1.5 Linear Independence

MATH 294 FALL 1987 MAKE UP PRELIM ? # 2 294FA87MUPxQ2.tex

1.5.1 In parts (a) - (g), answer true or false.

a) $Span(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \vec{v}_4) = \Re^3$, where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},\ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \vec{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

b) The 4 vectors in (a) are independent.

c) Referring to (a) again, all vectors, $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $Span(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \vec{v}_4)$ satisfy a linear equation $ax_1 + bx_2 + cx_3 = 0$ for scalars a,b,c not all 0.

d) The rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ is 3.

e) In $\Re^n$, n distinct vectors are independent.

f) n + 1 distinct vectors always span $\Re^n$, for n ¿ 1.

g) If the vectors $\vec{v}_1,\ \vec{v}_2,\ \ldots, \vec{v}_n$ span $\Re^n$, then they are a basis for $\Re^n$.

MATH 294 SPRING 1989 PRELIM 2 # 3 294SP89P2Q3.tex

1.5.2 Consider the system of equations,

$$\begin{array}{rcrcrcr} -x_1 & + & 2x_2 & + & 3x_3 & = & -1 \\ 2x_1 & + & 5x_2 & - & 3x_3 & = & 2 \\ 11x_1 & + & 14x_2 & - & 21x_3 & = & 11 \end{array}$$

a) Find all solutions, if any exist, of the system.

b) Is the set of vectors given by,

$$\begin{bmatrix} -1 \\ 2 \\ 11 \end{bmatrix},\ \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix},\ \text{and}\ \begin{bmatrix} 3 \\ -3 \\ -21 \end{bmatrix}$$

linearly independent or dependent?

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1.5.3 Consider the following two vectors in $\Re^3$:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ and } \mathbf{v_2} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

a) Find a third non-zero vector $\mathbf{v_3}$ so that the set $\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}$ is linearly dependent. (explain)

b) Find a third vector $\mathbf{v_3}$ so that the set $\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}$ is linearly **independent**. (explain)

MATH 293 FALL 1995 PRELIM 2 # 4 293FA95P2Q4.tex

1.5.4 Consider the following wing set of vectors in $\Re^3$:

$$v_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} -4 \\ 6 \\ 7 \end{bmatrix},\ v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

a) Is the set linearly independent or dependent? Give reasons for your answer.
b) Find $W = span\{v_1,\ v_2,\ v_3\}$, i.e., give any correct formula for a typical element of W.
c) What geometrical object is W? (e.g., point, line, plane, space, etc.

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1.5.5 Let

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix},\ \mathbf{v_2} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix},\ \mathbf{v_3} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

The value of h for which $\mathbf{y}$ is in $Span\{\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3}\}$ is:
a) -5
b) 5
c) 0
d) 1
e) none of the above.

MATH 293 SPRING 1996 FINAL # 5 293SP96FQ5.tex

1.5.6 Let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix},\ \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix},\ \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix},\ \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{z} = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}.$$

Which one of the following is true?
a) $\{\mathbf{u},\ \mathbf{z}\}$ is linearly dependent.
b) $\{\mathbf{v},\ \mathbf{w},\ \mathbf{y}\}$ is linearly independent.
c) $\{\mathbf{v},\ \mathbf{w},\ \mathbf{z}\}$ is linearly dependent.
d) $\{\mathbf{u},\ \mathbf{v},\ \mathbf{w},\ \mathbf{z}\}$ is linearly dependent.
e) None of the above.

MATH 293 SPRING 1996 FINAL # 31 293SP96FQ31.tex

1.5.7 Let $S = \{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$ be a set of three vectors in $\Re^3$. If none of the vectors in S in a multiple of another vector in S, then S is linearly independent. True or false.

MATH 293 SPRING 1996 FINAL # 32 293SP96FQ32.tex

1.5.8 In special cases, it is possible for a set of four vectors in $\Re^6$ to span $\Re^6$. True or false.

MATH 294 SPRING 1997 PRELIM 2 # 2 294SP97P2Q2.tex

1.5.9 For what value(s) of the parameters s and t is the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} s \\ t \\ 2 \end{bmatrix} \right\}$$

linearly dependent?

MATH 294 SPRING 1997 PRELIM 2 # 3 294SP97P2Q3.tex

1.5.10 Which of the following sets of vectors span all of $\Re^2$? You do *not* need to explain your answers.

$$S_1 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\},\ S_2 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\},\ S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\},$$

$$S_4 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\},\ S_5 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right\}.$$

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1.5.11 Consider three vectors in $\Re^4$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 5 \\ -2 \\ a \\ b \end{pmatrix}.$$

a) For what values of a and b does v_3 lie in $span\{v_1, v_2\}$?

b) For what values of a and b is the set $S = \{v_1, v_2, v_3\}$ linearly independent in $\Re^3$?

c) For what values of a and b does

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ a \\ b \end{pmatrix}$$

have at least one solution x?

MATH 294 FALL 1998 PRELIM 1 # 2 294FA98P1Q2.tex

1.5.12 **a)**

$$\text{Let } A = \begin{bmatrix} 1 & -7 & 2 & 2 \\ -6 & 5 & 8 & 12 \\ 12 & 0 & -4 & 12 \end{bmatrix}$$

Are the columns of A linearly independent? Why or why not.

b) Determine if the columns of the given matrix form a linearly dependent set. **Hint:** one way to do this is by row operations.

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 3 & -5 & 5 \\ -2 & 6 & -6 \end{bmatrix}$$

c) Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}. \text{ Given that } \vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is a solution to $A\vec{x} = \vec{b}$, is this solution unique?

d) For the matrix in (c) is there a solution to $A\vec{x} = \vec{b}$ for all $\vec{b}$ in $\Re^3$? Why or why not?

MATH 294 FALL 1998 PRELIM 3 # 3 294FA98P3Q3.tex

1.5.13 If A is a (possibly not square) matrix with $A^T A$ invertible, are the columns of A linearly independent? (yes, no, maybe).

MATH 293 Unknown FINAL # 5 293xxFQ5.tex

1.5.14 **a)** Let A be an n x n matrix. Show that if $A\mathbf{x} = \mathbf{b}$ has a solution then $\mathbf{b}$ is a linear combination of the column vectors of A.

b) Let A be a 4 x 4 matrix whose column space is the span of vectors $\mathbf{v} = (\mathbf{v_1},\ \mathbf{v_2},\ \mathbf{v_3},\ \mathbf{v_4})^t$, satisfying $v_1 - 2v_2 + v_3 - v_4 = 0$. Let $\mathbf{b} = (1,\ b_2,\ b_3,\ 0)^t$. Find all values of $\mathbf{b_2},\ \mathbf{b_3}$ for which the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

MATH 294 SPRING 1996 PRELIM 2 # 2b 294SP96P2Q2b.tex

1.5.15 The following matrix applies to the next 2 questions

Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Are the columns of B linearly independent? Explain why or why not.

MATH 293 SPRING 1996 PRELIM 2 # 3 293SP96P2Q3.tex

1.5.16 Is the span of the columns of A equal to all of $\Re^3$? Explain why or why not.

MATH 294 SPRING 1999 PRELIM 1 # 2 294SP99P1Q2.tex

1.5.17 **a**) Is the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} \right\}$ linearly independent (why or why not)?

b) Is the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix} \right\}$ linearly independent (why or why not)?

c) What is the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

d) What (in vector form) is the general solution to $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 3 & 6 & 2 \\ 3 & 4 & 3 & 8 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$?

The Matlab dialogue below may or may not be useful to you.

>> **B** = [

1 2 3 0 4;

4 3 6 2 1;

3 4 3 8 6;

];

>> rref(B)

ans =

$$\begin{matrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & -2 & 0 \end{matrix}$$