1.4 Linear Transformation I

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1.4.1 **a**) Consider the vector transformation

$$\mathbf{y} = f(\mathbf{x}) \text{ from } V_2 \text{ to } V_2 \text{ such that if } \mathbf{y} = (y_1, y_2),\ \mathbf{x} = (x_1, x_2),$$

$$y_1 = \frac{(x_1 + x_2)}{\sqrt{2}} \quad y_2 = \frac{(x_1 + x_2)}{\sqrt{2}}.$$

Verify that $\mathbf{y} = A(\mathbf{x})$ is <u>linear</u> and find a matrix A such that

$$f(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } V_2.$$

b) Consider the linear transformation $\mathbf{z} = g(\mathbf{y})$ from V_2 to V_2 with matrix

$$B = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Find a matrix for the composite transformation $z = g(f(\mathbf{x}))$ ("function of a function").

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1.4.2 Let T be the linear transformation

$$T = \frac{d^2}{dx^2}$$

acting on the space spanned by $B_1 = \{1, \sin(x), \cos(x)\}$.

a) Find the matrix T_{B_1} which represents T in the basis B_1.

b) If B_2 is the basis $B_2 = \{1, \sin(x) + \cos(x), \sin(x) - \cos(x)\}$, find the matrix T_{B_2}, which represents T in the basis B_2.

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1.4.3 Consider the linear transformation, T, of the plane to itself, which is represented, in the standard basis, by the non-singular matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The equation of a certain curve, (straight line), in the (x, y)-coordinate system is given by $y = mx + h$.

a) Find the equation of this same curve in the (x', y')-coordinate system.

b) What is the shape of this curve in the (x', y')-coordinate system?

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1.4.4 Consider physical vectors

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

where a, b and c are scalars and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular unit vectors. A linear transformation is defined as

$$T(\mathbf{v}) = (\mathbf{i} - 2\mathbf{j}) \text{ x } \mathbf{v}$$

Find the matrix representation of T in the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ basis.

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1.4.5 Consider the vector space $V_3$with the standard basis $B_1 = S = (\mathbf{i}, \mathbf{j}, \mathbf{k})$. Now consider a second basis B_2 which is obtained by rotating the basis B_1 by 30 degrees (anticlockwise) about the z axis.
Also, consider the linear transformation $T : V_3 \to V_3$ which reflects any vector $v \epsilon V_3$ about the x-z plane.

a) Find$(B_2 : B_1)$.

b) Find matrix representations of T_{B_1} and T_{B_2} of T in the bases B_1 and B_2.

Hint: $\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$ is a vector in B_2. Also, check if $(B_2 : B_1)^{-1} = (B_2 : B_1)^t$.

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1.4.6 Every vector $\vec{v}$ in two dimensional physical space can be written as $\vec{v} = x\hat{i} + y\hat{j}$ where $\hat{i}$ and $\hat{j}$ are unit vectors on the positive x and y axes respectively. In each of the following cases, find the matrix representing the linear transformation indicated and state whether or not it is invertible.

a) T_1 is the transformation which reflects each vector about the y axis.

b) T_2 is the transformation which rotates each vector about the origin by an angle of 60^o in a counterclockwise direction.

c) T_3 is the transformation which transforms each vector into its vector projection on the x axis.

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1.4.7 Consider linear transformation in $\Re^2$ and the standard basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

a) Find the matrix U of the linear transformation that stretches the x component of each vector by a factor of 2 and keeps the y component unchanged.

b) Find the matrix R of the linear transformation that rotates each vector by 45 degrees in the counterclockwise direction.

c) Do the above transformations commute, i.e. is RU equal to UR?

d) If yes, stop. If no, find the matrix V such that

$$RU = VR$$

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1.4.8 Consider linear transformations $L : V \to V$, where V is the vector space of all real 3 x 3 matrices, and consider the specific transformation defined by

$$L(A) = A - A^T \qquad (1)$$

where $A \epsilon V$ is a 3 x 3 real matrix and A^T is the transpose of A.

a) Show that L, as defined in (1) above, is a linear operator (transformation).

b) Now consider $N(L)$, the null space of L = the set of all 3 x 3 matrices B such that $L(B) = 0$. Check the following matrices to see if they are in the null space of L.

$$B_1 = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 3 & 0 & 4 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

c) Find the null space of L.

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1.4.9 Which of the following transformations is linear?

a) $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c) $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

d) $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

e) $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_2 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

MATH 293 SPRING 1995 PRELIM 3 # 2 293SP95P3Q2.tex

1.4.10 **a)** Are the vectors

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

linearly independent? Do they span $\Re^2$?

b) Are the vectors

$$\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

linearly independent? Do they span $\Re^3$?

c) Let $C(\Re)$ be the vector space of continuous functions on $\Re$. Are the three elements $\sin(x),\ \sin(x+1),\ \sin(x+2)$ linearly independent? Do they span $C(\Re)$?

MATH 293 FALL 1995 PRELIM 2 # 3 293FA95P2Q3.tex

1.4.11 Consider the linear transformation $T : \Re^3 \to \Re^3$ that reflects vectors through the plane $y = z$. (You can think of the plane as a two-sided mirror.)

a) Find the standard matrix A of T. You may use the fact that $A = [T_{(e_1)}, T_{(e_2)}, T_{(e_3)}]$ if you wish.

b) Is the transformation onto $\Re^3$? Give reasons for your answer.

c) Is the transformation one-to-one? Give reasons for your answer.

d) Determine A^2. Explain your result in geometric or physical terms.

MATH 293 FALL 1995 FINAL # 4 293FA95FQ4.tex

1.4.12 **a)** Show that translation in $R^2 \to R^2$, i.e.

$$\mathbf{T} = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix} \qquad (1)$$

is **not** a linear transformation.

b) Now consider homogeneous coordinates $\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $\mathbf{T} = \left(\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} x_1 + h \\ x_2 + k \\ 1 \end{bmatrix}$.

Find the 3 x 3 matrix for $\mathbf{T}$.

c) The shear transformation $\mathbf{S} : \Re^2 \to \Re^2$ along the x_1 axis has the following matrix

$$\begin{bmatrix} 1 & \tan\gamma \\ 0 & 1 \end{bmatrix}$$

This transformation rotates the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by an angle. Find this angle.

d) Do the transformations T and S, commute in general? What happens in the special case $k = 0$? Give reasons for your answer.

MATH 293 SPRING 1996 PRELIM 2 # 2a 293SP96P2Q2a.tex

1.4.13 The following matrices apply to the next 3 questions:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

a) Is B, viewed as a linear transformation, one-to-one? If yes, explain why; if no, explain why and find the solution set of $B\mathbf{x} = \mathbf{0}$.

MATH 293 SPRING 1996 PRELIM 2 # 3 293SP96P2Q3a.tex

1.4.14 **a)** Is A, viewed as a linear transformation, onto? Explain why or why not.

MATH 293 SPRING 1996 PRELIM 2 # 7 293SP96P2Q7.tex

1.4.15 Which of the following functions are linear transformations?

i) $f : \Re^1 \to \Re^1,\ f(x) = 2x$

ii) $g : \Re^1 \to \Re^1,\ g(x) = 2x + 1$

iii) $S : \Re^1 \to \Re^2,\ S(x) = \begin{bmatrix} 2x \\ 2x+1 \end{bmatrix}$

iv) $T : \Re^3 \to \Re^4,\ T(\mathbf{x}) = C\mathbf{x}$, where C is the matrix above (in question 13).

v) $R : \Re^n \to \Re^m,\ R(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$

MATH 293 SPRING 1996 FINAL # 6 293SP96FQ6.tex

1.4.16 Let $T : \Re^5 \to \Re^3$ be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, where A is a matrix and $\mathbf{x}$ is a vector in $\Re^5$. Then A has dimensions (rows, columns):

a) 3 x 5
b) 5 x 3
c) 5 x 5
d) 3 x 3
e) none of the above

MATH 293 SPRING 1996 FINAL # 7 293SP96FQ7.tex

1.4.17 Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The best description of $\mathbf{x} \to A\mathbf{x}$ is:

a) a rotation about the origin.
b) a reflection through the x-axis.
c) a reflection through the y-axis.
d) a reflection through the origin.
e) a reflection through the line $y = x$.

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1.4.18 Suppose that matrix A sends $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Then the matrix for A is:

a) $\begin{pmatrix} 4 & -1 \\ -8 & 3 \end{pmatrix}$

b) $\begin{pmatrix} -8 & 3 \\ 4 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 8 & -3 \\ -4 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 8 & 3 \\ 4 & 1 \end{pmatrix}$

e) none of the above

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1.4.19 If A is 5 x 4, then $\mathbf{x} \to A\mathbf{x}$ cannot map $\Re^4$ onto $\Re^5$. True or false.

MATH 294 SPRING 1997 PRELIM 2 # 9 294SP97P2Q9.tex

1.4.20 Suppose $T : \Re^n \to \Re^m$ is linear, and suppose that $T(\vec{x}_1) = \vec{b}_1$ and $T(\vec{x}_2) = \vec{b}_2$. Find a vector $\vec{x} \in \Re^n$ such that $T(\vec{x}) = 1.1\vec{b}_1 - 2.3\vec{b}_2$.

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1.4.21 Which of the following functions are linear transformations? You do *not* need to explain your answers.

i) $T_1 : \Re^2 \to \Re^2$, given by

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+\frac{1}{10} \end{bmatrix}$$

ii) $T_2 : \Re^2 \to \Re^2$, T_2 is reflection in line $y = x+1$.

iii) $T_3 : \Re^2 \to \Re^2$, T_3 is reflection in line $x = 0$.

iv) $T_4 : \Re \to \Re^3$, given by

$$T_4(x) = \begin{bmatrix} \cos\left(\frac{\pi}{7}\right)x \\ 0 \\ x - x\cos\left(\frac{\pi}{7}\right) \end{bmatrix}$$

MATH 294 SPRING 1997 PRELIM 2 # 5 294SP97P2Q5.tex

1.4.22 Find the matrix A for the linear transformation $T : \Re^2 \to \Re^2$ which is the composition of *first* applying a rotation by angle $\frac{\pi}{2}$ *clockwise*, followed by *then* applying reflection in the line $y = x$.

MATH 294 SPRING 1997 PRELIM 2 # 6 294SP97P2Q6.tex

1.4.23 Find the matrix A for the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ x \\ y \end{bmatrix},$$

and find the inverse of A.

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1.4.24 (All parts are independent problems.)

a) If the det $A = 2$. Find the det A^{-1}, det A^T.

b) From $PA = LU$ find a formula for A^{-1} in terms of P, L and U. Assume P, L, U, A are invertible n x n matrices.

c) Find the rank of matrix A:

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} [2 - 12].$$

d) Find a 2 x 2 matrix E such that for *every* 2 x 2 matrix A, the second row of EA is equal to the sum of the first two rows of A, e.g., if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ then } EA = \begin{bmatrix} 1 & 2 \\ 3+1 & 4+2 \end{bmatrix}$$

e) Write down a 2 x 2 matrix P which projects every vector on to x_2 axis. Verify that $P^2 = P$.

MATH 294 SPRING 1997 FINAL # 7.1 294SP97FQ7p1.tex

1.4.25 Suppose A is a 6 row by 7 column matrix for which $NulA = Span\{\vec{x}_o\}$ for some $\vec{x}_o \neq \vec{0}$ in $\Re^7$. Which of the following are always TRUE of A? (NO Justification is necessary.) Express your answers as e.g: TRUE: a,b,c,d; FALSE: e

a) The columns of A are linearly dependent.
b) The linear transformation $\vec{x} \to A\vec{x}$ is onto.
c) $A\vec{x} = \vec{0}$ has only the trivial solution.
d) The columns of A form a basis for $\Re^6$.
e) The columns of A span all of $\Re^6$.

MATH 294 FALL 1997 PRELIM 2 # 6 294FA97P2Q6.tex

1.4.26 Let V be the vector space of 2 x 2 matrices.

a) Find a basis for V.
b) Determine whether the following subsets of V are subspaces. If so, find a basis. If not, explain why not.
1. $\{A \text{ in } V | \det A = 0\}$
2. $\{A \text{ in } V | A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$.

c) Determine whether the following are linear transformations. Give a short justification for your answers.
1. $T : V \to V$, where $T(A) = A^T$,
2. $T : V \to \Re^1$, where $T(A) = \det A$,

MATH 294 SPRING 1998 PRELIM 2 # 4 294SP98P2Q4.tex

1.4.27 **a**) Determine the 2 x 2 matrix that corresponds to a clockwise rotation through an angle ϕ.
b) Using homogeneous coordinates find the 3 x 3 matrix that describes the 2D composite transformations of reflecting in the y axis and then translating (3,3).

MATH 294 FALL 1998 PRELIM 1 # 1 294FA98P1Q1.tex

1.4.28 **a**) A transformation, $T : \Re^n \to \Re^n$ is defined as $T(\underline{u}) = \underline{u} + \underline{s}$, $\underline{s}$ = constant vector. Is this a linear transformation? Why or why not?
b) Sketch the image of the unit box, drawn below, after being mapped by the transformation

$$\underline{x} \to A\underline{x}, \quad A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Clearly label on your sketch the images of the points labeled a, b, c, and d.
Give a geometric interpretation of this transformation in words.

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1.4.29 A linear transformation $T : \Re^2 \to \Re^2$ maps the square shown (ABCD) to the parallelogram shown (A'B'C'D'). (The answers to the questions below do not depend on each other. You will not get credit for an incorrect answer to one part based on an incorrect answer to another part.)

a) Find the matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

b) Is the map one to one (why or why not)?

c) Is the map onto (why or why not)?

d) Describe in words the geometry of the transformation $T_2(x) = AA\mathbf{x}$. (Use one or more words, like 'stretch', 'rotate', 'reflect', 'expand', 'project', 'shear' or 'translate' and describe the amount and/or orientation of such distortion.)