2.10 Orthogonal Projection / Gram Schmidt

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2.10.1 **a**) Find the orthogonal (scalar) projection of the vector $\vec{v} = \vec{i} + \vec{j} + \vec{k}$ in the direction of the vector $\vec{w} = 5\vec{i} + 12\vec{j}$

b) Consider the two vectors

$$\vec{a} = 3\vec{i} - 4\vec{j}$$

$$\vec{b} = 3\vec{i} + 4\vec{j}$$

The vector $\vec{u}$ has orthogonal projections $-\frac{1}{5}$ and $\frac{7}{5}$ along the vectors $\vec{a}$ and $\vec{b}$, respectively. Find $\vec{u}$.
Hint: Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$

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2.10.2 **a**) What is the formula for the scalar orthogonal projection of a vector $\vec{v} \in \Re^n$ onto the line spanned by a vector $\vec{w}$.
Let

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Suppose $\vec{v}_1$ has orthogonal projection 3 and 7 onto the lines spanned by $\vec{b}_1$ and $\vec{b}_2$ respectively.

b) Find $\vec{v}_1$.

c) Suppose $\vec{v}_2$ has orthogonal projections -6 and -14 onto the lines spanned by $\vec{b}_1$ and $\vec{b}_2$ respectively. Find $\vec{v}_2$.

d) Are $\vec{v}_1$ and $\vec{v}_2$ linearly independent.

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2.10.3 As part of their plan to take over the world, lab assistant Pinky has collected 100 points of data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{100}, y_{100},$$

(which represent some devious no-good data) which his partner, Brain, will analyze. A computer program boils down this data into the following set of numbers:

$$\sum_1^{100} x_i = 10, \sum_1^{100} x_i^2 = 20, \sum_1^{100} x_i^3 = 100, \sum_1^{100} x_i^6 = 200,$$

and

$$\sum_1^{100} y_i = 200, \sum_1^{100} x_i y_i = 230, \sum_1^{100} x_i^2 y_i = 250, \sum_1^{100} x_i^3 y_i = 300.$$

Brain has determined that the data is probably of the form $y = a + bx^3$. Your job is to find the least-squares solution to this problem (i.e. find the a and b that gives the least-squares solution).

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2.10.4 Consider $\mathcal{W}$, a subspace of $\Re^4$, defined as $\sqsupset\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$\mathcal{W}$ is a "plane" in $\Re^4$.

a) Find a basis for a subspace $\mathcal{U}$ of $\Re^4$ which is orthogonal to $\mathcal{W}$.

Hint: Find *all* vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ that are perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

b) What is the geometrical nature of $\mathcal{U}$?

c) Find the vector in $\mathcal{W}$ that is closest to the vector $\vec{y} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

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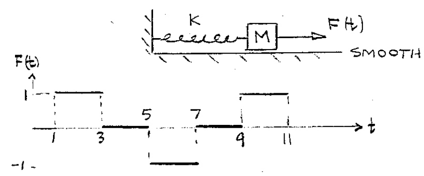
2.10.5 The following figure shows numerical results y_i, for $i = 1, 2, \ldots, n$. It is known that the exact solution of the problem is a formula of the form $y = c$, for some constant c. Find the least squares solution for the constant c in terms of $y_1, y_2, \ldots, y_n$, and n.



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2.10.6 **a)** Find orthonormal eigenvectors $\{\vec{v}_1, vecv_2\}$ of A. [Hint: do not go on to parts d-e below until you have double checked that you have found two orthogonal unit vectors that are eigenvectors of A.

b) Use the eigenvectors above to diagonalize A.

c) Make a clear sketch that shows the standard basis vectors $\{\vec{e}_1, \vec{e}_2\}$ of $\Re^2$ and the eigenvectors $\vec{v}_1, \vec{v}_2$ of A.

d) Give a geometric interpretation of the change of coordinates matrix, P, that maps coordinates of a vector with respect to the eigen basis to coordinates with respect to the standard basis.

e) Let $\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Using orthogonal projection express $\vec{b}$ in terms of $\{\vec{v}_1, \vec{v}_2\}$ the eigenvectors of A.

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2.10.7 Consider the following three vectors in $\Re^3$:

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

[Note: $\vec{u}_1$ and $\vec{u}_2$ are orthogonal.].

a) Find the orthogonal projection of $\vec{y}$ onto the subspace of $\Re^3$ spanned by $\vec{u}_1$ and $\vec{u}_2$.

b) What is the distance between $\vec{y}$ and $span\,\{\vec{u}_1, \vec{u}_2\}$?

c) In terms of the standard basis for $\Re^3$, find the matrix of the linear transformation that orthogonally projects vectors onto $span\,\{\vec{u}_1, \vec{u}_2\}$.

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2.10.8 The vectors $\{(1, 0, 0, -1), (1, -1, 0, 0), (0, 1, 0, 1)\}$ are linearly independent and span a subspace S of $\Re^4$. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of S that is orthogonal to the first vector of the given set, $(1, 0, 0, -1)$.

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2.10.9 **a)** Find an orthonormal basis for the space of vectors in $\Re^3$ having the form $\begin{bmatrix} c_1 - c_2 \\ c_2 \\ 2c_2 \end{bmatrix}$. You may use Gram-Schmidt or any other method.

b) If $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\right\}$ is an orthonormal basis for $\Re^4$,

$$\begin{bmatrix} 1 \\ -9 \\ 0 \\ \sqrt{5} \end{bmatrix} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4, \text{ and } \vec{b}_2 = \begin{bmatrix} 0.5 \\ 0 \\ \alpha \\ 0 \end{bmatrix},$$

(where c_i are real constants), find the possible values of c_2 and α

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2.10.10 Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}.$$

a) Find an orthogonal basis for the null space of A.

b) Find a basis for the orthogonal complement of $Nul(A)$, i.e. find $(Nul(A))^T$.

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2.10.11 Let $A = [\vec{v}_1 \vec{v}_2]$ be a 1000×2 matrix, where $\vec{v}_1, \vec{v}_2$ are the columns of A. You aren't given A. Instead you are given only that

$$A^T A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Find an **orthonormal**basis $\{\vec{u}_1, \vec{u}_2\}$ of the column space of A. Your formulas for $\vec{u}_1$ and $\vec{u}_2$ should be written as linear combinations of $\vec{v}_1, \vec{v}_2$. (Hint: what do the entries of the matrix $A^T A$ have to so with dot products?)

MATH 293 SPRING ? FINAL # 2

2.10.12 **a)** Find a basis for the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

b) Find the rank of A and a basis for its column space, noting that $A = A^T$.

c) Construct an orthonormal basis for the row space of A.