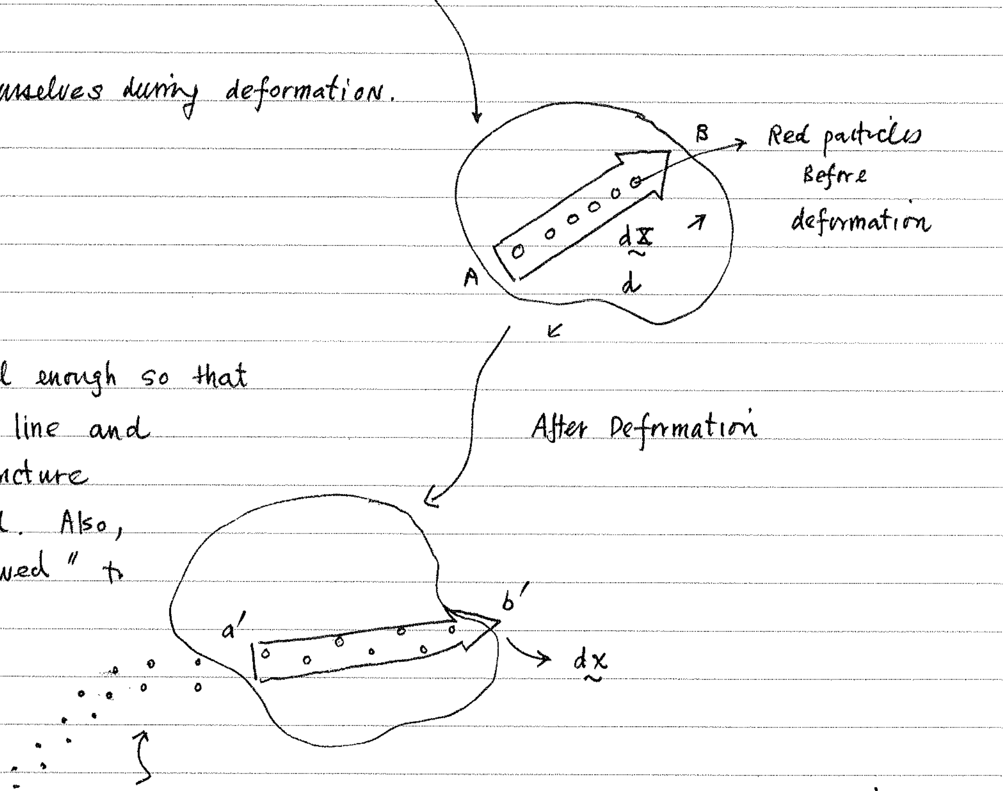
$d\underset{\sim}{x} = \underset{\approx}{F} \cdot d\underset{\sim}{X} \qquad dx_j = F_{ij}\, dX_j$

$\underset{\approx}{F}$ characterises motion of particles near the ref. pt.

Why use $\underset{\approx}{F}$?



Micro-structures rearrange themselves during deformation.



too large a length scale give a curve. (Not a straight line)

Big Assumption

∃ a length scale d small enough so that straight line go to straight line and Big enough that the micro-structure deformation can be averaged. Also, microstructure deformation is "slaved" to average deformation.

∴ $\underset{\approx}{F}$ characterize the average motion of the micro-structures. ⇒ $\underset{\approx}{F}$ determines the interparticle forces ⇒ forces averaged to give traction & stress.

⇒ $\underset{\approx}{F}$ $\underbrace{\text{determines}}_{\text{depends on material.}}$ stress.

Sept, 21, 1992.

We need only to study $\underset{\approx}{F}$ to keep track of deformation.

Facts About $\underset{\approx}{F}$ or the linear map $d\underset{\sim}{X} \longrightarrow d\underset{\sim}{x}$

straight line ⟶ straight lines

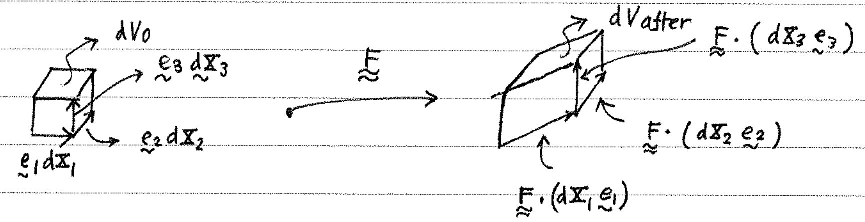
cubes ⟶ parallelipide

sphere ⟶ ellipsoids

$\underset{\sim}{a} + \underset{\sim}{b} \longrightarrow \underset{\approx}{F}\cdot\underset{\sim}{a} + \underset{\approx}{F}\cdot\underset{\sim}{b}$

* $\text{Vol} \xrightarrow{\underset{\approx}{F}} \det[\underset{\approx}{F}]\ \text{Vol}.$ ①   * $\text{density} \xrightarrow{\underset{\approx}{F}} \dfrac{\text{density}}{\det \underset{\approx}{F}}$

To prove ①, consider the cube with sides $dX_i \underset{\sim}{e_i}$ No sum.



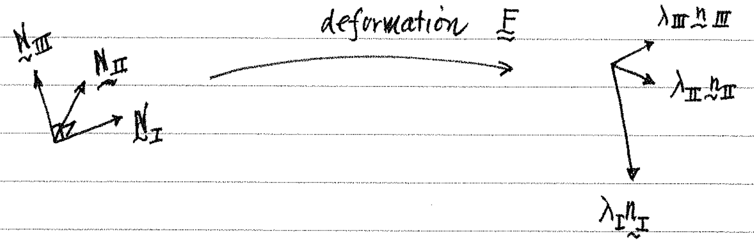
$$dV_{\text{after}} = (\underset{\approx}{F}\cdot dX_1\,\underset{\sim}{e_1} \times \underset{\approx}{F}\cdot dX_2\,\underset{\sim}{e_2}) \cdot \underset{\approx}{F}\, dX_3\,\underset{\sim}{e_3}$$

$$= \underbrace{dX_1\, dX_2\, dX_3}_{dV_0}\ \det \underset{\approx}{F}$$

* $\underset{\approx}{F}$ satisfies Polar Decomposition.

Polar Decomposition Theorem

Geometric statement: ∃ three mutually orthogonal unit vectors $\underset{\sim}{N_I}, \underset{\sim}{N_{II}}, \underset{\sim}{N_{III}}$ that get mapped into 3 mutually orthogonal vectors $\lambda_I \underset{\sim}{n_I}$, $\lambda_{II} \underset{\sim}{n_{II}}$, $\lambda_{III} \underset{\sim}{n_{III}}$ for all $\underset{\approx}{F}$ $\det \underset{\approx}{F} \neq 0$



$\underset{\sim}{N_I}, \underset{\sim}{N_{II}}, \lambda_I, \cdots, \underset{\sim}{n_I}$ depends on $\underset{\approx}{F}$.

$$\underset{\approx}{F}\cdot\underset{\sim}{N_I} = \lambda_I \underset{\sim}{n_I} \qquad \underset{\approx}{F}\cdot\underset{\sim}{N_{II}} = \lambda_{II} \underset{\sim}{n_{II}} \qquad \underset{\approx}{F}\cdot\underset{\sim}{N_{III}} = \lambda_{III} \underset{\sim}{n_{III}}$$

page 3

Mathematical statement

For any $\underset{\approx}{F} \ni \det \underset{\approx}{F} \neq 0$, $\exists$ an $\underset{\approx}{R}$ and a $\underset{\approx}{U}$ so that

$$\underset{\approx}{F} = \underset{\approx}{R} \cdot \underset{\approx}{U}$$

$$\underset{\approx}{R}^{T} \cdot \underset{\approx}{R} = \underset{\approx}{R} \cdot \underset{\approx}{R}^{T} = \underset{\approx}{I}$$

$\underset{\approx}{U}$ is symmetric & positwe definite.

From Geometric statement to Math. statement.

Let $\underset{\approx}{R} = \underset{\sim}{n}_{I} \underset{\sim}{N}_{I} + \underset{\sim}{n}_{II} \underset{\sim}{N}_{II} + \underset{\sim}{n}_{III} \underset{\sim}{N}_{III}$

$\underset{\approx}{U} = \lambda_{I} \underset{\sim}{N}_{I} \underset{\sim}{N}_{I} + \lambda_{I} \underset{\sim}{N}_{II} \underset{\sim}{N}_{II} + \lambda_{III} \underset{\sim}{N}_{III} \underset{\sim}{N}_{III}$

$\underset{\approx}{U}$ is symmetric and positive definite λ_{I}'s > 0

$$\underset{\sim}{F} \cdot d\underset{\sim}{X} = \underset{\sim}{F} \cdot \left[dX_{I} \underset{\sim}{N}_{I} + dX_{II} \underset{\sim}{N}_{II} + dX_{III} \underset{\sim}{N}_{III} \right]$$

$$= R \cdot \left[dX_{I} \lambda_{I} \underset{\sim}{N}_{I} + dX_{II} \lambda_{II} \underset{\sim}{N}_{II} + dX_{III} \lambda_{III} \underset{\sim}{N}_{III} \right]$$

$\underset{\approx}{R} \cdot \underset{\approx}{U}$

$$= \lambda_{I} dX_{I} \underset{\sim}{n}_{I} + \lambda_{II} dX_{II} \underset{\sim}{n}_{II} + \lambda_{III} dX_{III} \underset{\sim}{n}_{III}.$$

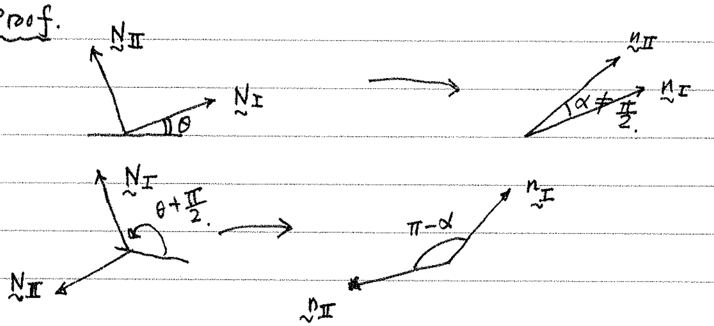
special case $dX_{I} = 1 \quad dX_{2} = dX_{3} = 0$

$\underset{\sim}{N}_{I} \rightarrow \lambda_{I} \underset{\sim}{n}_{I}$

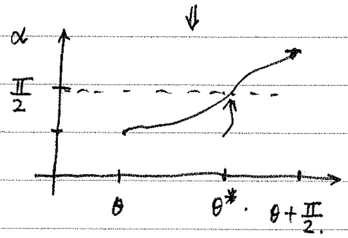
$\underset{\sim}{N}_{II} \rightarrow \lambda_{II} \underset{\sim}{n}_{II}$

$\underset{\sim}{N}_{III} \rightarrow \lambda_{III} \underset{\sim}{n}_{III}$

2D Proof.



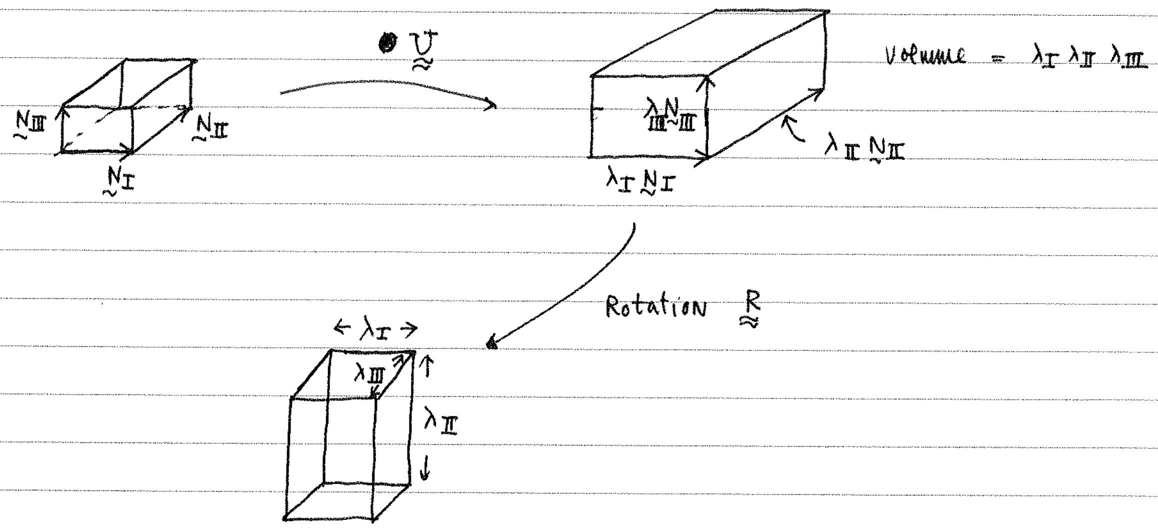
Note.
α is a smooth of θ.
$\Downarrow$



General Interpretation of $\underset{\approx}{F} = \underset{\approx}{R}\cdot\underset{\approx}{U} = \left[\underset{\sim}{n}_I \underset{\sim}{N}_I + \underset{\sim}{n}_{II} \underset{\sim}{N}_{II} + \underset{\sim}{n}_{III} \underset{\sim}{N}_{III}\right]\cdot\left[\lambda_I \underset{\sim}{N}_I \underset{\sim}{N}_I + \lambda_{II} \underset{\sim}{N}_{II} \underset{\sim}{N}_{II} + \lambda_{III} \underset{\sim}{N}_{III} \underset{\sim}{N}_{III}\right]$.

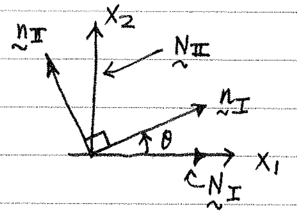
$$\underset{\approx}{F}\cdot d\underset{\sim}{X} = \underbrace{\underset{\approx}{R}}_{\text{Rotation}}\cdot\underbrace{\left[\underset{\approx}{U}\cdot d\underset{\sim}{X}\right]}_{\text{stretch}}$$

∃ a cube
~~Also~~ Aligned
with $\underset{\sim}{N}_I$'s



Example. A

Rotation about $x_3 = z$ axis   No stretch



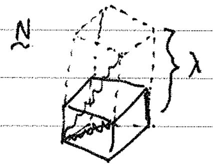
$$\underset{\approx}{U} = \underset{\approx}{I}$$

$$\underset{\approx}{R} = \underset{\sim}{n}_I \underset{\sim}{N}_I + \underset{\sim}{n}_{II} \underset{\sim}{N}_{II} + \underset{\sim}{N}_{III} \underset{\sim}{N}_{III}$$

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example B.

Uniaxial stretch in $\underset{\sim}{N}$ direction (No Rotation $\underset{\approx}{R} = \underset{\approx}{I}$)



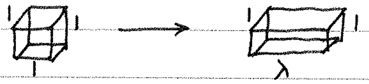
$$U = \underset{\approx}{I} + (\lambda - 1)\underset{\sim}{N}\underset{\sim}{N}$$

$$d\underset{\sim}{X} \perp \underset{\sim}{N} \Rightarrow d\underset{\sim}{x} = d\underset{\sim}{X}$$

$$d\underset{\sim}{X} \,//\, \underset{\sim}{N} \Rightarrow d\underset{\sim}{x} = \lambda\, d\underset{\sim}{X}$$

But No Rotation, i.e. $\underset{\approx}{R} = \underset{\approx}{I}$

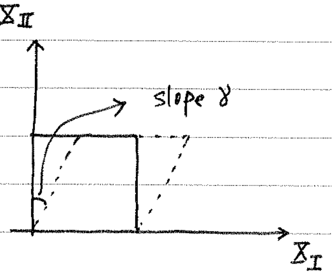
$$\Rightarrow \quad \underset{\sim}{e}_1 \to \lambda \underset{\sim}{e}_1 \qquad \underset{\sim}{e}_j \to \underset{\sim}{e}_j \qquad j = 2, 3. \qquad \underset{\approx}{R} = \underset{\approx}{I} \qquad [U] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



C) Isotropic Expansion (No Rotation)

$$\underset{\approx}{R} = \underset{\approx}{I} \qquad \underset{\approx}{U} = \lambda \underset{\approx}{I}$$

d) Simple shear.



$$\underset{\sim}{x}(\underset{\sim}{X}) \quad \Rightarrow \quad \begin{aligned} X_2 &= x_2 \\ x_1 &= X_1 + \gamma X_2. \end{aligned}$$

Hint : Look at $\underset{\approx}{F}^T \cdot \underset{\approx}{F} = \underset{\approx}{U} \cdot \underset{\approx}{V}$

Lecture. 9/24/92 page 1.

Given $\underset{\sim}{x}(\underset{\sim}{X})$, how to find $\underset{\approx}{R}$ & $\underset{\approx}{U}$?

Steps : 1) Find $\dfrac{\partial x_i}{\partial X_j} = F_{ij}$

2) Find $\underset{\approx}{C} = \underset{\approx}{F}^T \cdot \underset{\approx}{F} = \underset{\approx}{U} \cdot \underset{\approx}{U}$ Note $\underset{\approx}{F}^T \cdot \underset{\approx}{F} = (\underset{\approx}{R}\underset{\approx}{U})^T \cdot (\underset{\approx}{R}\underset{\approx}{U}) = \underset{\approx}{U}^T\underset{\approx}{U} = \underset{\approx}{U}\cdot\underset{\approx}{U}$

3) Find eigenvalues & eigenvectors of $\underset{\approx}{C}$

$\lambda_I^2, \lambda_{II}^2, \lambda_{III}^2$ $\underset{\sim}{N}_I, \underset{\sim}{N}_{II}, \underset{\sim}{N}_{III}$

4) Find $\underset{\approx}{U} = \lambda_I \underset{\sim}{N}_I \underset{\sim}{N}_I + \lambda_{II} \underset{\sim}{N}_{II} \underset{\sim}{N}_{II} + \lambda_{III} \underset{\sim}{N}_{III} \underset{\sim}{N}_{III} = \sqrt{\underset{\approx}{C}}$

5) Find $\underset{\approx}{U}^{-1} = \frac{1}{\lambda_I} \underset{\sim}{N}_I \underset{\sim}{N}_I + \frac{1}{\lambda_{II}} \underset{\sim}{N}_{II} \underset{\sim}{N}_{II} + \frac{1}{\lambda_{III}} \underset{\sim}{N}_{III} \underset{\sim}{N}_{III}$

6) Find $\underset{\approx}{R} = \underset{\approx}{F} \cdot \underset{\approx}{U}^{-1} = \underset{\approx}{R} \cdot \underset{\approx}{U} \cdot \underset{\approx}{U}^{-1} = \underset{\approx}{R}$

What do we care about for stresses ? that part of the deformation that changes the distance between atoms / grain. ⇒ Ignore $\underset{\approx}{R}$. (a Rigid Rotation). So that $\underset{\approx}{U}$ characterizes changes in length & change in shape.

It is nice to have something that is easy to calculate : e.g. $\underset{\approx}{C} = \underset{\approx}{U} \cdot \underset{\approx}{U}$ which also characterize the deformation.

What's wrong with $\underset{\approx}{C}$: Nothing, but it is nice to have a deformation = 0 when there's no deformation. ⇒

$$\underset{\approx}{C} - \underset{\approx}{I}$$

Define $$\underset{\approx}{E} = \tfrac{1}{2}\left[\underset{\approx}{C} - \underset{\approx}{I}\right] = \tfrac{1}{2}\left[\underset{\approx}{F}^T\underset{\approx}{F} - \underset{\approx}{I}\right] = \text{Green strain}$$

= a measure of finite strain.

lots of finite strain measures.

An interpretation of Finite strain measure $\underset{\approx}{E}$.

$$|d\underset{\sim}{x}|^2 - |d\underset{\sim}{X}|^2 = d\underset{\sim}{x} \cdot d\underset{\sim}{x} - d\underset{\sim}{X} \cdot d\underset{\sim}{X}$$

$$= (\underset{\approx}{F} \cdot d\underset{\sim}{X}) \cdot (\underset{\approx}{F} \cdot d\underset{\sim}{X}) - d\underset{\sim}{X} \cdot d\underset{\sim}{X}$$

$$= d\underset{\sim}{X} \cdot (\underset{\approx}{F}^T \cdot \underset{\approx}{F} - \underset{\approx}{I}) \cdot d\underset{\sim}{X} = 2 d\underset{\sim}{X} \cdot \underset{\approx}{E} \cdot d\underset{\sim}{X}$$

Sept. 24, 92 page 2.

Most classical solid mechanics makes further assumptions about the deformation:

1) Small
2) Two dimensional. $\Rightarrow$ to get some linear equations.

<u>Small deformation</u>

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial (x_i - X_i)}{\partial X_j} + \frac{\partial X_i}{\partial X_j} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$$\text{or} \quad \underset{\approx}{F} = \nabla_{\underset{\sim}{X}} \underset{\sim}{u} + \underset{\approx}{I}$$

Assume $\nabla_{\underset{\sim}{X}} \underset{\sim}{u}$ is small, $\underset{=}{\text{or}}$ $|u_{i,j}| \equiv \left|\frac{\partial u_i}{\partial X_j}\right| \ll 1$.

<u>or</u> Difference in displacement between two near by particles $\ll$ difference in position of those particles.

Define: $\underset{\approx}{\epsilon} = \text{sym}(\nabla_{\underset{\sim}{X}} \underset{\sim}{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$ small strain matrix.

$$\underset{\approx}{\omega} = \text{Antisym}(\nabla_{\underset{\sim}{X}} \underset{\sim}{u}) = \frac{1}{2}(u_{i,j} - u_{j,i})$$

$$\therefore \quad u_{i,j} = \underset{\underset{\text{symmetric}}{\uparrow}}{\epsilon_{ij}} + \underset{\underset{\text{anti-symmetric.}}{\uparrow}}{\omega_{ij}}$$

~~$\varepsilon_{ij}\omega_{jk} + \varepsilon_{ij}\omega_{kj} = 0$~~

Note:

$$F_{ik} \cong (\delta_{ij} + \varepsilon_{ij})(\delta_{jk} + \omega_{jk}) \quad \sim O(\varepsilon_{ij}^2) \lll 1 \qquad \varepsilon_{ij}\omega_{jk} \cong 0. \quad \varepsilon_{ij}[\cancel{\omega_{jk}} + \omega_{kj}] = 0.$$

$$= \delta_{ij}\delta_{jk} + (\varepsilon_{ik} + \omega_{ik}) + \epsilon_{ij}\omega_{jk} \qquad \epsilon_{ij}\omega_{jk} = \cancel{}\epsilon_{ji}\omega_{jk}$$

$$= -\epsilon_{ji}\omega_{kj}$$

$$= \delta_{ik} + u_{i,k} = x_{i,k}. = \frac{\partial x_i}{\partial X_k}$$

$$\underset{\approx}{F} = \underset{\underset{\approx}{U}}{(\underset{\approx}{I} + \underset{\approx}{\epsilon})}\ \underset{\underset{\approx}{R}}{(\underset{\approx}{I} + \underset{\approx}{\omega})} = \underset{\underset{\underset{\approx}{R}}{\wr\wr}}{(\underset{\approx}{I} + \underset{\approx}{\omega})}\ \underset{\underset{\underset{\approx}{U}}{\wr\wr}}{(\underset{\approx}{I} + \underset{\approx}{\epsilon})} = \underset{\approx}{R} \cdot \underset{\approx}{U}$$

$$(\underset{\approx}{I} + \underset{\approx}{\omega})^T \cdot (\underset{\approx}{I} + \underset{\approx}{\omega}) = \underset{\approx}{I} + \underset{\approx}{\omega}^T + \underset{\approx}{\omega} + \underset{\approx}{\omega}^T\underset{\approx}{\omega} = \underset{\approx}{I} + \underset{\ll 1.}{O(u_{i,j})^2}$$

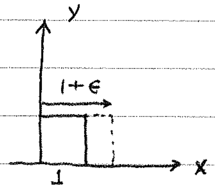
$$\Rightarrow \quad \underset{\approx}{I} + \underset{\approx}{\omega} = \underset{\approx}{U}$$

So, ~~we have, after~~ had we linearised, $\underset{\approx}{F} = \underset{\approx}{R} \bullet \underset{\approx}{U}$, we get.

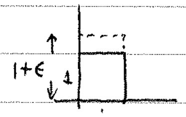
$$\underset{\approx}{F} = \underbrace{(\underset{\approx}{I} + \underset{\approx}{\omega})}_{\underset{\approx}{R}} \cdot \underbrace{(\underset{\approx}{I} + \underset{\approx}{\epsilon})}_{\underset{\approx}{U}}$$

e.g. $[R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \overset{\sim\theta}{-\sin\theta} \\ \sin\theta & 0 \end{bmatrix}$ $\theta \ll 1$

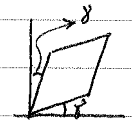
$\epsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$



$[\epsilon] = \begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix}$ $[\omega] = \underset{\sim}{0}$



$[\epsilon] = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon \end{bmatrix}$ $[\omega] = [\underset{\sim}{0}]$



$[\epsilon] = \begin{bmatrix} 0 & 2\gamma \\ 2\gamma & 0 \end{bmatrix} \frac{1}{2}$ $[\omega] = [0]$

Bulk strain $\epsilon_{ii} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \sim \det \underset{\approx}{F} - 1$

$\therefore$ Volume $= (1 + \epsilon_{ii})$ Vol (Initial)

$\sim (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33})$ Vol (Initial)

~~$\sim (1+\epsilon_{ii})$ Vol. (Initial).~~

Strain displacement Relation (compatibility has more detailed meaning).

2D simplications: 1) Plane strain $\epsilon_{33} = \epsilon_{23} = \epsilon_{13} = 0$

$\underset{\approx}{\epsilon}(x_1, x_2, x_3) = \underset{\approx}{\epsilon}(x_1, x_2)$

2) Anti-plane shear $\epsilon_{13} = \epsilon_{23} \neq 0$ $\epsilon_{13}(x_1, x_2)$ $\epsilon_{23}(x_1, x_2)$

↑ only one non-trivial displacement.

Sept. 28, 1992. TAM 663. page 1.

Principle of virtual Work

Linear momentum Balance ⇒ $\underset{\sim}{t} = \underset{\sim}{n}\cdot\underset{\approx}{\sigma}$

Angular " " ⇒ Antisym $\underset{\approx}{\sigma} = \underset{\sim}{0}$ ⇒ $\underset{\approx}{\sigma}^T = \underset{\approx}{\sigma}$

$$\Bigg\} \Rightarrow \underset{\sim}{t} = \underset{\approx}{\sigma}\cdot\underset{\sim}{n} = \underset{\sim}{n}\cdot\underset{\approx}{\sigma}$$

Linear momentum balance ⇒ $\underset{\sim}{\nabla}_{\underset{\sim}{x}}\cdot\underset{\approx}{\sigma} + \underset{\sim}{F} = \rho\,\underset{\sim}{a}$

$$\sigma_{ij,j} + b_i = \rho\,\ddot{u}_i$$

true stress. Body force / unit current volume current density

Geometry of Deformation (Kinematics)

$$\underset{\approx}{F} = F_{ij}\underset{\sim}{e}_i\underset{\sim}{e}_j = \underset{\sim}{\nabla}_X\,\underset{\sim}{x} = x_{i,j}\underset{\sim}{e}_i\underset{\sim}{e}_j = (\delta_{ij} + u_{i,j})\underset{\sim}{e}_i\underset{\sim}{e}_j$$

$$x_{i,j} = \delta_{ij} + \epsilon_{ij} + \omega_{ij}$$

where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $\omega_{ij} = \frac{1}{2}[u_{i,j} - u_{j,i}]$

$$\therefore \quad \underset{\approx}{F} = \underset{\approx}{I} + \underset{\approx}{\varepsilon} + \underset{\approx}{\omega} \leftarrow \text{"small Rotation"}$$

"small strain"

$$= \underset{\approx}{R}\cdot\underset{\approx}{U}$$

$$\underset{\approx}{R} = \sum_{\beta=I}^{III} \underset{\sim}{n}_\beta\,\underset{\sim}{N}_\beta \qquad \underset{\approx}{U} = \sum_{\beta=I}^{III} \lambda_\beta\,\underset{\sim}{N}_\beta\,\underset{\sim}{N}_\beta$$

Small strain & small displacement (*) → include momentum balance equation.

$u_{i,j} \ll 1$ $x_i \sim X_i$ except in $\underset{\sim}{u} = \underset{\sim}{x} - \underset{\sim}{X}$

Small strain ⇒ $\underset{\approx}{R} = \underset{\approx}{I} + \underset{\approx}{\omega}$ $\underset{\approx}{\varepsilon} \approx \underset{\approx}{E} \equiv \frac{1}{2}(\underset{\approx}{F}^T\underset{\approx}{F} - \underset{\approx}{I})$

$\underset{\approx}{U} = \underset{\approx}{I} + \underset{\approx}{\varepsilon}$

so that $\boxed{\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})}$ primary interest.

Sept. 28, 92. page 2

Aside on "Compatibility."

If you think of $\varepsilon_{ij}(\underset{\sim}{x}) = \varepsilon_{ij}(\underset{\sim}{X})$ since $\underset{\sim}{x} \sim \underset{\sim}{X}$ is arbitrary, can it be any function of $\underset{\sim}{x}$? Ans: NO !

For example $\varepsilon_{xx} = y^2$ $\varepsilon_{yy} = 0$ $\varepsilon_{xy} = 0$.

strain Displacement relation in 2D. $\varepsilon_{xx} = u_{x,x}$ $\varepsilon_{yy} = u_{y,y}$ $\varepsilon_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x})$

$$2\varepsilon_{xy,xy} = \varepsilon_{xx,yy} + \varepsilon_{yy,xx}$$ 2D compatibility Equation.

example: $0 \neq 2 + 0$. [necessary condition for existence of $\underset{\sim}{u}$].

The principle of virtual Work or Principle of virtual Power (velocities). (PVW)

The "PVW" means different things to different people at different times (Beware !)

Possible Meanings

- → As a mechanical principle which replaces Linear Momentum Balance.
- → As an equation which conveniently expresses the divergence thm.
- → As a statement of stationary potential energy in Elasticity.
- → As an approximate statement of Momentum-balance in Numerical Methods
- → As a principle to replace strain displacement relation
- → As an equation to be used for various purposes. (the PVW Eqn.)

The PVW Eqn.

$$\int_V \sigma_{ij}\varepsilon_{ij}\,dV = \int_V [b_i - \rho\, a_i]u_i\,dV + \int_S t_i u_i\,dS$$

Schematic MAP: In all case, we assumed that $\begin{cases} 1) & V \text{ and } S \text{ are given} \quad S = \partial V \\ 2) & \sigma_{ij} = \sigma_{ji} \end{cases}$

$$\boxed{\text{PVW Eqn} + u_i \text{ arbitrary} + \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})} \Leftrightarrow \boxed{\sigma_{ij,j} + b_i = \rho a_i \quad \sigma_{ij}n_j = t_i}$$

linear Momentum balance

$$+\ \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

Note that $\underset{\sim}{\sigma}$ doesn't mean the actual stress in the problem, $\underset{\sim}{\varepsilon}$ does not mean actual $\underset{\sim}{\varepsilon}$.

$\underset{\sim}{\sigma}$ & $\underset{\sim}{\varepsilon}$ does not have to be related to each other (i.e., not related by Material properties).

$$\underbrace{\text{PVW} + u_i \text{ arbitary} + \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \Rightarrow \text{Linear Momentum Balance.}}_{\text{The Principle of virtual work.}}$$

$$\underset{\text{IVW}}{\text{Internal virtual work}} = \overset{\text{EVW}}{\text{External virtual work}} \text{ for arbitary virtual Displacement}$$

$$\text{IVW} = \int_V \sigma_{ij}\varepsilon_{ij}\, dV$$

$$\text{EVW} = \int_V [b_i - \rho \overset{a_i}{}] u_i dV + \int_S t_i u_i dS$$

Assume IVW = EVW for $\boxed{\underline{\underline{\text{all}}}}$ u_i's that satisfies $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

arbitary virtual displacements

<u>Side Calculations.</u> $\sigma_{ij}\varepsilon_{ij} = \sigma_{ij}[u_{i,j} + u_{j,i}]/2 = \sigma_{ij} u_{i,j} = (\sigma_{ij}u_i)_{,j} - \sigma_{ij,j} u_i$

$$\therefore \int_V \sigma_{ij}\varepsilon_{ij}\, dV = \int_V (\sigma_{ij}u_i)_{,j}\, dV - \int_V \sigma_{ij,j} u_i\, dV$$

$$= \int_{S} \sigma_{ij} n_j u_i\, dS - \int_V \sigma_{ij,j} u_i\, dV = \text{IVW} = \text{EVW}$$

$$= \int_V (b_i - \rho a_i) u_i dV + \int_S t_i u_i dS$$

$$\Rightarrow \int_V \underbrace{[\sigma_{ij,j} + (b_i - \rho a_i)]}_{\underset{\sim}{f}(\underset{\sim}{x})} u_i dV + \int_S \underbrace{[-\sigma_{ij} n_j + t_i]}_{\underset{\sim}{g}(\underset{\sim}{x})} u_i dS = \underset{\sim}{0} \quad \forall u_i$$

Since u_i is arbitary: $\Rightarrow$ $\sigma_{ij,j} + (b_i - \rho a_i) = \underset{\sim}{0}$

$$\sigma_{ij} n_j = t_i$$

Sept.28, 1992 page 4.

$\therefore$ PVW + $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ + arbitary displ. replaces Linear Momentum Balance.

<u>Application</u>: <u>Numerical Methods.</u> $\Rightarrow$ PVW eqns + arbitary u_i somewhat approximately replace Linear Momentum Balance Equation.

Next class: Start with Linear Momentum Balance + $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ + $\sigma_{ij} = \sigma_{ji}$ $\Rightarrow$ PVW is true.

[Again, $\underset{\approx}{\sigma}$ need Not have physical meaning].

$\underset{\approx}{\epsilon}$ need " " " "

$\underset{\approx}{\sigma}$ & $\underset{\approx}{\epsilon}$ need Not be related]

<u>Sept 30, 1992</u> <u>page 1.</u>

<u>Today : Unprepared lecture on PVW.</u>

Ave. stress : $\frac{1}{V}\int_V \sigma_{ij}\,dV$, think about $t_i x_j$ & divergence thm.

Principle of virtual work.

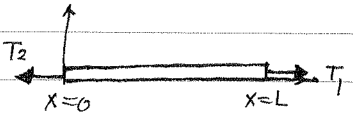
PVW is true if equilibrium is true & strain-displacement relation holds.

Given $\sigma_{ij,j} + b_i = \rho a_i$ & $\sigma_{ij} = \sigma_{ji}$ & $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\Rightarrow \int_V \sigma_{ij}\varepsilon_{ij}\,dV = \int_S (b_i - \rho a_i)\,u_i\,dS + \int_S t_i u_i\,dS.$$

<u>A crude Example.</u> A rod in 1dim, no Body force, no dynamics.

$\underset{\approx}{\sigma} = T\underset{\sim}{e_1}\underset{\sim}{e_1}$



$\varepsilon_X = u_{X,X} = \epsilon$

What does PVW say ? $\int_V \sigma_{ij}\varepsilon_{ij}\,dV = \int_0^L T\epsilon\,dl = \text{IVW}$

$\text{EVW} = u(L)T_1 - u(0)T_2$

What does this said about T_1 & T_2 & $T(x)$ for <u>arbitary</u> $u(x)$

Sept. 30, 1992. page 2.

Imagine $\underset{u_0}{\overset{}{u(x) = \underset{\parallel}{\text{constant}}}} \Rightarrow u_{x,x} = 0 \Rightarrow \varepsilon = 0$

$$\therefore \quad 0 = u[T_1 - T_2] \quad \Rightarrow \quad T_1 = T_2.$$

Now $\quad T u_{x,x} = (T u_x)_{,x} - T_{,x} u_x$

$$\int_0^L (T u_{x,x})\, dl = \int_0^L (T u_x)_{,x}\, dx - \int_0^L T_{,x} u_x\, dx = \cancel{T_1 = T_2}\ T_1 u(L) - T_2 u(0)$$

$$\Rightarrow \int_0^L T_{,x} u_x\, dx = \cancel{T u_x \Big|_0^L - \int_0^L (T u_{x,x})\, dl} = T u_x \Big|_0^L + T_1 u(L) - T_2 u(0)$$

$\Rightarrow \quad T_{,x} = 0 \qquad$ T does not change along length.

<u>PVW and Real Work</u> (statics only)

A body is subjected to load and deform, Real external work $= \sum \underset{\text{path of force}}{\int} \underset{\sim}{F} \cdot d\underset{\sim}{l}$

Motion of material pt where F is applied.

the actual applied Body force

$$dW_{ext} = \int_S t_i\, du_i\, dS + \int_V b_i\, du_i\, dV$$

the real traction

du_i = actual displacement increment.

Note that if $\quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$d\varepsilon_{ij} = \tfrac{1}{2}\left[(du_i)_{,j} + (du_j)_{,i}\right]$$

$\Rightarrow$ can use $d\varepsilon_{ij}$ and du_j as the virtual strain & virtual displacement in PVW. Since the real stress satisfies $\sigma_{ij,j} + b_i = 0$

$$\Rightarrow \int_V \sigma_{ij}\, d\varepsilon_{ij}\, dV = \int_V b_i (du_i)\, dV + \int_S t_i\, du_i\, dS$$

Assume First Law of thermodynamics (No heat flow)

work $\overset{\text{in}}{\wedge}$ = increase in Internal energy

$$dE_{int} = \int_V \sigma_{ij}\, d\varepsilon_{ij}\, dV \rightarrow$$

This form of energy depends on the material not necessary recoverable.

Sept. 30, 1992 page 3.

Total int. energy $= \int dW$.

How to keep track of $\int dW$?

Net work in a process $= \Delta W = \int_{t_1}^{t_2} \dot{W}\, dt$

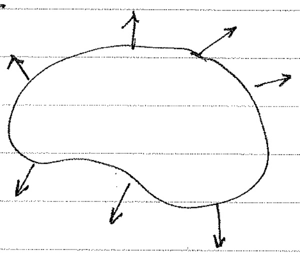
where $\dot{W}$ [redacted] $= \int_V \sigma_{ij} \frac{d\varepsilon_{ij}}{dt}\, dV$

$$= \int_V b_j \dot{u}_i \, dV + \int_S \dot{u}_i t_i \, dS$$

$$W_2 - W_1 = \Delta W = \int_{t_1}^{t_2} \dot{W}\, dt = \int_{t_1}^{t_2} \left[\int_V \sigma_{ij} \dot{\varepsilon}_{ij}\, dV \right] dt.$$

Example

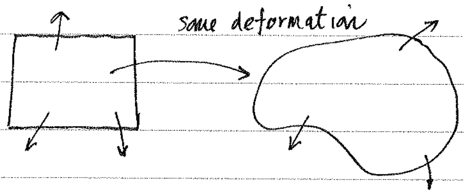
some Load.



Have some t_{ij}, b_i, σ_{ij}

PVW equation is true.

and



u_i, ε_{ij}

Today: 6:30 RM 101 Monday. Oct. 5, 1992. page 1.

A) Momentum balance ⇒ force ⇒ $\underset{\approx}{\sigma}$ & $\underset{\sim}{t}$,

B) Deformation ⇒ $\underset{\approx}{\epsilon}$

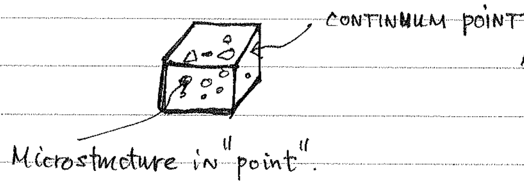
C) PVW. Eqn. uses the concept of stress & strain

What does the material have to say about relation between stress & strain?

Assumption : $\underset{\approx}{\sigma}$ describes an average microscopic internal forces

$\underset{\approx}{\epsilon}$ " " " microscopic motions

then $\underset{\approx}{\sigma}$ determines $\underset{\approx}{\epsilon}$ OR vis-versa.



Average motion ⇒ details of micro-structural motion
⇒ details of the forces are
⇒ average traction ⇒ average stress } Rarely executed in details.

Most General constitutive law : Imagine given a material and you know it's initial state (e.g. you know where every atom is) Imagine $0 < t' < t$ where t is the time of interest, Imagine in this time interval, $\epsilon(t')$ is given
↑ strain history.

$$\underset{\approx}{\sigma}(t) = \underset{\text{functional}}{\mathcal{F}}\left(\underset{\approx}{\epsilon}(0 \leq t' \leq t), \text{ initial state}\right)$$

OR conversely,

$$\underset{\approx}{\epsilon}(t) = \mathcal{G}\left(\underset{\approx}{\sigma}(t'), 0 \leq t' \leq t, \text{ initial state}\right)$$

"Additional state variable evolution equation is needed."

More practically, but still general, $\underset{\approx}{\sigma}(t) = F(\alpha_i(t), \dot{\underset{\approx}{\epsilon}})$
↑ state variable.

adequately describe internal state variable of material

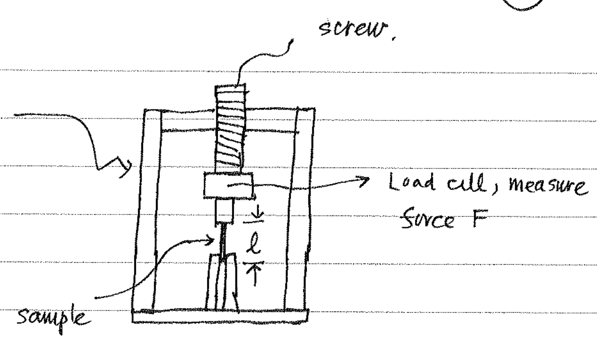
$$\dot{\alpha}_i = g_i(\underset{\sim}{\alpha}, \underset{\approx}{\epsilon}) \qquad i = 1, \cdots n = \text{no. of state variable.}$$

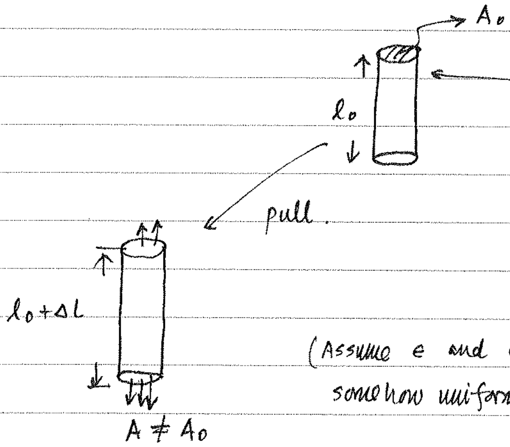
still too general to be of much practical used.

<u>Time, history & rate effect.</u> Two not quite distinct questions : 1) Rate or time or history effects.
2) Effects associated with stress and strain being tensors, how does σ_{23} depends on ϵ_{12} ?

page 2 [redacted] Oct, 5, 1992 Monday.

1D constitutive Laws; Mechanical testing. (Testing machine)





Define $\epsilon = \frac{\Delta l}{l_0}$

$\sigma = F/A_0$ = nominal stress

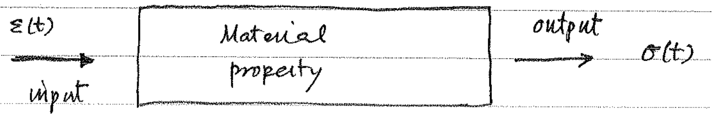
(Assume ε and σ are somehow uniform in sample).

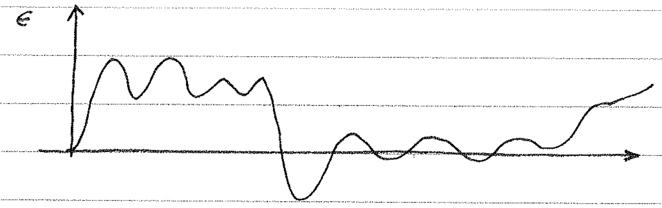
$\neq$ true stress $= \frac{F}{A}$

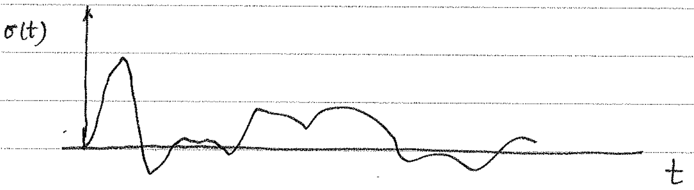
unless $A \approx A_0$

Constitutive law is relation between ε(t) and σ(t). in (1D).

For a given initial state. (Black Box).

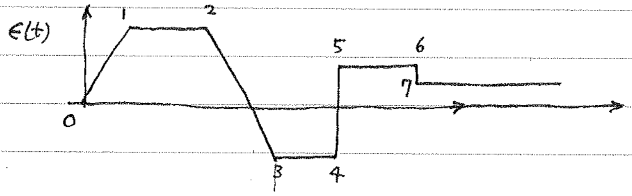






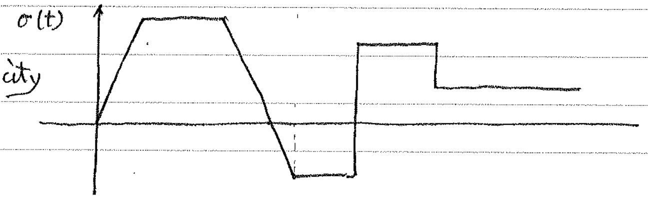
Models: Examples of 1D Constitutive laws:

(I) Elasticity: (A) linear elasticity



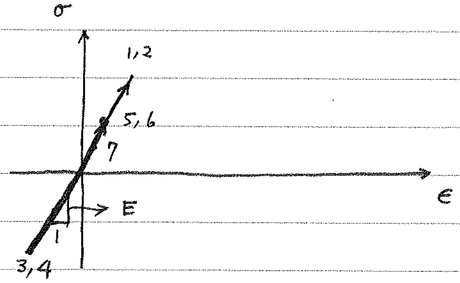
$\sigma = E \epsilon$

Young's modulus of elasticity

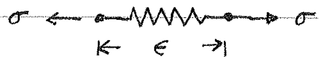


page 3.

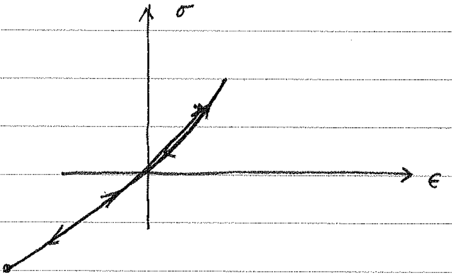
Cross plot. A)



A curve of σ vs ε describe the whole constitutive Law:

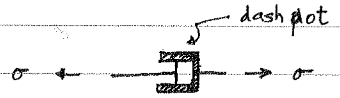


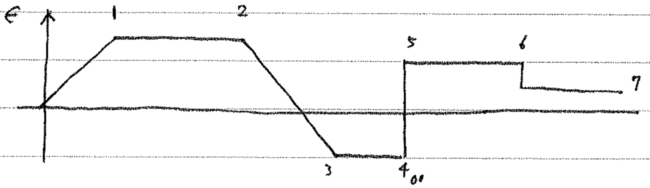
B Nonlinear Elasticity. $\sigma = f(\epsilon)$

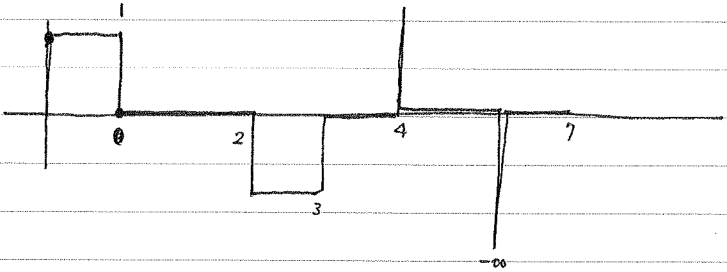


II Viscous :

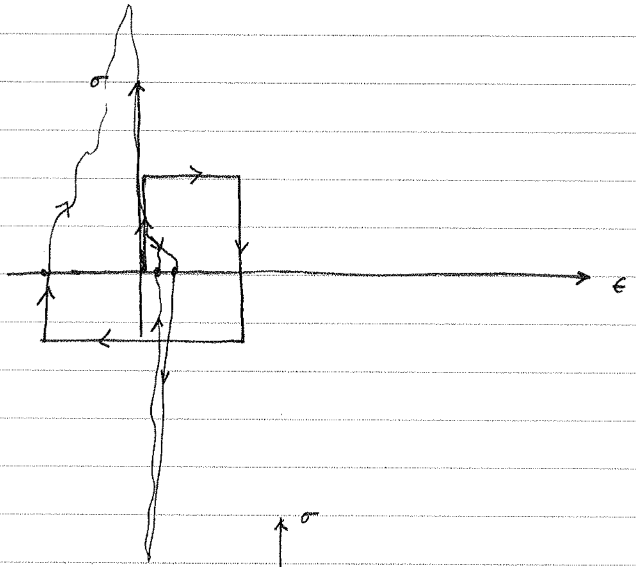
(A) Linear viscous



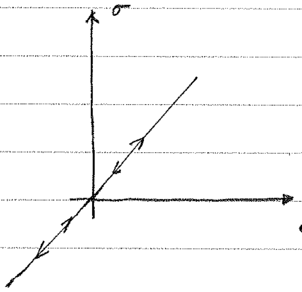




Cross-plot.



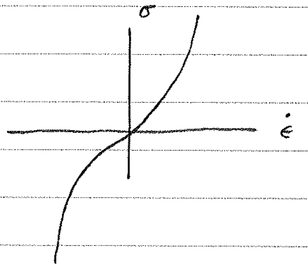
Cross-plot

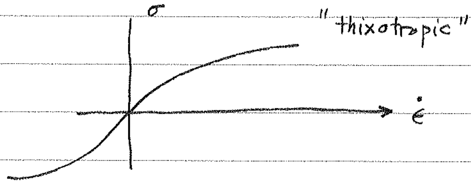


page 3.

Non-linear Dash pot. (viscous) $\sigma = f(\dot{\epsilon})$

"Anti-thixotropic"



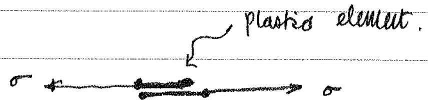


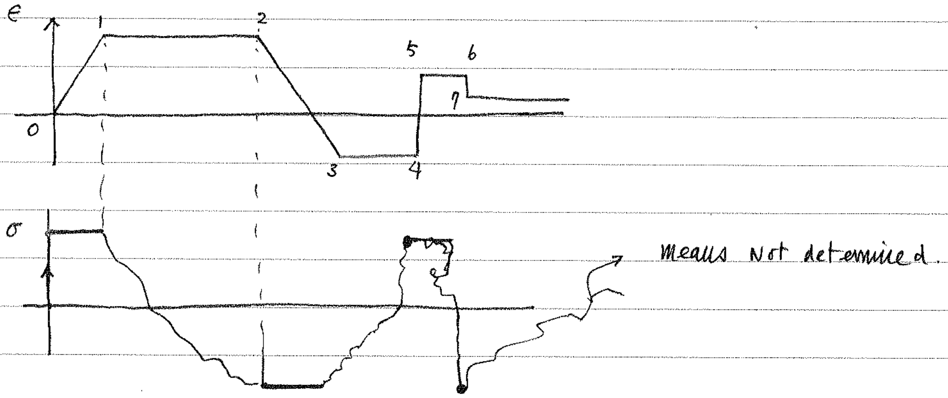
(III) Plasticity (None of them are accurate for any materials).

Momentum balance: Extremely accurate.
Linear Elasticity very accurate for some material under small Load (10^{-5})

Non-linear and inelastic (OK 5% - 80% ERROR).

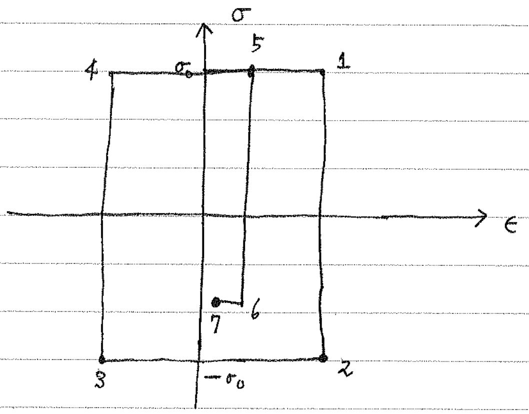
(A) CRUDEST LAW: Rigid-perfectly plastic





$$\sigma = \sigma_0 \quad \text{if} \quad \dot{\epsilon} > 0$$
$$\Rightarrow |\sigma| < \sigma_0 \quad \text{if} \quad \dot{\epsilon} = 0$$
$$= -\sigma_0 \quad \text{if} \quad \dot{\epsilon} < 0$$

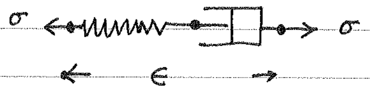
CROSS-Plot.

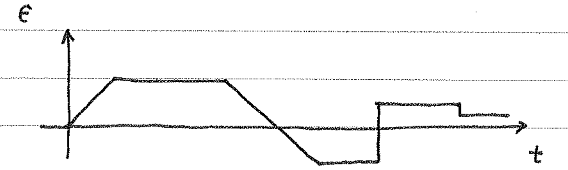


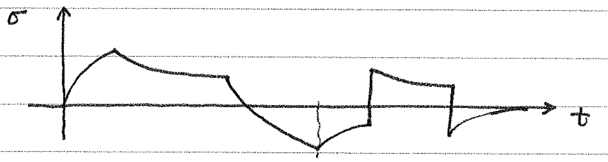
Oct. 7, 1992 page 2

Linear Visco-elasticity. Laws.

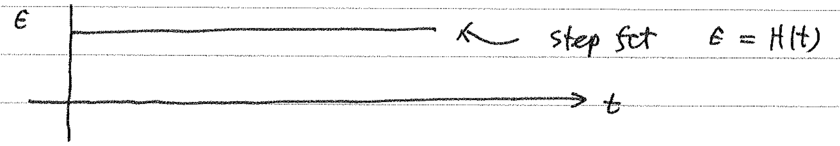
A. ~~Kelvin - Voigt~~ Maxwell Models.

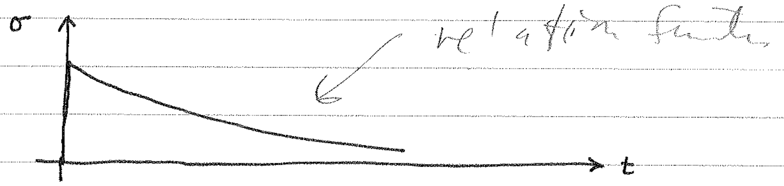




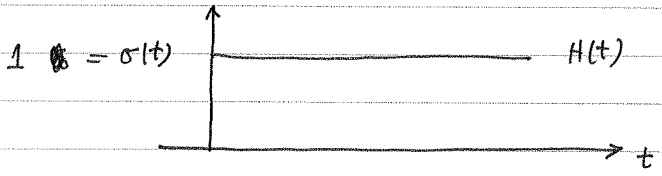


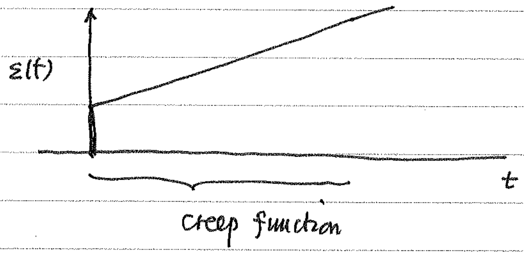
Because the law is linear, all one needed to draw is the response due to a step input.





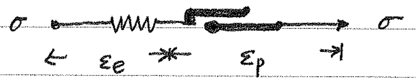
Voigt.
~~B~~.A ~~Maxwell~~ Model . Instead of step input in strain, we could use a step input in stress.



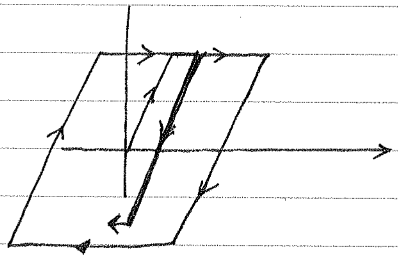


page 4. Oct. 5, 92.

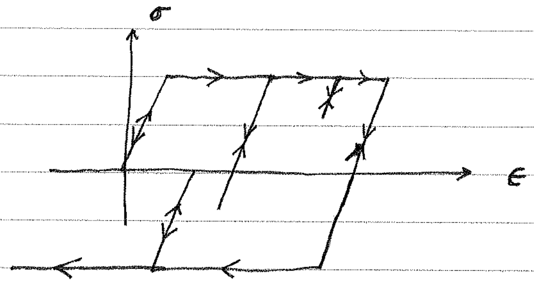
B. Elastic – perfectly plasticity.



$\varepsilon = \varepsilon_e + \varepsilon_p$



Elastic perfectly plasticity curve in another test.



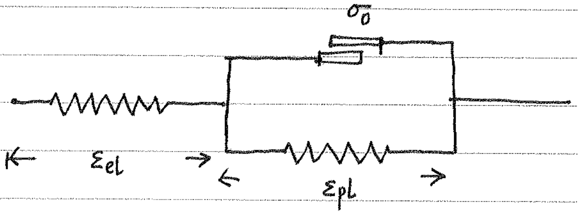
page 1 Oct, 7, 1992 TAM 663.

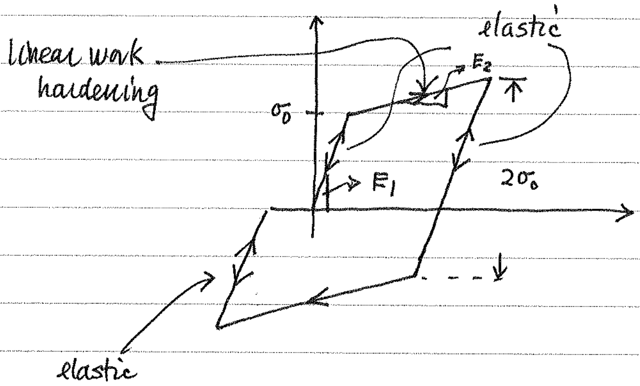
1 Day constitutive law. & Introduction to visco-elasticity.

C.) Elastic – linear work hardening = plastic

"Kinematic hardening" (compression yield stress decreases as tension yield stress increases).

↑ yield surface moves rigidly in stress field.

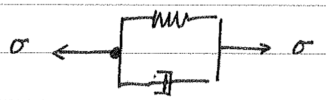


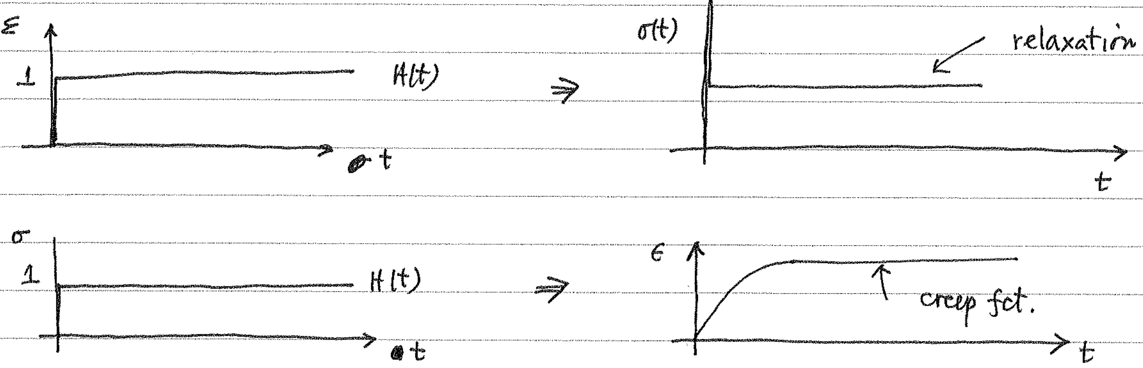


D) Plasticity laws gets very elaborate e.g. (Hart).

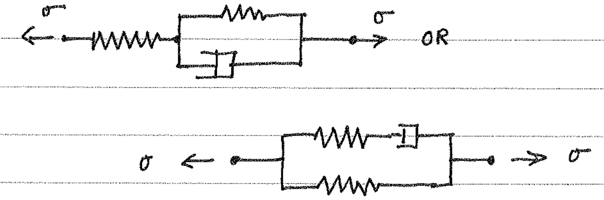
page 3.   Oct. 7, 92

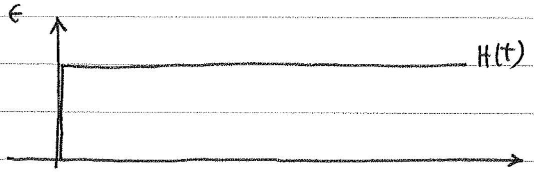
~~Maxwell model~~
Kelvin Voigt

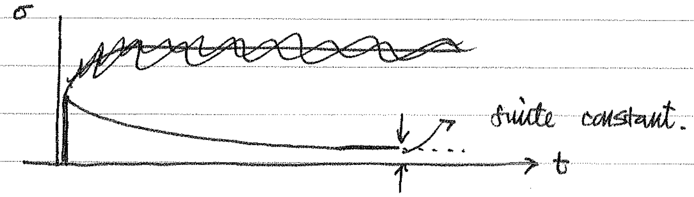


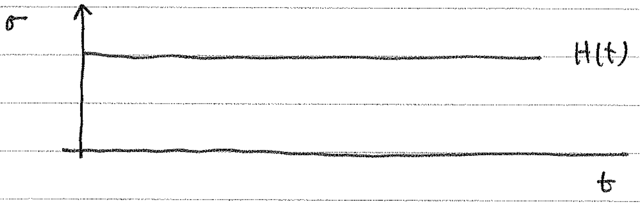


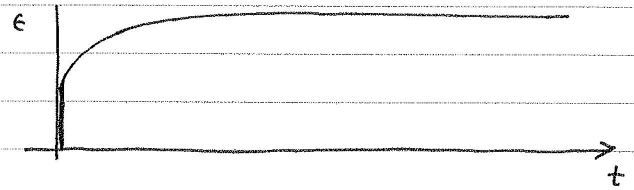
c. 3 Element model, the standard model.











Two elastic moduli   Long term and short term.

Ratio of Hts at $t=0$ short time modulus

" " " at $t=\infty$ Long time modulus.

page.4 Oct. 7, 92.

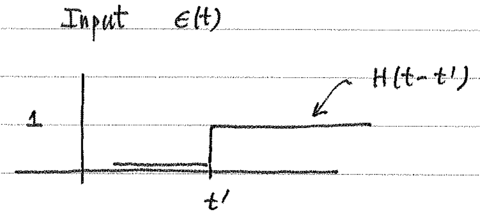
D. General, linear, time independent.

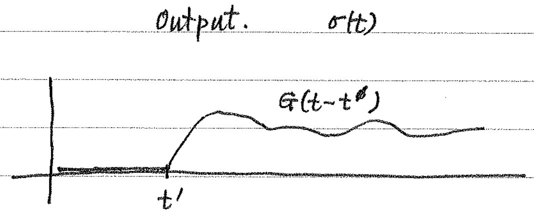
Linearity ⇒ If $\epsilon_1(t) \rightarrow \sigma_1(t)$ and $\epsilon_2(t) \rightarrow \sigma_2(t)$, then
$\epsilon = A\epsilon_1(t) + B\epsilon_2(t) \Rightarrow \sigma(t) = A\sigma_1(t) + B\sigma_2(t)$ where A & B are scalars.

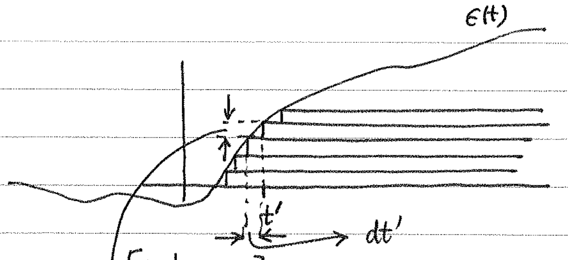
time independent : homogeneous in time OR autonomous. $\epsilon(t) \rightarrow \sigma(t)$
then : $\epsilon(t-t') \rightarrow \sigma(t-t')$

excludes curing, age, absorption of moisture OR material constants are independent of time.

Let's use this two assumptions and derive the general law :







$$\left[\frac{d\epsilon}{dt}\bigg|_{t'} dt'\right] H(t-t')$$

$$\int_{-\infty}^{t} h(t-t')\,\frac{d\varepsilon}{dt'}\,dt' = \overset{\epsilon}{\sigma}(t)$$

$$\sigma(t) = \int_{-\infty}^{t} G(t-t')\,\frac{d\varepsilon}{dt'}\,dt'$$

convolution of $G * \epsilon'(t)$

step response.

[Elasticity, (Linear),] (Constitutive model.) (6) eqns.

$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ e.g $\sigma_{xy} = C_{xykl}\,\epsilon_{kl}$. at a point in space

There are $3 \times 3 \times 3 \times 3 = 81$ material constants or 81 elastic moduli.

$\underset{\sim}{C} = C_{ijkl}\,\underset{\sim}{e}_i\,\underset{\sim}{e}_j\,\underset{\sim}{e}_k\,\underset{\sim}{e}_l$ is a fourth order tensor which describes a linear relation between two second order tensor.

page 5 Oct. 7, 92.

How many independent constants are there ?

$\sigma_{ij} = \sigma_{ji} \Rightarrow \underset{\approx}{\sigma}$ has only 6 independent components.

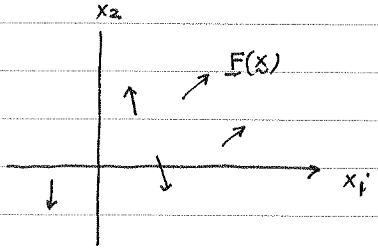
$\therefore \quad C_{ijkl} = C_{jikl}$ $\qquad 6 \times 9 = 54$ constant.

$\varepsilon_{ij} = \varepsilon_{ji} \quad C_{ijkl} = C_{ijlk}$, we now have $36 = 6 \times 6$ independent constant.

Is $C_{ijkl} \overset{?}{=} C_{klij}$ If this is true, then we can get rid of 15 constant (since the six by six matrix is symmetric) this gives. $36 - 15 = 21$ constants.

Something to think about ?

$$\underset{\text{force}}{F_i} = A_{ij} \underset{\text{position}}{x_j}$$



under what circumstance will $A_{ij} = A_{ji}$.

page 1 Oct. 13, 1992.

Momentum Balance $\sigma_{ij,j} = -b_i + \rho \ddot{a}_i \underset{\text{Statics + No Body forces.}}{\Rightarrow} \sigma_{ij,j} = 0$ (3 Equations.)

$\sigma_{ij} = \sigma_{ji}$

$\sigma_{ij} n_j = t_i$

(6 eqns)

$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ $\qquad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}$. Constitutive Law. (6 eqns.)

Strain displacement relation (Geometry) $\qquad$ (Material properties)

What to do with this equations :

Ans: Solve ∂ value problems (BVPS).

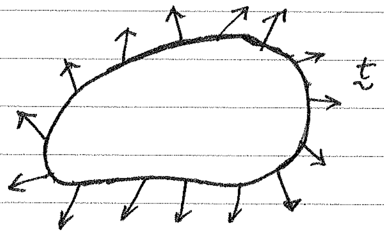
Given information about the Loads & displacements on the outside of some body, find other information about forces & displacement inside & outside the body.

page 2.

Simplest example: Traction Boundary value problem.

Given : Geometry of Body, given material law and $\underset{\sim}{t}$ everywhere on the boundary of the Body.

What is σ_{ij}, $\varepsilon_{ij}(\underset{\sim}{x})$, $\underset{\sim}{u}(\underset{\sim}{x})$



Simplest examples is : given $\underset{\sim}{t} = \underset{\approx}{\sigma}^{\circ}\cdot\underset{\sim}{n}$

$\uparrow$ constant independent of $\underset{\sim}{n}$, $\underset{\approx}{\sigma}^{\circ}$ is a symmetric tensor.

$\underset{\sim}{b} = \underset{\sim}{a} = 0$

If C_{ijkl} is a constant independent of position.

then

$$\sigma_{ij} = \sigma^{0}_{ij}$$

$$\varepsilon_{ij} = C^{-1}_{ijkl}\,\sigma^{0}_{ij}$$

Clearly, for $\underset{\approx}{\sigma}^{\circ} = \underset{\approx}{\sigma}^{\circ T}$, the momentum equation is valid.

Constitutive law, $\underset{\sim}{\varepsilon} = \underset{\approx}{C}\,\underset{\sim}{\sigma}$ $\Rightarrow$ $\varepsilon_{ij} = C^{-1}_{ijkl}\,\sigma_{kl}$

$$= C^{-1}_{ijkl}\,\sigma^{0}_{kl} = L_{ijkl}\,\sigma^{0}_{kl}$$

$$\varepsilon_{ij} = L_{ijkl}\,\sigma^{0}_{kl} \equiv \varepsilon^{0}_{ij}$$

Strain displacement equation $(u_{i,j} + u_{j,i})/2 = \varepsilon_{ij}$

~~Does~~ Do such u_i exist $\ni$ $\left(\frac{u_{i,j} + u_{j,i}}{2}\right) = \varepsilon^{0}_{ij}$

Answer let $u_i = \varepsilon^{0}_{ij}x_j$ $= \varepsilon^{0}_{ik}x_k$, then $\frac{u_{i,j} + u_{j,i}}{2} = \varepsilon^{0}_{ij}$ ✓ (Assuming that $L_{jikl} = L_{ijkl}$)

Note that, in general $\sigma_{ij}(\underset{\sim}{x}) \Rightarrow \underset{\sim}{\varepsilon}(\underset{\sim}{x}) \Rightarrow \underset{\sim}{u}(\underset{\sim}{x})$

$\uparrow$ this step is hard to do. Assured it is possible if ε_{ij} satisfies the six compatible equations.

page 3.

So we have a solution

$$\sigma_{ij} = \overset{o}{\sigma}_{ij}$$

$$\varepsilon_{ij} = L_{ijkl}\overset{o}{\sigma}_{kl}$$

$$u_i = L_{ijkl}\overset{o}{\sigma}_{kl}\, x_j$$

satisfies all equations.

solution is not unique since

$$u_i = L_{ijkl}\overset{o}{\sigma}_{kl}\, x_j + \underbrace{\underbrace{w_{ij}}_{\text{anti-symmetric}}}_{\text{Rigid Body Rotation.}} x_j + \underbrace{\overset{o}{u}_i}_{\text{constant. translation.}}$$

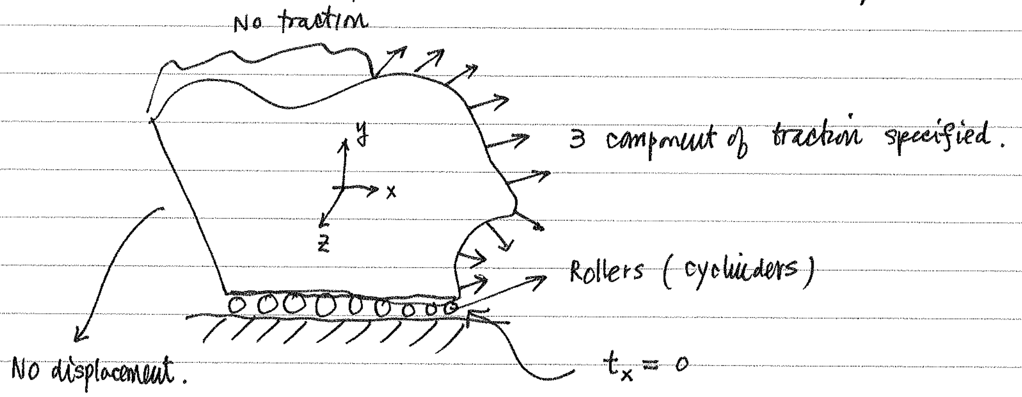
positive definiteness of the L_{ijkl} tensor ?

Fact: stress & strains are unique, displacements unique, up to a small rigid Body motion.

other boundary value problems:

Can specify positions or displacements on other parts of the Body.

The most general common case is: Everywhere on the ∂, make a 3D orthogonal coord. system. For each of the three components: must specify traction or displacement.



$t_x = 0$

$u_y = 0$

$u_z = 0$ } two displacements are specified.

page 4.

More complicated boundary conditions are possible that mixed-up the displacements & traction, e.g. put springs on $\partial \cdot$ or another elastic body on the boundary.

Let's look at work as a given body deforms. (Homogenous deformation)

$$dW = \underset{\text{Power}}{P}\,dt = \left[\int_S t_i \dot{u}_i \, dS\right] dt$$

$$\overset{\text{PVW.}}{\overset{\cancel{\text{PVB}}}{=}} \left[\int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV\right] dt.$$

If we assume Homogenous deformation

$$P\,dt = \sigma_{ij} \dot{\varepsilon}_{ij} V \, dt.$$

$$\therefore \quad P = \sigma_{ij} \dot{\varepsilon}_{ij} V$$

Suppose $\underset{\sim}{t_i} = \underset{\text{traction}}{\underset{\sim}{t_i}}(t)$ $\quad t =$ time $\quad$ Homogenous deformation

$$\Delta W = \text{Net work} = \int_{t_1}^{t_2} \sigma_{ij} \dot{\varepsilon}_{ij} V \, dt \qquad \left[\text{Note that } \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = \sigma_{ij}(\underset{\approx}{\varepsilon})\right]$$

= path integral in strain space (6 dimensional)

$$= \int_{\underset{\approx}{\varepsilon}_1}^{\underset{\approx}{\varepsilon}_2} \sigma_{ij} \, d\varepsilon_{ij} \quad \text{on } \underline{\text{some path}}. \qquad \text{where } \sigma_{ij}(\underset{\approx}{\varepsilon})$$

always true whether σ_{ij} is a function of $\underset{\approx}{\varepsilon}$ or not.

Notation to simplify thinking:

$$\sigma_{ij} \rightarrow \sigma_i$$

$$[\sigma] = \begin{bmatrix} \sigma_1 & \sigma_4 & \sigma_5 \\ \sigma_4 & \sigma_2 & \sigma_6 \\ \sigma_5 & \sigma_6 & \sigma_3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_6 \end{bmatrix}$$

Similarly for. $[\varepsilon]$

p.5.

Then C_{ijkl} has now only two subscript.

$$C_{ijkl} \Rightarrow C_{ij} = \begin{bmatrix} C_{11} \cdots\cdots & C_{16} \\ C_{21} \cdots\cdots & C_{26} \\ \vdots & \\ C_{61} \cdots\cdots & C_{66} \end{bmatrix}$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \Rightarrow \sigma_i = C_{ij}\,\varepsilon_j$$

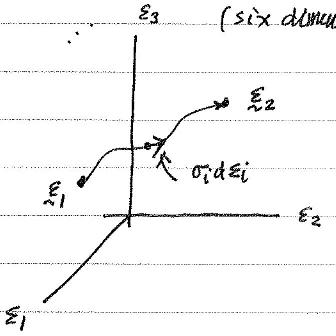
Ask student to identify C_{ij} with C_{ijkl}.

Back to work path integral.

$$W = \int P\,dt = \int_{path} \sigma_{ij}\,d\varepsilon_{ij} \Rightarrow W = \int_{path} \sigma_i\,d\varepsilon_i$$

$$W = \int_{\underset{\sim}{\varepsilon_1}}^{\underset{\sim}{\varepsilon_2}} C_{ij}\,\varepsilon_j\,d\varepsilon_i$$

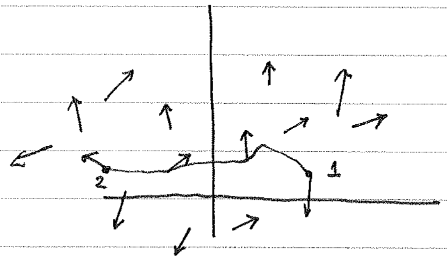
(six dimensional strain space)



Aside on conservative vector fields.

Assume every point in space a vector field is defined. $\underset{\sim}{F}(\underset{\sim}{x},\underset{\sim}{y})$

Force.



Work ~~on~~ from ① to ②

$$\int_{\textcircled{1}}^{\textcircled{2}} \underset{\sim}{F}\cdot d\underset{\sim}{x} = \int_{\textcircled{1}}^{\textcircled{2}} F_x\,dx + \int_{\textcircled{1}}^{\textcircled{2}} F_y\,dy.$$

page 6.

The work is a function of <u>end points</u> & <u>Path</u> given $F(\underset{\sim}{x})$

$$W(\underset{\sim}{x}) = \int_{\underset{\sim}{x}_0}^{\underset{\sim}{x}} \underset{\sim}{F}(\underset{\sim}{x})\cdot d\underset{\sim}{x}$$

Two possibility: (I) the kind of vector field $\underset{\sim}{F}$ ∋ W depends on path as well as end pts
(II) Independent of path but depends on End points.

case (II) $\Rightarrow$ $dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy.$

$$= F_x dx + F_y dy$$

$$\therefore \quad \frac{\partial W}{\partial x} = F_x \qquad \frac{\partial W}{\partial y} = F_y. \qquad (1)$$

Note, in general, if we are <u>Given</u> $F_x = F_x(x,y)$ and $F_y = F_y(x,y)$, then in general (1) is Not. unless

$$\boxed{\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}}$$

If W does not exist, then going round & round gives infinite work. ⇒ Force is "conservative"
does not imply an ∞ amount of work.

page 1. Sept. 19, 1992. TAM 663.

Today: Elastic constants.

Recall: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \rightarrow \sigma_i = C_{ij}\varepsilon_j$ $\quad i \leq 6$, $j \leq 6$.

For homogeneous Deformation $W = V\int_{\underset{\sim}{\varepsilon}_1}^{\underset{\sim}{\varepsilon}_2} \underset{\sim}{\sigma}\cdot d\underset{\sim}{\varepsilon}$ or $W = V\int_{\varepsilon_{ij}^{(1)}}^{\varepsilon_{ij}^{(2)}} \sigma_{kl}\, d\varepsilon_{kl}$ Volume.

Note that W is dependent on loading history.

If we assume that <u>W is independent</u> of Loading history OR path, then

$$\frac{\partial W}{\partial \varepsilon_i} = \sigma_i \qquad \text{by (1) in previous lecture.}$$

or

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}$$

page 2. Oct. 19, 1992.

since the W is independent of path, let $\varepsilon_{ij} = t\overset{\circ}{\varepsilon}_{ij}$ where we start with zero strain, and ends up in $\overset{\circ}{\varepsilon}_{ij}$

then $$\int_0^{\overset{\circ}{\varepsilon}_{ij}} \sigma_{ij}\, d\varepsilon_{ij} = \int_0^1 C_{ijkl}\varepsilon_{kl}\, d\varepsilon_{ij}$$

$$= C_{ijkl}\int_0^1 \overset{\circ}{\varepsilon}_{kl}\overset{\circ}{\varepsilon}_{ij}\; t\, dt$$

$$= \frac{C_{ijkl}\overset{\circ}{\varepsilon}_{kl}\overset{\circ}{\varepsilon}_{ij}}{2} = \frac{\overset{\circ}{\sigma}_{ij}\overset{\circ}{\varepsilon}_{ij}}{2}$$

$$\therefore \quad W = \frac{\overset{\circ}{\sigma}_{ij}\overset{\circ}{\varepsilon}_{ij}}{2} = \frac{C_{ijkl}\overset{\circ}{\varepsilon}_{ij}\overset{\circ}{\varepsilon}_{kl}}{2}$$

~~Let~~ so that $\underline{C_{klij}} = C_{ijkl}$. OR $C_{ij} = C_{ji}$

Consider Non-linear elasticity : $\sigma_i = \sigma_i(\underset{\sim}{\varepsilon}) = \sigma_i(\varepsilon_i)$

If a potential exist (and it better) $\dfrac{\partial\sigma_i}{\partial\varepsilon_j} = \dfrac{\partial\sigma_j}{\partial\varepsilon_i}$

Linear Elasticity. Work $= V\int \sigma_i\, d\varepsilon_i$ is conservative

$$\frac{\partial C_{ij}\varepsilon_j}{\partial\varepsilon_k} = \frac{\partial C_{kj}\varepsilon_j}{\partial\varepsilon_i}$$

$\Rightarrow \quad C_{ik} = C_{ki}$ e.g. $C_{14} = C_{41}$

or $C_{1112} = C_{1211}$

So we only have $\dfrac{\overset{\overset{(36-6)}{\parallel}}{30}}{2} + 6 = 15 + 6 = 21$ terms are independent.

<u>Cartoon</u> Imagine 6 experiment with 6 measurements for <u>each</u> experiments

$\varepsilon_1 \neq 0$ all others zero measure 6 stress components

$\varepsilon_2 \neq 0$ " " " Ditto

$\varepsilon_3 \neq 0$ " " " ⋮

$\varepsilon_{12} \neq 0$ " " " ⋮

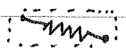
$\varepsilon_{13} \neq 0$ " " " ⋮

$\varepsilon_{23} \neq 0$ " " " ⋮

of these 36 measurements, only 21 are independent.

Example. composite made with fibers, say imposed was $\varepsilon_{11} = 0.001$ and measured was $\sigma_{22} = 10^{8}$ MPa $\Rightarrow$ $\varepsilon_{22} = .001$, measure would be $\sigma_{11} = 10^{8}$ MPa.

$$C_{12} = C_{21}$$

[Think of ]

$$\sigma_{11} = C_{12}\varepsilon_{22} = C_{21}\varepsilon_{11} = \sigma_{22}$$

$$\underline{\text{or}} \quad C_{1122} = C_{2211}.$$

Define strain energy W = Work per unit volume that you have to do to the material to strain it from 0 to $\underset{\approx}{\varepsilon}$.

$$W = \int_{0}^{\underset{\approx}{\varepsilon}} \sigma_{ij}' \, d\varepsilon_{ij}' \qquad \text{where} \quad \sigma_{ij}' = \sigma_{ij}'(\varepsilon_{ij}')$$

$$= \sigma_{ij}\varepsilon_{ij}/2 \qquad \text{as Before.}$$

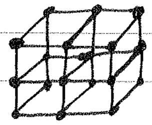
$$= \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = W(\underset{\approx}{\varepsilon})$$

Net work for Homogenous deformation from 0 to $\underset{\approx}{\varepsilon}$ = $VW(\underset{\approx}{\varepsilon})$

21 constants is too big! There will be less if material has symmetry or the micro-structures are random (which implies isotropy)

Two Examples. (cubic symmetry + isotropy)

Cubic symmetry: cubic crystal lattice

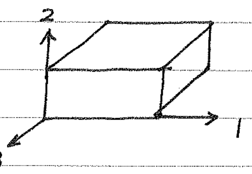


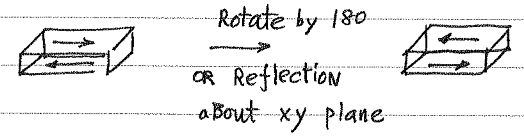
Assume coordinate axis aligned with the cubic axis

Do six experiments, where one strains are Not zero, all others are zero.

Experiment 1

ε_1 or ε_{11} $\longrightarrow$ $\sigma_1 = C_{11}\varepsilon_1$, $\sigma_2 = C_{21}\varepsilon_1$, $\sigma_3 = C_{31}\varepsilon_1 = C_{21}\varepsilon_1$ (Rotate 90° about x axis has to get same result), $\sigma_{12} = \sigma_4 = C_{41}\varepsilon_1 \Rightarrow 0 = C_{41}$ since Rotation about x axis by $\pi = 180^\circ$)





$$\therefore \quad C_{41} = C_{51} = C_{61} = 0$$

By material symmetry

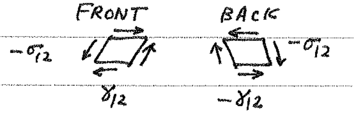
ε_2 or ε_{22} $\quad \sigma_2 = C_{22}\varepsilon_2 = C_{11}\varepsilon_2$ $\quad \sigma_1 = C_{12}\epsilon_2 = C_{21}\epsilon_2$ $\quad \sigma_3 = C_{32}\epsilon_2 = C_{21}\epsilon_2$.

90° Rotation about x_3 axis

$$\sigma_4 = \sigma_5 = \sigma_6 = 0 \quad \Rightarrow \quad C_{42} = C_{52} = C_{62} = 0$$

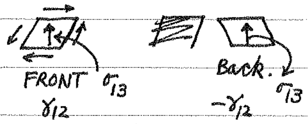
ϵ_3 = just the same

$\epsilon_4 = \epsilon_{12} \rightarrow \sigma_4 = C_{44}\epsilon_4$ (Note : since shear strain cannot cause elongation (since elongation cannot cause shear and by existence of strain energy, so that all others $\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6 = 0$ or $C_{14}, C_{24}, C_{34}, C_{54}, C_{64} = 0$)



No contradiction

if



$C_{13} = 0$

$\sigma_5 = C_{44}\epsilon_5$

$\sigma_6 = C_{44}\epsilon_6$

$$\therefore \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{21} & 0 & 0 & 0\\ C_{21} & C_{11} & C_{21} & 0 & 0 & 0\\ C_{21} & C_{21} & C_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{44}\end{bmatrix} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

$\Rightarrow$ 3 independent constants

3 orientation variables.

Not an isotropic material

page 4.

What if form of C was persevered for all orientations ? ⇒ isotropic.

Isotropic case.

10/20/92.   Lecture TAM 663.   page 1.

Isotropic material. $\sigma_{ij} = C_{ijkl} u_{k,l}$ $3 \times 3 \times 3 \times 3 = 81$ constants in general.

$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl} = C_{jikl} \Rightarrow 6 \times 9 = 54$ constants

Rotation does not cause stress $\varepsilon_{ij} = \epsilon_{ji} \Rightarrow C_{ijkl} = C_{ijlk} \Rightarrow 6 \times 6 = 36$ constant.

Can get work out of a closed cycle in strain cycle in strain space $\Rightarrow C_{ijkl} = C_{klij}$

or $C_{12} = C_{21}$, $\Rightarrow \frac{6 \times 6}{2} + \frac{6}{2} = 21$ constants.

Various symmetries reduce the elastic constant.

e.g. Transversely isotropic. (see H.W)

cubic symmetry (last time)   3 elastic constants.

Isotropy   2 constants.

Isotropic, linear, elastic constitutive law.

Pick coordinate axis aligned with principal direction of stress.

Find $\underset{\approx}{\epsilon}$ caused by $\underset{\approx}{\sigma}$

Isotropy ⇒ principal axis of $\underset{\approx}{\epsilon}$ same as principal axis of $\underset{\approx}{\sigma}$

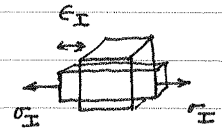
Now $\underset{\approx}{\sigma} = \sigma_1 \underset{\sim}{n}_I \underset{\sim}{n}_I + \sigma_2 \underset{\sim}{n}_{II} \underset{\sim}{n}_{II} + \sigma_3 \underset{\sim}{n}_{III} \underset{\sim}{n}_{III}$

$\underset{\approx}{\epsilon} = \epsilon_1 \underset{\sim}{n}_I \underset{\sim}{n}_I + \epsilon_2 \underset{\sim}{n}_{II} \underset{\sim}{n}_{II} + \epsilon_3 \underset{\sim}{n}_{III} \underset{\sim}{n}_{III}$

Now σ_1 and ε_1, σ_2 and ϵ_2, ϵ_3 and σ_3 has to relate ~~th~~ to each other in exactly one way – that's one elastic constants.

σ_1 has to related to ϵ_2, ϵ_3 in exactly one way so is σ_2 to ϵ_1, ϵ_3 etc. that's the other elastic constant.

Tension test.



$$\epsilon_I, \epsilon_{II}, \epsilon_{III}$$

$$\epsilon_I = \frac{\sigma_I}{E}$$

poisson's Ratio $\nu = -\frac{\epsilon_{II}}{\epsilon_I} = -\frac{\epsilon_{III}}{\epsilon_I}$ in tension test.

<u>Strain - stress Relation in principal Axis</u>

$$\epsilon_I = \frac{\sigma_I}{E} - \frac{\nu\sigma_{II}}{E} - \frac{\nu}{E}\sigma_{III}$$

$$\epsilon_{II} = \frac{\sigma_{II}}{E} - \frac{\nu\sigma_I}{E} - \frac{\nu\sigma_{III}}{E}$$

$$\epsilon_{III} = \frac{\sigma_{III}}{E} - \frac{\nu\sigma_I}{E} - \frac{\nu\sigma_{II}}{E}$$

E and ν are the two elastic constants.

To write the strain - stress relations in a non-principal directions. TRICK.

$$\epsilon_I = \frac{\sigma_I(1+\nu)}{E} - \frac{\nu}{E}\left(\sigma_I + \sigma_{II} + \sigma_{III}\right)$$

$$\epsilon_{II} = \frac{\sigma_{II}(1+\nu)}{E} - \frac{\nu}{E}\left(\sigma_I + \sigma_{II} + \sigma_{III}\right)$$

$$\epsilon_{III} = \frac{\sigma_{III}(1+\nu)}{E} - \frac{\nu}{E}\left(\sigma_I + \sigma_{II} + \sigma_{III}\right)$$

But $\sigma_{KK} = \sigma_I + \sigma_{II} + \sigma_{III}$ is an invariant.

This means that $\epsilon_I - \frac{\sigma_I(1+\nu)}{E} = -\frac{\nu}{E}\left(\sigma_I + \sigma_{II} + \sigma_{III}\right) = -\frac{\nu}{E}\sigma_{KK}$

$\epsilon_{II} - \frac{\sigma_{II}(1+\nu)}{E}$ etc is independent of coord. system, i.e.,

$$\underset{\approx}{\epsilon} = \frac{(1+\nu)}{E}\underset{\approx}{\sigma} - \frac{\nu\sigma_{KK}}{E}\underset{\approx}{I}$$

So that

$$\epsilon_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu}{E}\sigma_{KK}\delta_{ij}$$

page 3 Oct. 28

We could have done the same thing if we start with a strain experiments, we have

$$\sigma_{ij}' = 2G\varepsilon_{ij}' + \lambda\varepsilon_{KK}\delta_{ij}$$

G, λ, Lamé constants.

Relation between constants. E, ν, G, λ.

Note $$\varepsilon_{KK} = \frac{\sigma_{KK}(1+\nu)}{E} - \frac{\nu}{E}\sigma_{KK}\delta_{KK}$$

$$= \frac{\sigma_{KK}(1+\nu)}{E} - \frac{3\nu}{E}\sigma_{KK} = \frac{(1-2\nu)\sigma_{KK}}{E}.$$

$$\therefore \quad \frac{E\varepsilon_{KK}}{(1-2\nu)} = \sigma_{KK}.$$

$$\Rightarrow \quad \varepsilon_{ij} = \frac{(1+\nu)\sigma_{ij}'}{E} - \frac{\nu}{E}\cdot\frac{E}{(1-2\nu)}\varepsilon_{KK}\delta_{ij}$$

$$\Rightarrow \quad \varepsilon_{ij}' + \frac{\nu}{(1-2\nu)}\varepsilon_{KK}\delta_{ij}' = \frac{(1+\nu)}{E}\sigma_{ij}'$$

$$\text{or} \quad \sigma_{ij}' = \frac{E}{(1+\nu)}\varepsilon_{ij}' + \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{KK}\delta_{ij}'$$

$$\left[\ \frac{\Delta V}{V} = \varepsilon_{KK} = \left(\frac{\sigma_{KK}}{3}\right)\left[3(1-2\nu)/E\right] = \frac{-P}{K}\right.$$

where K = Bulk Modulus $= \dfrac{E}{3(1-2\nu)}$]

Note $\frac{1}{2} = \nu \Rightarrow K \to \infty$.

$$\therefore \quad 2G = \frac{E}{(1+\nu)} \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Longitudinal Modulus. $= \dfrac{\sigma_I}{\epsilon_I}$ when $\epsilon_I \neq 0$ all other $\epsilon_{ij} = 0$

Let $i = j = 1$ with NO SUM

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda\epsilon_{11}$$

$$\Rightarrow \quad \frac{\sigma_{11}}{\epsilon_{11}} = (2G+\lambda) = \frac{E}{1+\nu} + \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$= \frac{E}{(1+\nu)}\left[1 + \frac{\nu}{1-2\nu}\right]$$

$$= \frac{E}{(1+\nu)}\left[\frac{\overset{1-\nu}{\cancel{2}-2\nu}}{1-2\nu}\right] = \frac{E}{(1-2\nu)}\left(\frac{1-\nu}{1+\nu}\right) > E$$

page 4. Oct. 22. 92.

How big are elastic constants ?

$1 \text{ Atm} = 15 \text{ Psi}$

$\cong \frac{1}{10} \text{ MPa} = \frac{1}{10} \text{ N/m}^2$

Fundamentally

A real material

a) Holds together

b) is not a Bomb.

$\Rightarrow$ strain energy is positive definite

$$C_{ij} x_i x_j > 0$$

$$\forall \; x_i \neq 0$$

Every experiment has to give a positive stiffness.

$$\begin{matrix} K > 0 \\ G > 0. \end{matrix} \Rightarrow \begin{matrix} E > 0 \\ -1 \leq \nu \leq \frac{1}{2} \end{matrix}$$

In Practice

Steel : $E = 30 \times 10^6 \text{ lb/in}^2$

$\cong 2 \times 10^{11} \text{ Pa} = 2 \times 10^5 \text{ MPa.}$

$$0.3 \leq \nu \leq \underset{\underset{\text{Rubber}}{\uparrow}}{\frac{1}{2}}$$

$$0 \leq \nu \leq \frac{1}{2}$$

~~Sept~~ Oct. 26, 1992 page 1.

Today: Linearity, supperposition, Reciprocal theorem.

$$\sigma_{ij,j} = -b_i \quad \text{statics} \qquad (1)$$

$$\sigma_{ij} = \sigma_{ji} \qquad (2)$$

$$\underset{\approx}{\sigma}\cdot\underset{\sim}{n} = \underset{\sim}{t} \quad (5) \qquad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3)$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad (4) \qquad C_{ijkl} = C_{jikl} = C_{klij} = C_{ijlk}$$

$$C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} > 0 \qquad \forall\ \varepsilon_{ij}, \varepsilon_{kl} \neq 0.$$

$\uparrow$
$\underset{\approx}{C}$ is positive definite

For Isotropy: Replace $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$ with

$$\varepsilon_{ij} = \sigma_{ij}\frac{(1+\nu)}{E} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

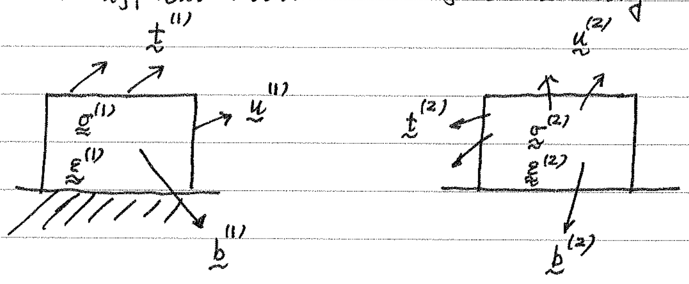
or

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\delta_{ij}$$

Consider a given linear elastic object:

$$C_{ijkl} = C_{ijkl}(\underset{\sim}{x}) \qquad \text{(Not necessarily homogeneous).}$$

Consider 2 different solns to the full elasticity equations.



Fact: all the governing eqns are "linear"

$$\underset{\sim}{u}^{(3)} = C_1\underset{\sim}{u}^{(1)} + C_2\underset{\sim}{u}^{(2)}$$
$$\underset{\approx}{\varepsilon}^{(3)} = c_1\underset{\approx}{\varepsilon}^{(1)} + C_2\underset{\approx}{\varepsilon}^{(2)}$$
$$\underset{\approx}{\sigma}^{(3)} = c_1\underset{\approx}{\sigma}^{(1)} + C_2\underset{\approx}{\sigma}^{(2)}$$

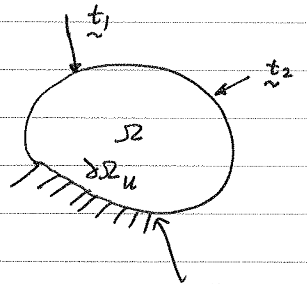
It is clear that.
$\downarrow$
$\Rightarrow$ $\underset{\sim}{u}^{(3)}$, $\underset{\approx}{\varepsilon}^{(3)}$, $\underset{\approx}{\sigma}^{(3)}$ satisfies all the governing equations (1),(2)(3),(4),(5).

This is called linearity & superposition.

Oct. 26, 1992. page2.

Use of these principles: If we are given a problem, we can break it into simpler problems & add results.

Example. 1



$\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega_u$ on $\partial\Omega - \partial\Omega_u$, we have two loads $\underset{\sim}{t}_1$ and $\underset{\sim}{t}_2$.

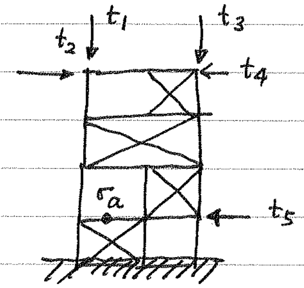
Look at the full solution if $\underset{\sim}{t}_1$ alone is applied, say $\underset{\sim}{u}_1(\underset{\sim}{x})$
" " " " " " $\underset{\sim}{t}_2$ alone " " say $\underset{\sim}{u}_2(\underset{\sim}{x})$

The full solution $\underset{\wedge}{\overset{\underset{\sim}{u}(\underset{\sim}{x})}{}}$ when both $\overset{\underset{\sim}{t}_1 \,\&\, \underset{\sim}{t}_2 \text{ are}}{\overbrace{\text{applied}}}$ are:

$$\underset{\sim}{u}_1(\underset{\sim}{x}) + \underset{\sim}{u}_2(\underset{\sim}{x})$$

Note that the homogenous boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega_u$ are identical satisfied.

Ex.2



$t_1 \rightarrow \sigma_a^{(1)}$
$t_2 \rightarrow \sigma_a^{(2)}$
$t_3 \rightarrow \sigma_a^{(3)}$
$t_4 \rightarrow \sigma_a^{(4)}$
$t_5 \rightarrow \sigma_a^{(5)}$

Then the stresses due to a combined loading of $t_1, \dots, t_5$ is

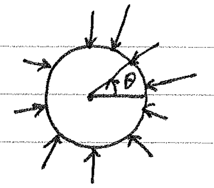
$$\sigma_a^{(1)} + \cdots + \sigma_a^{(5)}$$

Indeed, we need only work out the solution for $t_1 = 1$, $t_2 = 1$, ... etc. then the case of loading of arbitary magnitude $c_1, \dots, c_5$ are given by

$$\sigma_a = c_1 \sigma_a^{(1)} + \cdots + c_5 \sigma_a^{(5)}.$$

page. 3.

Example. Fourier series solution.



$p = p(\theta)$

look at

$p_0 = 1$

$p_{1C} = \cos\theta$

$p_{1S} = \sin\theta$

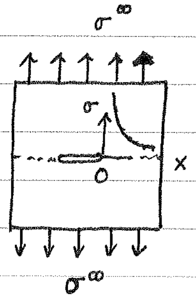
$p_{2C} = \cos 2\theta$

$p_{2S} = \sin 2\theta$

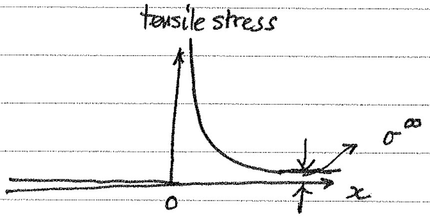
$\vdots$

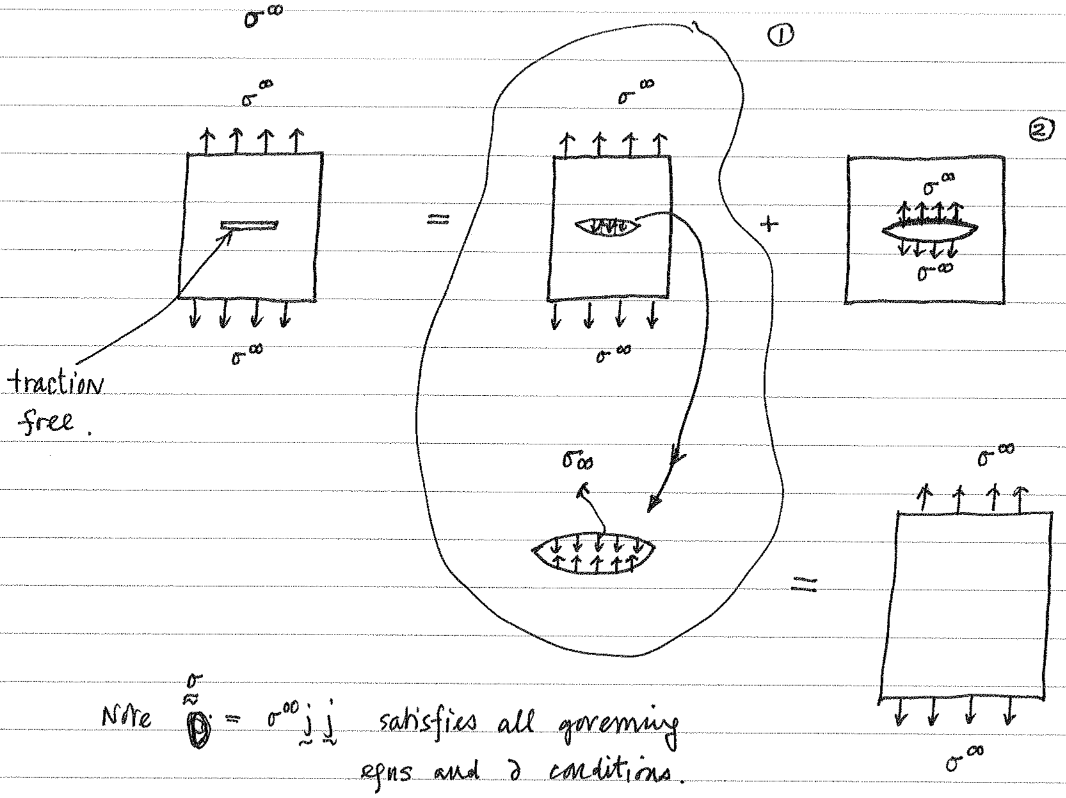
Any function $P(\theta) = \sum_{n=0}^{\infty} A_n \cos n\theta + \sum_{n=0}^{\infty} B_n \sin n\theta$

Example Fracture Mechanics.



FRACTURE.





$\underset{\sim}{t} = (\sigma^\infty \underset{\sim}{j}\,\underset{\sim}{j}) \cdot \underset{\sim}{n}$

Note $\underset{\approx}{\sigma} = \sigma^\infty \underset{\sim}{j}\,\underset{\sim}{j}$ satisfies all governing eqns and ∂ conditions.

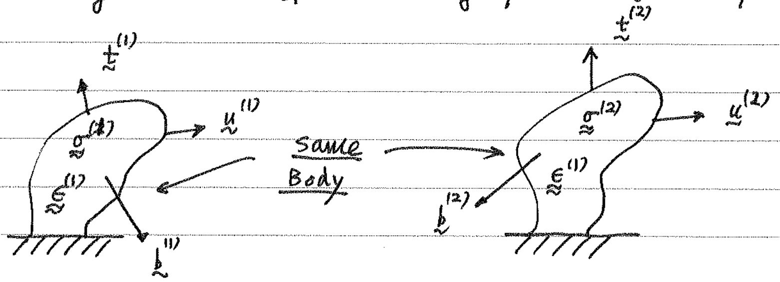
page 4.

$$\overset{\text{fracture}}{\sigma_{ij}}(\underset{\sim}{x})\underset{\sim}{e}_i\underset{\sim}{e}_j = \sigma_{ij}^{(2)}(\underset{\sim}{x})^{\underset{\sim}{e}_i\underset{\sim}{e}_j} + \sigma^{\infty}\, \underset{\sim}{j}\underset{\sim}{j} \qquad \underset{\sim}{j} = \underset{\sim}{e}_2$$

~~Recipro~~ Reciprocal theorem.

A given body with two different loading systems. (traction, displacements, etc.)

①



Both satisfies the same elasto-statics governing equation.

Write PVW.

$$\int_V \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)}\, dV = \int_S t_i^{(1)} u_i^{(2)}\, dS + \int_V b_i^{(1)} u_i^{(2)}\, dV \qquad (1)$$

Also we have

$$\int_V \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)}\, dV = \int_S t_i^{(2)} u_i^{(1)}\, dS + \int_V b_i^{(2)} u_i^{(1)}\, dV \qquad (2)$$

subtract. (1) from (2)

$$\int_V \left[\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} - \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}\right] dV = \int_S \left[t_i^{(2)}u_i^{(1)} - t_i^{(1)}u_i^{(2)}\right] dS + \int_V \left[b_i^{(2)}u_i^{(1)} - b_i^{(1)}u_i^{(2)}\right] dV \qquad (3)$$

Now $\varepsilon_{ij}^{(1)} = L_{ijkl}\,\sigma_{kl}^{(1)}$

$\sigma_{ij}^{(2)} L_{ijkl}\,\sigma_{kl}^{(1)} = L_{ijkl}\,\sigma_{kl}^{(1)}\sigma_{ij}^{(2)}$

$\sigma_{ij}^{(1)} L_{ijkl}\,\sigma_{kl}^{(2)} = L_{ijkl}\,\sigma_{ij}^{(1)}\sigma_{kl}^{(2)}$ since $L_{ijkl} = L_{ijkl}$. $\leftrightarrow$ [By the existence of strain energy function]

We have $\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)} = \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}$ so that the 1st on the LHS of (3) vanishes.

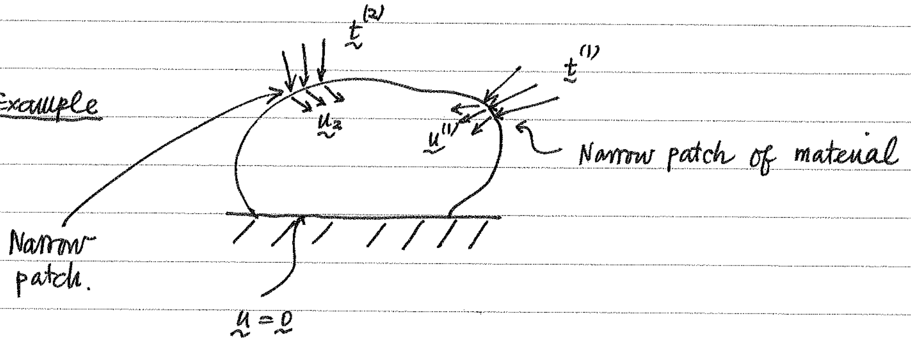
$$\therefore \int_S [t_i^{(2)} u_i^{(1)} - t_i^{(1)} u_i^{(2)}]\, dS + \int_V [b_i^{(2)} u_i^{(1)} - b_i^{(1)} u_i^{(2)}]\, dS. = 0$$

$$\therefore \int_S t_i^{(1)} u_i^{(2)}\, dS + \int_V b_i^{(1)} u_i^{(2)}\, dS = \int_S t_i^{(2)} u_i^{(1)}\, dS + \int_V b_i^{(2)} u_i^{(1)}\, dV$$

(Betti's Reciprocal theorem)

Work of force system ① when moves through displacement ② = Work of force system ② when moves through displacement ①.

Example

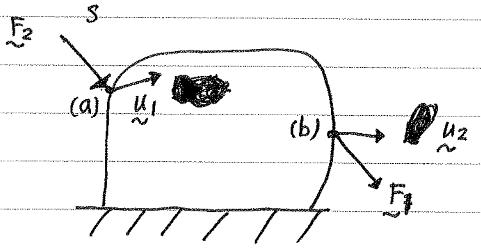


Suppose $\underset{\sim}{u}^{(1)}$ ~ constant on its patch.
$\underset{\sim}{u}^{(2)}$ ~ " " its "

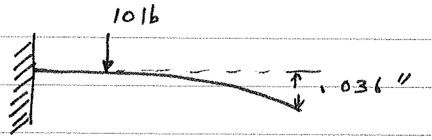
$$\int_S t_i^{(1)} u_i^{(2)}\, dS \approx \underset{\sim}{u}_i^{(2)} \underbrace{\int t_i^{(1)}\, dS}_{F_i^{(1)}} = u_i^{(2)} F_i^{1}$$

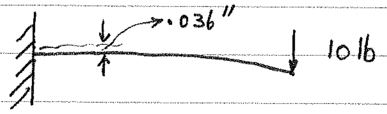
$$\int_S t_i^{(2)} u_i^{(1)}\, dS \sim u_i^{(1)} F_i^{(2)}$$

Another problem :



$$\underset{\sim}{F}_{1a} \cdot \underset{\sim}{u}_{2a} = \underset{\sim}{F}_{2b} \cdot \underset{\sim}{u}_{1b}$$



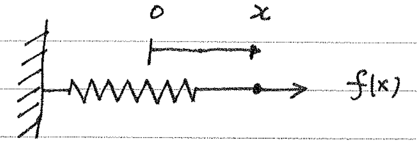


~~Oct.~~ Nov. 2. page 1. TAM 663.

Today : Stationary potential Energy.

Look at springs & potential energy :

Non-linear spring W = potential energy $= W(x) = \int_0^x f(x')dx'$

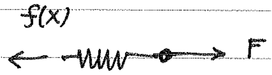


Potential energy when a force is applied.

$$W^{total} = W^{int} + W^{applied}$$
$$= W(x) + -Fx$$

Given F, find x.

Method 1. Use Equilibrium



$$F = f(x)$$

Method ; Use Method of stationary potential energy. "Minimum potential energy" [Foget Equilibrium].

$$\delta W^{total} = \delta W - F\,\delta x = \delta W + \delta W^{applied}$$
$$= \left(\frac{\partial W}{\partial x} - F\right)\delta x$$

$$\therefore \quad F = \frac{\partial W}{\partial x} = f(x)$$

3D Elasticity $\quad W_{int} = \int_V w\,dV = \int_V w(\underset{\sim}{\varepsilon})\,dV \qquad w(\underset{\sim}{\varepsilon}) = \int_0^{\underset{\sim}{\varepsilon}} \underset{\sim}{\sigma}(\underset{\sim}{\varepsilon}')\,d\underset{\sim}{\varepsilon}'$

in elasticity. $\rightarrow = \frac{1}{2} C_{ijkl}\,\varepsilon_{ij}\varepsilon_{kl}$

specified.

$$W_{ext} = -\int t_i u_i \, dS$$

surface traction
(traction ∂ condition)

Principle of Stationary potential energy.

$$\delta W^{total} = \delta W^{int} + \delta W^{applied.} = 0$$

$\forall$ arbitary variation in the displacement.

Forget $\sigma_{ij,j} + b_i = 0$ Also $\overset{\text{forget}}{\wedge}$ $\sigma_{ij} n_j = t_i$.

calculus of variation

$$\underset{\text{scalar}}{W^{total}} = \underset{\text{functional.}}{\mathcal{F}}(\underbrace{\underset{\sim}{u}(\underset{\sim}{x})}_{\text{function of } \underset{\sim}{x}})$$

δW^{total} with variation in $\underset{\sim}{u}$, i.e. $\delta\underset{\sim}{u}$ is equal to zero.

$$\Rightarrow W[\overset{*}{\underset{\sim}{u}}(x)] - W[\overset{*}{\underset{\sim}{u}}(\underset{\sim}{x}) - \epsilon\delta\underset{\sim}{u}] = O(\varepsilon^2) \quad \text{if} \quad \delta W^{total} = 0 \quad \text{at} \quad \underset{\sim}{u} = \underset{\sim}{u}^*(\underset{\sim}{x})$$

$$\forall \underbrace{\delta u(x)}_{\text{variations.}}$$

$\delta\underset{\sim}{u}(x)$ is different function than $\underset{\sim}{u}$ but is $\underset{\sim}{0}$ whenever $\underset{\sim}{u}$ is identified. or fixed.

Proceed. $0 = \delta W^{total} = \delta W^{int} + \delta W^{applied}$

$$= \delta\int_V W(\underset{\approx}{\epsilon})\,dV - \delta\int_{S_T} t_i u_i\,dS - \delta\int_V b_i u_i\,dV$$

$$\delta\int_V W(\underset{\approx}{\epsilon})dV = \int_V \delta W(\underset{\approx}{\epsilon})\,dV \qquad \overset{\|}{-\int_{S_T}} t_i\delta u_i\,dS - \int_V b_i\delta u_i\,dV$$

$$\delta W(\underset{\approx}{\epsilon}) = \delta\int_0^{\underset{\approx}{\epsilon}} \underset{\approx}{\sigma}(\underset{\approx}{\epsilon}')\,d\underset{\approx}{\epsilon}' = \underset{\approx}{\sigma}(\underset{\approx}{\epsilon}):\delta\underset{\approx}{\epsilon}$$

$$= \sigma_{ij}\,\delta u_{i,j} = \sigma_{ij}(\delta u_i)_{,j} = (\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij,j}\,\delta u_i$$

$$= (\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij,j}\,\delta u_i$$

$$\therefore \int_V \delta W(\underset{\approx}{\epsilon})dV = \int_{S_T} (\sigma_{ij}\delta u_i n_j)\,dS + \int_V b_i\delta u_i\,dV$$

$\uparrow$ since $\delta u_i = 0$ at S_u where displacement is perscribed.

$$\therefore \quad 0 = \delta W = \int_{S_T} (\sigma_{ij} n_j - t_i)\, \delta u_i \, ds \;-\; \int_V (\sigma_{ij,j} + b_i)\, \delta u_i \, dV$$

$\forall$ $\underline{\underline{\delta u_i}}$ so that

$$\sigma_{ij} n_j = t_i \qquad \text{on } S_T$$
$$\sigma_{ij,j} = -b_i \qquad \text{on } V$$

$\therefore$ Principle of stationary potential energy $\Rightarrow$ Equilibrium & $\sigma_{ij} n_j = t_i$.

It looks like the Principle of virtual work:

$$IVW = \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV$$

→ some stress field which do not have to do with ε_{ij} and $\delta\varepsilon_{ij}$

Now we have $\delta W^{int} = \underbrace{\int_V \delta W \, dV}_{\text{Existence of strain energy density.}}$

Note: (Not prove here) The equilibrium solution is not just a point of stationary energy but a point of minimum total potential Energy. so long as C_{ijkl} is $\underline{\underline{\text{positive definite}}}$.

Why Bother? PVW or Principle of Stationary potential Energy is the basis of Numerical methods.

How does it work:

1. Throw away Equilibrium equations.
2. Parameterize three set of all possible displacements with the finite set of parameters u_i
3. Calculate W using u_i
4. $\underbrace{\text{Set } \frac{\partial W}{\partial u_i} = 0}_{\text{Approximation Solution of equilibrium eqns.}} \quad \forall i$.

One approach is to pick $f^i_j(\underset{\sim}{x})$ ← which function, ← which component. $= \underset{\sim}{f}^i(\underset{\sim}{x})$ which satisfies the displacement conditions. $[\ \underset{\sim}{u} = 0 \text{ on } S_{\underset{\sim}{u}}\ ]$

page 4.

$$\underset{\sim}{u}(\underset{\sim}{x}) = \sum_{i=1}^{N} u^i \underset{\sim}{f}^i(\underset{\sim}{x}) \Rightarrow \underset{\sim}{\epsilon}(\underset{\sim}{x}) \Rightarrow W(u^i) \Rightarrow W^{total}(u^i)$$

Set $\frac{\partial W^{total}}{\partial u^i} = 0$. In linear elasticity,

$\frac{\partial W^{total}}{\partial u^i} = 0$ are a set of linear algebraic equation in u^i

Today: Finite element Method. (cont.d) Nov. 4, 92. page 1.

→ "strength of Material"

Recall $W^{total} = \int_V W(\underset{\sim}{\epsilon}(\underset{\sim}{u}))\, dV - \int_{S_T} t_i u_i\, dS - \int_V b_i u_i\, dV$

surface where traction is prescribed.

Can use Principle of virtual work. or Principle of stationary potential energy.

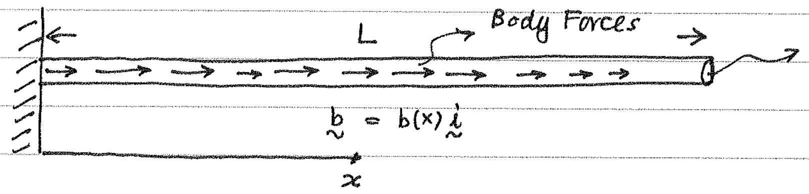
Assume $\underset{\sim}{u} = \sum_{i=1}^{N} u^i \underset{\sim}{f}_i(\underset{\sim}{x})$

Some functions which satisfies displacement ∂ conditions.

$$\Rightarrow W^{total}(\underset{\sim}{u}) = W^{total}(u_i)$$

$$\delta W^{total} = 0 \Rightarrow \frac{\partial W^{total}}{\partial u_i} = 0$$

Example:



Area of cross-section = A

E = modulus.

Find $\underset{\sim}{u}(\underset{\sim}{x})$ $W^{total} = \int_V W(\underset{\sim}{\epsilon})\, dV - \int_S b\, u\, dV$

$$W(\underset{\sim}{\epsilon}) = \frac{1}{2}\sigma\epsilon = \frac{1}{2}E\epsilon^2$$

$$\int W(\underset{\sim}{\epsilon})\, dV = \int \frac{E\epsilon^2}{2} A\, dx \quad \therefore \quad W^{total} = \int_0^L \frac{EA\epsilon^2 dx}{2} - \int_0^L b(x)\, u(x)\, A\, dx$$

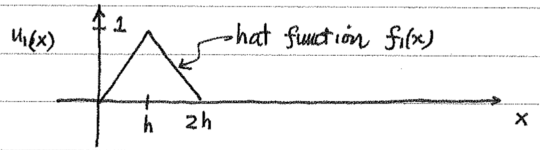
page 2. Nov. 4, 1992.

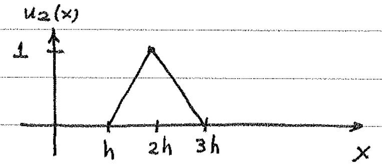
$$= \int_0^L \frac{EA\, u_{,x}^2}{2}\, dx - \int_0^L A\, b(x) u(x)\, dx$$

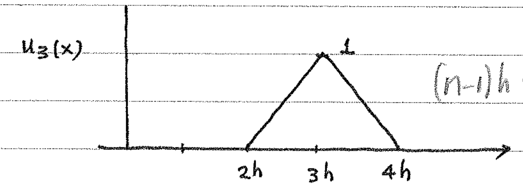
Taking 1st variation $\Rightarrow$ $E u_{,xx} + b = 0$

Lets say u(x) described by u_i : need to pick function $f_i(x)$

A simple function to use are hat functions.





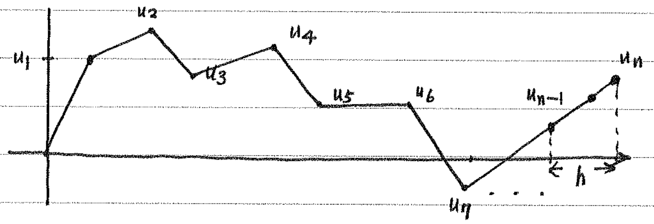


$(n-1)h < x < nh$

$$u(x) = u_{(n-1)} + (x-(n-1)h)(u_n - u_{n-1})/h$$

$$u_{,x} = u_n - u_{n-1}/h$$

approximate u(x) with



$$u_{,x} = \frac{u_n - u_{n-1}}{h}$$

= constant in each region

Divide $\int_0^L$ into $\int_0^h + \int_h^{2h} + \dots + \int_{(n-1)h}^{nh}$

$$W^{total}(u_i) = \sum_{i=1}^{N} \frac{EA}{2} \frac{(u_i - u_{n-1})^2}{h^2} h - A \sum_{n=1}^{N} \int_0^h \overset{b((n-1)h+x')}{\cancel{b(x+h)}} \left[u_{n-1} + \frac{(u_n - u_{n-1})}{h} x' \right] dx'$$

$$= \sum_{i=1}^{N} \frac{EA}{2} \left(\frac{u_i - u_{i-1}}{h} \right)^2 h - A \sum_{i=1}^{N} \underbrace{\int_0^h b((i-1)h + y) \left[u_{i-1} + \frac{(u_i - u_{i-1}) y}{h} \right] dy}_{b_i u_i}.$$

b_i comes from $\int_{(i-1)h}^{ih} b(x)\, dx$ and $\int_{(i-1)h}^{ih} x\, b(x)\, dx$

page 3. Nov. 4, 1992.

$$W^{total} = \frac{EA}{2h}\Big[\cdots + \underbrace{(u_7-u_6)^2 + (u_8-u_7)^2}_{u_7 \text{ shows up in two places}} + \cdots\Big]$$

$$+ \; A\sum_i u_i b_i$$

The alegbraic equation resulting from the 7th unknown. [there are in general n linear equations].

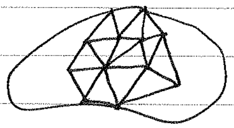
$$\frac{\partial W^{total}}{\partial u_7} = \left[2(u_7-u_6) + 2(u_7-u_8)\right] - A b_7.$$

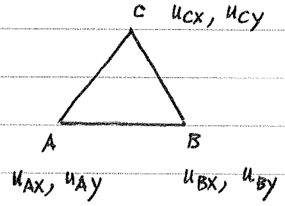
$$= \frac{EA}{2h}(4u_7 - 2u_6 - 2u_8) - Ab_7. = \frac{E\cancel{A}}{h}\left[2u_7 - u_6 - u_8\right] - \cancel{A}b_7 = 0.$$

2D Elasticity.

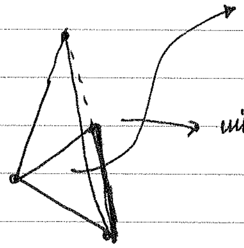
One finite element approach is called the constant strain triangles

Divide the region into Δ's





Linearly interpolate the function in the triangle



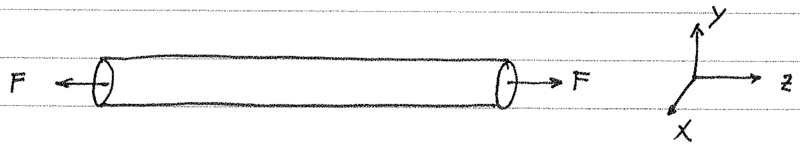
unit spline function.

u_x and u_y are linear function of position $\Rightarrow$ ε_{xx}, ε_{yy}, ε_{xy} are constant in each triangle.

Special case in 3D Elasticity : Strength of Material.

" Tension, Torsion of Round Bars, Pure Bending."

Simple tension



$$\sigma = F/A \qquad \varepsilon_l = \sigma/E \qquad \varepsilon_t = -\nu\frac{\sigma}{E}$$

page 5. November. 4

Another approach: Make a kinematic assumption.

Assume: $\underbrace{\text{plane sections}}_{}$ remain plane & remain ⊥ to z direction

⊥ to z direction distort.

and they can ~~reform~~ in their plane.

$$u_z(x,y,z) = u_z(z)$$
$$u_x(x,y,z) = u_x(x,y,z)$$
$$u_y(x,y,z) = u_y(x,y,z)$$

$$\therefore \quad u_z = C_z + \epsilon z = C_z + \frac{F}{AE} z$$

$$u_x = C_x - \frac{\nu F}{AE} x + \text{Rotation}$$

$$u_y = C_y - \frac{\nu F}{AE} x. + \text{Rotation}.$$

For isotropic elasticity.

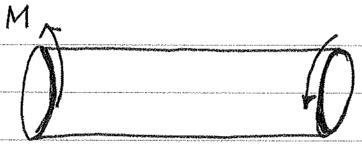
$$\Rightarrow \quad \epsilon_{zz} = \frac{F}{AE} \qquad \epsilon_{xx} = \epsilon_{yy} = -\frac{\nu F}{AE}$$

$$\epsilon_{xy} = \epsilon_{yz} = \epsilon_{xz} = 0$$

$\Rightarrow \quad \sigma_{zz} = \dfrac{F}{AE}$ all others are zero so that $\sigma_{ij,j} = 0$ is satisfied. ⇒ Bar theory is an exact three Dimensional theorem. No tractions on sides and the wrong traction at the ends.

page 6.

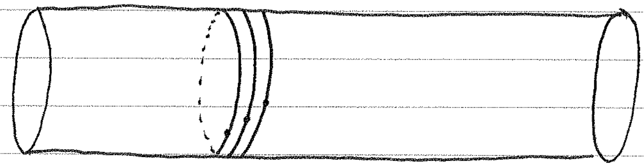
Torsion of Round Rods:



Kinematic assumption

plane section ⊥ to z axis remains ⊥ and do not distort in plane.

Picture: A bar is a stack of disks. Each disk is rigid, shear between disks



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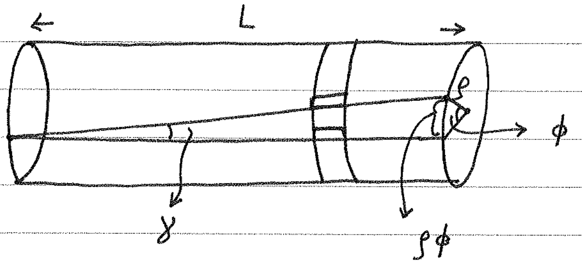
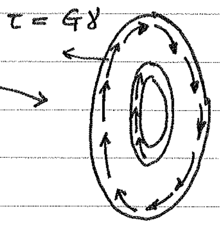


Shear strain $\gamma = \frac{\rho\phi}{L}$

Look at stress



Net torque = $T = \int_A \tau\rho \, dA$

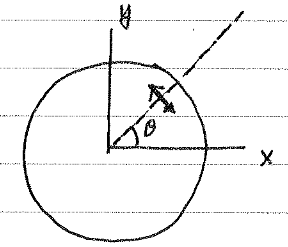
$$= \frac{G\phi}{L}\int_A \rho^2 \, dA \quad = \text{polar area of inertia} \cdot \frac{G\phi}{L}$$

$dA = 2\pi\rho \, d\rho$

Look at traction x-y component.

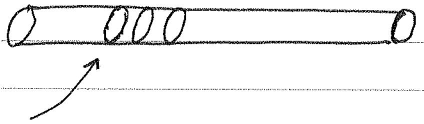
$$\tau_x = \tau \sin\theta = \tau \cdot \frac{y}{\rho} = \frac{G\phi y}{L} = \tau_{xz}$$

$$\tau_y = +\tau\cos\theta = +\tau\frac{x}{\rho} = \frac{G\phi x}{L} = \tau_{yz}$$



page 1. November. 9.

Rod : Long narrow structures :



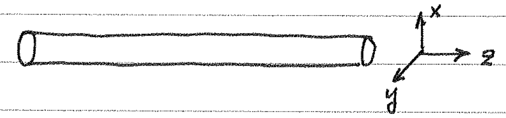
Cut into imagainary planar slices

There are three simple deformations which keep each plane planar and normal to the axis of the rod.

1.) stretching (Rod in tension)
2.) twisting (torsion) circular Bar ?
3.) Curving the axis.

Page 2 . Nov. 9.

1.) Tension in a Bar : Crude, approximate but luckily accurate theory.



$u_z = u_z(z)$ $\qquad\qquad$ $\epsilon_{zz} = u_{z,z}$.

$u_x = 0 \qquad u_y = 0$ $\qquad\qquad$ Note $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{zy} = 0$

" Bad constitutive law : " $\qquad \sigma_z = E\varepsilon_z \qquad (1)$

Note :
$$\varepsilon_{ij} = \frac{\sigma_{ij}(1+\nu)}{E} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\Rightarrow \qquad \sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{E\ \nu\ \varepsilon_{kk}}{(1+\nu)(1-2\nu)}\delta_{ij}$$

$$\Rightarrow \quad \therefore \quad \sigma_{zz} = \left[\frac{E}{(1+\nu)} + \frac{\nu E}{(1+\nu)(1-2\nu)}\right]\varepsilon_{zz}$$

$$= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{zz}$$

so that (1) can't be valid unless $\nu = 0$!

Better (but not in accuracy) use same kinematic assumption

$$F = \sigma A$$
$$= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_z \ A$$

Also predict $\sigma_{xx}, \sigma_{yy} \neq 0$ which violates traction free conditions on the sides.

e.g $\qquad \sigma_{xx} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{zz}$

Best theory : Exact three D solution ,

$\qquad \sigma_z \neq 0 \qquad$ all other stresses $= 0$

$$\Rightarrow \qquad \varepsilon_{zz} = \frac{\sigma_{zz}}{E} \qquad\qquad \epsilon_{xx} = \epsilon_{yy} = \frac{-\nu\sigma_{zz}}{E}$$

$$u_z = \frac{\sigma_{zz}}{E}z \qquad u_x = -\frac{\nu\sigma_{zz}}{E}x \qquad u_y = -\frac{\nu\sigma_{zz}}{E}y$$

+ Rigid Body terms .

page 3.

The previous solution satisfies traction ∂ solns.

An alternative approach, use Kinematic assumption

$$\delta W^{total} = 0 \qquad \Rightarrow \qquad u_{z,z} A \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \text{Body force / unit length.}$$

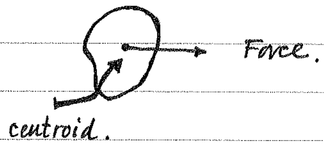
What's the use ?

(Do NOT know details of the traction)

St. Venant's Principle:

σ is uniform across the cross-section far from the ends.

i.e. the resultant force F acts through the centroid of the cross-section area



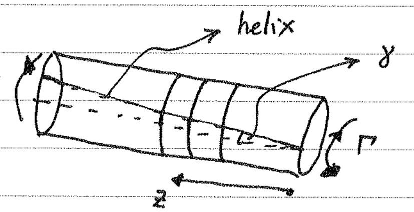
so that resultant moment = 0. [Sloppy description].

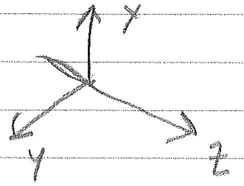
The Bar theory is not accurate at the ends of the bar where the boundary conditions is not clearly specified.

i.e. $\frac{d}{l} \ll 1.$ where d = diameter of Bar.

l = length of Bar.

Torsion. Twisting of the planes.

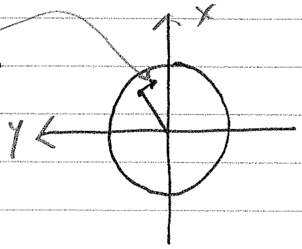




Planes twist: $u_z = 0$

$u_x = yz \cdot \phi/L$

$u_y = -xz\frac{\phi}{L}$



$\tau_{xz} = Gy\frac{\phi}{L}$

$\tau_{yz} = -Gx\frac{\phi}{L}$

$\Rightarrow \varepsilon_{yz} = y\phi/L$

$\varepsilon_{xz} = -x\phi/L$ ~~$\tau_{xz} = Gyz\,\phi/L$~~

~~$\tau_{yz} = -Gxz\,\phi$~~

$\sigma_{zz} = \sigma_{xx} = \sigma_{yy} = \sigma_{xy}$

$$\Rightarrow \Gamma = \frac{\phi}{L} G \underbrace{\int \rho^2 dA}_{J}$$

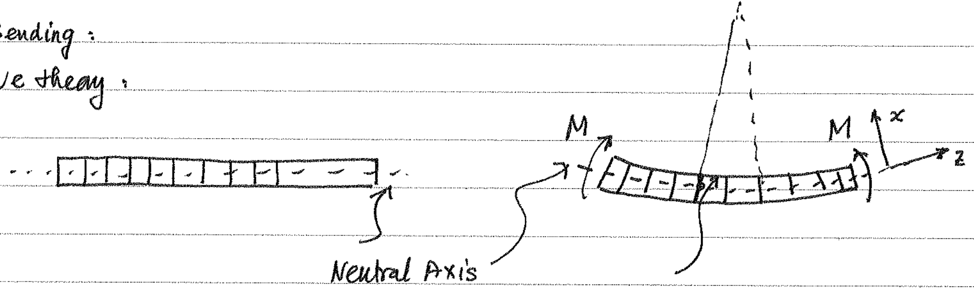
$\Rightarrow$ "$T = \frac{\phi J G}{L}$"

$$\underline{T} = \int_A \underline{r} \times \underline{t}\, dA = \int_A (x\underline{i} + y\underline{j}) \times \left[(Gy\phi/L)\underline{i} - (Gx\phi/L)\underline{j} \right] dA = \left[\iint_A (x^2+y^2)\, dA \right] \frac{-\phi G}{L} \underline{k}$$

page 4. November 9.

Bending :

Naive theory :



$\epsilon_z = -xX$ $K =$ Radius of curvature

Bad constitutive law $\sigma_{zz} = E\varepsilon_{zz}$

$\sigma_{33} = 0$ all other stresses $= 0$

$\Rightarrow \quad M = EIX \qquad I = \iint x^2 dA$

cross-section

More Rational But worst theory :

Use same kinematic assumption But correct constitutive law :

1st problems : Does not give traction free sides

: Does not agree with experiments.

Rational sophisticate theory : 3D exact theory :

$\sigma_{zz} = Cx \neq 0$ where the constant $C = -KE$.

$\sigma_{xy} = \sigma_{xx} = \sigma_{yy} = \sigma_{xz} = \sigma_{yz} = 0$

$\Rightarrow \quad \varepsilon_{zz} = E\sigma_{zz}$ and $\varepsilon_{zz} = -Kx$

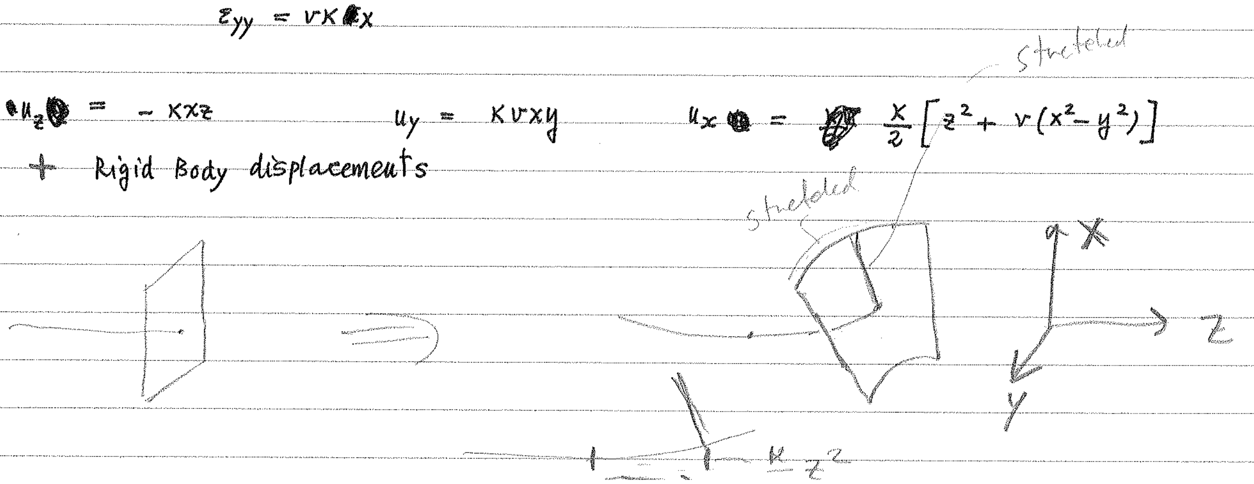
$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{xy} = 0$

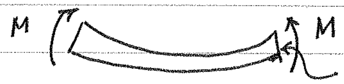
$\varepsilon_{xx} = \nu Kx$

$\varepsilon_{yy} = \nu Kx$

$u_z = -Kxz \qquad u_y = K\nu xy \qquad u_x = \frac{K}{2}\left[z^2 + \nu(x^2 - y^2)\right]$

+ Rigid Body displacements





Don't know the detail but resultant Force = 0

St. Venant's principle ⇒ exact solution to be accurate far from the ends.

For Round shafts.

We can use our three theory with good accuracy for Bars with tension, bending and torsion so long as T, F, M, A changes slowly along length.

i.e., the theory is good if you can cut out length ΔL where it is reasonably accurate to assume constant T, A, F and M.

Lecture, November 11, 1992.

Anti-plane $u = v = 0$ $W = w(x,y)$

$\Rightarrow \epsilon_{zz} = \frac{\partial w}{\partial z} = 0$ all component $= 0$ except $\gamma_{xz} = \frac{\partial w}{\partial x} = 2\varepsilon_{xz}$

$\gamma_{yz} = \frac{\partial w}{\partial y} = 2\varepsilon_{yz}$

Isotropy elasticity ⇒

$$\tau_{xz} = G\frac{\partial w}{\partial x} = 2G\varepsilon_{xz}$$

$$\tau_{yz} = 2G\varepsilon_{yz} = G w_{,y}$$

all other components $= 0$

Equilibrium Equations

$$\sigma_{ij,j} + b_i = \rho a_i$$

No Body forces and statics ⇒ $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$

Putting in constitutive law ⇒ $G\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] = 0 \Rightarrow \nabla^2 w = 0$

at every point w fct curve up in the x direction as much as curve down in the y direction, w looks saddle like.

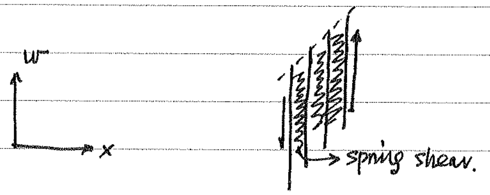
w is the real or imaginary part of some analytic function of $z = x + iy$.

Nov. 11, 1992, page. 2.

A Model for the kinematic assumption $w = w(x, y)$ $u = v = 0$

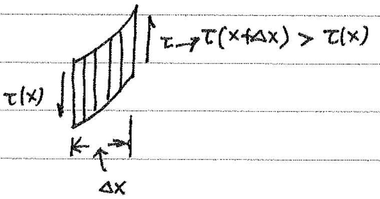
Anti-plane shear deformation is the deformation of a stack of $_\wedge$ inextensible needle pointed in the z direction and can only move along their length.

side view:

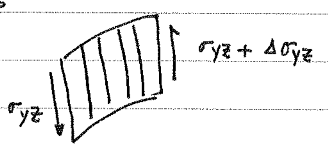


net upward force from $\pm n_x$ faces if $w_{,xx} > 0$

Net downward force from $\pm n_y$ faces if $(w_{,y})_{,y} < 0$

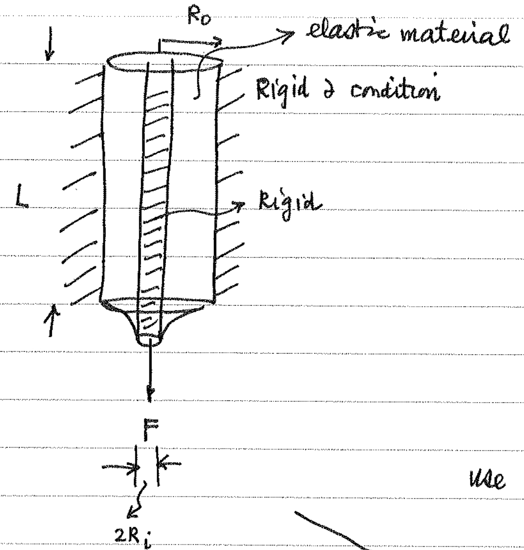


Look at side view at y axis



Simple Example

Rigid Rod pull-out of the center of a rigid pipe that is filled with elastic material.



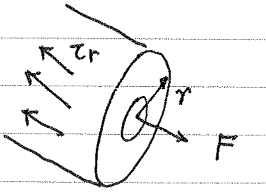
Assume an ~~a~~ axissymmetric solution

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0$$

w is independent of θ.

$$\Rightarrow \overset{w}{\cancel{\theta}} = A \ln r + C$$

Use Equilibrium & ∂ condition to find constants.



$$\tau_r \cdot 2\pi r L = F$$

$$\Rightarrow \tau_r = \frac{-F}{2\pi r L}$$

$$\tau_r = G\frac{\partial w}{\partial r} = \frac{-F}{2\pi r L}$$

$$w_{,r} = -\frac{GF}{2\pi r L}$$

$$\Rightarrow w = \frac{-GF}{2\pi L}\ln r + C$$

page 3

$$\frac{\partial w}{\partial r} = \frac{-F}{2\pi L r G} \quad \Rightarrow \quad w = \frac{-F}{2\pi L G} \ln r + C$$

$$\text{or} \qquad w = \frac{-F}{2\pi L G} \ln\left(\frac{r}{r_0}\right)$$

This model is not good at the ends.

B.C. TRACTION FREE. But anti-plane shear soln has $\tau_{rz} \neq 0$.

<u>Model II.</u> Plane strain.

$$\underset{\sim}{u} = u(x,y)\,\underset{\sim}{i} + v(x,y)\,\underset{\sim}{j} + \underbrace{\epsilon_{zz} z}_{\uparrow}$$

Generalised plane strain.

11/16/92 page 1

Plane stress deformation $\sigma_{ij} = \sigma_{ij}(x,y)$

$$\varepsilon_{ij} = \varepsilon_{ij}(x,y)$$

$$\sigma_{zx} = \sigma_{zz} = \sigma_{zy} = 0$$

$$\Rightarrow \nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \qquad \sigma_{xx,x} + \sigma_{xy,y} = 0$$

$$\sigma_{xy,x} + \sigma_{yy,y} = 0$$

2D compatibility

Note that plane stress is NOT an exact theory, it does not satisfy all compatibility equation.

Note $\varepsilon_{ij} = \dfrac{\sigma_{ij}(1+\nu)}{E} - \dfrac{\nu}{E}\sigma_{kk}\delta_{ij}$

$$\Rightarrow \varepsilon_{zz} = \frac{-\nu(\sigma_{xx}+\sigma_{yy})}{E}$$

$$u_z = \int u_{z,z}\,dz = \int \varepsilon_{zz}\,dz = \varepsilon_{zz} z + \underbrace{C(x,y)}$$

If we consider $\varepsilon_{xz} = \frac{1}{2}(u_{z,x} + u_{x,z}) = \frac{1}{2}u_{z,x}$.

$$\varepsilon_{xz} = \frac{u_{z,x}}{2} = \frac{z}{2}\frac{\partial\varepsilon_{zz}}{\partial x} + \underbrace{\frac{1}{2}\frac{\partial C(x,y)}{\partial x}}$$

Forget it because there is no Bending e.g. $u_z = 0$ on the center line. $z=0$

$$\therefore \frac{1}{2}u_{z,x} = \frac{z}{2}\left[\frac{-\nu}{E}\right]\left[\sigma_{xx,x} + \sigma_{yy,x}\right] = \varepsilon_{xz}$$

$$\Rightarrow \therefore \text{ since } \sigma_{xz} = 2G\varepsilon_{xz}$$

$$= \overbrace{2G \underbrace{\frac{z}{2}}_{\sim\text{thickness}} \left(\frac{-\nu}{E}\right)}^{\sim 1} \left[\sigma_{xx,x} + \sigma_{yy,x}\right] \neq 0 \text{ in general. } (*)$$

Question: How big is the error.

Let t = thickness of the plate in (*) $O(t) \sim z$ $\dfrac{G\nu}{E} = O(1)$

$$\sigma_{xx,x} + \sigma_{yy,x} \sim \frac{\text{change in stress}}{\text{characteristic distance of change}}$$

$$\therefore \sigma_{xz} \sim \text{characteristic change in in plane stress} \cdot \frac{\text{plate thickness}}{\text{characteristic in plane dimension}}.$$

page 2. Nov, 16, 92

so that error is small as long as $t \ll$ characteristic distance of stress change $= a$ where a = size of plate, any characteristic distance in loading.

There are two way to get around it:

1) Errors are small if $a \gg t$ so don't worry about it.

2.) Generalized plane stress.

(2) Consider the full-3D Eqns, consider a flat plate with faces in $\pm z$ directions and average 3D elastic equations through plate thickness.

Assume mid-surface no z displacement and that all stresses are symmetric wrt the midsurface.
Also, $\sigma_{zz} = 0$ on $z = \pm h/2$.
In these cases the plate can be "thick". Let the thickness of the plate by h.

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} = 0 \qquad (1)$$

$$\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{yz,z} = 0 \qquad (2)$$

$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0 \qquad (3) \qquad \text{This eqn ??}$$

Integrate these eqns from $-h$ to h in z, $\Rightarrow \int_{-h/2}^{h/2} \sigma_{ij,j}\,dz = 0 \quad \forall i$

Define $\bar{\sigma}_{ij} = \int_{-h/2}^{h/2} \sigma_{ij}\,dz \qquad \bar{\epsilon}_{ij} = \int_{-h/2}^{h/2} \varepsilon_{ij}\,dz$

$$\bar{u}_i = \int_{-h/2}^{h/2} u_i\,dz \qquad ??$$

$$\overline{\sigma_{xz,z}} = \int_{-h/2}^{h/2} \sigma_{xz,z}\,dz = \sigma_{xz}\Big|_{h/2}^{h/2} = 0$$

TRACTION free.

Likewise $\overline{\sigma_{yz,z}}$, $\overline{\sigma_{zz,z}} = 0$ so that the average of the (3) terms over z is identically satisfied.

(1) & (2) $\bar{\sigma}_{xx,x} + \bar{\sigma}_{xy,y} = 0$

$\bar{\sigma}_{xy,x} + \bar{\sigma}_{yy,y} = 0$

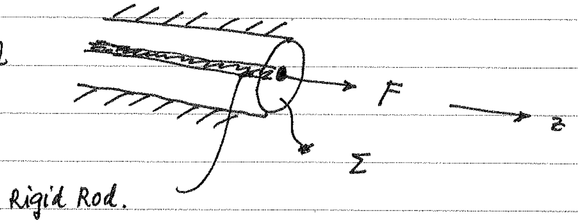
strain displacement relation $\Rightarrow$ $\bar{\varepsilon}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i})$ $i = 1, 2.$

$\bar{u}_z = 0$?

$$\bar{\sigma}_{ij} = 2G\,\bar{\varepsilon}_{ij} + \lambda\,\bar{\varepsilon}_{kk}\,\delta_{ij}$$

True for thick plates too. ??

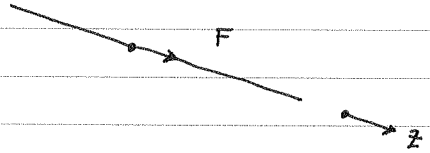
Anti-plane shear solution : Recall



$w = A \ln r + B.$ [inexact, as solution is NOT TRACTION FREE at Σ]

Another way of looking at this soln is for this 3D problem.

Simplest Exact non-constant three D Elastic problem is : Line Load in a full space.



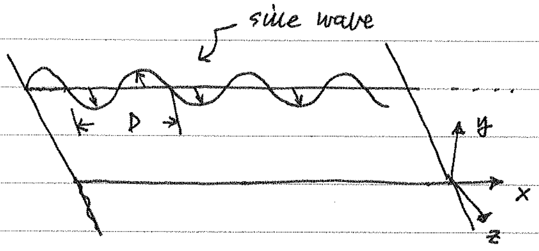
F = force per unit length on a line in z direction through the origin of a full space.

$$w = A \ln r + B$$

$$A = -\frac{F}{2\pi G}$$

$$w = \underbrace{\frac{-F}{2\pi G}}_{\text{Global equilibrium}} \ln r + \underbrace{B}_{\text{arbitary constant}}.$$

Sine Wave on a Half-space.



$\tau_{yz} = \tau_{yz}(x, y)$

$\tau_{yz}(x, y=0) = A \sin(\kappa x)$ $\qquad \kappa D = 2\pi$

$\underset{\sim}{u} = w \underset{\sim}{k}$ $\qquad \tau_{yz} = G \frac{\partial w}{\partial y} = A \sin \kappa x$ at $y = 0$

$\nabla^2 w = 0$

Let $W = A \sin \kappa x \, f(y) = \frac{A}{\kappa^{2}} \sin \kappa x \, e^{-\kappa|y|}$ $\quad \kappa > 0$ OR

$+ \kappa^{-1} A \sin \kappa x \, e^{-|\kappa||y|}$

Where we have get rid of the exponential growing solution.

$\therefore \quad w = \frac{A}{G\kappa} \sin \kappa x \, e^{\kappa y}$ $\qquad$ for $y < 0$. $\quad \kappa > 0$

or $\quad w = \frac{A \sin \kappa x}{G\kappa} e^{-|\kappa y|}$

$\therefore$ ~~w =~~ $\boxed{\tau = G\kappa w}$! everywhere on the surface.

$\tau = G w / \kappa^{-1}$ $\quad$ OR $\quad$ Big $\kappa \Rightarrow$ Big stiffness if we define stiffness as $\left(\frac{\tau}{w}\right)$

Note w and τ_{yz} is in phase.

TAM 663   Nov. 18, 1992.   page 1.

Comments on plane stress:

1) Assume symmetry about $z = 0$

$$u_x(x,y,z)\,\underset{\sim}{i} + u_y(x,y,z)\,\underset{\sim}{j} + u_z(x,y,z)\,\underset{\sim}{k}$$
$$= u_x(x,y,-z)\,\underset{\sim}{i} + u_y(x,y,-z)\,\underset{\sim}{j} - u_z(x,y,-z)\,\underset{\sim}{k}$$

$$\therefore \quad u_{x,z}(z) = -u_{x,z}(-z) \qquad\qquad u_{y,z} = -u_{z,x}(-z)$$

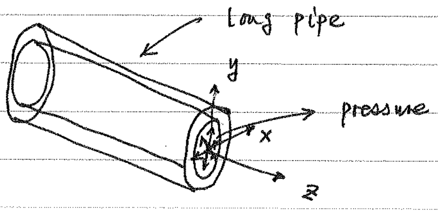
Now $\sigma_{xz} = 2G\,(u_{x,z} + u_{z,x}) \Rightarrow \displaystyle\int_{-h/2}^{h/2} \sigma_{xz}\,dz = 0$

Likewise $\displaystyle\int_{-h/2}^{h/2} \sigma_{yz}\,dz = 0$

2) Assume $\sigma_{zz} = 0$ Approximate assumption. (But we don't need it, do we?)

Another 2D problem:

Pressure vessel problem:

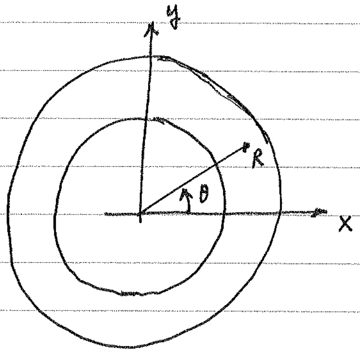


What is the stress in the pipe wall.

1) Stretch in the z direction $\sigma_{zz} = \cancel{P/\pi R_{ou}}\ (P \cdot \pi R_i^2)/(\pi R_o^2 - \pi R_i^2)$

2) deformation in xy plane. [plane strain problem in the xy plane] → Long Cylinder.

Look at 2D problem:



Use polar coordinates.

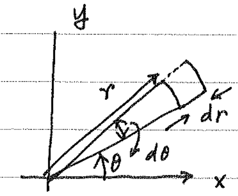
Aside on Governing equation in polar coordinates.

Stress Equilibrium:

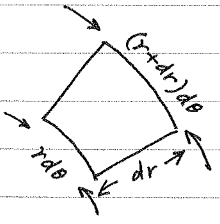
Instead of using

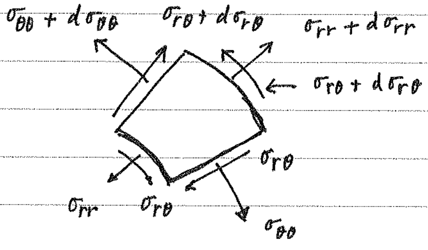
$$\underset{\approx}{\nabla} \cdot \underset{\approx}{\sigma} = \underset{\sim}{0} \text{ in}$$

polar coordinates.



We draw a FBD.





page 2. Nov, 18, 1992. pg. 2.

Force Balance in r direction

$$(\sigma_{rr}+d\sigma_{rr})(r+dr)d\theta \;-\; \sigma_{rr}(rd\theta) \;=\; -\sigma_{r\theta}\,dr + (\sigma_{r\theta}+d\sigma_{r\theta})\,dr. - (\sigma_{\theta\theta}+d\sigma_{\theta\theta})\,dr\,d\theta.$$

$$\wr$$

$$\sigma_{rr}\,dr\,d\theta + d\sigma_{rr}\,r\,d\theta = d\sigma_{r\theta}\,dr. - \sigma_{\theta\theta}\,dr\,d\theta.$$

Divide by $rdrd\theta$

$$\frac{\sigma_{rr}}{r} + \frac{\partial\sigma_{rr}}{\partial r} = \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} - \frac{\sigma_{\theta\theta}}{r.} \Rightarrow \boxed{\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \qquad (1)}$$

$$\sin(d\theta)$$

$$\Sigma F_\theta = 0$$

$$-\sigma_{\theta\theta}\,dr + (\sigma_{\theta\theta}+d\sigma_{\theta\theta})\,dr - \sigma_{r\theta}(rd\theta) + (\sigma_{r\theta}+d\sigma_{r\theta})(r+dr)d\theta + (\sigma_{r\theta}+d\sigma_{r\theta})\,d\theta\,dr$$

Again, divided by $rd\theta dr$.

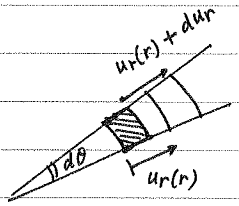
$$\boxed{\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} + \frac{\partial\sigma_{r\theta}}{\partial r} = 0 \qquad (2)}$$

Axial symmetry $\Rightarrow$ $\sigma_{r\theta} = 0$ so that (2) is satisfied identically

① becomes $\dfrac{\partial\sigma_r}{\partial r} + \dfrac{\sigma_r-\sigma_\theta}{r} = \dfrac{d\sigma_r}{dr} + \dfrac{\sigma_r-\sigma_\theta}{r} = 0$

Strain and displacements.

Assume Axial symmetry: $u_r(r)$ and $u_\theta(r) = 0$



$$\epsilon_{r\theta} = 0$$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{\cancel{rd}[r+u_r(r)]d\theta - rd\theta}{rd\theta}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r}$$

Constitutive law:

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij}$$

$\varepsilon_{zz} = 0$

$\varepsilon_{r\theta} = 0$

$$\sigma_{rr} = 2G[\varepsilon_{rr}] + \lambda(\varepsilon_{rr}+\varepsilon_{\theta\theta})$$

$$\sigma_{rr} = (2G+\lambda)\,\varepsilon_{rr} + \lambda\,\varepsilon_{\theta\theta}$$

$$\sigma_{\theta\theta} = (2G+\lambda)\,\varepsilon_{\theta\theta} + \lambda\,\varepsilon_{rr}.$$

⇒ If we knew $u_r(r)$ then we are done.

$$\sigma_{rr} = (2G+\lambda)\,u_{r,r} + \frac{\lambda u_r}{r}$$

$$\sigma_{\theta\theta} = (2G+\lambda)\,\frac{u_r}{r} + \lambda\, u_{r,r}$$

Now, put in Equilibrium equation give one O.D.E for $u(r)$

$$\sigma_r - \sigma_\theta = (2G+\lambda)\,u_{,r} + \frac{\lambda u}{r} - (2G+\lambda)\,\frac{u}{r} - \lambda u_{,r}$$

$$= 2G\,u_{,r} - 2G\,\frac{u}{r} = 2G\left[u_{,r} - \frac{u}{r}\right].$$

$$0 = \frac{\sigma_r - \sigma_\theta}{r} + \frac{d\sigma_r}{dr} = (2G+\lambda)\,u_{,rr} + \left(\lambda\frac{u}{r}\right)_{,r} + 2G\left[\frac{u_{,r}}{r} - \frac{u}{r^2}\right].$$

$$\Rightarrow \quad \boxed{\sigma_{rr} = \frac{A}{r^2} + C}$$

$u = r^s$

$\Rightarrow (2G+\lambda)\, s(s-1)\, r^{s-2}$

$+\lambda\,(s-1)\, r^{s-2}$

$+2G\left(s r^{s-2} - r^{s-2}\right)$

$\Rightarrow (2G+\lambda)\, s(s-1)$

$+\lambda\,(s-1)$

$+2G\,(s-1) \quad = 0$

$(s+1)(s-1) = 0$

$s = -1, 1$

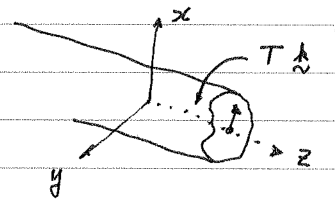
Nov, 23, 1992.

Torsion of Arbitrary cross-section   Page 1

(Linearly elastic - isotropy.)

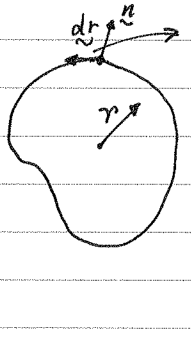
2. Reasons to study Torsion
1. 3D problem that can be done (almost)
2. Important to structures.

Reveal the guess that deformation is that of a stack of cards being twisted along the Axis. Look at this in the context of a non-round solid bar.



= torque at the end.

In a cross-section, we have $\underset{\sim}{r} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$



s = arc length on boundary curve

$$\underset{\sim}{dr} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j}$$

$$\underset{\sim}{dn} = \frac{dy}{ds}\,\underset{\sim}{i} - \frac{dx}{ds}\,\underset{\sim}{j}$$

page 2. Nov. 22, 1992

<u>Round</u> Torsion Soln. $\underset{\sim}{u} = \left[-\underset{\sim}{i}\, yz + xz\,\underset{\sim}{j} + \overset{\downarrow}{0}\,\underset{\sim}{k}\right]\frac{\phi}{L}$ (*)

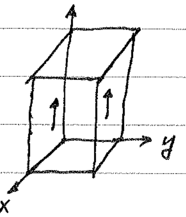
total twist

length

$$\underset{\sim}{w}' = \frac{\phi}{L}\underset{\sim}{k} = \text{rotation per unit length.}$$

(*) solves all the 3D linear elastic equation in an isotropic solid.

Consider the stresses on the ∂ of Bar,



$$\varepsilon_{xz} = -y w'/2 \qquad \varepsilon_{yz} = x w'/2$$

$$\Rightarrow \sigma_{xz} = \frac{-G w'}{2} y \qquad \sigma_{yz} = \frac{G w'}{2} x$$

OR

$$\tau_{xz}\underset{\sim}{i} + \tau_{yz}\underset{\sim}{j} = G\underset{\sim}{w}' \times \underset{\sim}{R} \qquad \underset{\sim}{R} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

$$\underset{\sim}{t} = \underset{\sim}{\sigma}\cdot\underset{\sim}{n}$$

<u>Sides</u>

$$\underset{\sim}{\sigma} = -Gyw'(\underset{\sim}{i}\underset{\sim}{k} + \underset{\sim}{k}\underset{\sim}{i}) + Gxw'(\underset{\sim}{j}\underset{\sim}{k} + \underset{\sim}{k}\underset{\sim}{j})$$

$$\underset{\sim}{n} = \frac{dy}{ds}\underset{\sim}{i} - \frac{dx}{ds}\underset{\sim}{j}$$

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{n} = -Gyw'\frac{dy}{ds}\underset{\sim}{k} + Gw'x\frac{dx}{ds}\underset{\sim}{k}$$

$$= \tau_{xz}\frac{dy}{ds} + \tau_{yz}\frac{dx}{ds}$$

$$= -Gyw'\left(\frac{dR}{ds}\right)\cdot\underset{\sim}{R}$$

$\therefore\ \underset{\sim}{\sigma}\cdot\underset{\sim}{n} = 0$ unless $\underset{\sim}{R}\cdot\frac{d\underset{\sim}{R}}{ds} = 0$ or Boundary curve is a circle.

At the ends, we have $\underset{\sim}{\sigma}\cdot\underset{\sim}{n} = \underset{\sim}{\sigma}\cdot\underset{\sim}{k}$

$$= -Gyw'\underset{\sim}{i} + Gxw'\underset{\sim}{j} = \underset{\sim}{t}$$

$$T\underset{\sim}{k} = \int_A \underset{\sim}{R}\times\underset{\sim}{t}\,dA = Gw'\int \underset{\sim}{R}\times(-\underset{\sim}{i}y + x\underset{\sim}{j})\,dA$$

$$= Gw'\int(x\underset{\sim}{i} + y\underset{\sim}{j})\times(-\underset{\sim}{i}y + x\underset{\sim}{j})\,dA$$

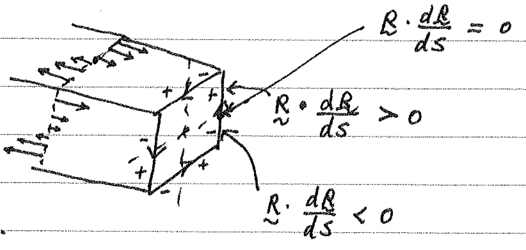
$$= Gw'\underset{\sim}{k}\int_A R^2\,dA$$

page 3.

We have exact solution to some 3D problems, except it does not satisfy the wrong ∂ condition.

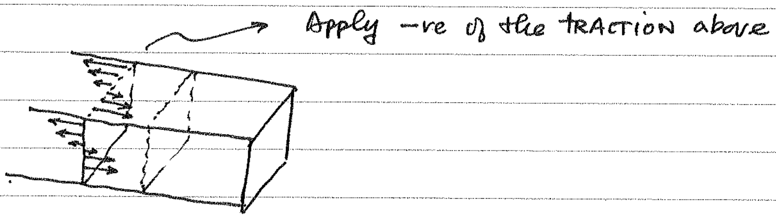
e.g. for a square cross-section

$$\underset{\sim}{t} = -\underset{\sim}{k}\, w' G \left(\underset{\sim}{R}\cdot\frac{d\underset{\sim}{R}}{ds}\right)$$



If we applied these tractions, we will be O.K.

What to do to get the exact solution, we solve the problem below.



OR $\underset{\sim}{t} = \underset{\sim}{k}\, G w' \underset{\sim}{R}\cdot\frac{d\underset{\sim}{R}}{ds}$, then use supperposition.

This is a problem in anti-plane shear. [since there is no net stretch in the z direction]

There is an exact solution to Anti-plane

$$\nabla^2 w = 0 .$$

i.e, $\underset{\sim}{u} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + w\underset{\sim}{k}$ $\qquad w = w(x,y)$

Need to find w ∋ B.C. conditions are satisfy

$$\underset{\sim}{t} = G\underset{\sim}{k} w' \underset{\sim}{R}\cdot\frac{d\underset{\sim}{R}}{ds} = \underset{\sim}{k}\, G w' \frac{dR}{ds}\cdot \underset{\sim}{R} \qquad \text{Known}$$

$$\underset{\sim}{t} = \left(G w_{,x}(\underset{\sim}{i}\underset{\sim}{k} + \underset{\sim}{k}\underset{\sim}{i}) + G w_{,y}(\underset{\sim}{j}\underset{\sim}{k} + \underset{\sim}{k}\underset{\sim}{j})\right)\cdot\left[\frac{dy}{ds}\underset{\sim}{i} - \frac{dx}{ds}\underset{\sim}{j}\right]$$

$$= \left(G w_{,x}\frac{dy}{ds} \cancel{+} - G w_{,y}\frac{dx}{ds}\right)\underset{\sim}{k} = G\frac{dw}{dn}\underset{\sim}{k}$$

$$\nabla^2 w = 0 \qquad \text{and} \qquad \frac{\partial w}{\partial n} = \text{given} = f(s) = G w' \frac{dR}{ds}\cdot R$$

a ~~Neumann problem~~

(84)

page 3. November 22, 1992.

Theory of Torsion: the Usual Way.

Approach 2. a) Make a better guess to start with, b) use stress function.

$$\psi(x,y) = w(x,y)$$

Guess

$$\underset{\sim}{u} = (-yz\,\underset{\sim}{i} + xz\,\underset{\sim}{j} + \psi\,\underset{\sim}{k})\,w'$$

$\psi(x,y)$ is called the warping function

calculate strains & stresses:

$$\varepsilon_{xz} = w'(-y + \psi_{,x})/2 \qquad \sigma_{xz} = G\,\varepsilon_{xz}\,w'$$

$$\varepsilon_{yz} = w'(x + \psi_{,y})/2 \qquad \sigma_{yz} = G\,\varepsilon_{yz}\,w'$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0$$

Our boundary condition are $\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = 0$ so that

$$(-y + \psi_{,x})\,n_x + (x + \psi_{,y})\,n_y = 0$$

$$(-y + \psi_{,x})\,\frac{dy}{ds} - (x + \psi_{,y})\,\frac{dx}{ds} = 0$$

OR

$$\frac{\partial \psi}{\partial n} = \underset{\sim}{R} \cdot \frac{d\underset{\sim}{R}}{ds}$$

we have

$$\nabla^2 \psi = 0$$

"stress function" approach:

Equilibrium Equations: Look at Equilibrium Eqns.

$$(*) \quad \sigma_{xz,x} + \sigma_{yz,y} = 0 \quad \Rightarrow \quad \sigma_{xz} = +\frac{\partial \phi}{\partial y} \qquad \sigma_{yz} = -\frac{\partial \phi}{\partial x}$$

which automatically solve (*).

$$2G\,\varepsilon_{xz} = \sigma_{xz} = \frac{\partial \phi}{\partial y} \qquad 2G\,\varepsilon_{yz} = -\frac{\partial \phi}{\partial x}$$

~~$\varepsilon_{xz} = \frac{\partial w}{\partial x}$~~ ~~$\varepsilon_{yz} = \frac{\partial w}{\partial y}$~~.

$\therefore$ ~~$2\hat{G}\frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial y}$ and~~ ~~$2\hat{G}\frac{\partial w}{\partial y} = -\frac{\partial \phi}{\partial x}$~~.

so that

$f(z) = 2\hat{G}w + i\phi$ is an ~~at~~ analytic function since

Strain displ :

$$G\left[w'(-y + \psi_{,x}) \right] = \phi_{,y} \qquad (1)$$

$$-G\left[w'(x + \psi_{,y}) \right] = -\phi_{,x} \qquad (2)$$

Differentiate (1) by y and (2) by x , substract $\Rightarrow$

$$\phi_{yy} + \phi_{,xx} = 2Gw'$$

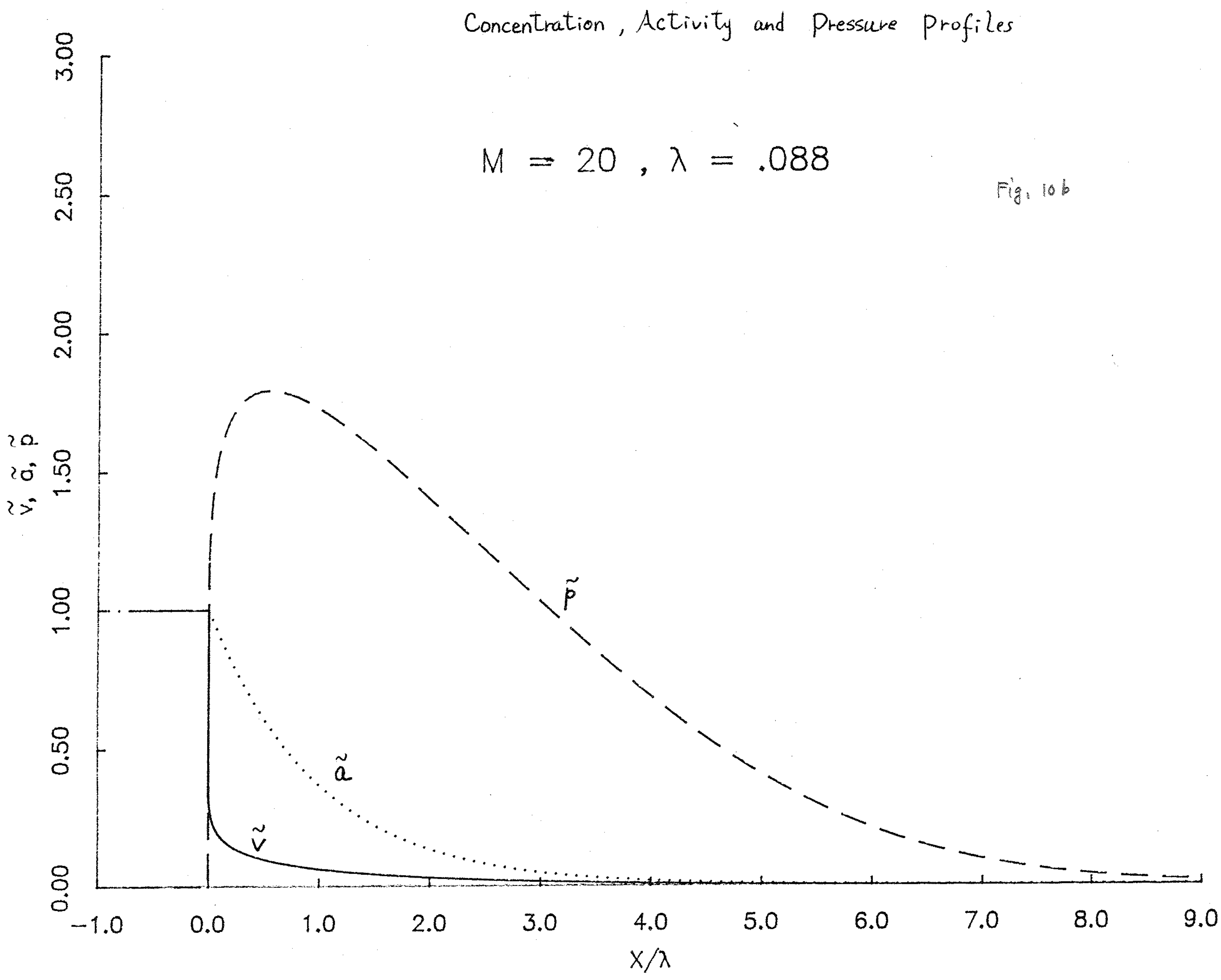
and we have $\sigma_{xz} n_x + \sigma_{yz} n_y = 0 \Rightarrow \phi_{,y} \frac{dy}{ds} + \phi_{,x} \frac{dx}{ds} = 0$

$$\cong \boxed{\phi_{,t} = 0} \quad \text{on } \partial .$$

We have, therefore

$$\nabla^2 \phi = 2G\omega'$$

$$\phi_{,t} = 0 \qquad \partial$$



Concentration, Activity and Pressure Profiles

$M = 20 , \lambda = .088$

Fig. 10b

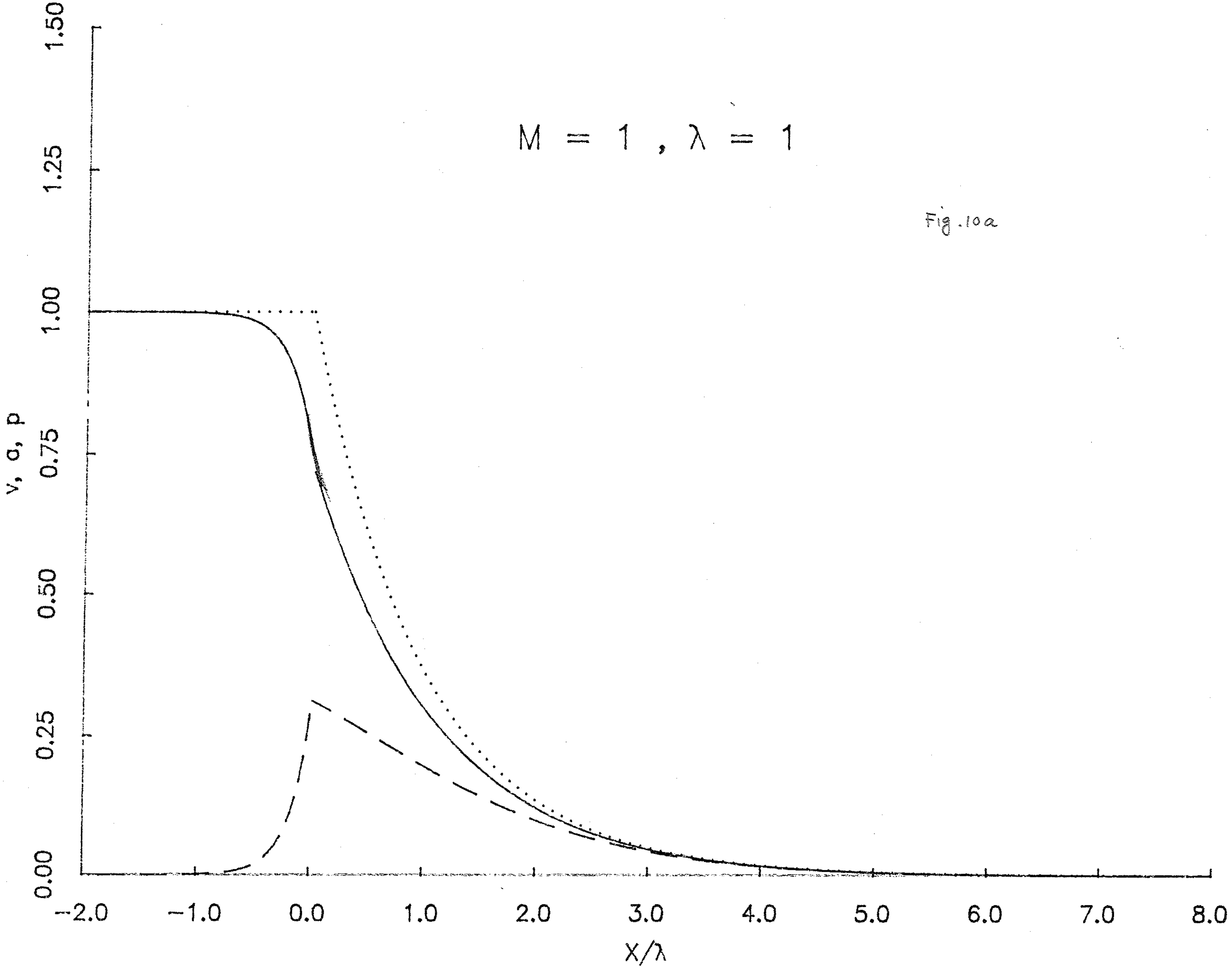

M = 1 , λ = 1
v, a, p
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−2.0
−1.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
X/λ

Fig. 10a

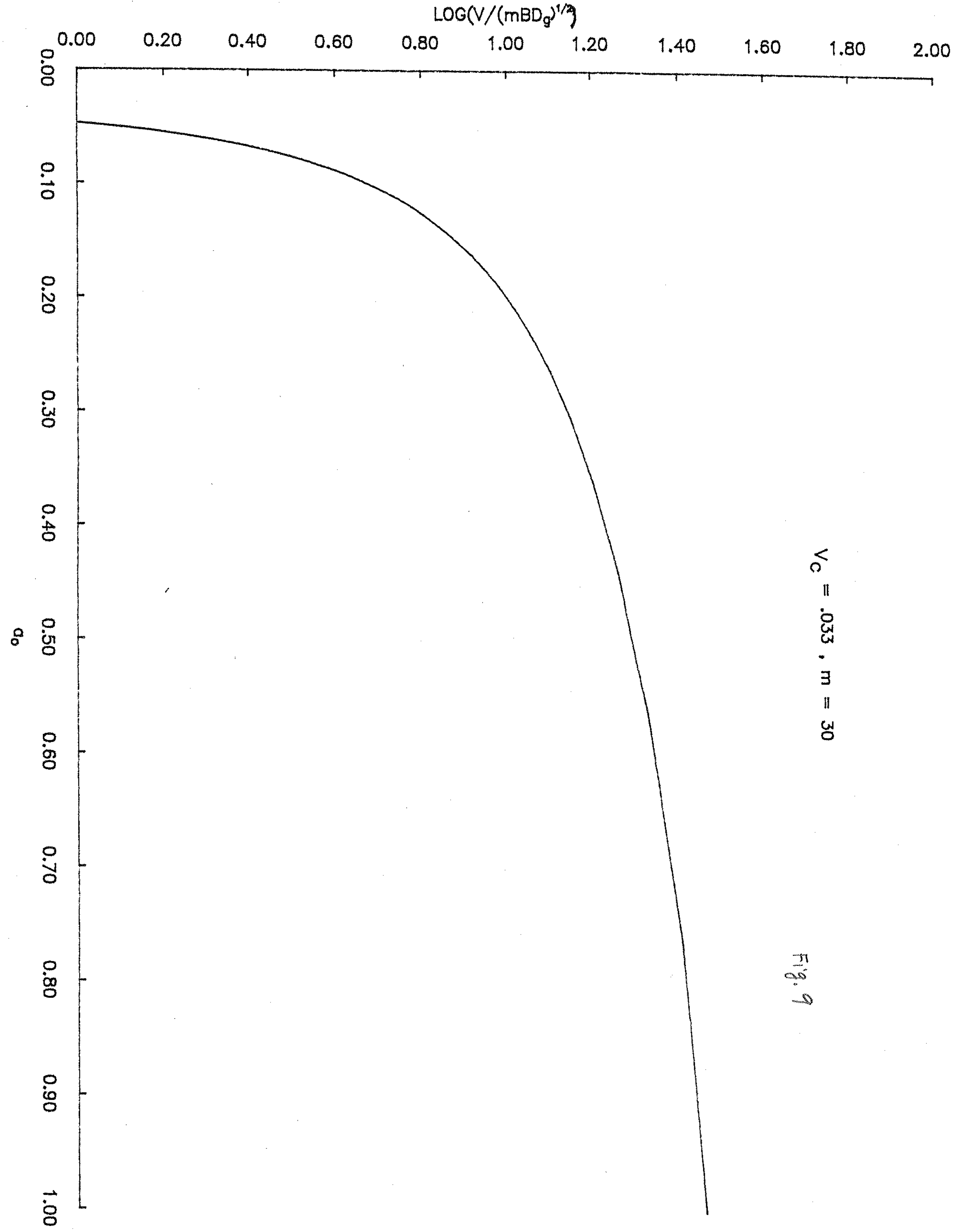

LOG(V/(mBD$_g$)$^{1/2}$)
0.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
a_o
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
V_c = .033 , m = 30

Fig. 9

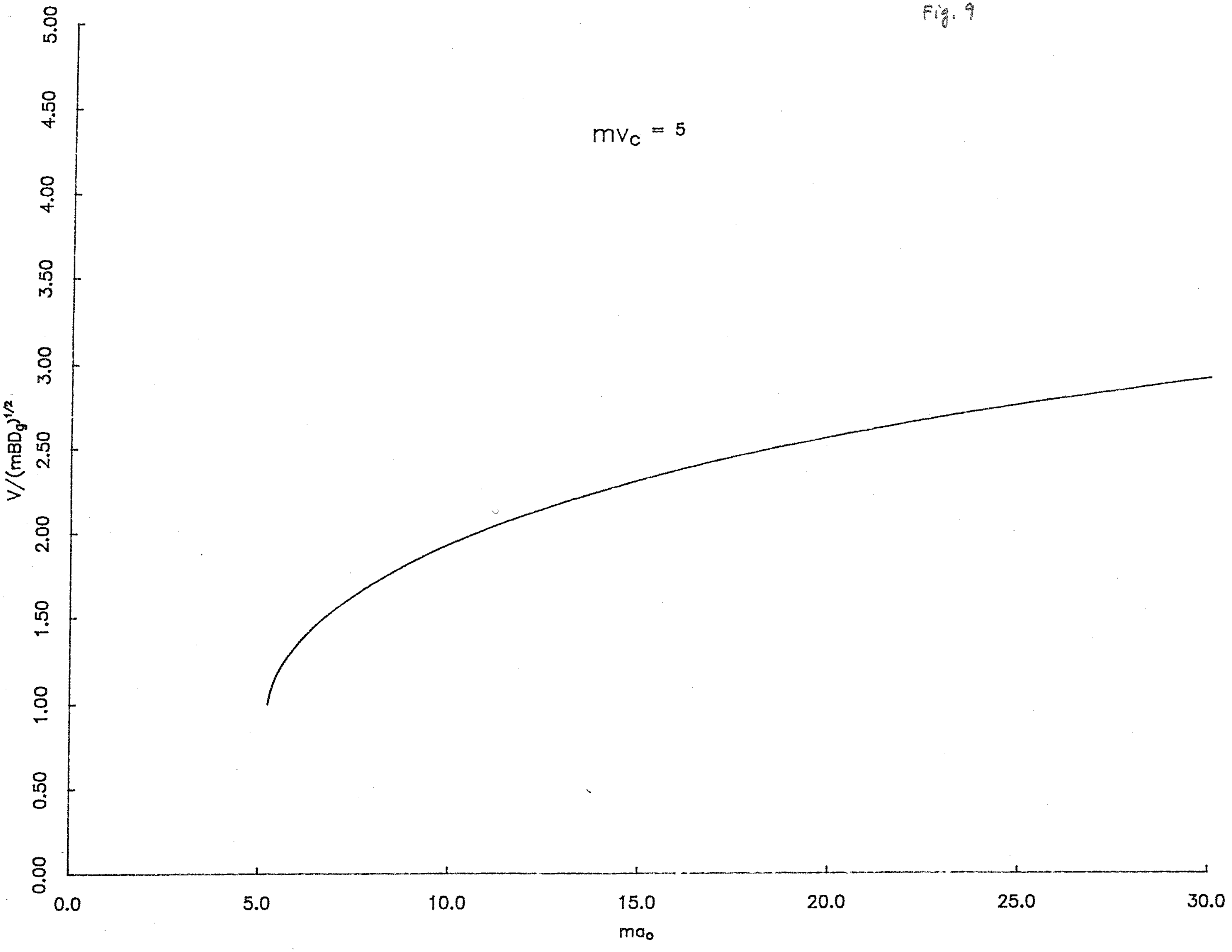

$mv_c = 5$
$V/(mBD_g)^{1/2}$
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
0.0
5.0
10.0
15.0
20.0
25.0
30.0
ma_o

Fig. 9

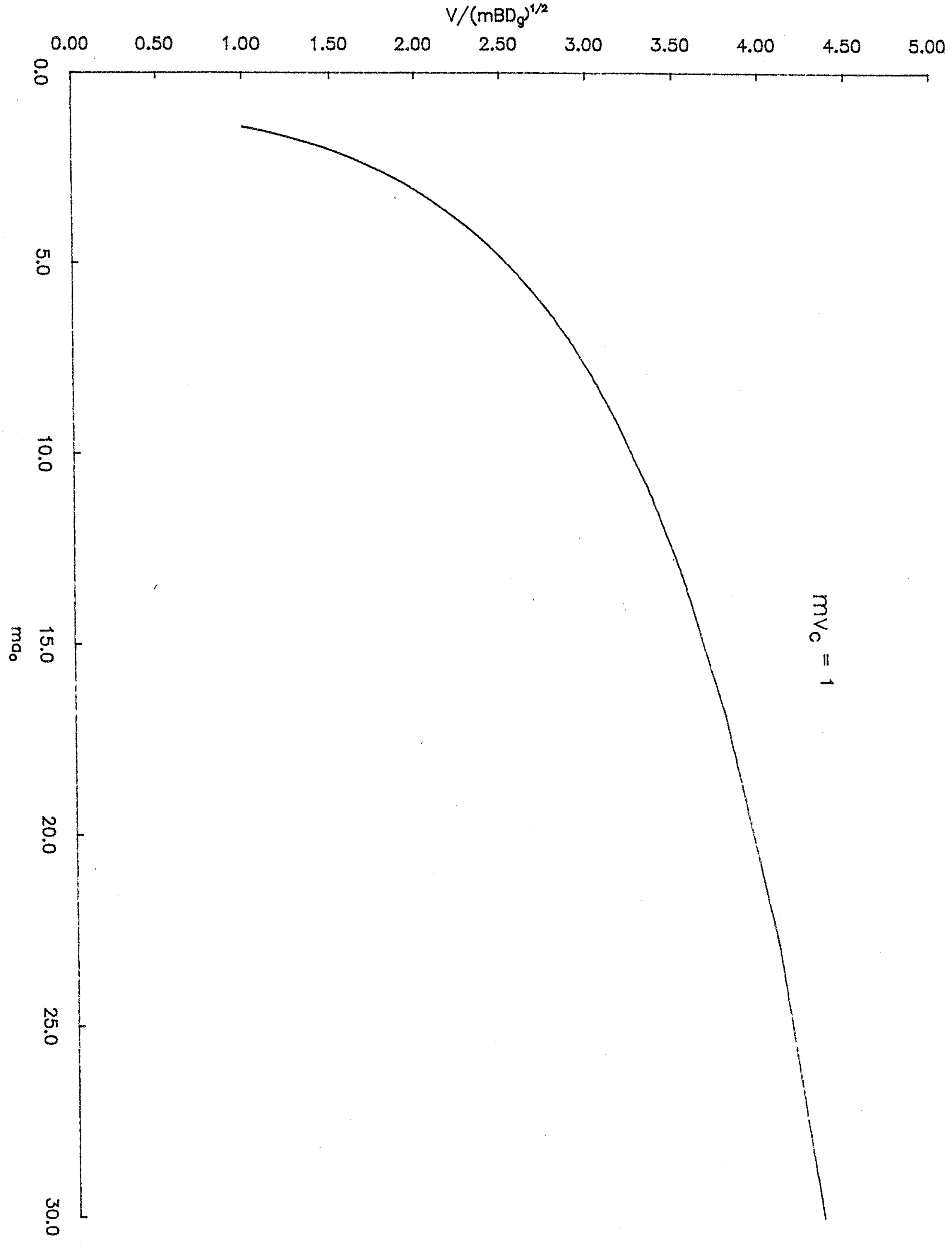

$V/(mBD_g)^{1/2}$
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
0.0
5.0
10.0
15.0
20.0
25.0
30.0
ma_o
$mV_c = 1$

Fig 8

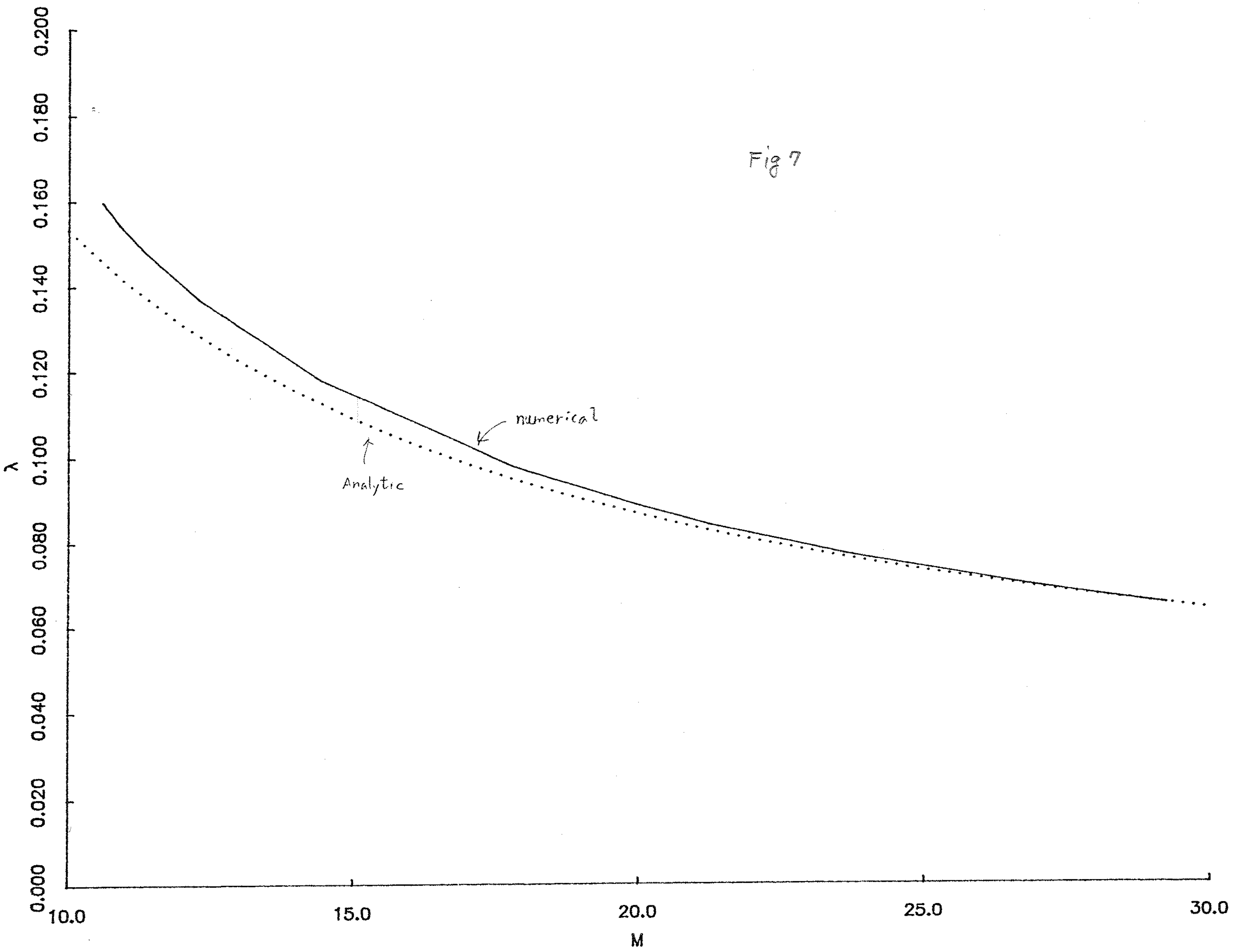


Fig 7

0.200
0.180
0.160
0.140
0.120
0.100
0.080
0.060
0.040
0.020
0.000
λ
10.0
15.0
20.0
25.0
30.0
M
numerical
Analytic

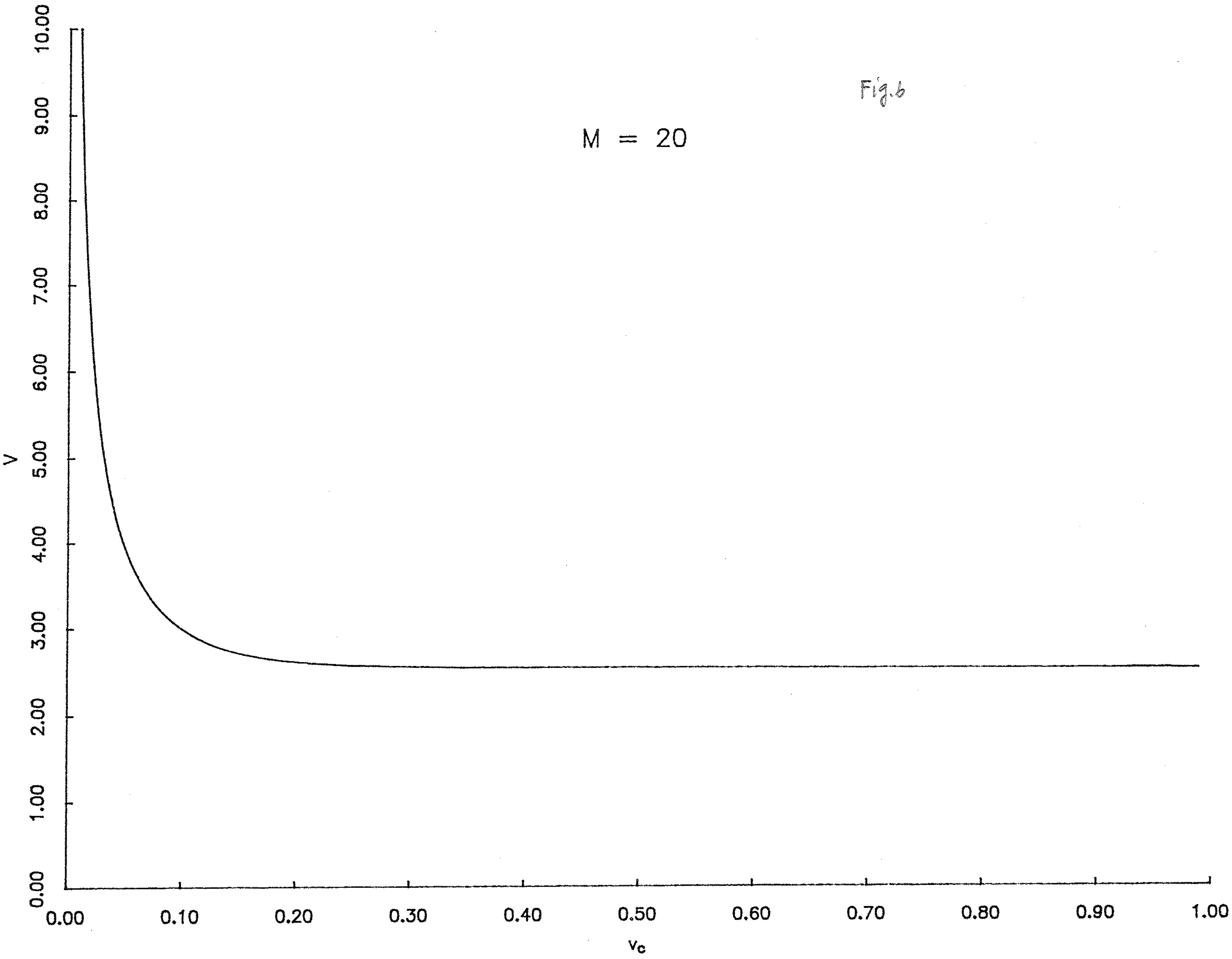

M = 20
V
10.00
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
v_c

Fig.6

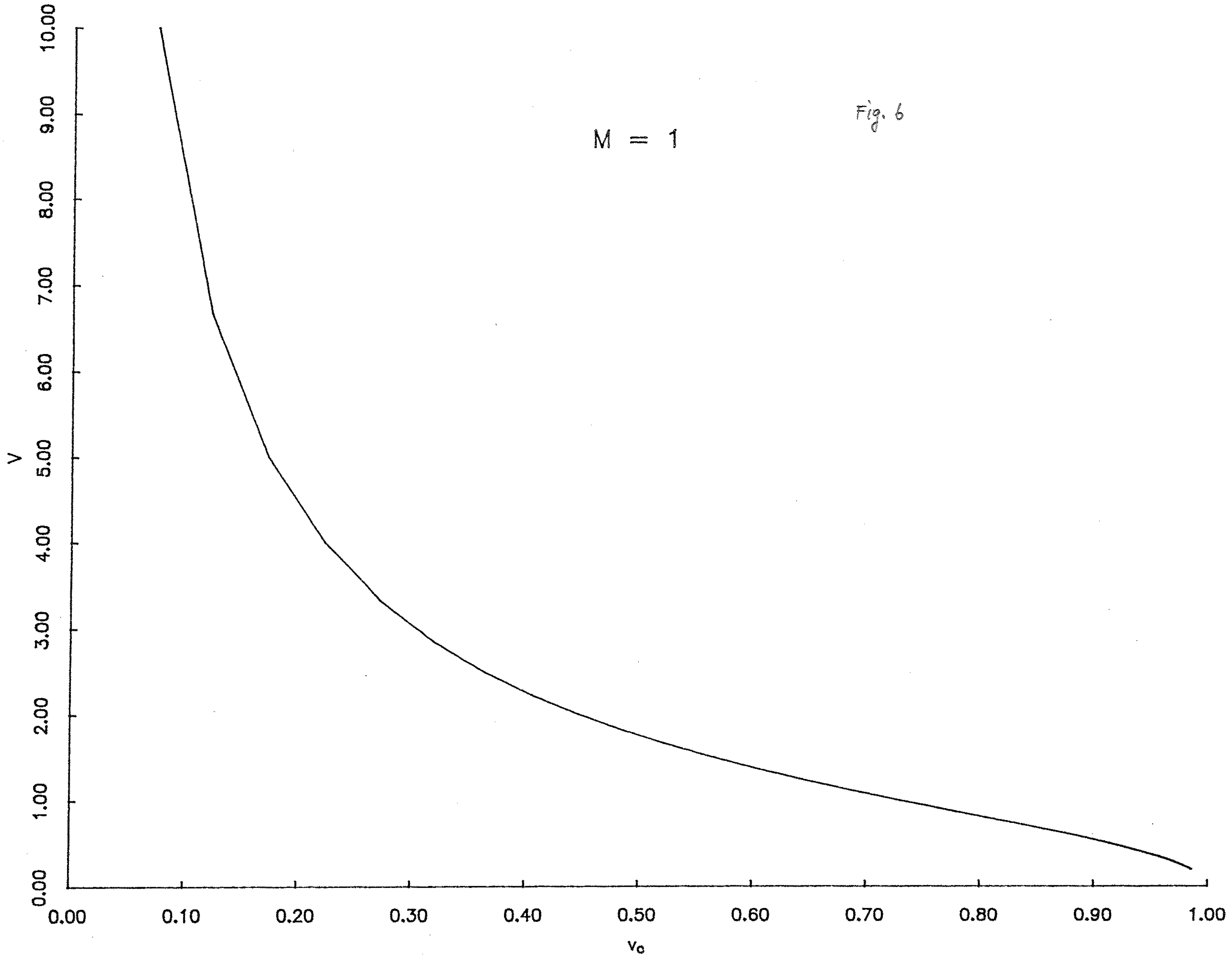
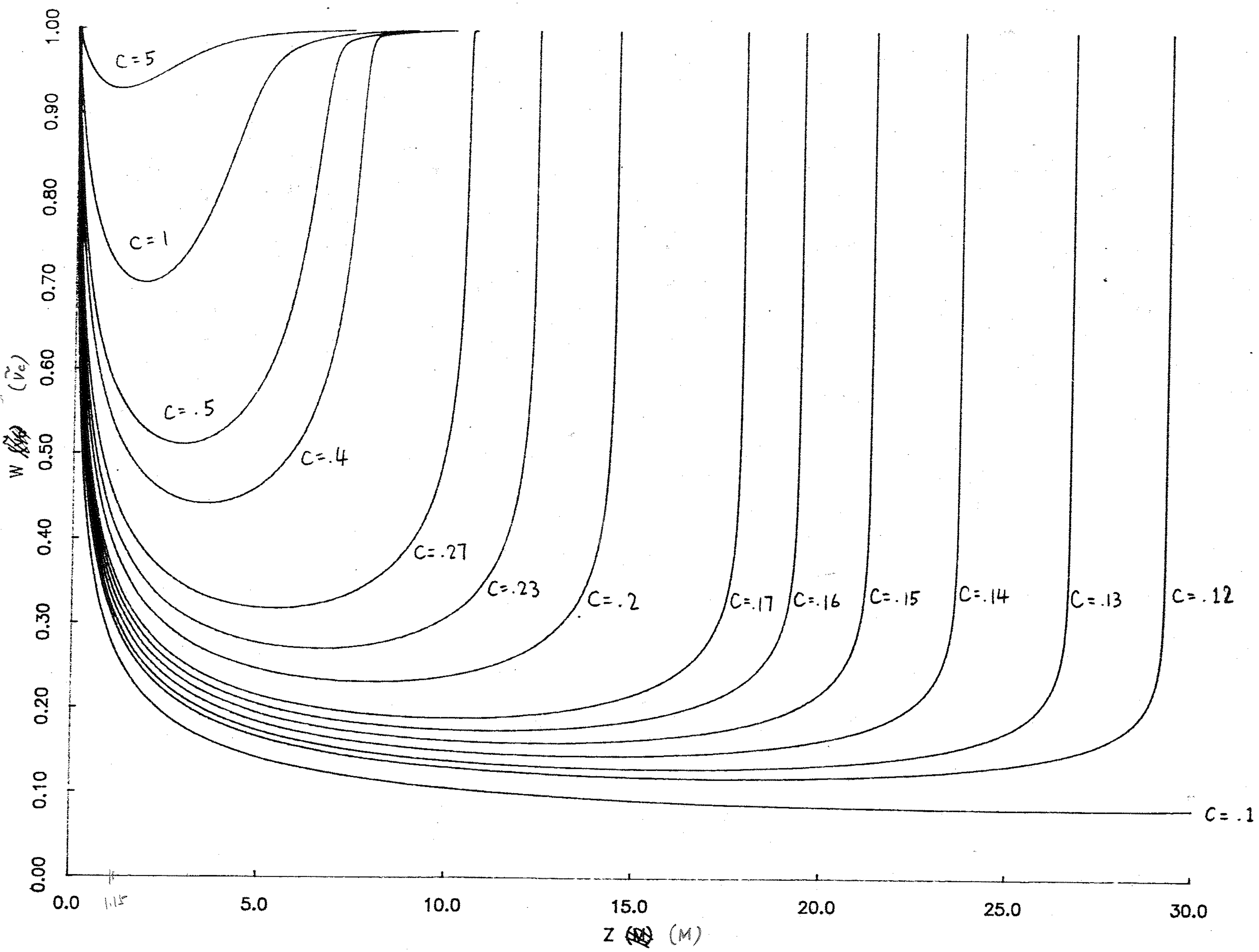

M = 1
V
10.00
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
v_c

Fig. 6

Fig.5.

(Fig.5)

$C = M\lambda^2$



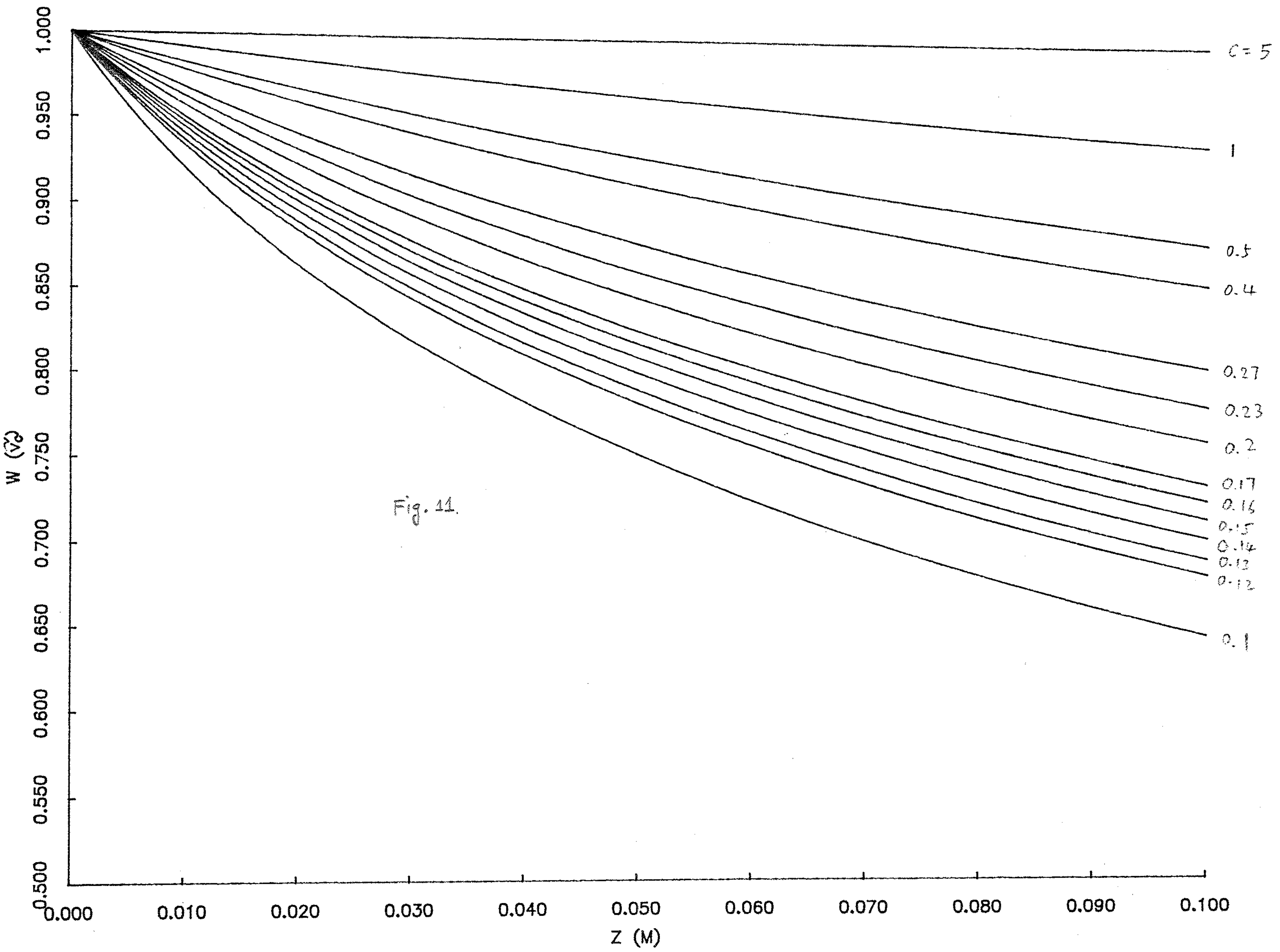
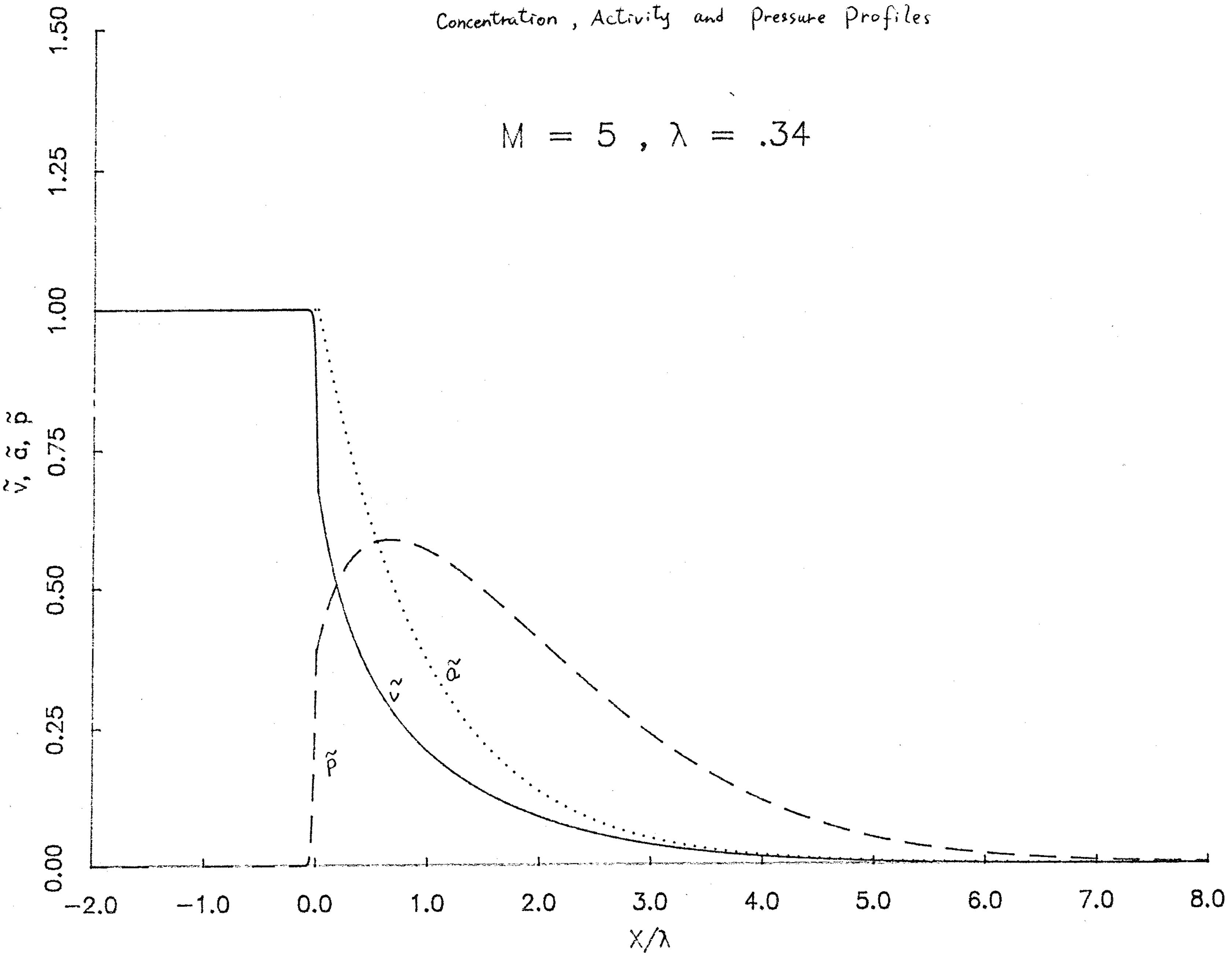
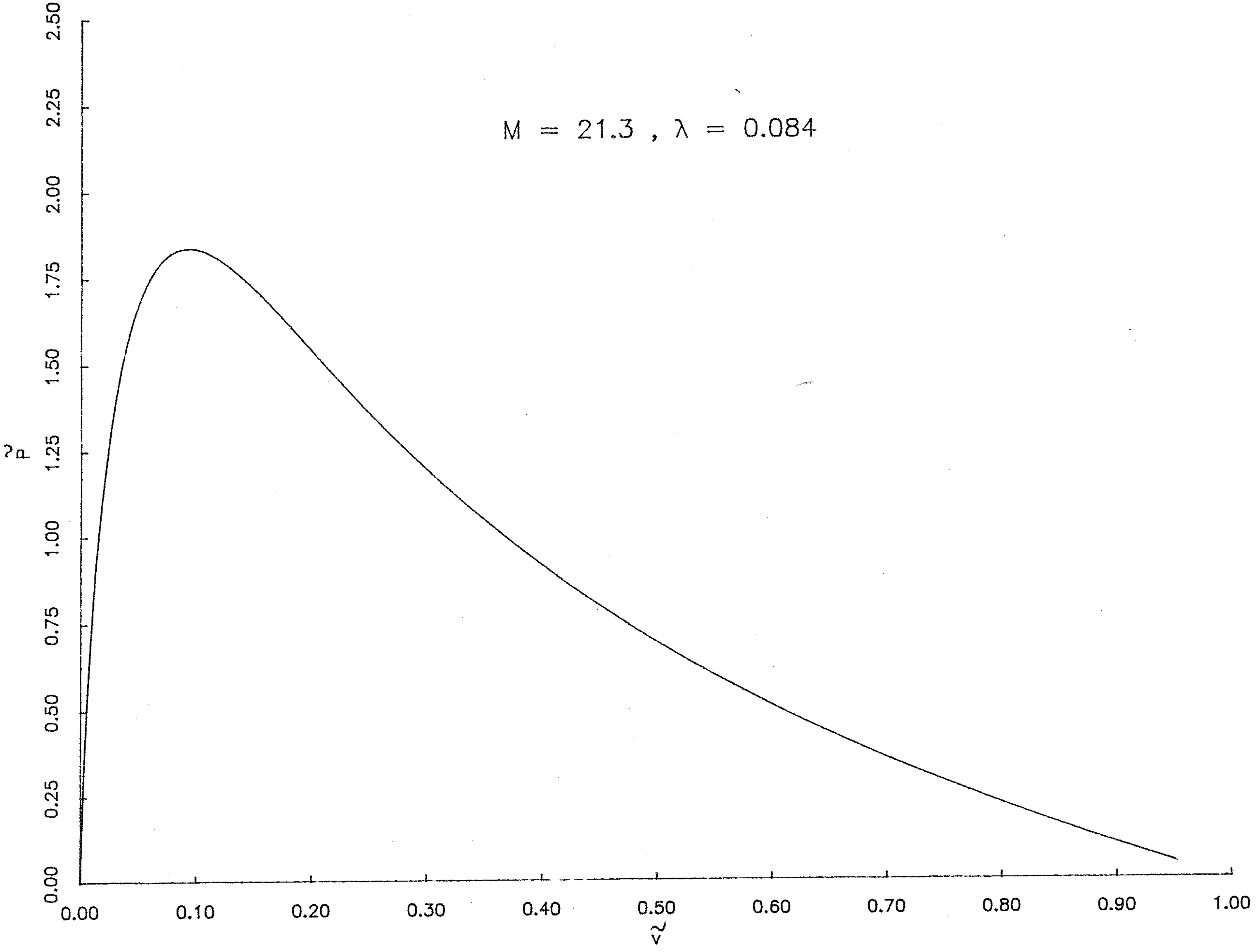

1.000
0.950
0.900
0.850
0.800
0.750
0.700
0.650
0.600
0.550
0.500
W (Vo)
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
0.080
0.090
0.100
Z (M)
C = 5
1
0.5
0.4
0.27
0.23
0.2
0.17
0.16
0.15
0.14
0.13
0.12
0.1

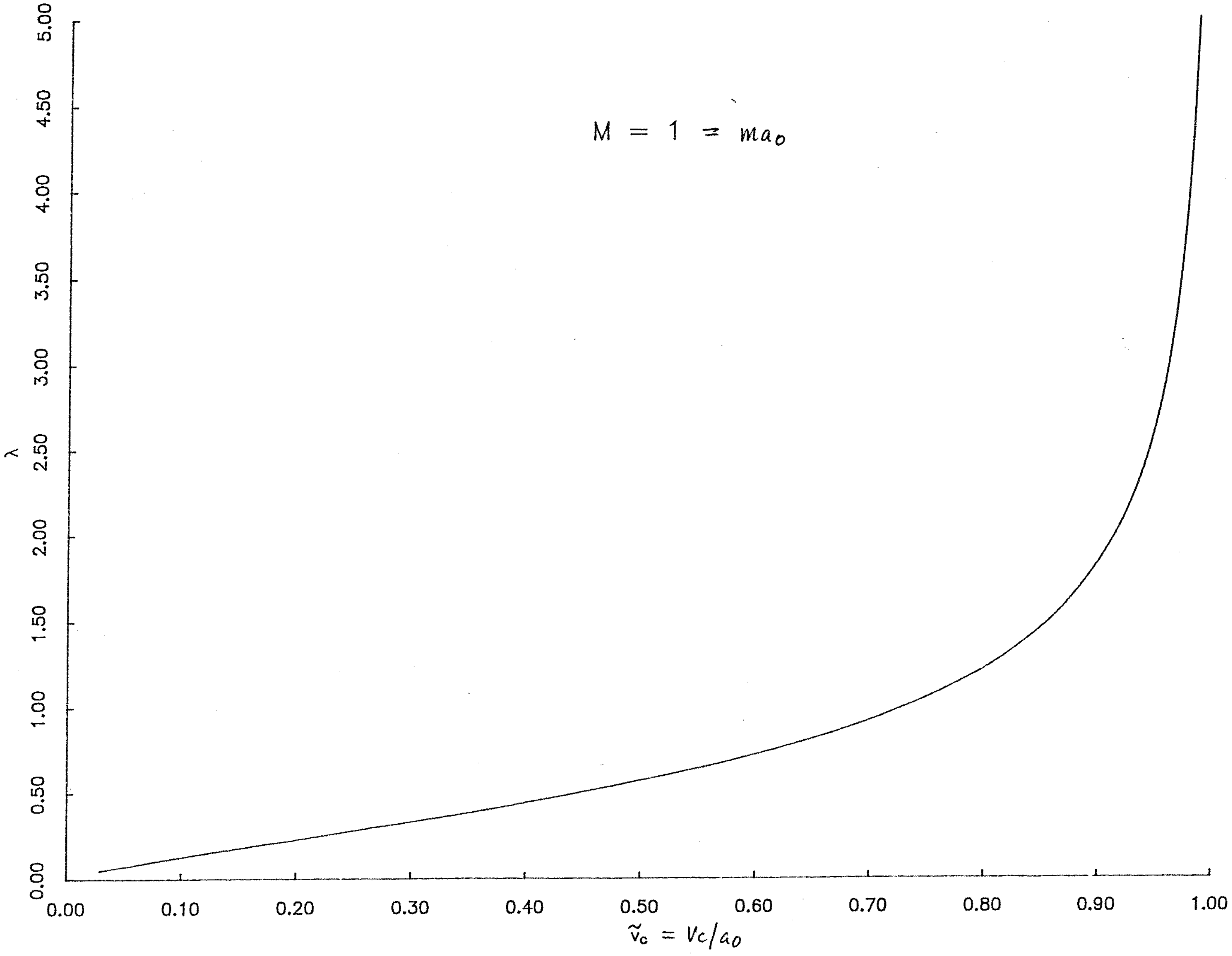
Fig. 11.

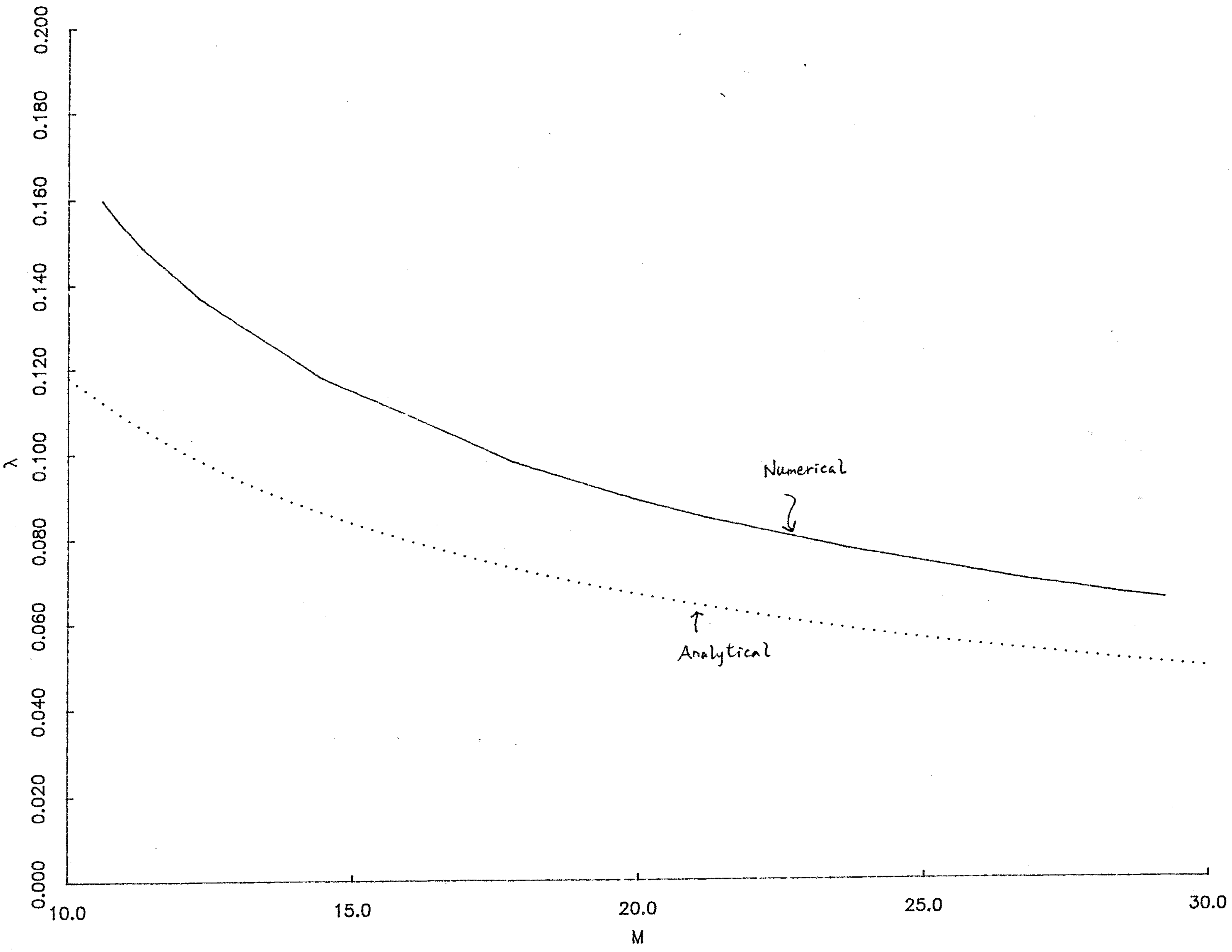


Concentration, Activity and Pressure Profiles

$M = 5 , \lambda = .34$


M = 21.3 , λ = 0.084
$\tilde{p}$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
2.25
2.50
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
$\tilde{v}$


$M = 1 = ma_0$
λ
5.00
4.50
4.00
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
$\tilde{v}_c = V_c/a_0$


0.200
0.180
0.160
0.140
0.120
0.100
0.080
0.060
0.040
0.020
0.000
λ
10.0
15.0
20.0
25.0
30.0
M
Numerical
Analytical

Nov. 11 Wendesday. last part note.

2D problem Model 2: plain strain

kinematic assumption : $\underline{u} = u(x,y)\underline{i} + v(x,y)\underline{j} + \overset{\nearrow}{0}\,\underline{k}$

(for generalized plain strain $\varepsilon_{zz}\underline{z}$ = constant).

$$\therefore \quad \varepsilon_{xx} = u_{,x} \qquad \varepsilon_{yy} = u_{,y} \qquad \varepsilon_{xy} = \frac{1}{2}(u_{,x} + u_{,y})$$

$$\varepsilon_{zz} = 0 \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0$$

Defination in plain strain is that of inextensible needles pointed in Z direction, that can only move in X & Y directions.

Constitutive Law: Isotropic elasticity

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\delta_{ij}\,\varepsilon_{kk}$$

$$\begin{cases} \sigma_{xx} = (2G+\lambda)\,\varepsilon_{xx} + \lambda\,\varepsilon_{yy} \\ \sigma_{yy} = (2G+\lambda)\,\varepsilon_{yy} + \lambda\,\varepsilon_{xx} \\ \sigma_{xy} = 2G\,\varepsilon_{xy} \end{cases}$$

$$\sigma_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\varepsilon_{ij} = \frac{\sigma_{ij}(1+\nu)}{E} - \frac{\nu}{E}\,\delta_{ij}\,\sigma_{kk}$$

$$\varepsilon_{zz} = 0 \Rightarrow \sigma_{zz}(1+\nu) - \nu(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = 0$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

2D Equilibrium equs. w/ $\underline{b} = \underline{a} = 0$.

$$\boxed{\begin{aligned} &\sigma_{xx,x} + \sigma_{xy,y} = 0 \\ &\sigma_{xy,x} + \sigma_{yy,y} = 0. \end{aligned}}$$

2D Compatability Equs.

$$\frac{\partial^2}{\partial x \partial y} \left\{ \varepsilon_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x}) \right\}$$

$$* \quad \varepsilon_{xy,xy} = \frac{1}{2}\left[(u_{x,x})_{,yy} + (u_{y,y})_{,xx}\right]$$

any ε_{xx}, ε_{yy}, ε_{xy} must satisfy this equ.

Fact: * implies $u_x(x, y)$ & $u_y(x, y)$ exist.

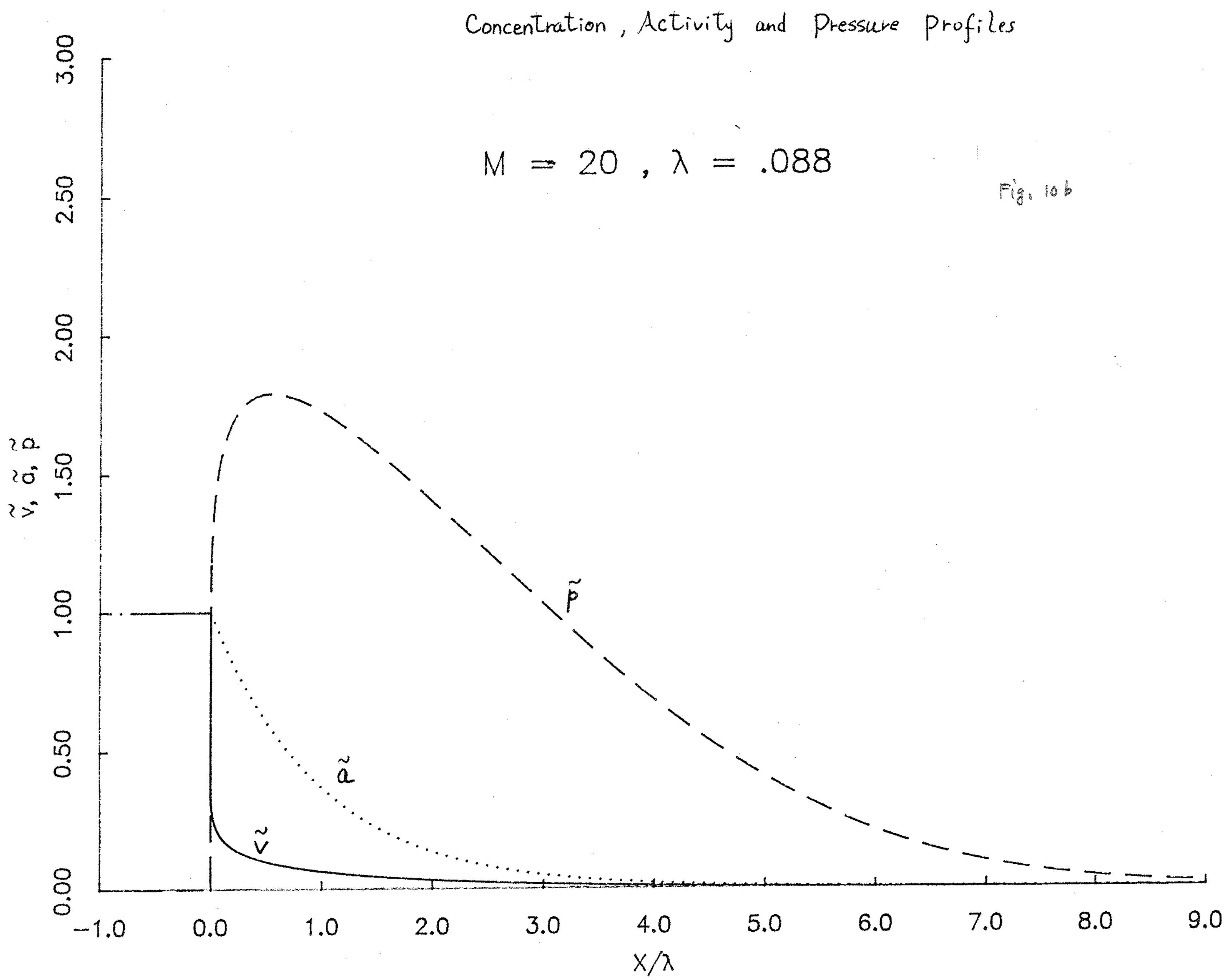
Apply constitutive law and equilibrium equs. to compatability equ

$$\Rightarrow: \quad \boxed{\nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0}$$

$$\boxed{\begin{array}{l} \sigma_{xx,x} + \sigma_{xy,y} = 0 \\ \sigma_{xy,x} + \sigma_{yy,y} = 0, \end{array}}$$

plain strain governing equs. written in terms of stress.

BVP: Solve these equs. w/ given traction BC's.



Concentration, Activity and Pressure Profiles

$M = 20$, $\lambda = .088$

Fig. 10b

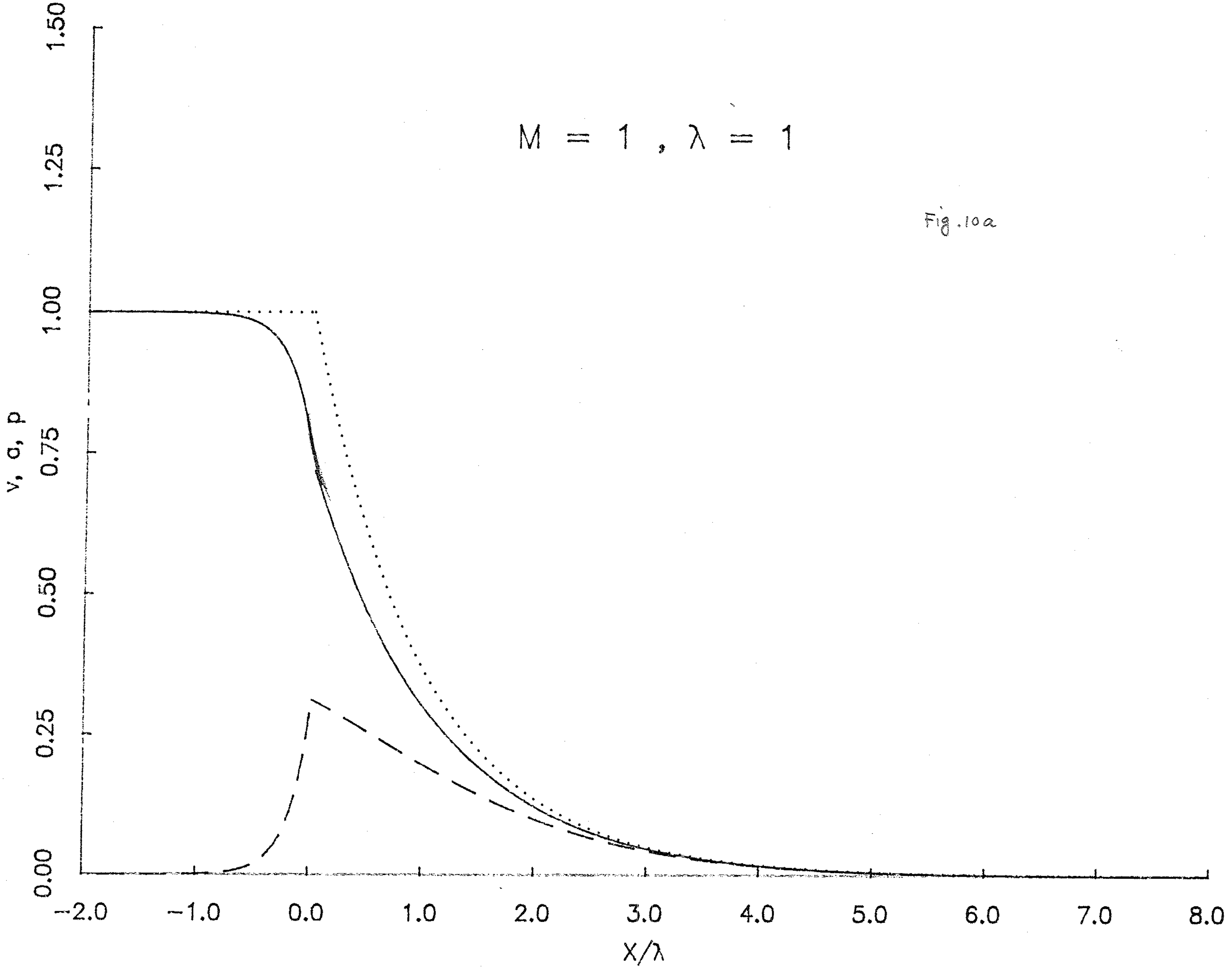

M = 1 , λ = 1
v, a, p
0.00
0.25
0.50
0.75
1.00
1.25
1.50
−2.0
−1.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
X/λ

Fig.10a

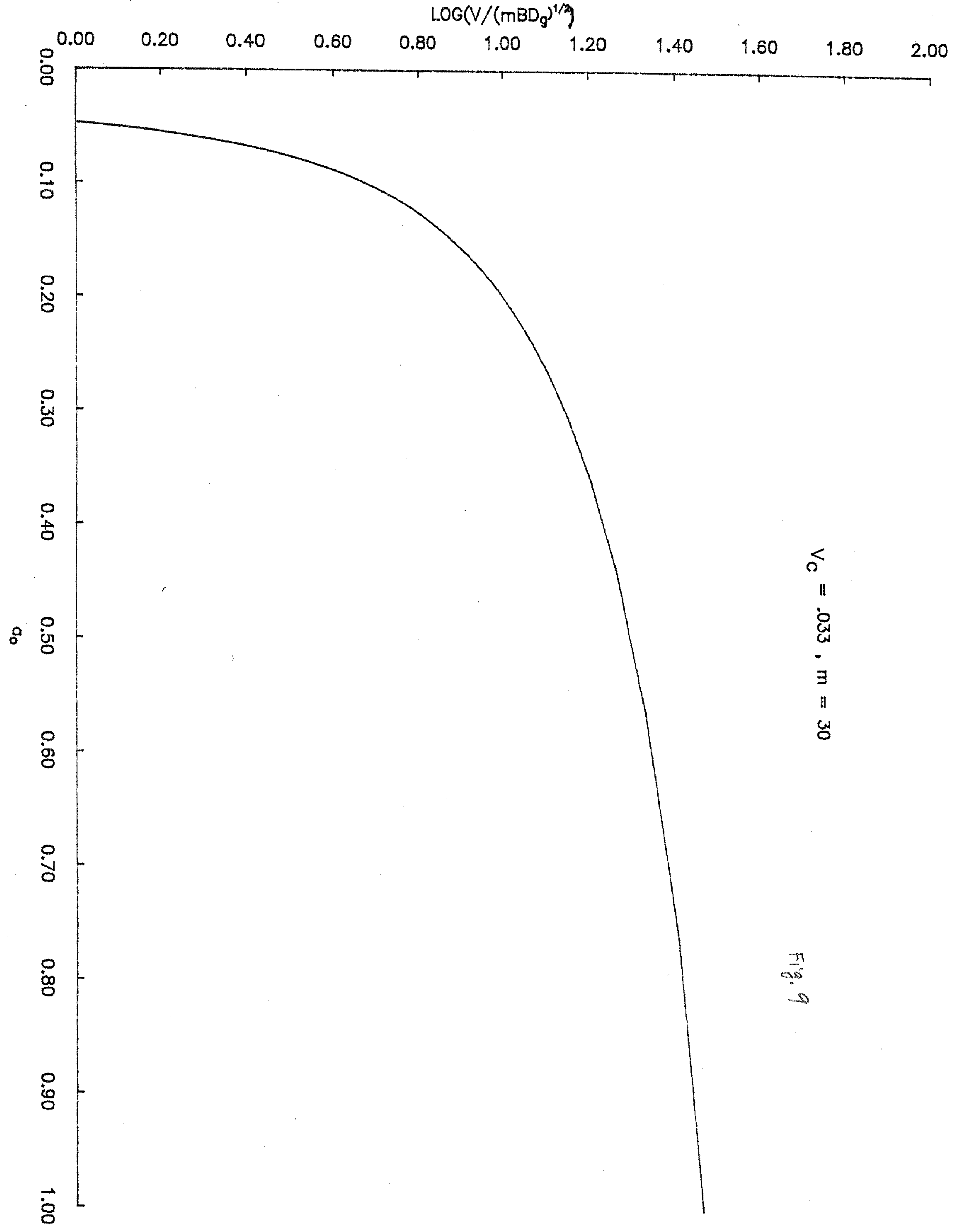

LOG(V/(mBDg)^1/2)
0.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
a₀
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
Vc = .033 , m = 30

Fig. 9

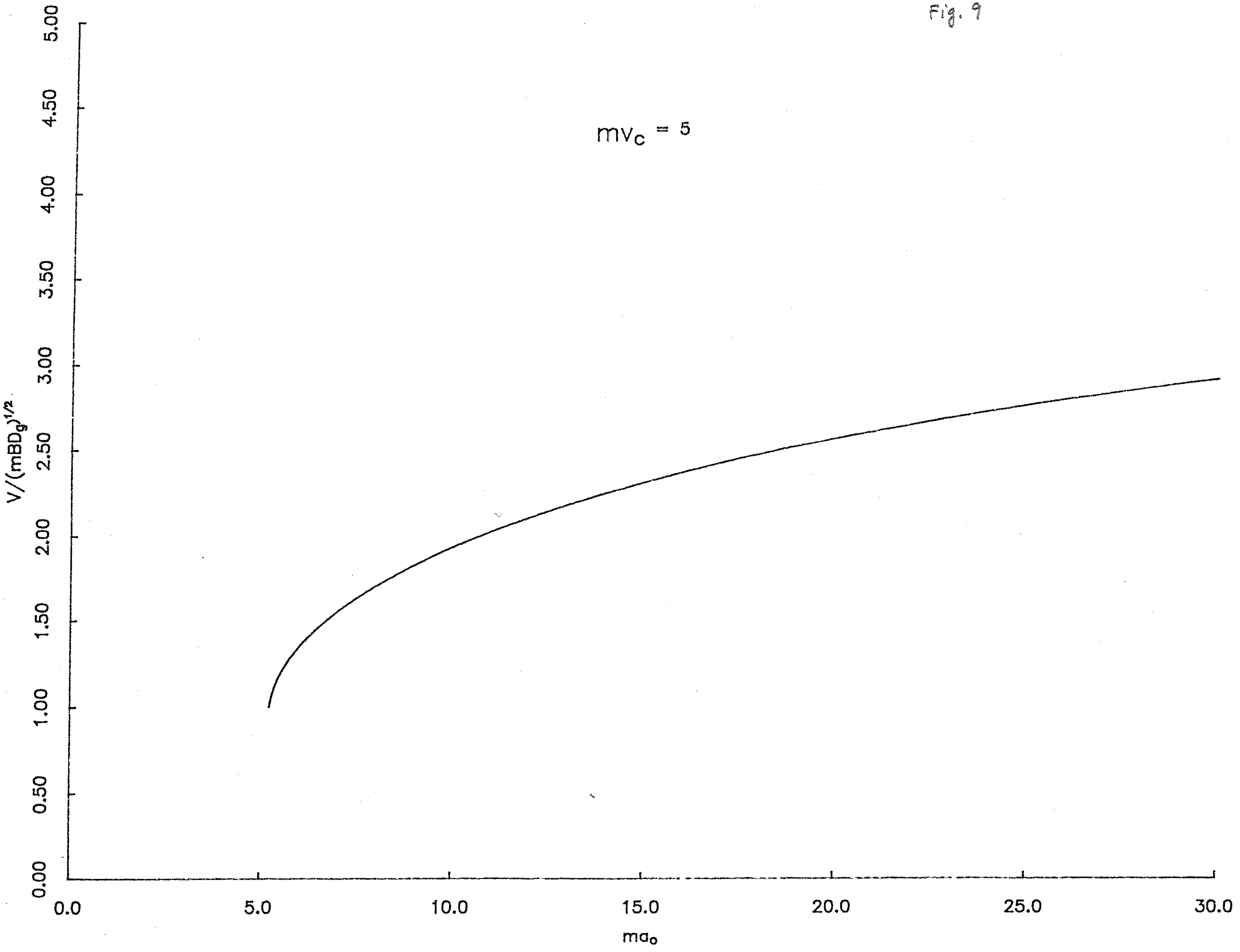

$mv_c = 5$
$V/(mBD_g)^{1/2}$
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
0.0
5.0
10.0
15.0
20.0
25.0
30.0
ma_o

Fig. 9

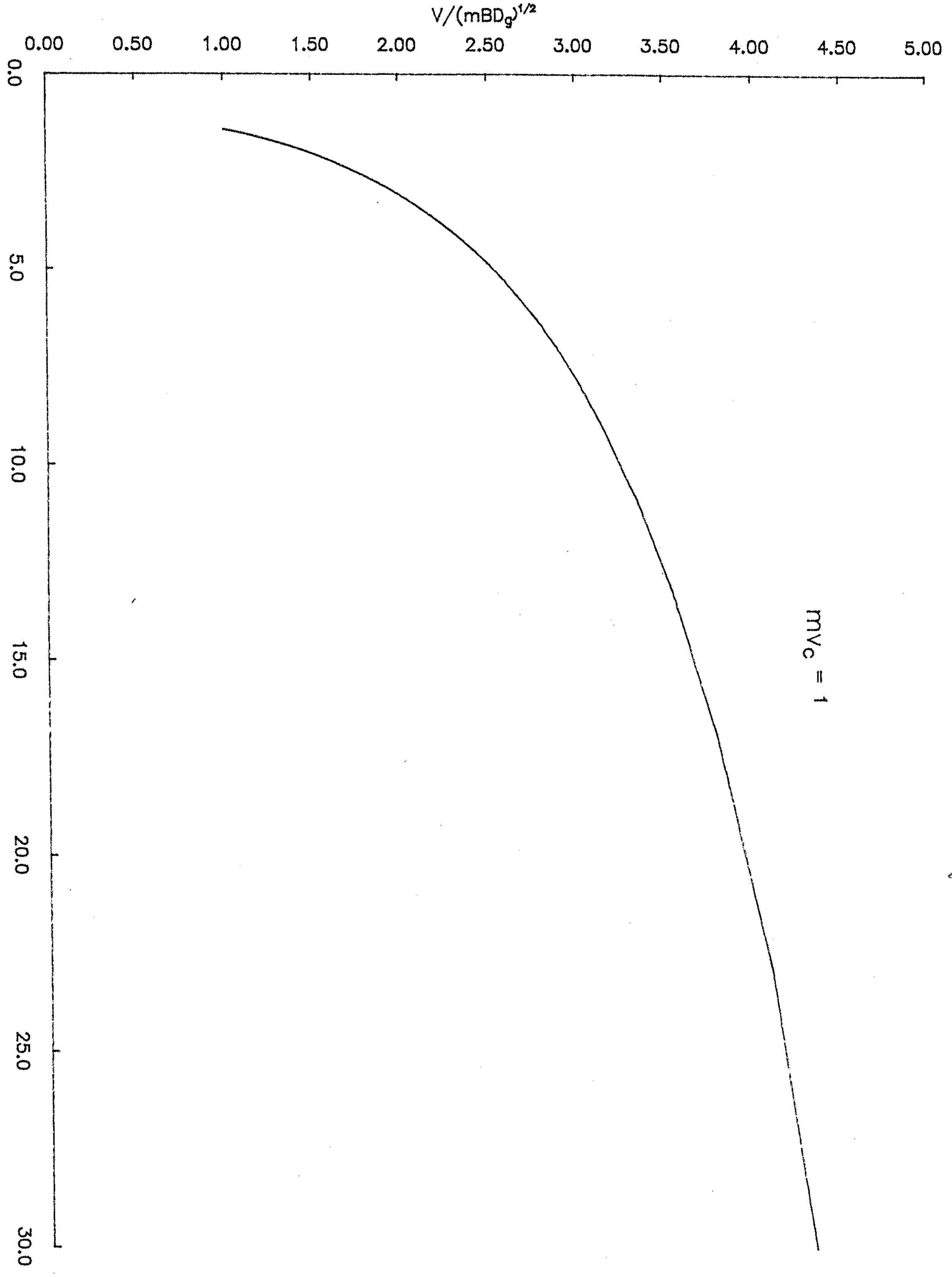

V/(mBD$_g$)$^{1/2}$
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
0.0
5.0
10.0
15.0
20.0
25.0
30.0
ma$_o$
mV$_c$ = 1

Fig 8

Fig 7

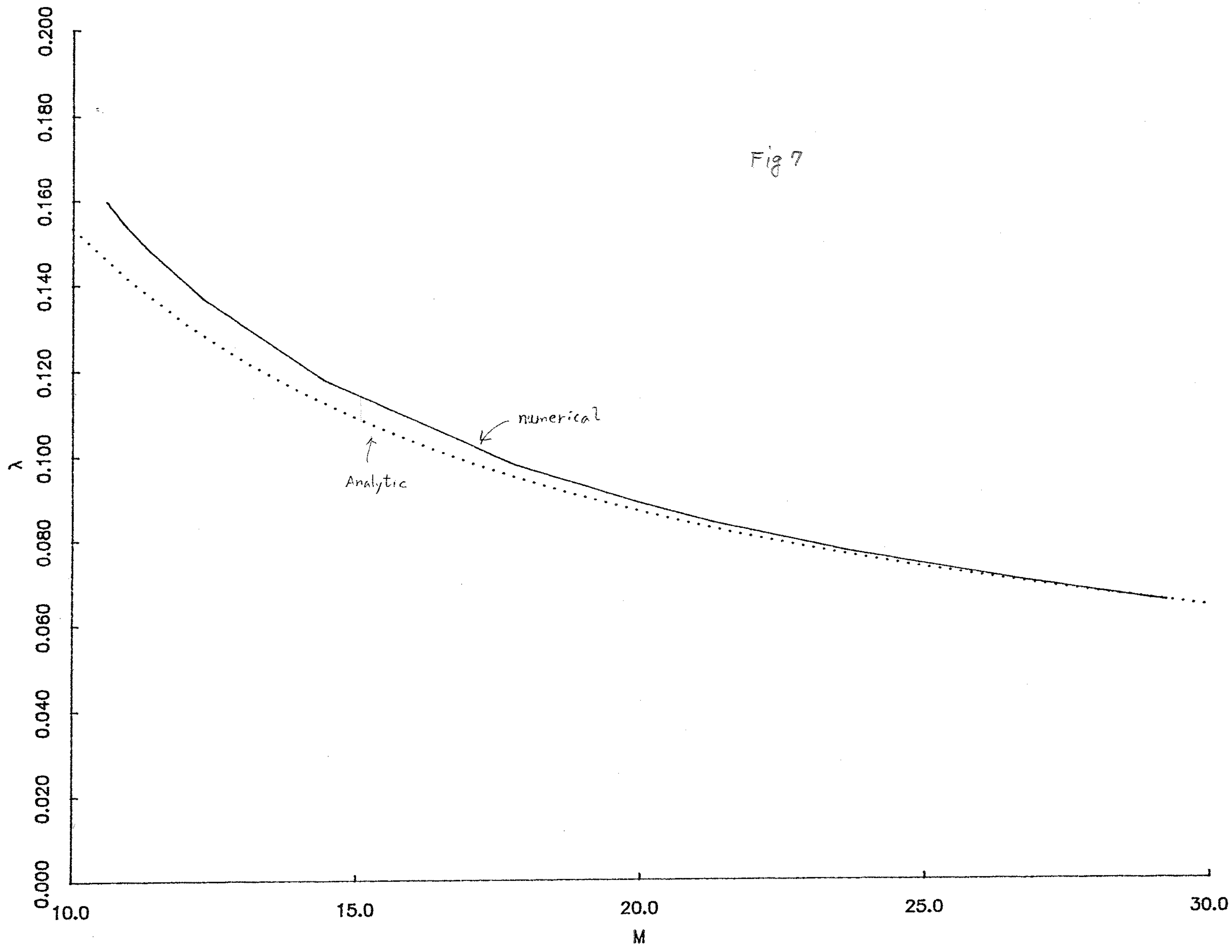
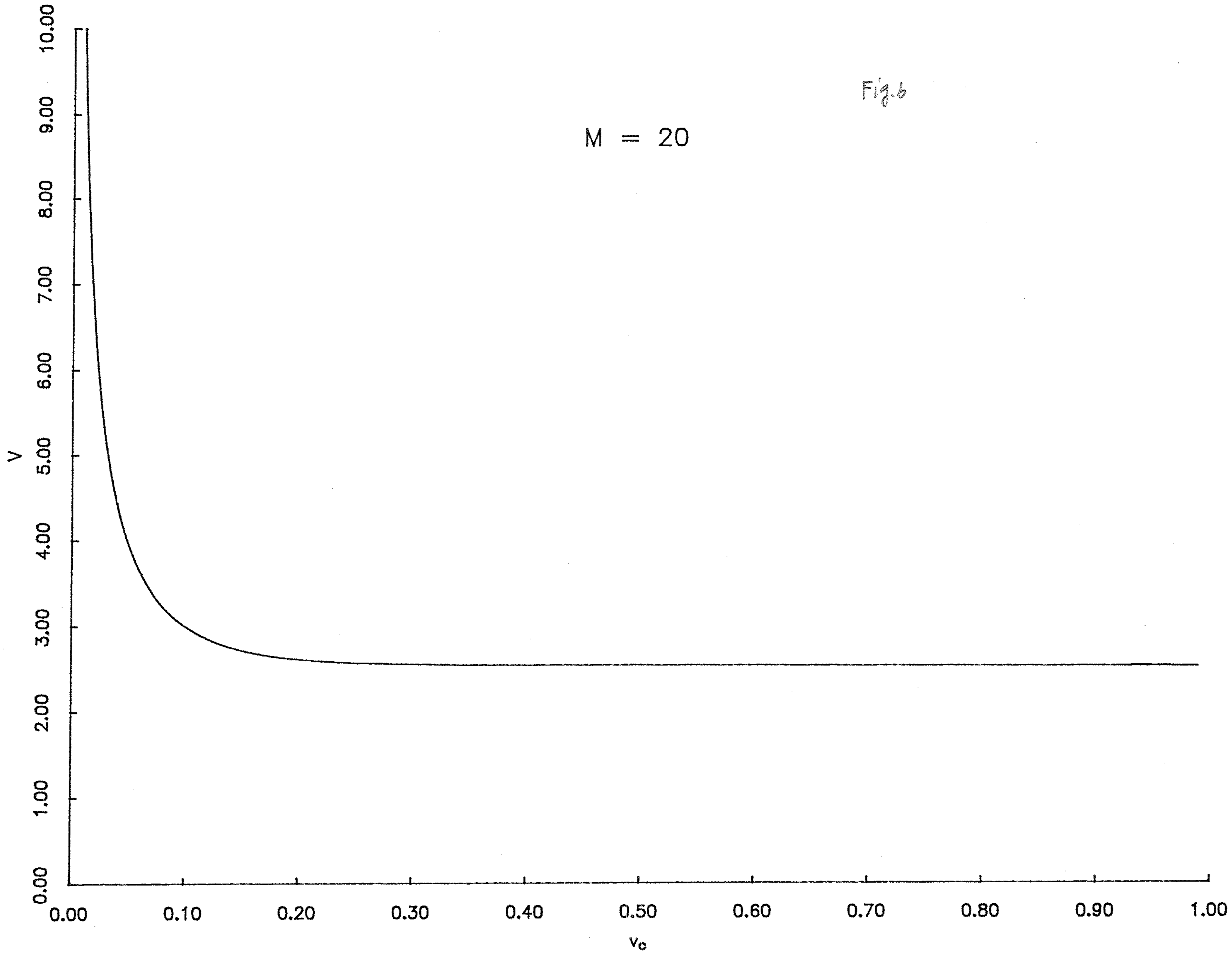

0.200
0.180
0.160
0.140
0.120
0.100
0.080
0.060
0.040
0.020
0.000
λ
10.0
15.0
20.0
25.0
30.0
M
numerical
Analytic

Fig.6


M = 20
V
10.00
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
v_c

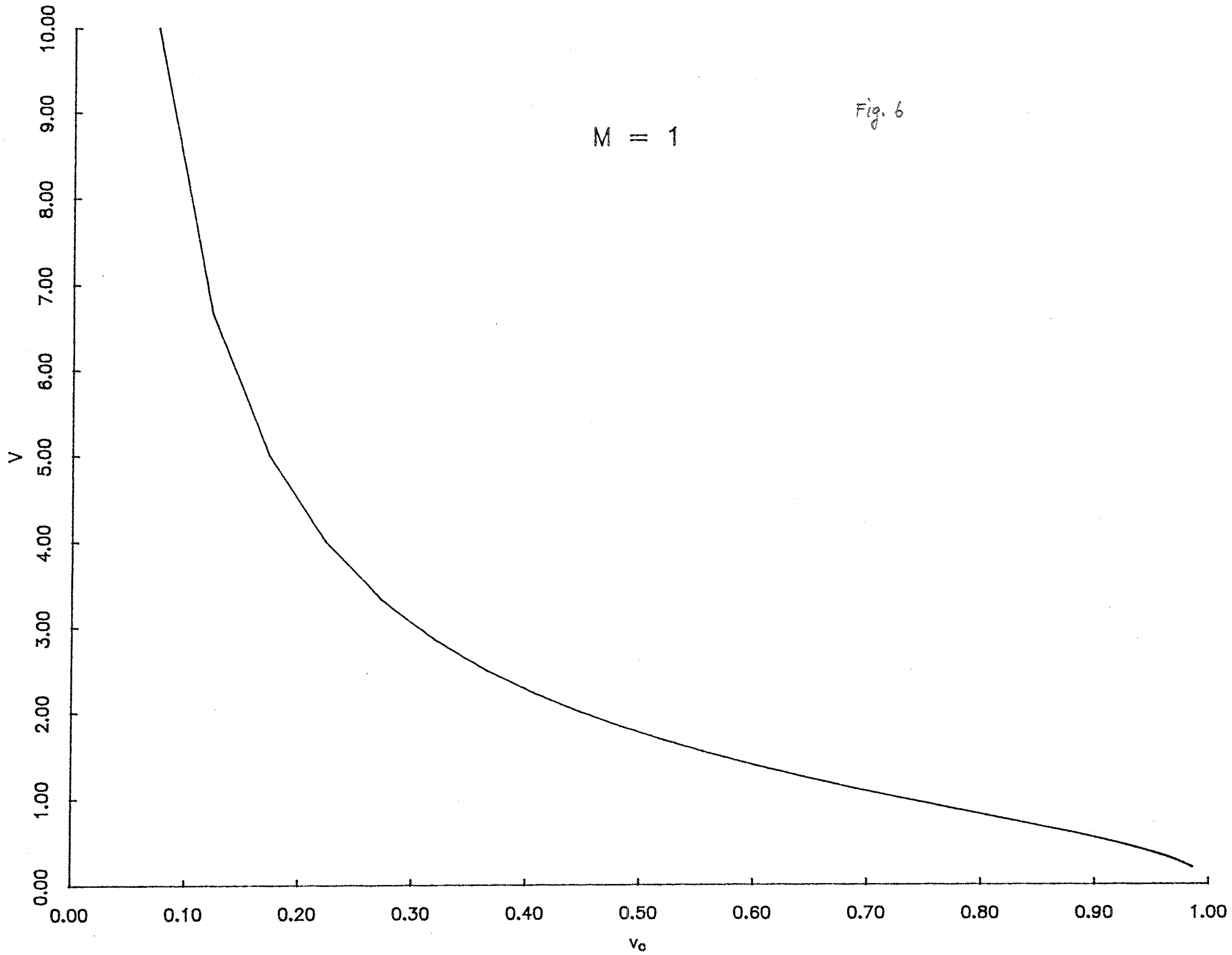
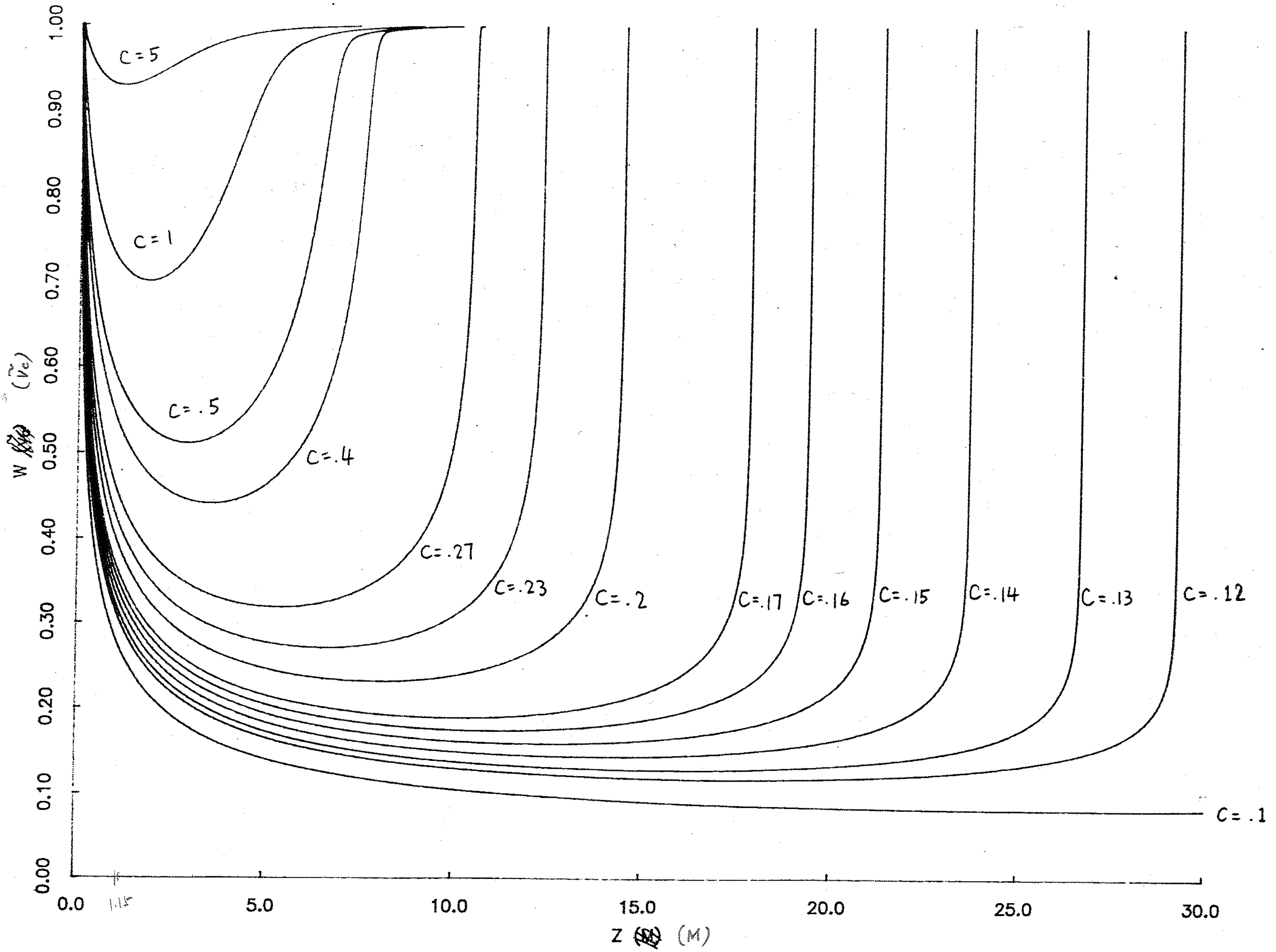

M = 1
V
10.00
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
v_o

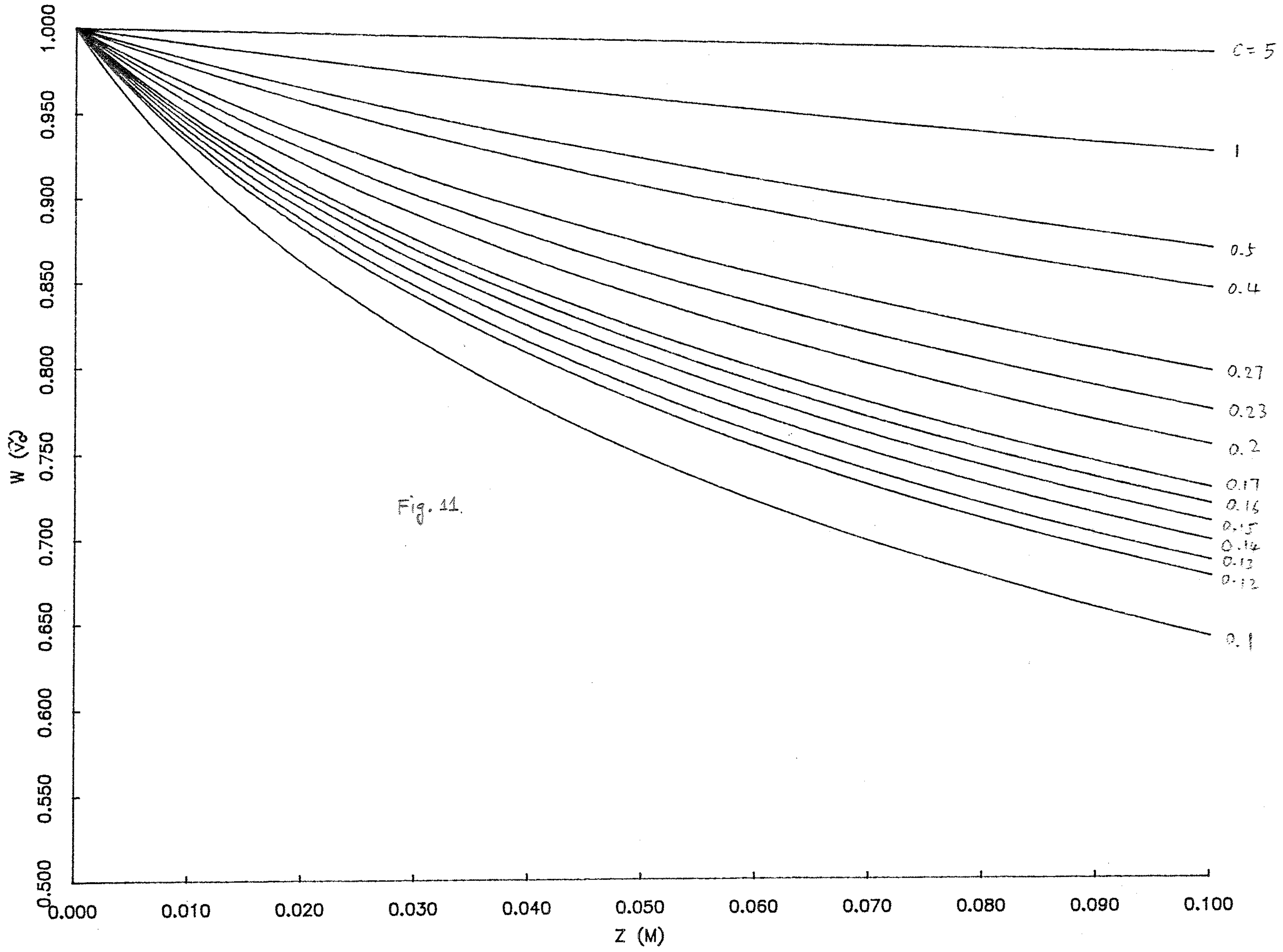
Fig. 6

Fig.5.

(Fig.5)

$C = M\lambda^2$

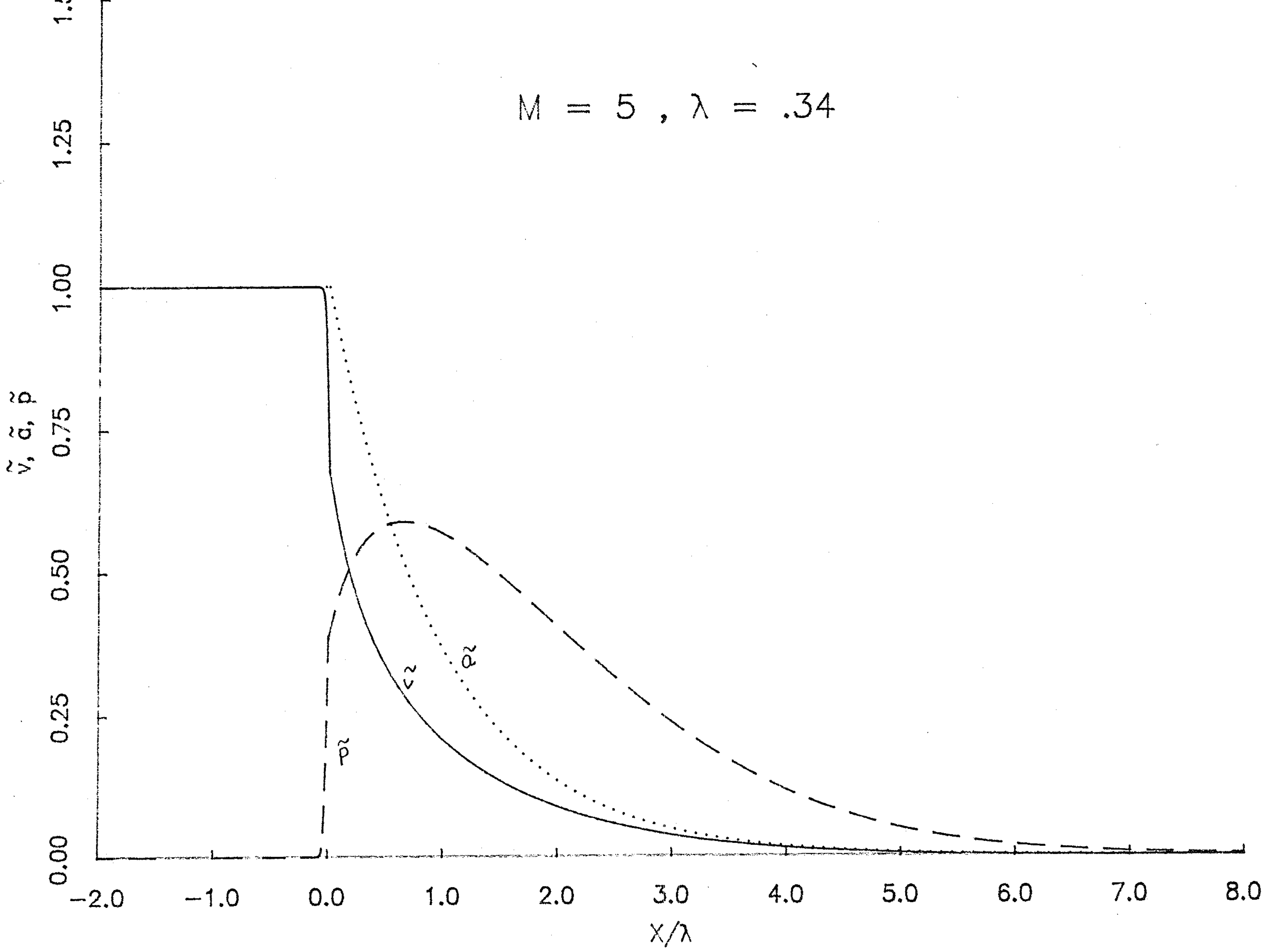


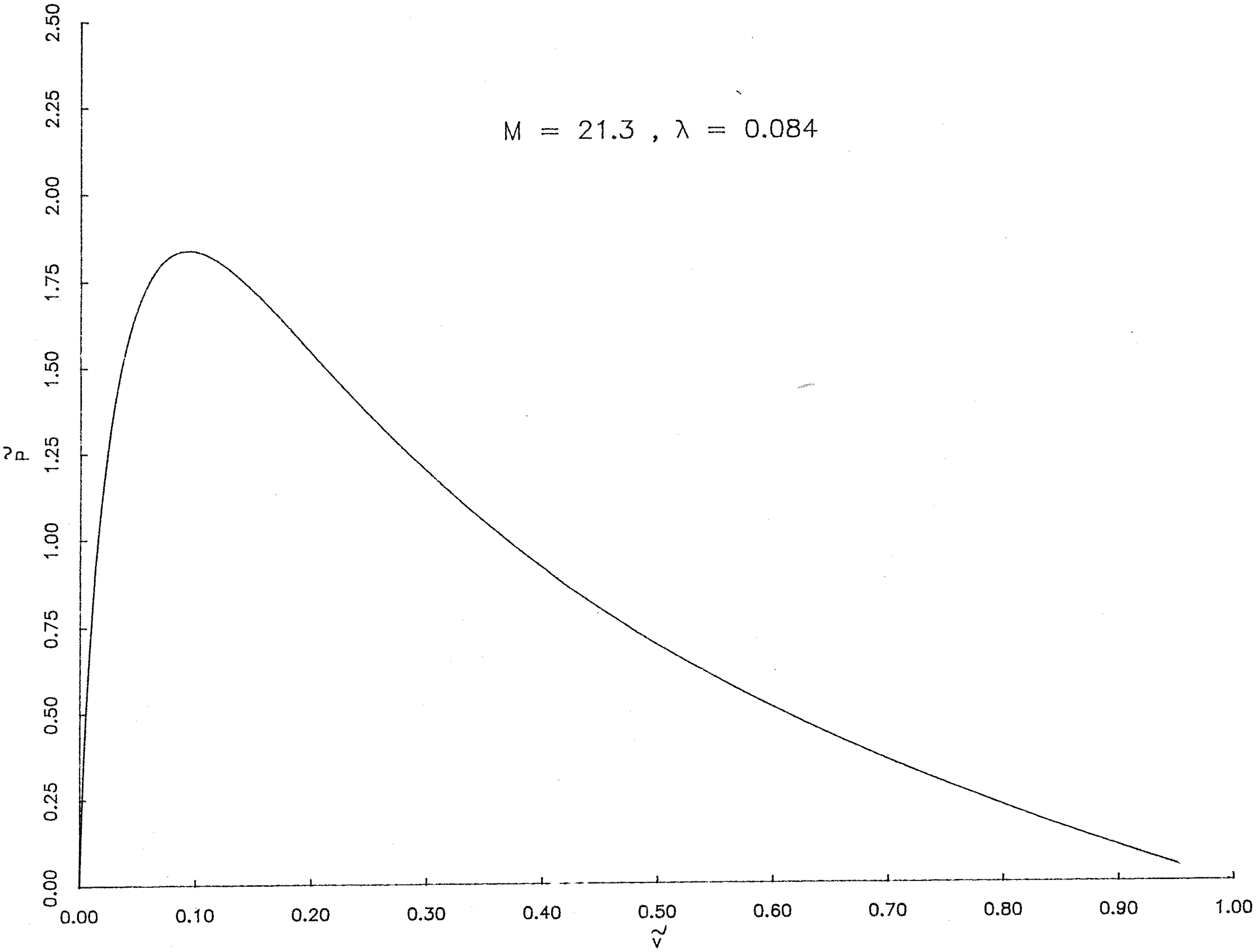


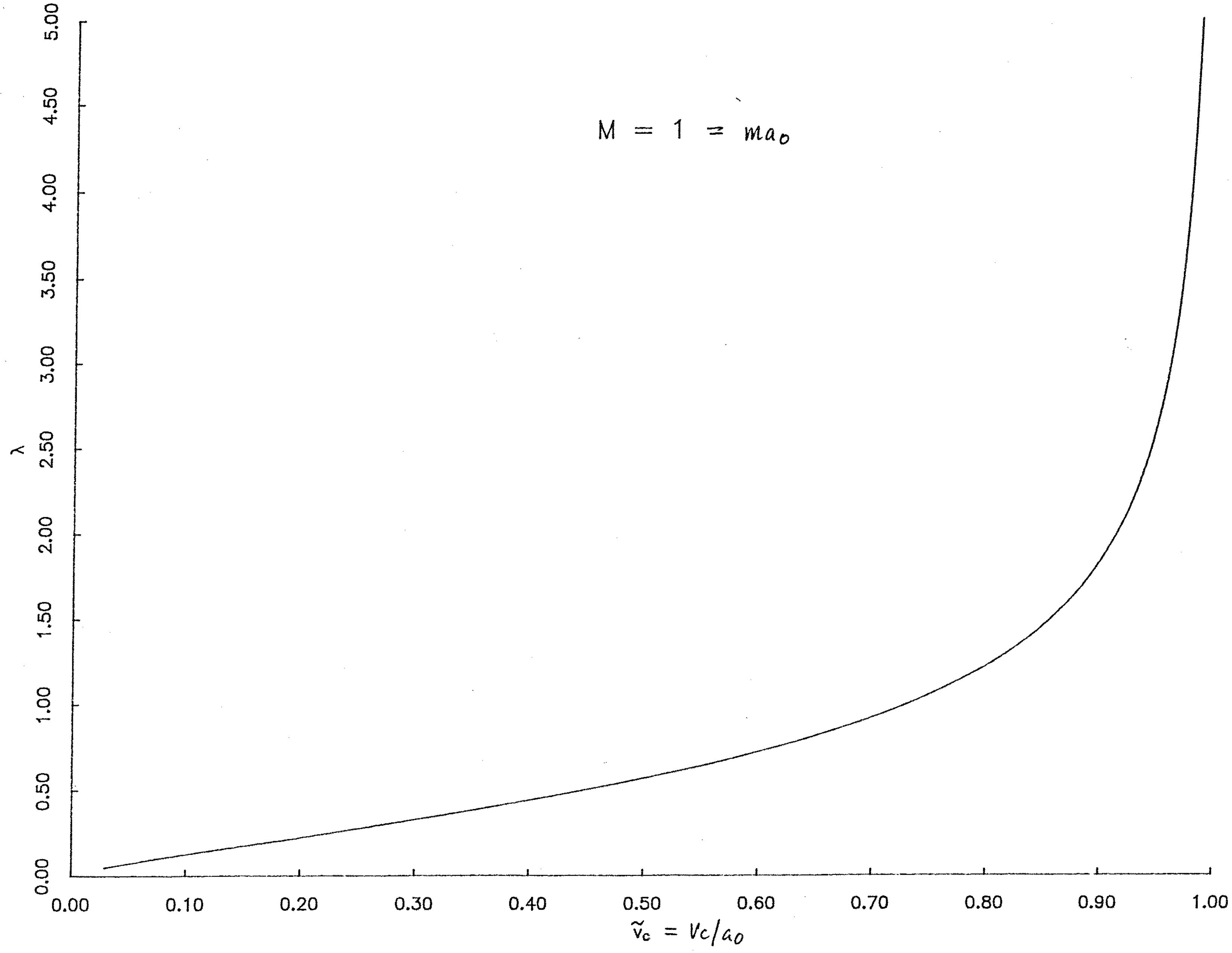
1.000
0.950
0.900
0.850
0.800
0.750
0.700
0.650
0.600
0.550
0.500
W (V)
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
0.080
0.090
0.100
Z (M)
C = 5
1
0.5
0.4
0.27
0.23
0.2
0.17
0.16
0.15
0.14
0.13
0.12
0.1

Fig. 11.

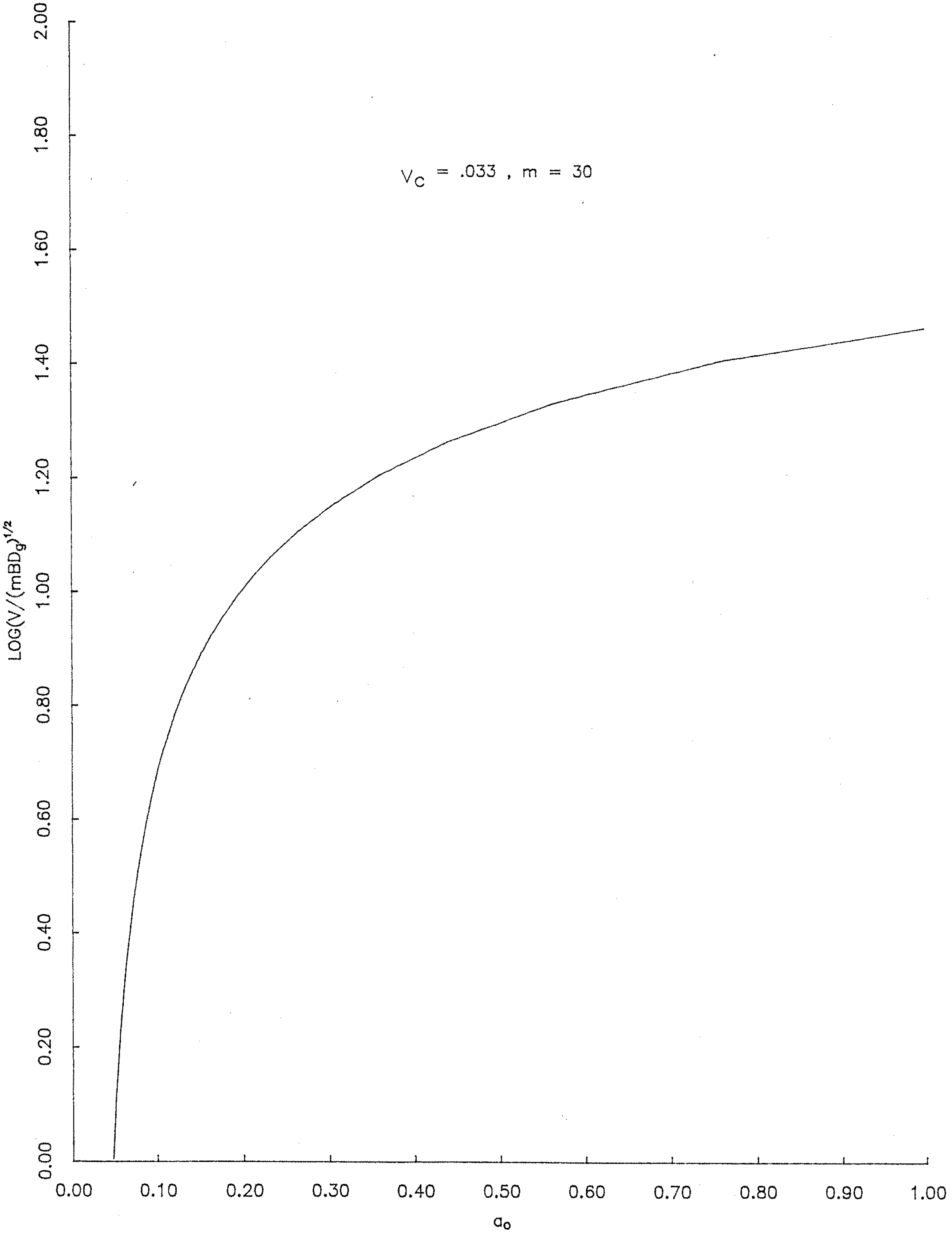
Concentration, Activity and Pressure Profiles

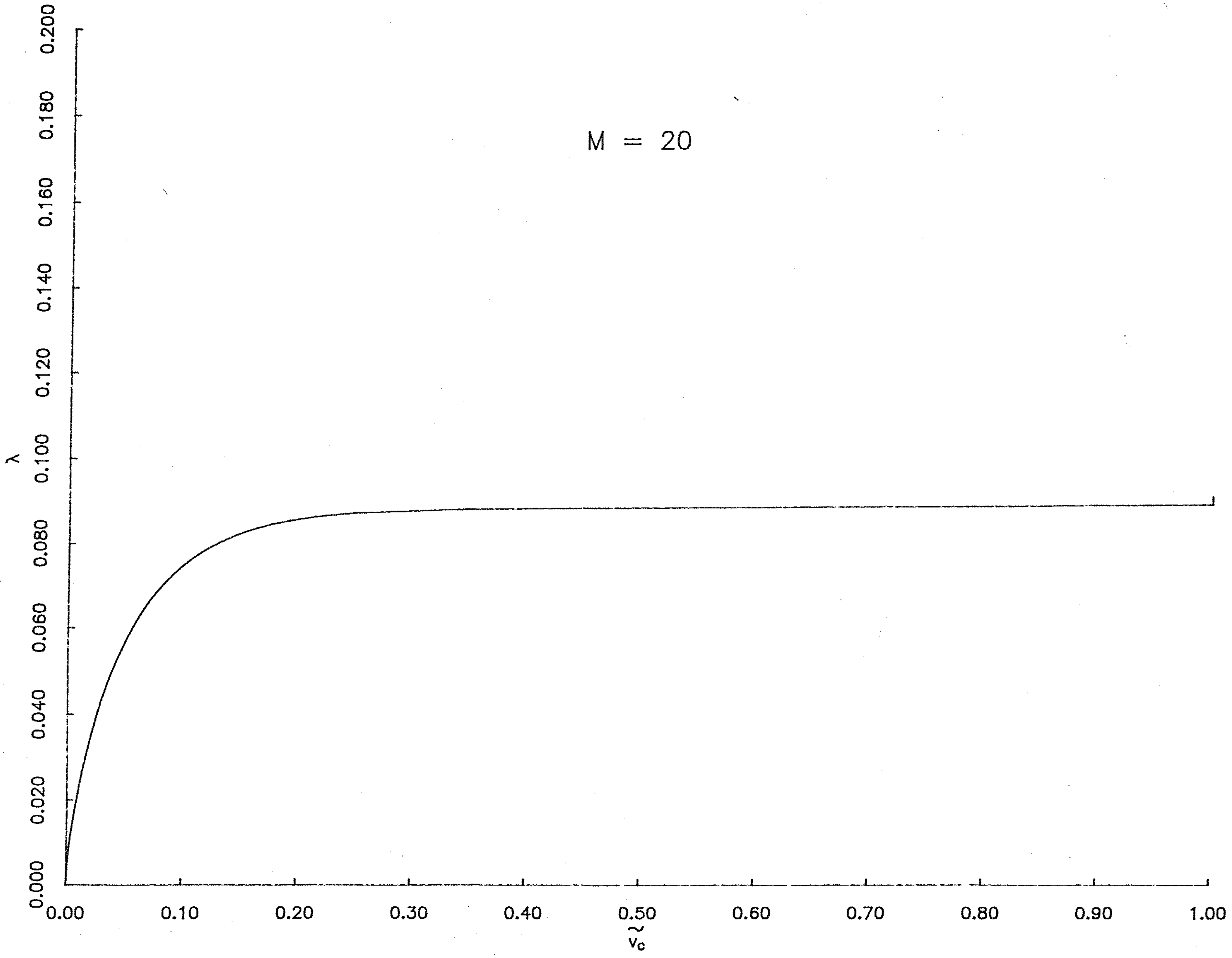



M = 21.3 , λ = 0.084
p̃
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
ṽ


M = 1 = ma_o
λ
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
5.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
$\tilde{v}_c = Vc/a_0$

(100)


V_C = .033 , m = 30
LOG(V/(mBD_g)^1/2
2.00
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
a_o


M = 20
λ
0.000
0.020
0.040
0.060
0.080
0.100
0.120
0.140
0.160
0.180
0.200
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
$\tilde{v}_c$