**MATH 294 SPRING 1996 PRELIM 2 # 8**

**5.2.1** Consider the PDE $u_t = -6u_x$

**a)** What is the most general solution to this equation you can find?

**b)** Consider the initial condition $u(x,0) = \sin(x)$. What does $u(x,t)$ look like for a very small but not zero t?

**MATH 294 SPRING 1983 PRELIM 3 # 5**

**5.2.2** Consider $u_t = u_x$. Which of the functions below are solutions to this equation? (Show your reasoning.)

**a)** $3e^{-\lambda t}\sin\sqrt{\lambda t}$

**b)** $3\ e^{-3t}e^{-3x} + 5e^{-5t}e^{-5x}$

**c)** $ae^{-3t}e^{-5x}$

**d)** $\sin(x)\cos(t) + \cos(x)\sin(t)$

**e)** $sinh^{-1}\left[(x+t)^3\right]$.

**MATH 294 SPRING 1984 FINAL # 12**

**5.2.3** Determine if the following equation is of the form of a linear partial differential equation. If not, explain why.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0.$$

**MATH 294 SPRING 1984 FINAL # 13**

**5.2.4** Verify that the given function is a solution of the given partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0, u(x,y) = f\left(\frac{y}{x}\right), x = 0$$

$f(\cdot)$ is a differentiable function of **one** variable.

**MATH 294 FALL 1991 PRELIM 2 # 1**

**5.2.5** **a)** Find the general solution of
$(y + 2x\sin y dx + (x + x^2\cos y)dy = 0$.

**b)** Determine the solution of the initial-value problem
$(x^2+4)\frac{dy}{dx} + 2xy = x$, with $y(0) = 0$.

**MATH 294 SPring 1992 FINAL # 9**

**5.2.6** Consider the initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < L, t > 0$$

with the boundary conditions

$$u(0,t) = u(L,t) = 0, t \geq 0$$

and the initial condition

$$u(x,0) = f(x), 0 \leq x \leq L.$$

**a**) Use the method of separation of variables to derive the solution of this problem. You may use the fact that the equation $\dot{X} + \lambda X = 0, 0 < x < L$ with the boundary condition $X(0) = X(L) = 0$ has nontrivial solutions only for an interface number of constants $lambda_n = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \ldots$. This corresponding solutions are of the form $X_n = A_n \sin \frac{n\pi x}{L}$.

**b**) Find the solution when $L = 1$ and $f(x) = -6 \sin 4\pi x + \sin 7\pi x$.

**MATH 294 FALL 1992 FINAL # 4**

**5.2.7** For the PDE $u_x + 4u_y = 0$:

**a**) Solve it by separation of variables.

**b**) Show that any function of the form $u(x, y) = f(ax + y)$ is a solution if $f$ is differentiable and the constant $a$ is chosen correctly.

**c**) Solve the PDE with boundary condition $u(x, 0) = \cos x$. (You may use (b) rather than (a).)

**MATH 294 SPRING 1994 FINAL # 3**

**5.2.8** Let $D$ be a region in the $(x, y)$ plane and $C$ be its boundary curve with counter-clockwise orientation. If the function $u(x, y)$ satisfies $u_{xx} + u_{yy} = 0$ in $D$, show that

$$\oint_C uu_x dy - uu_y dx = \int\int_D \left(u_x^2 + u_y^2\right) dxdy.$$

**MATH 294 SPRING 1984 FINAL # 15**

**5.2.9** Show that the partial differential equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu\right)$$

can be reduced to

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$$

by setting $u(t, x) = e^{\alpha x + \beta t} v(t, x)$ and choosing the constants $\alpha$ and $\beta$ appropriately.

**MATH 294 SPRING 1988 PRELIM 2 # 3**

**5.2.10** Find **one** non-zero solution to the equation below. Do not leave any free constants in your solution (that is, assign some specific numerical values to any constants in your solution). Note that you do not have to satisfy any specific initial conditions or boundary conditions.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

)

**MATH 294 FALL 1993 PRELIM 3 # 3**

**5.2.11** **a**) Find the two **ordinary** differential equations that arise from the **partial** differential equation
$\alpha^2 u_{xx} = utt$ for $0 < x < \ell, t \geq 0$ when the equation is solved by separation of variables using a separation constant $\sigma = -\lambda^2 < 0$.

**b**) Solve the ordinary differential equation which gives the $x$ dependence. Use the boundary conditions $u(0,t) = u(\ell,t) = 0$ for $t \geq 0$

**c**) Solve the ordinary differential equation which gives the time dependence.

**MATH 294 FALL 1994 PRELIM 3 # 2**

**5.2.12** Given the partial differential equation (P.D.E)

$$u_x + u_{yy} + u = 0,$$

**a**) Use separation of variables to replace the equation with two ordinary differential equations.

**b**) Find a non-zero solution to the P.D.E.

**MATH 294 SPRING 1995 FINAL # 4**

**5.2.13** Consider the *first order* partial differential equation

$$u_t + cu_x = 0, -\infty < x < \infty, o < t < \infty, (*)$$

where $c$ is a constant. We wish to solve this in two different ways.

**a**) Find a general solution to (*) by first writing the equation with the change of variables, $\xi = x - ct, \eta = t$.

**b**) Now solve (*) using a separation of variables technique. What are the units of $c$ if $x$ is in meters and $t$ is in second?

**c**) Find $u(x,t)$ if $u(x,0) = ke^{-x^2}$.

**d**) Discuss the nature of your solution.

**MATH 294 FALL 1995 PRELIM 3 # 3**

**5.2.14** While separating variables for a PDE, Professor X was faced with the problem of finding positive numbers $\lambda$ and functions $X$ which are not identically zero and

$$X''(x) = -\lambda X(x)$$

$$X(0) = 0, X'(1) = 0$$

Find the values of $\lambda$ and corresponding functions $X$ which will solve the professor's problem.

**MATH 294 FALL 1995 PRELIM 3 # 4**

**5.2.15** For the PDE $yu_{xx} + u_y = 0$, (which is not the heat equation)

**a**) Assuming the product form $u(x, y) = X(x)Y(y)$, find ODE's satisfied by $X$ and $Y$.

**b**) Find solutions to the ODE's.

**c**) Write down at least one non-constant solution to the PDE.

**MATH 294 SPRING 1996 PRELIM 3 # 2**

**5.2.16** **a**) Consider the function $f$ defined by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ 3 & \text{if } 0 \leq x < \pi \\ f(x + 2\pi) & \text{for all } x. \end{cases}$$

Calculate the Fourier series of $f$. Write out the first few terms of the series explicitly. Make a sketch showing the graph of the function to which the series converges on the interval $-4\pi < x < 4\pi$. To what values does the series converges at $x = 0$?

**b**) Consider the partial differential equation $u_t + u = 3u_x$ which is not the heat equation). Assuming the product form $u(x, t) = X(x)T(t)$, find ordinary differential equations satisfied by $X$ and $T$, (You are not asked to solve them.)

**MATH 294 SPRING 1996 FINAL # 6**

**5.2.17** Consider the equation for a vibrating string moving in an elastic medium

$$a^2 u_{xx} - b^2 u = u_{tt}$$

where $a$ and $b$ are constants. ($a$ would be the wave speed if not for the elastic constant $b$.) Assume the ends are fixed at $x = 0, L$ and initially the string is displaced by $u(x, 0) = f(x)$, but not moving $u_t(x, o) = 0$.

**a**) Find a general solution for these conditions. (If you need help, you may wish to work part (b) first.)

**b**) If the first term in the general solution to part (a) is

$$u_1(x, t) = c_1 \cos(\lambda_1 t) \sin\left(\frac{\pi x}{L}\right)$$

where $(\lambda_1)^2 = \left(\frac{\pi a}{L}\right)^2 + b^2$, find the solution when the string starts from $u(x, 0) = 2\sin\left(\frac{\pi x}{L}\right)$

**MATH 294 SPRING 1997 FINAL # 9**

**5.2.18** Consider the PDE

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t);\ 0 \le x < \pi;\ t \ge 0 \tag{7}$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{8}$$

$$u(x,0) = \sin(x) \tag{9}$$

**a**) Define

$$v(x,t) = e^t u)x,t) \tag{10}$$

If $u(x,t)$ satisfies equations (7, 8 ,9), show that $v(x,t)$ satisfies the standard heat equation

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2}\frac{\partial v(x,t)}{\partial t} \tag{11}$$

with boundary conditions and initial conditions

$$v(0,t) = v(\pi,t) = 0 \tag{12}$$

$$v(x,0) = \sin x \tag{13}$$

**b**) The general solution of equations (11,12) is

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin nx \tag{14}$$

Find the unique solution $v)x,t)$ of equations (11,12,13).

**c**) Now find the unique solution of equations (7, 8, 9).

**MATH 294 FALL 1992 FINAL # 7**

**5.2.19** For each of the following Fourier series representations,

$1 = \frac{4}{\pi}\sum^i nfty_{n=1}\frac{1}{2n-1}\sin[(2n-1)x],\ 0 < x < \pi,$

$x = 2\sum^i nfty_{n=1}\frac{(-1)^{n+1}}{n}\sin nx,\ -\pi < x < \pi,$

$x = \frac{\pi}{2} - \frac{4}{\pi}\sum^i nfty_{n=1}\frac{1}{(2n-1)^2}\cos[(2n-1)x],\ 0 \le x < \pi.$

**a**) Find the numerical value of the series at $x = -\frac{\pi}{3}, \pi$ and $12\pi + 0.2$ (9 answers required).

**b**) Find the Fourier series for $|x|,\ -\pi < x < \pi$. (Think - this is easy!).

**MATH 294 FALL 1992 FINAL # 8**

**5.2.20** Solve the initial-boundary-value problem

$$T_t = T_{xx},\ 0 < x < \pi,\ t > 0,$$

$$T_x(0,t) = T_x(\pi,t) = 0,$$

$$T(x,0) = 3x.$$

(You may use information from problem 7 if this helps.)

**MATH 294 SPRING 1998 PRELIM 1 # 5**

**5.2.21** Consider the partial differential equation for $u(x,t)$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with the initial conditions

$$u(x,0) = 1 - x,\ 0 \le x \le 1,\ u(x,0) = 0 \text{ elsewhere.}$$

(No boundary conditions are necessary).
Use a centered difference approximation for the space derivative:

$$\frac{\partial u}{\partial x}(x,t) \approx \frac{1}{2h}[u(x+h,t) - u(x-h,t)],$$

and a forward difference approximation for the time derivative:

$$u(x,t+k) \approx u(x,t) + k\frac{\partial u}{\partial t}(x,t).$$

Then introduce a grid with $N+1$ spatial points $x_i = ih$, with $i = 0,1,2,\ldots N$ (where $h$ is the grid spacing) and times $t_j = jk$ (where $k$ is the time step). Let $u(x_i,t_j) \equiv u[i,j]$.

**a**) With $h = 0.25$, and $k = 0.25$, write down the values of the initial condition at each grid point for $0 \le x \le 2$, i.e. $u[i,0],\ i = 0,\ldots,8$.

**b**) Obtain the expression relating $u[i,j+1]$ to $u[i-1,j]$ and $u[i+1,j]$.

**c**) Use the initial data (with $h = 0.25$ and $k = 0.25$) to determine the approximate the value of $u$ at $x_i = 1.0,\ t_k = 0.5$.

**MATH 294 FALL 1998 FINAL # 3**

**5.2.22** Consider the partial differential equation $\frac{\partial u}{\partial x} - \beta \frac{\partial u}{\partial t} = 0$ (Note that this is not the heat equation.) with the initial condition $u(x, 0) = x^2$. In an approximate solution $u$ is to be evaluated on a grid of points spaces by $h$ on the $x$ axis and $\delta t$ on the $t$ axis: $x_i = (i-1)h$ and $t_j = (j-1)\delta t$. The values of $u(x_i, t_j)$ are contained in the array $\hat{u}_{ij} \equiv \hat{u}(i, j) \equiv u(x_i, t_j)$. Here are the forward difference approximations: $\frac{\partial u(x,t)}{\partial x} = \frac{1}{h}[u(x+h, t) - u(x, t)]$ and $\frac{\partial u(x,t)}{\partial t} = \frac{1}{\delta t}[u(x, t+\delta t) - u(x, t)]$

**a)** Derive a finite difference algorithm for this equation. That is, find an expression for $\hat{u}(i, j+1)$ in terms of $\hat{u}(i, j)$ and $\hat{u}(i+1, j)$.

**b)** Let $h = 1, \delta t = \frac{1}{2}$, and $\beta = 1$. Use your approximate scheme above and the given initial condition to find approximate values for $u\left(2, \frac{1}{2}\right)$ , $u\left(3, \frac{1}{2}\right)$, and $u(2, 1)$.

**c)** Find the exact solution to the partial differential equation and given boundary condition. (Do not waste time with Fourier series formulae).

**MATH 294 SPRING 1999 PRELIM 3 # 2**

**5.2.23** Parts a), b), and c) are related, but each part can be done independently of the other parts. Consider the following problem consisting of a PDE for $u = u(x, t)$, two B.C.'s and an I.C.:

$$\frac{\partial^2 u}{\partial x^2} = u + \frac{\partial u}{\partial t}$$

$$\text{B.C.'s: } u(0, t) = 0, \frac{\partial u(2, t)}{\partial x} = 0, t > 0$$

$$I.C.: u(x, 0) = \sin \frac{\pi x}{4}, \; 0 \le x \le 2.$$

**a)** Use separation of variables on the PDE to obtain two ODE's for $X(x)$ and $T(t)$, and the B.C. for $X(x)$.

**b)** In some separation of variables problem, a student obtained the following ODE plus B.C.'s on $X(x)$:

$$\frac{d^2X}{dx^2} + \lambda X = 0 \; X(0) = 0, \frac{dX(2)}{dx} = 0.$$

Find all nontrivial solutions to the ODE with these B.C.'s.

**c)** Which, if any, of the equations given below is a solution to the PDE's, B.C.'s and I.C. at the top of the page? (Justification of your answer is required to get credit. Note that you may have to check several boundary conditions as well as the PDE.)

**i)** $u(x, t) = e^{\left(-1-\frac{\pi^2}{16}\right)t} \sin \frac{\pi x}{4}$

**ii)** $u(x, t) = e^{-\frac{\pi^2 t}{16}} \sin \frac{\pi x}{4}$

**iii)** $u(x, t) = e^{-\frac{\pi^2 t}{16}} \cos \pi x$

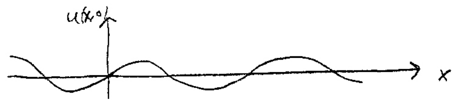
**iv)** $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2\pi^2 t}{16}} \sin \frac{n\pi x}{4}, \; b_n = \int_0^2 sin\left(\frac{\pi x}{4}\right) sin\left(\frac{n\pi x}{2}\right)$

**MATH 294 SPRING 1983 PRELIM 3 # 2**

**5.2.24** Consider the PDE $u_t = -6u_x$.

**a**) What is the most general solution to this equation you can find?

**b**) Consider the initial condition $U(x,0) = \sin(x)$. What does $u(x,t)$ look like for a very small but not zero $t$?




**MATH 294 FALL 1987 PRELIM 2 # 5 MAKE-UP**

**5.2.25** Find the solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{array}{c} 0 < x < 1 \\ 0 < y < 1 \end{array},$$

$$u(0,y) = u(x,0) \equiv 0,$$

$$u(1,y) = u(x,1) \equiv 1.$$

**MATH 294 SPRING 1996 FINAL # 7 MAKE-UP**

**5.2.26** Solve the problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$u(1,\theta) = 4\sin(\theta) - 3\cos(2\theta)$$

$$\lim_{r\to 0} u(r,\theta) = 0$$

You may use the fact that $r^{\pm n}\cos(n\theta)$ and $r^{\pm n}\sin(n\theta)$ are some of the solutions to this partial differential equation.

**MATH 294 FALL 1996 PRELIM 3 # 3 MAKE-UP**

**5.2.27** **a**) Find all solutions of the form $u(x,t) = X(x)T(t)$ for

$$xu_x = 2u_t \;;$$

subscripts indicate differentiation with respect to that variable.

**b**) If $u(x,0) = 2x - 3x^2$, find $u(x,t)$.

**MATH 294 SPRING 1996 FINAL # 7**

**5.2.28** In the problem below, choose the solution that corresponds to the given physical problem. Justify your choice. (Note that a sketch is required with (a), and that to make the right choices you will probably have to check several of the boundary/initial conditions as well as the appropriate partial differential equation.)

**a**) A taut string stretching to infinity in both directions has a wave speed $a$ and an initial displacement $y(x,0) = \frac{1}{(1+8x^2)}$ but no initial velocity.

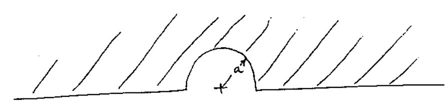
**i**) $y(x,t) = \frac{\frac{1}{2}}{1+8(x-at)^2} + \frac{\frac{1}{2}}{1+8(x+at)^2}$

**ii**) $y(x,t) = \frac{1}{1+8x^2} + \frac{\frac{1}{2}}{1+8(x-at)^2} + \frac{\frac{1}{2}}{1+8(x+at)^2}$

**iii**) $y(x,t) = \sum_{i=1}^{n} c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right)$ where $c_n = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

**iv**) (no choice, do this) Plot the solution initially and when $at = 1$

**b**) The steady state temperature exterior to a semicircular hole $(r > a, 0 < \theta < \pi)$ with boundary conditions $u(r,0) = 0\ and u(r,\pi) = 0$ for $a < r < \infty$ and $u(a,\theta) = f(\theta)$ and $\lim_{r\to\infty} u(r,\theta) = 0$ for $0 \leq \theta \leq \pi$.




In all of the choices, $c_n = \frac{2a^n}{\pi}\int_0^\pi f(\theta)\sin(n\theta)d\theta$.

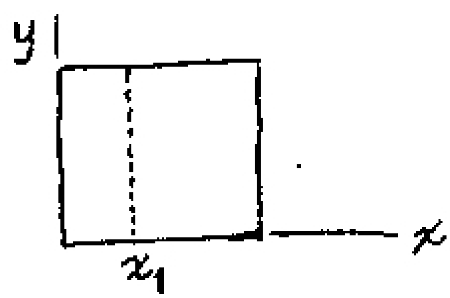
**i**) $u(r,\theta) = \sum_{i=1}^{n} c_n r^{-n}\sin(n\theta)$

**ii**) $u(r,\theta) = \sum_{i=1}^{n} c_n r^{-n}\cos(n\theta)$

**iii**) $u(r,\theta) = \sum_{i=1}^{n} c_n r^{n}\sin(n\theta)$

**iv**) $u(r,\theta) = \sum_{i=1}^{n} c_n r^{n}\cos(n\theta)$

**c**) A square copper plate with sides $L$ has all four edges maintained at 0°




A line across the plate at $x = x_1, 0 < y < L$ is heated to $T_1$ by an external heat source until a steady state results. The temperature in the plate is:

**i**) $T = \begin{cases} T_1 \frac{x}{x_1} & \text{if} \quad x \leq x_1 \\ T_1 \frac{L-x}{L-x_1} & \text{if} \quad x > x_1 \end{cases}$

**ii**) $T = \sum_{i=1}^{n} b_n e^{-\frac{n^2 \pi^2 k t}{L}}$ where $b_n = \frac{1}{L} \int_0^L \frac{T_1 x}{x_1} \sin\left(\frac{n\pi x}{L}\right) dx$

**iii**) $T = \frac{4T_1}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(L-y)}{L}\right)}{n \sinh(n\pi)}$

**iv**) None of the above.