1st order ODEs

M294 P II FA90 #4

2) (a) $\frac{dy}{dx} = x^2(e^{-x^3} - 3y)$

put into normal form: $\frac{dy}{dx} + 3x^2 y = x^2 e^{-x^3}$. $R = e^{\int p\,dx} = e^{\int 3x^2 dx} = e^{x^3}$

So: $ye^{x^3} = \int x^2 dx + C = \frac{x^3}{3} + C$, $y = e^{-x^3}\left(\frac{x^3}{3} + C\right) = \frac{x^3}{3}e^{-x^3} + Ce^{-x^3}$

(b) $\frac{dy}{dx} - y = 3 + 2e^x$. $R = e^{\int -1\,dx} = e^{-x}$, so $ye^{-x} = \int (3e^{-x} + 2)\,dx + C$,

$y = e^x(-3e^{-x} + 2x + C) = -3 + 2xe^x + Ce^x$.

M294 P I FA92 #1

9)

a) $y' + 2xy = e^{-x^2}$ integrating factor is e^{x^2}:

$(e^{x^2} y)' = 1$

$e^{x^2} y = c + x$

$\boxed{y = (c+x)e^{-x^2}}$ gen. soln.

b) $y(0) = 2 = (c+0)e^0$, $c = 2$, $\boxed{y = (2+x)e^{-x^2}}$

M294 P II FA93 #1

14) $y^{-3}dy = dx$, $-\frac{1}{2}y^{-2} = x + C$, $y = \frac{\pm 1}{\sqrt{-2x - 2C}}$.

For $y(0) = -4$ choose $C = -\frac{1}{32}$ and the minus sign

$\boxed{y = \frac{-1}{\sqrt{-2x + \frac{1}{16}}}}$ recheck: $y' = -\left(-\frac{1}{2}\right)\left(-2x + \frac{1}{16}\right)^{-3/2}(-2) \stackrel{\checkmark}{=} y^3$.

$-2x + \frac{1}{16} \geq 0$ makes $\boxed{x \leq \frac{1}{32}}$

M294 P II SP95 #1a.

21) $xy' + 2y = \sin x$

or $y' + \frac{2}{x}y = \frac{\sin x}{x}$ —(1) to turn into standard form for 1ST order linear ODE
restrict $x > 0$ so that $\frac{2}{x} \not\to \infty$ @ $x = 0$

Find integrating factor

$$\int \frac{2\,dx}{x} = 2\ln x + \text{const} = \ln x^2 \qquad \therefore \text{ multiply by } e^{\ln x^2} = x^2$$

So (1) is $x^2y' + 2xy = x\sin x$

or $\frac{d}{dx}(x^2y) = x\sin x$

$\therefore\ x^2y = \int x\sin x + k = -x\cos x + \sin x + k$

-- .....So $\boxed{y = (k - x\cos x + \sin x)/x^2}$

To match I.C.

$$y(\tfrac{\pi}{2}) = 1 = (k - 0 + 1)/(\tfrac{\pi}{2})^2 \Rightarrow k = \frac{\pi^2}{4} - 1$$

$$\therefore\ \boxed{y = x^{-2}\left[(\tfrac{\pi^2}{4}) - 1 - x\cos x + \sin x\right]}$$

M293 P I FA95 #5

22) (a) $\left.\begin{array}{l}\frac{dy}{dx} = y^3 \\ y(0) = 1/2\end{array}\right\}$ separable

$y^{-3}dy = dx$ separate

$\frac{-y^{-2}}{2} = x + C$ integrate

$y = \pm(-2x - 2C)^{-1/2}$

$y(0) = 1/2$ forces the + sign and $C = -$

$$\boxed{y = \frac{1}{\sqrt{4-2x}}}$$

(b) $y = (4-2x)^{-\frac{1}{2}}$

$y(0) = 4^{-1/2} = \frac{1}{\sqrt{4}} = \boxed{\tfrac{1}{2}}$ ✓

$y' = -\frac{1}{2}(4-2x)^{-3/2}(-2) = (4-2x)^{-3/2}$ $\boxed{\overset{?}{=} y^3 \text{ yes}}$ ✓

(c) $\frac{dy}{dx} - e^{-x} = -4y$, $y(0) = 1$ is 1st order linear

$y' + 4y = e^{-x}$ multiply by $e^{\int 4\,dx} = e^{4x}$

$\underbrace{e^{4x}y' + 4e^{4x}y}_{= (e^{4x}y)'} = e^{3x}$

integrate $e^{4x}y = \int e^{3x}dx = \frac{1}{3}e^{3x} + C$

$y = \frac{1}{3}e^{-x} + Ce^{-4x}$

use I.C. $y(0) = 1 = \frac{1}{3} + C$, $C = \frac{2}{3}$, $\boxed{y = \frac{1}{3}e^{-x}}$

(d) $y' + 4y = -\frac{1}{3}e^{-x} - \frac{2}{3}\cdot 4e^{-4x} + \frac{4}{3}e^{-x} + \frac{8}{3}e^{-4x}$ $\boxed{= \frac{3}{3}e^{-x}}$ ✓

$y(0) = \frac{1}{3} + \frac{2}{3}$ $\boxed{= 1}$ ✓

M294 P II FA95 #2

23)

(a) $y' - 3y = e^{-x}$ integrating factor is $e^{-3x}(y' - 3y) = e^{-3x}$

$(ye^{-3x})' = e^{-4x}$ $\qquad ye^{-3x} = \int e^{-4x}dx = -\frac{1}{4}e^{-4x}$

$y = -\frac{1}{4}e^{-x} + Ce^{3x}$ $\qquad y(0) = 5 = -\frac{1}{4} + C \;,\; C = \frac{21}{4}$

$$\boxed{y = -\frac{1}{4}e^{-x} + \frac{21}{4}e^{3x}}$$

M293 F SP96 #26

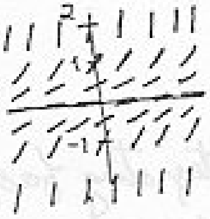
26) The answer is b).

M293 PI FA96 #4

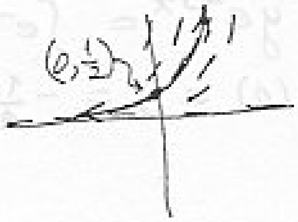
27) The only solution is $y(x) = \begin{cases} \frac{1}{4}(x-1)^2 & \text{if } 1 \le x \le 3 \\ 0 & \text{if } 0 \le x \le 1 \end{cases}$

M294 PI FA92 #2

30) $\boxed{a}$ $y' = y^2$



$\boxed{b}$



2c $y^{-2}y' = 1$ separable

$(-y^{-1})' = 1$

$-y^{-1} = x + c$

$-(\frac{1}{2})^{-1} = 0 + c, \quad c = -2$

$-y^{-1} = x - 2$

$\boxed{y = \frac{1}{2-x}}$

M294 F FA94 #1b

31b) $\underbrace{y' - y}_{p=-1} = \underbrace{2xe^{2x}}_{\text{"g"}}$ solve using integrating factors.

$\mu(x) = e^{-x}$ $\quad y(x) = \dfrac{\int \mu(x) g(x)\,dx + C}{\mu(x)} = \dfrac{\int e^{-x} 2xe^{2x}\,dx + C}{e^{-x}}$

$y(x) = \dfrac{2\int xe^{x}\,dx + C}{e^{-x}} = \dfrac{2(x-1)e^{x}}{e^{-x}} + \dfrac{C}{e^{-x}} = \boxed{2(x-1)e^{2x} + Ce^{x}}$

Check, $y' = 2e^{2x} + Ce^{x} + 4(x-1)e^{2x}$

$y' - y = 2e^{2x} + Ce^{x} + 4(x-1)e^{2x} - 2(x-1)e^{2x} - \cancel{Ce^{x}} = 2xe^{2x}$

$(2x - 2 + 2)e^{2x}$

$y(0) = 1 = 2(-1) + C \therefore C - 2 = 1 \therefore C = 3$

$\therefore \boxed{y(x) = 2(x-1)e^{2x} + 3e^{x}}$

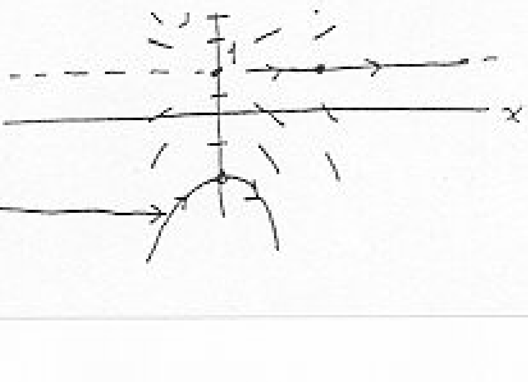
M294 PII FA94 #1

32)

) $y' = x(y-1)$

$y(3) = 1$ gives $\boxed{y = 1}$

$y(0) = -2$ gives $\boxed{y = 1 - 3e^{x^2/2}}$



M294 PII SP95 #33

GRAPH 1

33 a.) $y' = x - y \Rightarrow$ slope is 0 on $x - y = 0$

Also y' nowhere ∞

only graph 1 satisfies

We suspect iv.) since Graph 1 looks exponential

To check iv.) differentiate it

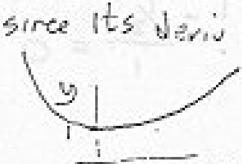
$\frac{d}{dx}[(x-y-1)e^x = c] \Rightarrow (1-y')\cancel{e^x} + (x-y-1)\cancel{e^x} = 0$

$1 - y' + x - y - 1 = 0$

$\therefore y' = x - y$

Can also easily solve eqn since it is 1st-order linear...

It is soln since its deriv is eqn



To solve for C

$(x-y-1)e^x = C = (0-1-1)e^0$

$\underline{\underline{C = -2}}$

b.) $y' + y^2 \sin x = 0$

slope is sinusoid & goes to 0 @ $x = 0, \pi, \ldots \Rightarrow$ Graph 4

Solution likely involves trig fn, so it is either i.) or iii)

iii) has soln $y = 0$, which fits graph 4 & DE ii.) $\therefore$ iii

Can also solve eqn since it is separable $y' = -y^2 \sin x \Rightarrow \frac{-dy}{y^2} = -\sin x\, dx$

$+\frac{1}{y} + \cos x = C$

$y^{-1} + \cos x = C = \frac{1}{1} + \cos 0 = C \quad \underline{\underline{C = 2}}$

M294 PII FA96 #1

35) ⓐ $y' = y^3$ is separable

$$\frac{dy}{y^3} = dx$$

$$-\frac{1/2}{y^2} = x + C$$

Applying initial condition,

$$-\frac{1/2}{1} = 0 + C$$

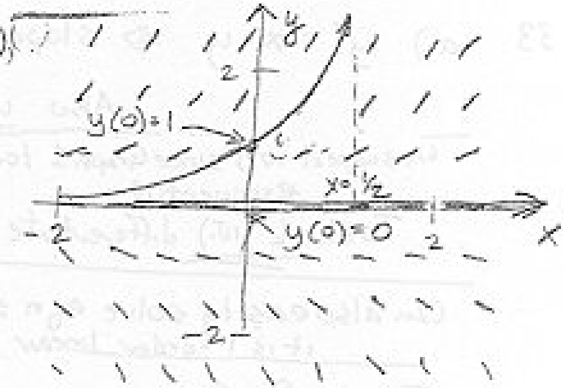
$$\therefore C = -1/2$$

$$\therefore \frac{-1/2}{y^2} = x - \frac{1}{2}$$

or

$$y^2 = \frac{-1/2}{x - 1/2} = \frac{1}{1-2x} \Rightarrow y = \pm\sqrt{\frac{1}{1-2x}}$$

blows up at $x = 1/2$

ⓑ 

The soln through $y(0)=0$ is $y=0$

ⓒ Solns exist & are unique since y^3 is continuous in neighborhood of (0,1) and (0,0)

$\frac{\partial f}{\partial y} = 3y^2$ also continuous

but soln for $y(0)=1$ only exists up to $x = 1/2$. Undefined for $x > \frac{1}{2}$

M294 PII SP96 #1

36 a) $x' + x = \cos 3t$ $\quad ' = \frac{d}{dt}$

Since this is a linear 1st order ODE, we can use an integrating factor to solve. Or we can solve the homogeneous equation and find a particular soln.

$x_0 = Ce^{-t}$

Try $x_p = A\cos 3t + B\sin 3t$
$x_p' = -3A\sin 3t + 3B\cos 3t$ $\Big\}$ $x_p' + x_p = (-3A + B)\sin 3t + 3B\cos 3t + A\cos 3t = \cos 3t$

$\therefore A = \frac{1}{3}B$ and $3B + \frac{B}{3} = 1$

$B = \frac{3}{10}$

$A = \frac{1}{10}$

$x = Ce^{-t} + \frac{1}{10}\cos 3t + \frac{3}{10}\sin 3t$

at $t=0$ $\quad x_0 = C + \frac{1}{10}$

$$\therefore \boxed{x = (x_0 - \tfrac{1}{10})e^{-t} + \tfrac{1}{10}\cos 3t + \tfrac{3}{10}\sin 3t}$$

For $t \to \infty$, this term vanishes and the solution is a simple oscillation. It is the same oscillation for all starting conditions since I.C. term $\to 0$ with period $3t = 2\pi$, $T = \frac{2\pi}{3}$

b) along $x = 0$ the slope field oscillates
at large $(\pm x)$, the slope field is $\binom{-}{+}$
at $x = 1$, slope $= 0$ for $t = 0$
$x = -1$ $\quad -2$ $\qquad$ Thus $\boxed{\text{ODE \#1 = FIELD A}}$

c. $x' + \frac{1}{2}x = 2\sin 6t$ $\quad$ Exponentially decaying soln decays to oscillation 2x as fast as $\cos 3t$

slope 0 @ $t = 0$, $x = 0$

$\boxed{\text{ODE \#2 = FIELD C}}$

$x' - x = \cos 3t$ has exponentially growing solutions.
slope field on $t = 0$ looks like the one in magnify a.)
so @ $x = -1$, $t = 0 \Rightarrow 0$ $\therefore$ $\boxed{\text{ODE \#3 = FIELD D}}$

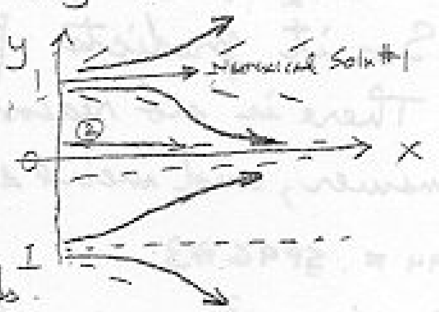
Solutions are clearly not linear
for FIELD B, it must be ODE #4

M294 PII FA96 #2

37) (a) $y' = y^3 - y = y(y^2 - 1) = y(y+1)(y-1)$

Equilibrium solutions when $y' = 0$

or $\begin{cases} y = 0 \\ y = \pm 1 \end{cases}$



$y = 0$ stable, $y = \pm 1$ unstable from slopefields.

(b) Euler Method $\qquad y_{n+1} = y_n + h(y_n^3 - y_n)$

$y(0) = 1$	$y(0) = .1$
$y_0(0) = 1$	$y_0(0) = 0.1$
$y_1(.5) = 1 + \frac{1}{2}(1^3 - 1) = 1$	$y_1(.5) = .1 + \frac{1}{2}(.001 - .1) = .050$
$y_2(1) = 1 + \frac{1}{2}(1^3 - 1) = 1$	$y_2(1) = .05 + \frac{1}{2}(.05^3 - .05) = .025$

≈ 0

M294 F FA92 #1

44) (a)
$$y' + ty^2 = 0$$
$$y^{-2}y' + t = 0$$
$$-y^{-1} + \frac{t^2}{2} = C$$
$$y = \left(\frac{t^2}{2} - C\right)^{-1}$$
$$y(0) = -C^{-1} = 1 \text{ so } C = -1$$
$$\boxed{y(t) = \left(\frac{t^2}{2} + 1\right)^{-1}}$$

(b) $y_{n+1} = y_n - n\Delta t y_n^2$

$y_0 = 1$

$y_1 = 1 - 0 = 1$

$y_2 = 1 - 1(.1)(1)^2 =$

y_0	y_1	y_2
1	1	

M294 F FA95 #1

51) $y' = -ry$, $y(t) = y(0)e^{-rt}$,

(a) $y(6000) = \frac{1}{2}y(0) = y(0)e^{-r \cdot 6000}$ so $\boxed{r = \frac{\ln 2}{6000}}$

(b) $\boxed{y(t) = y_0 e^{-\left(\frac{\ln 2}{6000}\right)t}}$

(c) $y(age) = \frac{10}{100}y_0 = y_0 e^{-\left(\frac{\ln 2}{6000}\right)age}$ so $\boxed{age = \frac{\ln 10}{\ln 2/6000}}$

M294 PI FA96 #2

53) (a)

```
x = 5
for k = 1 : 15
   x = x + math_is_fun(x);
end
'The answer is', x
```

```
function xdot = math_is_fun(x);
xdot = .2 * (2*x + 1)
```

other things work also

(b) $x_0 = 5$, $x_1 = 5 + .5*(2*5+1) = 10.5$

$x_2 = 10.5 + .5*(2*10.5 + 1) = 21.5$

So it predicts $\boxed{21.5000}$

(c) There is no reason why this should agree with the exact answer, and we've seen examples in homework where it doesn't.

M294 F, SP96 #3

55) (a) $y' = ay^2$, seperable, $y(t) = \frac{-1}{at + C}$ (b) $y(0) = 200$ gives $C = \frac{-1}{200}$ and $y(3) = 500$ gives $a = \frac{1}{1000}$

(c) $y(t) = \frac{1000}{5-t} \to \infty$ as $t \to 5^-$

M293 P I FA96 # 3

58) let $T(t)$ = coffee temperature at time t minutes after 10:00 AM.

$\frac{dT}{dt} = -k(T-65)$ is Newton's Law of Cooling.

The solution is of the form $T(t) = Ce^{-kt} + 65$. Then

$T(0) = 145 = C + 65$ ~~so~~ $C = 80$

$T(10) = 85 = 80e^{-k\cdot 10} + 65$ ~~so~~ $k = -\frac{1}{10}\ln\left(\frac{20}{80}\right)$

$T(5) = 80e^{-\left(-\frac{1}{10}\ln\left(\frac{20}{80}\right)\right)\cdot 5} + 65$

$= 80e^{\frac{5}{10}\ln\frac{1}{4}} + 65 = 80\left(\frac{1}{4}\right)^{\frac{5}{10}} + 65 = 80\cdot\frac{1}{2} + 65 = \boxed{105}$

M294 P II FA94 # 4

59) let $T(t)$ = temperature of pizza at t minutes after 6:00 pm

The assumption is that

$$\frac{dT}{dt} = k(T-100)$$

for some constant k.

Solution is $T(t) = 100 + Ce^{kt}$

We know at 6:00pm, $T(0) = 500 = 100 + Ce^{0}$, so $C = 400$.

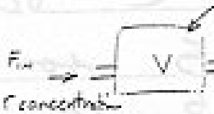
We know at 6:15, $T(15) = 400 = 100 + 400e^{k\cdot 15}$, so $k = \frac{1}{15}\ln\frac{300}{400}$

We know at 600°, $T(t_0) = 600 = 100 + 400e^{kt_0}$, so the unknown

time is $t_0 = \frac{1}{k}\ln\frac{500}{400} = \boxed{\frac{15\ln\frac{5}{4}}{\ln\frac{3}{4}} \text{ minutes before 6:00}}$

M294 P2 SP96 #3

62)



The lake's volume is constant, $\frac{dV}{dt}$, since flow in and out are the same.

Green $\boxed{\frac{dG}{dt} = \frac{-G}{V} F}$ (1) $G(t=0) = G_0$

$$\frac{kg}{sec} = \left(\frac{kg}{m^3}\right)\left(\frac{m^3}{sec}\right)$$

Red $\boxed{\frac{dR}{dt} = rF - \frac{R}{V}F}$ (2) $R(t=0) = 0$

Sol. to (1) $\boxed{G = G_0 e^{-\frac{F}{V}t}}$

Sol. to (2) $\frac{dR}{dt} + \frac{F}{V}R = rF$ 1st order linear with integrating factor $e^{\frac{F}{V}t}$

$$\frac{d}{dt}\left(Re^{\frac{F}{V}t}\right) = rFe^{\frac{F}{V}t}$$

or

$$Re^{\frac{F}{V}t} = rF\int e^{\frac{F}{V}t}dt = rVe^{\frac{F}{V}t} + C$$

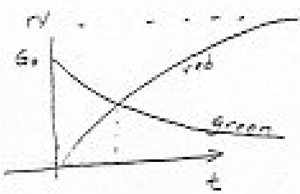
or

$$R = rV + Ce^{-\frac{F}{V}t}$$

$$0 = rV + C \Rightarrow C = -rV$$

$$\therefore \boxed{R = \left(1 - e^{-\frac{F}{V}t}\right)rV}$$

kg $\quad \frac{m^3}{sec\, m^3} sec \quad \frac{kg}{m^3} m^3$



Red > Green

when $\left(1 - e^{-\frac{F}{V}t}\right)rV > G_0 e^{-\frac{F}{V}t}$

or $e^{\frac{F}{V}t} - 1 > \frac{G_0}{rV}$

or $\boxed{t > \frac{V}{F}\ln\left(\frac{G_0}{rV} + 1\right)}$

M294 PII FA93 #2

63) Let $y(t)$ = the mass of salt in the tank after t minutes.

Then $\begin{cases} y(0) \text{ is unknown} \\ y(5\cdot 60) = 20\cdot 1{,}000 \text{ grams} \end{cases}$

And $y'_{\frac{g}{m}}$ = rate in − rate out $= 0 - 3\cdot\frac{y}{1000}\frac{g}{l}$. $\Bigg\}$ Find $\frac{y(0)}{1000}$.

So $y(t) = y(0)\, e^{-\frac{3t}{1000}}$

$y(5\cdot 60) = y(0)\, e^{-\frac{3\cdot 5\cdot 60}{1000}} = 20\cdot 1000$

→ $\boxed{\text{Initial concentration } \frac{y(0)}{1000} = 20\, e^{\frac{9}{10}} \text{ grams/liter}}$