MAE 6700 Advanced Dynamics 2013

Overview, Intro, Rotations 1/22

Outline of Class:

√ Rotations in 3D

√ Lagrange Eqs + non-conservative forces √

+ extra constraints (possibly nonholonomic)

Dealing with Constraints

Friction and Collisions

Multi-object 3D dynamics (Kane, Featherstone, Shabanna, TMT)

Hamilton's Principle (not equations)

Finding interesting solutions (root finding + numerical optimization → optimal or periodic solutions)

√ In-depth understanding of basic axioms + reasoning

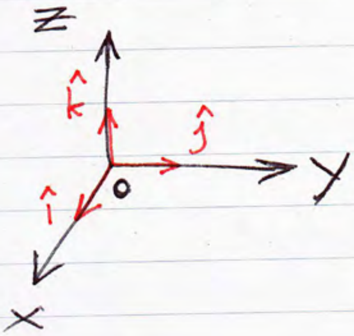
END GOAL (tentative): Simulate a bicycle

* No textbooks.

1/22

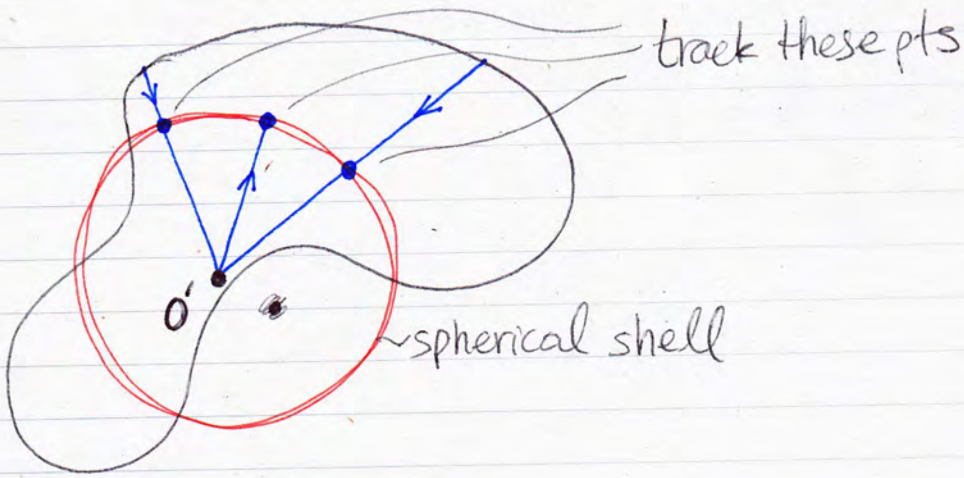
Rotations in 3D

Rigid object: "All distances between all pairs of material points are constant in time. All angles between all pairs of lines → constant. All shapes are congruent with the later versions of themselves" ⇒ no deformation



$$\vec{r}_{P/O}(t) = \underbrace{\vec{r}_{O'/O}(t)}_{\text{(displacement, translation)}} + \underbrace{\vec{r}_{P/O'}(t)}_{\text{rotation}}$$

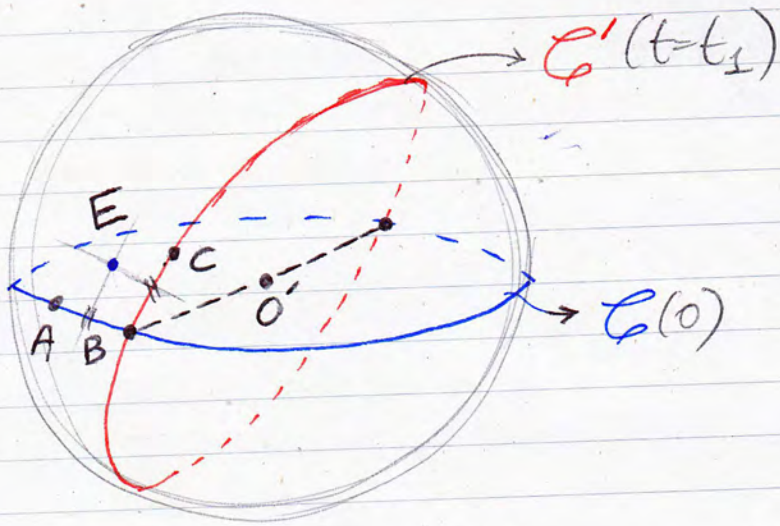
Consider pts on a spherical shell inside the object.



Euler's Thm: For any rotation, at any time t, there is some point E such that

$$\vec{r}_{E/O'}(t) = \vec{r}_{E/O'}(0)$$

"Proof": Before rotation, draw any great circle $\mathcal{C}$:



Rotation takes $\mathcal{C} \to \mathcal{C}'$ and $A \to B$, $B \to C$

$\mathcal{C}'$ and $\mathcal{C}$ intersect at two points. The line is a diameter.

↳ one of them is B

A and B are on $\mathcal{C}$ $\xrightarrow{\text{rigid}}$ $(AB) = (BC)$

The point E at intersection of $\perp$ bisectors of AB and BC doesn't move. But not constant for $0<t<t$

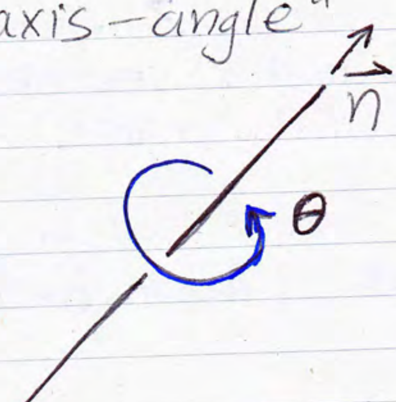
$\Downarrow$

$\boxed{\text{Rotation is about axis } O'E}$

Only $\vec{r}_{E/O'}(t_1) = \vec{r}_{E/O'}$

→ That was one representation of rotations:

"axis-angle"



Constraint!!!
$n_x^2+n_y^2+n_y^2 = 1$

Not unique!!!
$\vec{n}, \theta$ is same as
a) $-\vec{n}, -\theta$
b) $\vec{n}, \theta+2m\pi$

4 numbers:

n_x, n_y, n_z, θ

H/W Find a book or website that proves Euler's thm in a non formalistic/algebraic way.

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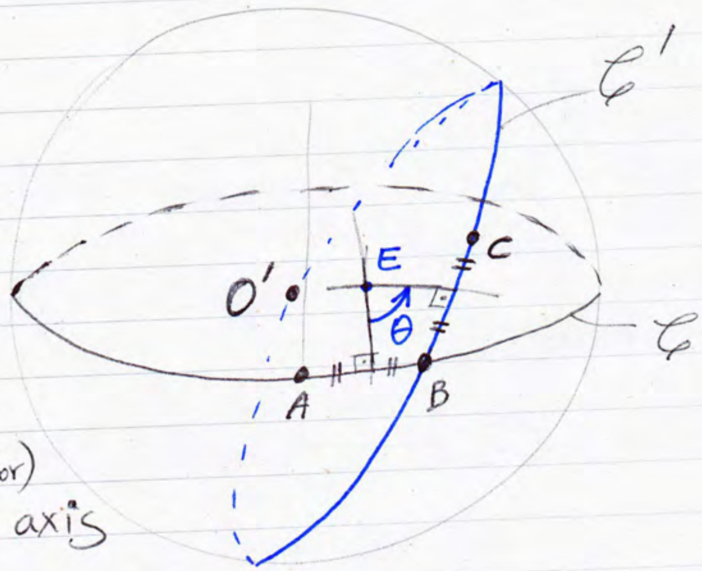
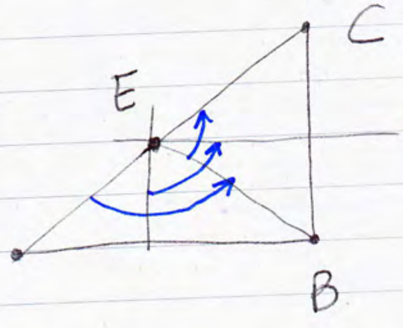
Euler's thm: Synge & Griffith book. for picture/proof.

$\mathcal{C} \to \mathcal{C}'$
$A \to B$
$B \to C$



$O'E$ or $\hat{n}$ (unit vector) is the rotation axis



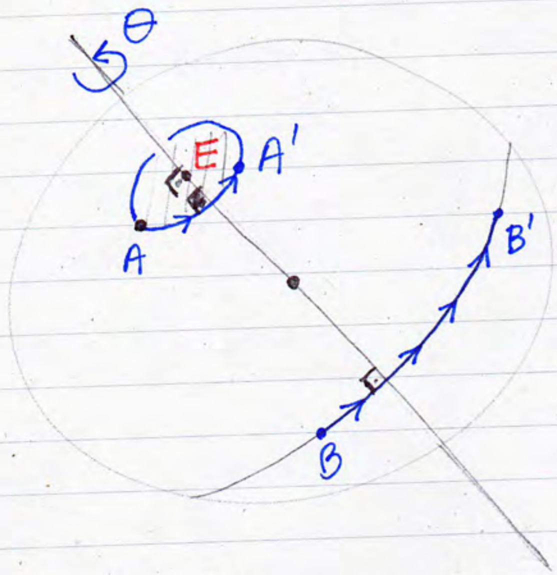
all these are the rotation angle θ.

- Representation #1: $\hat{n}, \theta$

#2: "vector" of rotation $\vec{N} = \theta \hat{n}$ (3 numbers: N_x, N_y, N_z)
↳ is it a VECTOR? *No
↳ is it unique? → NO

*Vector algebra does not transfer back to physical world.
Rotation is not commutative ($\vec{a}+\vec{b} \neq \vec{b}+\vec{a}$)

✱ All points move on lattitudinal arcs



E is on the bisector of both arcs AA' and BB'

#3: Look at rotated positions of two points

e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix}_A$; $\begin{bmatrix}0\\1\\0\end{bmatrix}_B$

At least 4 numbers. Redundancy $(A'B') = (AB)$

"Distance on a sphere = angle of arc of great circle"

★ How to add rotations:

Given $\theta_1, \hat{n}_1 (E_1)$ and $\theta_2, \hat{n}_2 (E_2)$ what is net $(E)\, \theta, \hat{n}$?

kinda equivalent

Rotation$_1$, then rotation$_2$

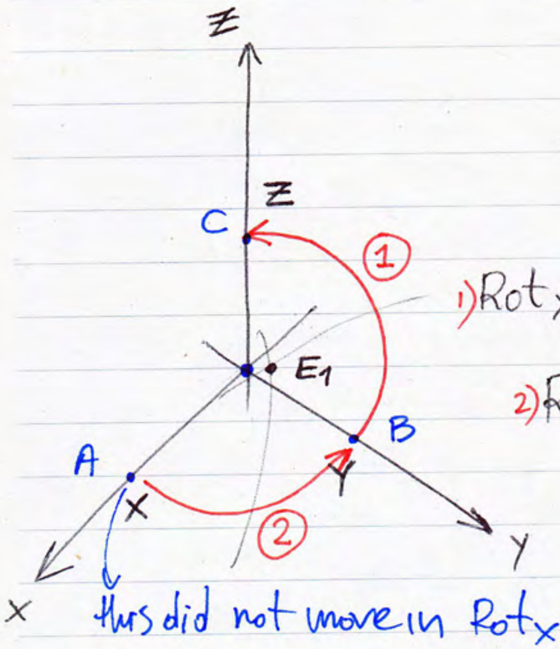
~~OE is not~~ E is not on the plane $O' E_1 E_2 \Rightarrow$

$\vec{N}$ is not $\vec{N}_1 + \vec{N}_2 \in$ plane $O' E_1 E_2$

Rot_1, then $Rot_2 \neq Rot_2$, then Rot_1

- "Net rotation depends on order of rotations"

Example: (a) Rot_x $90°$, then Rot_z $90°$



pts in space: X, Y, Z
material pts: A, B, C } initially coincide

1) Rot_x: A stays in X, B goes to Z, $C \to$ "$-Y$"

2) Rot_z: $A \to Y$, B stays at Z, $C \to X$

Net rotation is about E_1 $\to$

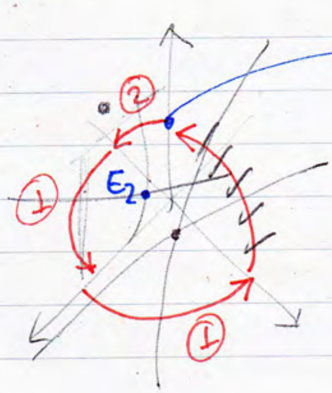
$$\hat{n} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \theta = \frac{2\pi}{3} = 120°$$

(b) Rot_z $90°$, then Rot_x $90°$

$A \to Y$, $B \to -A$, C stays

$A \to Z$, B stays at $-A$, $C \to -Y$

$$\hat{n}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} +1 \\ -1 \\ +1 \end{bmatrix} \checkmark, \quad \theta = 120°$$

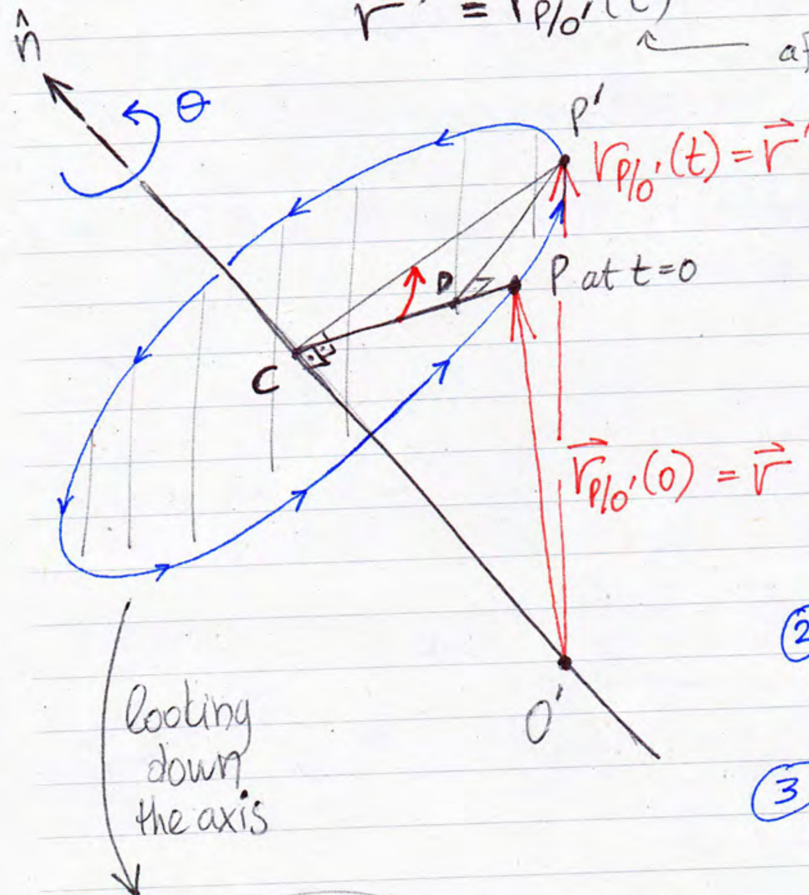


$\to$ b/c that pt did not move in Rot_z

Representation that we can calculate with!

Goal: Find formula that starts with $\hat{n}, \theta$ and $\vec{r}$ = position of material point P wrt O'. and gives

$$\vec{r}\,' = \vec{r}_{P/O'}(t)$$ ← after rotation



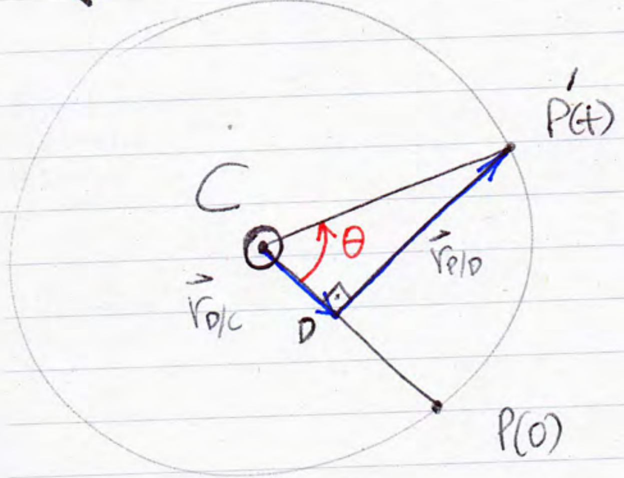
$$\vec{r}\,' = \overset{①}{\vec{r}_{C/O'}} + \overset{②}{\vec{r}_{D/C}} + \overset{③}{r_{P'/D}}$$

$$① \ \vec{r}_{C/O'} = \hat{n}(\hat{n}\cdot\vec{r}) = (\vec{r}\cdot\hat{n})\hat{n}$$

$$\vec{r}_{P/C} = \vec{r} - \hat{n}(\hat{n}\cdot\vec{r})$$

$$② \ \vec{r}_{D/C} = \cos\theta\left[\vec{r} - \hat{n}(\hat{n}\cdot\vec{r})\right]$$

$$③ \ r_{P'/D} = \sin\theta(\hat{n}\times\vec{r})$$



$$\boxed{\vec{r}\,' = (1-\cos\theta)\underbrace{\hat{n}\,\hat{n}}_{\text{"dyad"}}\cdot\vec{r} + \cos\theta\,\vec{r} + \sin\theta\,\hat{n}\times\vec{r}}$$

"dyad" = "nn^T"

(cont'd) 1/29

$$\vec{r}' = \left(\hat{n}\cdot\hat{n}\cdot + \cos\theta\left(I\cdot - \hat{n}\hat{n}\cdot\right) + \sin\theta\,\hat{n}\times\right)\underline{\vec{r}}$$

Observe: Rotation is linear in $\vec{r}$

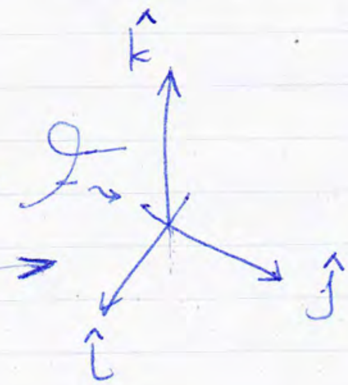
$$Rot(a_1\vec{r}_1 + a_2\vec{r}_2) = a_1 Rot(\vec{r}_1) + a_2 Rot(\vec{r}_2)$$

Why? Whole picture rotates due to rigidity.

Rotation algebra is independent of ref. point (O') (unlike translation)

Notation (vector vs. list of numbers)

$$[\vec{V}]_{\hat{i}\hat{j}\hat{k}} = [\vec{V}]_{\mathcal{F}} = [\quad]_{\text{(implicitly)}} \rightarrow$$



$$= \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \text{ (list of 3 numbers)}$$

Given:

$$\vec{r} = r_x\hat{i} + r_y\hat{j} + r_z\hat{k} = r_1\hat{e}_1 + \cdots = \sum_{i=1}^{3} r_i\hat{e}_i = r_i\hat{e}_i$$

Einstein/indicial summation notation...

subscript appears twice

$$\hat{n} = n_i\hat{e}_i \quad , \quad [\hat{n}] = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

and θ,

we want to calculate $[\vec{r}'] = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix}$ $\longrightarrow$

Quaternions

4 numbers: $\left\{ \sin\left(\frac{\theta}{2}\right)\hat{n} \,,\ \cos\left(\frac{\theta}{2}\right) \right\}$?

$$I \cdot \vec{r} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\left[\hat{n}\hat{n}\vec{r}\right] = \begin{bmatrix} n_1 (n_1 r_1 + n_2 r_2 + n_3 r_3) \\ n_2 (\quad - \text{''} - \quad) \\ n_3 (\quad - \text{''} - \quad) \end{bmatrix} = \begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\left[\hat{n} \times \vec{r}\right] = \begin{bmatrix} n_2 r_3 - n_3 r_2 \\ n_3 r_1 - n_1 r_3 \\ n_1 r_2 - n_2 r_1 \end{bmatrix} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\Rightarrow \left[\vec{r}\,'\right] = \left[(1-\cos\theta) \begin{bmatrix} n_1 n_1 & & \cdots \\ & n_2 n_2 & \\ \cdots & & n_3 n_3 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \right] \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$\longrightarrow$

Euler Angles (Gimball angles)

cool drawing here

"ZXZ rotations"

- 1st, rotate θ about $\underline{\underline{z}}$-axis

$$\left.\begin{array}{l}\hat{\imath} \to \hat{\imath}' \\ \hat{\jmath} \to \hat{\jmath}'\end{array}\right\} \to \text{in xy plane}$$

$$\hat{k} \to \hat{k}' = \hat{k}$$

- 2nd, rotate ϕ about <u>new</u> $\underline{\underline{x}}$-axis $= x'$

$$\hat{\imath}' \to \hat{\imath}'' = \hat{\imath}'$$

$$\hat{\jmath}' \to \hat{\jmath}''$$

$$\hat{k}' \to \hat{k}''$$

- 3rd, rotate ψ about new $\underline{\underline{z}}$-axis, called $z' = \hat{k}''$

$$\left.\begin{array}{l}\hat{\imath}'' \to \hat{\imath}''' \\ \hat{\jmath}'' \to \hat{\jmath}'''\end{array}\right\} \to \text{in } x' = \hat{\imath}' = \hat{\imath}'', \hat{\jmath}'' \text{ plane}$$

$$\hat{k}'' \to \hat{k}''' = \hat{k}''$$

<u>Unique?</u>, but <u>singular</u> parametrization

↑ Gimball lock

→

(cont'd from $[r'] = \dots$)

Matrix representation

$$[\vec{r}\,']_{\mathcal{E}} = [R]_{\mathcal{E}} \cdot [\vec{r}]_{\mathcal{E}} \quad \rightarrow \quad \text{"}\vec{r}\,' = R \cdot \vec{r}\text{"} \text{ (bad notation)}$$

Component repr.

$$r_i' = R_{ij} \cdot r_j$$

summation convention!!!

"kronecker delta" $\left\{ \begin{array}{l} =1, \text{ when } i=j \\ =0, \text{ otherwise} \end{array} \right\} = I_{ij}$

$$R_{ij} = (1-\cos\theta)\, n_i n_j + \cos\theta \cdot \underbrace{\delta_{ij}} - \sin\theta \cdot \underbrace{e_{ijk}}\, n_k$$

$$e_{ijk} = \begin{cases} 1, & ijk = \text{even permutation } \{123, 231, 312\} \\ -1, & ijk = \text{odd permutations } \{213, 132, 321\} \\ 0, & \text{for the 21 other cases } \{i=j \vee j=k \vee k=i \vee i=j=k\} \end{cases}$$

Direct Tensor Notation

$$\vec{r}\,' = R \cdot \vec{r} = (1-\cos\theta)\,\hat{n}\hat{n} + \cos\theta\, I + S^t(\hat{n}) \sin\theta$$

?

— Given R, can we find $\hat{n}$ and θ ?

$$\mathrm{trace}(R) = R_{ii} = R_{11} + R_{22} + R_{33} = (1-\cos\theta)\underbrace{(n_1n_1 + n_2n_2 + n_3n_3)}_{1 \,=\, n^T n \text{ (unit vect)}} + \cos\theta \cdot \underset{\delta_{ii}}{3} + \sin\theta \cdot 0$$

$$\Rightarrow \mathrm{trace}(R) = (1-\cos\theta) + 3\cdot\cos\theta$$
$$= 1 + 2\cos\theta \Rightarrow$$

$$\boxed{\cos\theta = \frac{tr(R)-1}{2}}$$

$\hat{n}, \theta$:)

$$\boxed{n_3 = -\frac{(R_{12}-R_{21})}{2\sin\theta}, \quad n_2 = \frac{R_{13}-R_{31}}{2\sin\theta}, \quad n_1 = -\frac{(R_{23}-R_{32})}{2\sin\theta}}$$

— <u>Rotation Matrix for ZXZ Euler angles</u>

$$R = R_\theta R_\phi R_\psi \quad , \vec{r}' = R\vec{r}$$

$$R_\psi = Rot_z(\psi) \quad , z = \text{original } z\text{-axis}$$

$$= [1-\cos\psi]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cos\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\psi \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\psi = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_\phi = \cdots \qquad R_\theta = \cdots$

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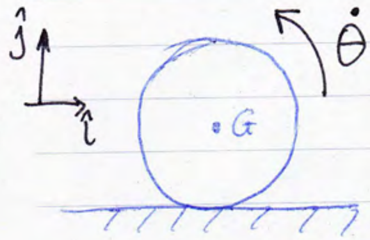
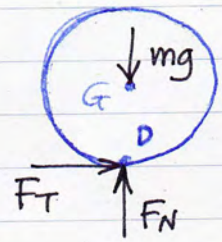
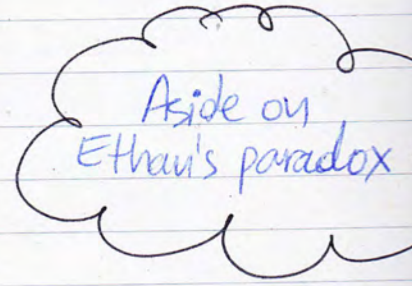


<u>FBD</u>





$\boxed{\vec{a}\cdot\hat{j} = 0}$ constraint

3 equations: $F_T,\ F_N,\ \ddot{\theta}$

$AMB_{/D} \Rightarrow \vec{r}_{G/D} \times \vec{a}_G m + I^G \ddot{\theta}\hat{k}$, $\vec{a}_G = -r\ddot{\theta}\hat{\imath} \Rightarrow \boxed{\ddot{\theta} = 0}$

$\Downarrow$

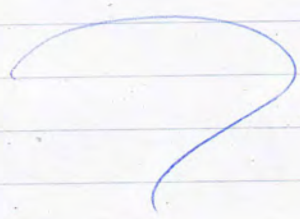
$\boxed{a_G = 0}$

$LMB \Rightarrow F_T\hat{\imath} + F_N\hat{\jmath} - mg\hat{\jmath} = m\vec{a}_G \Rightarrow$

$\{LMB\}\cdot\hat{\imath} \Rightarrow \boxed{F_T = 0}$

And $\boxed{F_N = mg}$

<u>Race between wheel & block</u> (H/W problem)



AGAIN

$$\vec{r}' = \underline{\underline{R}} \cdot \vec{r} \longrightarrow \underline{\underline{R}} = R_{ij}\hat{e}_i\hat{e}_j$$

$$= (1-\cos\theta)\hat{n}\hat{n} + \cos\theta \underbrace{\underline{\underline{I}}}_{\hat{e}_i\hat{e}_i} + \sin\theta \underbrace{\underline{\underline{S}}(\hat{n})}_{\epsilon_{ijk}\hat{e}_i n_j \hat{e}_k}$$

$ijk \qquad ijk$

<u>Check</u> $\underline{\underline{S}}(\hat{n})$

$$(\epsilon_{ijk}\hat{e}_i n_j e_k)\cdot\vec{r} \overset{?}{=} \hat{n}\times\vec{r}$$

$$\epsilon_{ijk} e_i n_j e_k (r_\ell \hat{e}_\ell) \overset{?}{=} (n_2 r_3 - n_3 r_2)\hat{e}_1 + (n_3 r_1 - n_1 r_3)\hat{e}_2 + (n_1 r_2 - r_2 n_1)\hat{e}_3$$

$$\epsilon_{ijk} e_i n_j r_k \quad ?$$

Check for $i = 1$ $\quad j,k = \{2,3\},\{3,2\}$
$= 2 \quad \dots$
$= 3 \quad \dots$
$\longrightarrow \checkmark$

$$\underline{\underline{R}} = \hat{e}_i' e_i \Rightarrow \underline{\underline{R}}\vec{r} = \hat{e}_i'\hat{e}_i(r_k \cdot \hat{e}_k) = r_i\hat{e}_i'$$

$$\boxed{\begin{aligned} \hat{e}_1' &= \mathrm{Rot}(\hat{e}_1) \\ \hat{e}_2' &= \mathrm{Rot}(\hat{e}_2) \\ \hat{e}_3' &= \mathrm{Rot}(\hat{e}_3) \end{aligned}}$$

$$[\underline{\underline{R}}]_{\underset{\hat{e}}{\parallel}} = \underbrace{\left[[\hat{e}_1']_e \,\middle|\, [\hat{e}_2']_e \,\middle|\, [\hat{e}_3']_e \right]}_{\text{components of } \mathrm{Rot}(\hat{e}) \text{ in } \hat{e} \text{ basis}}$$

MIQUELRIUS

Small Rotations

$$\underline{R}\cdot = (1-\cos\theta)\,\hat{n}\hat{n}\cdot + \underbrace{\cos\theta}_{= 1-\theta^2/2+\cdots}\,\underline{I}\cdot + \sin\theta\,\hat{n}\times$$

Say $\theta \ll 1$, keep terms of 1^{st} order in θ.

$$\Rightarrow \quad \boxed{\underline{R} \simeq \underline{I} + \theta\,\underline{S}(\hat{n})}$$

- Two (2) sequential small rotations:

$$R_2 R_1 \simeq \left[I + \theta_2 S(\hat{n}_2)\right]\cdot\left[I + \theta_1 S(\hat{n}_1)\right] \xrightarrow{\text{only } 1^{st} \text{ order}}$$

$$\simeq \underline{I} + \theta_2\,\underline{S}(\hat{n}_2) + \theta_1\,\underline{S}(\hat{n}_1)$$

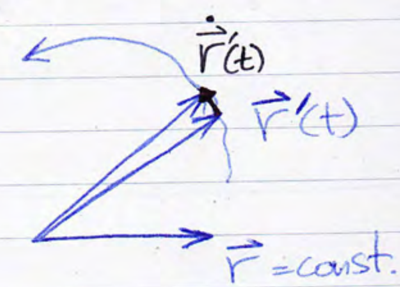
But $\underline{R}_1 \cdot \underline{R}_2$ = same thing

↓

small rotations **do** commute (to 1^{st} order)

$$\vec{r}\,' = \underline{R}\,\vec{r}$$

→ constant in that frame



- $\dot{\vec{r}}\,' = ? = \dot{\underline{R}}\,\vec{r} + \cancel{\underline{R}\,\vec{0}}$

Note: Now $\underline{R} = \underline{R}(t)$ → continuous time, NOT $t_0 \to t_f$

$$\begin{pmatrix}\text{rate of}\\ \text{change}\end{pmatrix} = \underbrace{\dot{\underline{R}}\cdot\underline{R}^{-1}}_{\text{some matrix}}\cdot\underbrace{\vec{r}\,'}_{\text{current position}}, \quad \underline{R}^{-1} = \underline{R}^T \quad \Rightarrow$$

$$\dot{\vec{r}}\,' = \omega\,\vec{r}\,', \quad \omega \equiv \dot{\underline{R}}\,R^{-1}$$

MIQUELRIUS

Small Rotations & Angular Velocities

1. Look at rotation wrt configuration at t:

$$\underline{\underline{R}}(\Delta t) = \underline{\underline{I}} + \theta \underline{S}(\hat{n})$$

What is the correct answer in the wheel vs block problem? → Wheel goes further...

$$\left.\begin{aligned}\dot{\vec{r}}_p &= \lim_{\Delta t \to 0} \frac{\vec{r}_p(t+\Delta t) - \vec{r}_p(t)}{\Delta t} \\ \vec{r}_p(t) &= \underline{\underline{I}}\, \vec{r}_p(t) \\ \vec{r}_p(t+\Delta t) &= (\underline{\underline{I}} + \Delta\theta\, \underline{S}(\hat{n}))\, \vec{r}_p\end{aligned}\right\} \dot{\vec{r}}_p = \frac{\Delta\theta}{\Delta t} S(\hat{n})\, \vec{r}_p$$

Define: $\boxed{\underline{\underline{\omega}} = \dot{\theta} \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}$

$$\boxed{\dot{\vec{r}}_p = \underline{\underline{\omega}} \cdot \vec{r}_p = \vec{\omega} \times \vec{r}_p \qquad \vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \dot{\theta} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}}$$

Answer did not depend on choice of $\vec{r}_p$

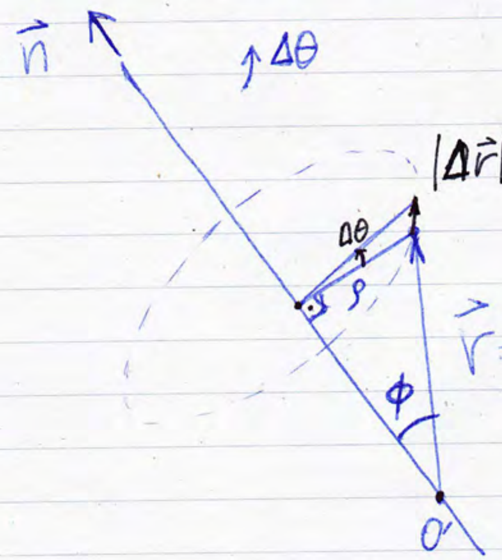
Most important formula: $\boxed{\dot{\vec{r}}_p = \vec{\omega} \times \vec{r}_p}$

MIQUELRIUS

2. Geometric Reasoning:

Recall: $\hat{n}\times\vec{r} = \cancel{|\hat{n}|}^{\,1}\cdot|\vec{r}|\cdot\sin\phi\,\hat{u}$

$\uparrow$ $\perp$ to $\hat{n},\vec{r}$ pla



$\vec{r}$ = fixed in object

$$|\Delta\vec{r}| = \Delta\theta\cdot\rho = \Delta\theta|\vec{r}|\sin\phi = \Delta\theta\,\hat{n}\times\vec{r}$$

$$\Downarrow$$

$$\Delta\vec{r} = \Delta\theta\,\hat{n}\times\vec{r}$$

$$\Downarrow$$

$$\boxed{\dot{\vec{r}} = \vec{\omega}\times\vec{r},\ \vec{\omega}\equiv\dot{\theta}\hat{n}}$$

3. "Third" method

$$\underbrace{\vec{r}'}_{\vec{r}(t)} = \underline{\underline{R}}\cdot\underbrace{\vec{r}_0}_{\vec{r}(t=0)} \Longrightarrow \dot{\vec{r}}' = \dot{\underline{\underline{R}}}\cdot\vec{r}_0$$

$$\vec{r}' = \underline{\underline{R}}\,\vec{r}_0 \Rightarrow \vec{r}_0 = \underline{\underline{R}}^T\vec{r}' \quad \Rightarrow$$

$$\boxed{\dot{\vec{r}}' = \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T\vec{r}'}$$

Define $\underline{\underline{\omega}} = \dot{\underline{\underline{R}}}\cdot\underline{\underline{R}}^T$,

$$\boxed{\underline{\underline{\omega}} = \dot{R}\cdot R^T}$$

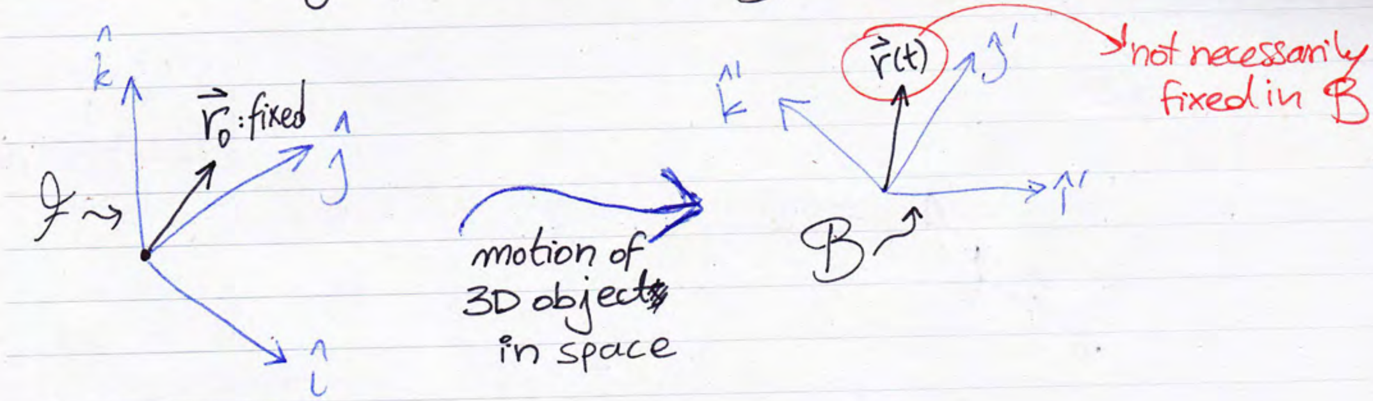
$\dot{\underline{\underline{I}}} = \underline{\underline{0}}$

$$\dot{(\underbrace{RR^T}_{I})} = \dot{R}R^T + R\cdot(\dot{R}^T)$$

$$0 = \dot{R}R^T + (\dot{R}R^T)^T \Rightarrow \dot{R}\cdot R^T = \underline{\underline{\omega}} = \text{skew-symmet}$$

$$\underline{\underline{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Change of Coordinates vs Rotation



Could also use "vectrix" notation

$$\vec{r} = \vec{r}$$

$$r_i^{\mathcal{F}} \cdot \hat{e}_i = r_i^{\mathcal{B}} \cdot \hat{e}_i'$$
$$= \hat{e}_i' \cdot r_i^{\mathcal{B}}$$

$$r_i^{\mathcal{F}} = R_{ij} \cdot r_j^{\mathcal{B}} \rightarrow [\vec{r}]_{\mathcal{F}} = [\underline{\underline{R}}]\,[\vec{r}]_{\mathcal{B}}$$

$$[\underline{\underline{R}}] = \left[\, [\hat{e}_1']_{\mathcal{F}} \;\middle|\; [\hat{e}_2']_{\mathcal{F}} \;\middle|\; [\hat{e}_3']_{\mathcal{F}} \,\right]$$

★ R was used to change coord. for a given set of frames $\mathcal{F}$ and $\mathcal{B}$.

<u>Recall</u>:

$$\hat{e}_i^{\mathcal{B}} = R_{ij}\, \hat{e}_j^{\mathcal{F}}$$
$$\hat{e}_i^{\mathcal{F}} = R_{ji}\, \hat{e}_j^{\mathcal{B}}$$

$$\dot{\vec{r}} = \dot{\vec{r}} \Rightarrow \dot{r}_i^{\mathcal{F}}\hat{e}_i = \dot{r}_i^{B}\hat{e}_i' + r_i\,{}^{B}\dot{\hat{e}}_i'$$

$$\boxed{\dot{\hat{e}}_i' = \vec{\omega}_{B/\mathcal{F}} \times \hat{e}_i'}$$

$$\dot{r}_i^{\mathcal{F}}\hat{e}_i = \dot{r}_i^{B}\hat{e}_i' + \vec{\omega}\times\vec{r}$$

$$\boxed{\frac{{}^{\mathcal{F}}d\vec{r}}{dt} = \frac{{}^{B}d\vec{r}}{dt} + \vec{\omega}\times\vec{r}}$$

Q-dot Formula / Transport thm

same thing

$$\boxed{{}^{\mathcal{F}}\dot{\vec{Q}} = {}^{B}\dot{\vec{Q}} + \vec{\omega}_{B/\mathcal{F}}\times\vec{Q}}$$

, for any $\vec{Q}$

Derivatives in a frame defined by differentiating components and not the base vectors.

~~2/12~~

Mechanics

"The 3 pillars":

I. Material Properties
II. Geometry / Kinematics
III. Laws of Mechanics
- $F = \dot{L}$
- $M = \dot{H}$
- action & reaction
- $P = \dot{E}_k$
- ...

AMB: For any point G and any system (closed), $\boxed{\sum \vec{M}^{ext}_{/C} = \dot{\vec{H}}_{/C}} = \cdots$

closed = fixed collection of material

$$\dot{\vec{H}}_{/C} = \int \vec{r}_{/C} \times \vec{a}\, dm, \quad \vec{a} = \vec{a}_{/F}$$

H/W 5735

$$= \frac{d}{dt}\left[\vec{r}_{G/C} \times \vec{V}_G\, m_{TOT} + \int \vec{r}_{G/C} \times \vec{V}_G\, dm\right] \circledast,$$

$\vec{V}_G = \vec{V}_{G/F}$; in the integral, $\vec{V}_G$ stands for $= \vec{V} - \vec{V}_G$ $= \vec{V}$ in a frame that moves w/ G & doesn't rotate.

$\vec{V} = \dot{\vec{r}} = \frac{d^F}{dt}\vec{r}_{/O}$

- $\vec{H}_{G/C} = \vec{r}_{G/C} \times (m_{TOT}\vec{V}_G)$

$$\vec{H}_{/G} = \int \vec{r}_{/G} \times \vec{V}_G\, dm$$

$$\dot{\vec{H}}_{/C} = \frac{d}{dt}\left(\vec{H}_{/C}\right), \quad \boxed{\vec{H}_{/C} = \vec{H}_{G/C} + \vec{H}_{/G}} \quad \text{(not vector addition)}$$

★ Special case: $G \equiv C$

$$\sum \vec{M}_{/G} = \frac{d}{dt}\left(\vec{H}_{/G}\right)$$

$$\vec{H}_{/G} = \int \vec{r}_{/G} \times \vec{V}_{/G}\, dm$$

$\circledast$ Also, $\dot{\vec{H}}_{/C} = \frac{d}{dt}\left[\vec{r}_{G/C} \times \vec{V}_G \left(\sum m_i\right) + \sum_{\text{all mass}} \left(\vec{r}_{i/G} \times \vec{V}_{i/G}\, m_i\right)\right]$

MIQUELRIUS

$AMB_{/G}$ of a rigid object B:

$$\sum \vec{M}^{ext}_{/G} = \frac{d}{dt}\left(\underline{\underline{\vec{H}_{/G}}}\right)$$

this is the tough part in 3D!

★ What is $\vec{H}_{/G}$? $\quad \vec{H}_{/G} = \int_B \vec{r}_{/G} \times \vec{v}_G\, dm =$

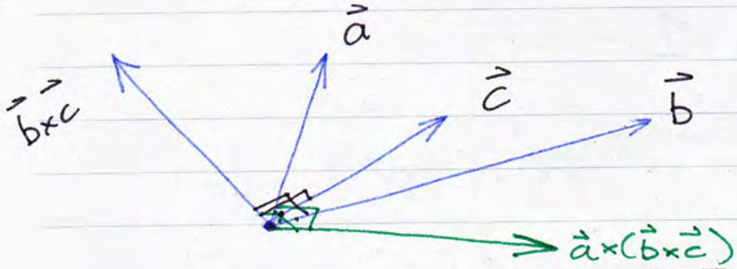
$$= \int_B \vec{r}_G \times (\underbrace{\vec{\omega} \times \vec{r}_{/G}}) \, dm$$

$\vec{v}_{/G} = \vec{\omega} \times \vec{r}_{/G}$

ASIDE: $\quad \vec{a} \times \vec{b} \times \vec{c}$ is ambiguous! because

$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

We have $\underline{\vec{a}\times(\vec{b}\times\vec{c})}$ = (Don Lewis & Andy Ruina method)



Distributive law of cross product:
$\vec{a}\times(\vec{d}+\vec{e}) = \vec{a}\times\vec{d} + \vec{a}\times\vec{e}$
(see Ruina & Pratap for derivation)

$\hookrightarrow$ in plane of $\vec{b}, \vec{c}$ $\Rightarrow$

$$\Rightarrow \boxed{\vec{a}\times(\vec{b}\times\vec{c}) = d_1 \vec{b}\,\underbrace{(\vec{a}\cdot\vec{c})}_{\text{number}} + d_2 \vec{c}\,\underbrace{(\vec{a}\cdot\vec{b})}_{\text{number}}},$$

where $\boxed{d_1 = +1 \quad , \quad d_2 = -1}$

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})}$$

"bac minus cab"

END ASIDE

Therefore, $\vec{H}_{/G} = \int \vec{r}_{/G} \times (\vec{\omega} \times \vec{r}_{/G})\, dm$

$$= \int \left[\vec{\omega}(\vec{r}_{/G} \cdot \vec{r}_{/G}) - \vec{r}_{/G}(\vec{r}_{/G} \cdot \vec{\omega}) \right] dm$$

Dynamics of a Rigid Objects in 3D wrt COM 2/14

$$\vec{H} = \vec{H}_{/G} = \int \vec{r} \times \vec{v}\, dm$$

$\vec{v} = \vec{v}_{/G} = \vec{\omega} \times \vec{r}$, $\vec{\omega} = \vec{\omega}_{B/F}$

$\vec{r} = \vec{r}_{/G}$

"bac minus cab" $= \int \left(\vec{\omega} \underbrace{(\vec{r} \cdot \vec{r})}_{\text{scalar}} - \underbrace{\vec{r}(\vec{r} \cdot \vec{\omega})}_{(\underline{\vec{r}\vec{r}})\vec{\omega}} \right) dm$

integral part does not depend on $\vec{\omega}$ $= \left[\int (\vec{r} \cdot \vec{r})\, dm\, \underline{\mathbf{1}} - \int \underline{\vec{r}\vec{r}}\, dm \right] \cdot \vec{\omega}$

Identity $\leftarrow$ "$I_{3\times3}$"

$\underline{\vec{r}\vec{r}} = \vec{r} \otimes \vec{r}$ ($=$ r*r')

$$[\vec{r}\vec{r}]_F = \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$$

$$\boxed{\vec{H} = \underline{\underline{I}} \cdot \vec{\omega}}$$

I for Inertia

$$\underline{\underline{I}} = \int \vec{r} \cdot \vec{r}\, dm\, \underline{\mathbf{1}} - \int \vec{r} \otimes \vec{r}\, dm$$

- $[I]_{ij} = \int \underset{\substack{\uparrow \\ r_k r_k}}{r^2} \delta_{ij}\, dm - \int r_i r_j\, dm$

MIQUELRIUS

- $$[I]_{\mathcal{E}} = \int \underbrace{\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}}_{\text{Inertia Matrix}} dm$$

Inertia Matrix (Symmetric) → existence of orthonormal eigenvectors.

- We can find coordinate system, say B, such that: → "prime system"

$$[I]_B = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$[I]_{i'j'} = I_1 \hat{e}_1' \hat{e}_1' + I_2 \hat{e}_2' \hat{e}_2' + I_3 \hat{e}_3' \hat{e}_3'$$

- Each bit of mass has a fixed-in-time coordinate x', y', z'.
- These $\hat{e}_1'$, $\hat{e}_2'$, $\hat{e}_3'$ are fixed to the body!

- $$\begin{cases} I_1 = \int (y'^2 + z'^2)\, dm \\ I_2 = \int (x'^2 + z'^2)\, dm \\ I_3 = \int (x'^2 + y'^2)\, dm \end{cases}$$

MIQUELRIUS

Properties of $\underline{\underline{I}}$

~~Properties~~

elements are sums of squares.

- $I_1 > 0,\ I_2 > 0,\ I_3 > 0 \Rightarrow$ (positive definite)
- $I_1 + I_2 = \int x'^2 + y'^2 + 2z'^2\, dm \geq I_3 = \int x'^2 + y'^2\, dm$

$$\Downarrow \; \vdots$$

$$\underbrace{\left\{\begin{array}{l} I_1 + I_2 \geq I_3 \\ I_1 + I_3 \geq I_2 \\ I_2 + I_3 \geq I_1 \end{array}\right\}}_{\text{"Triangle Inequality"}}$$

ex

OK ✓ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, NOT ok $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

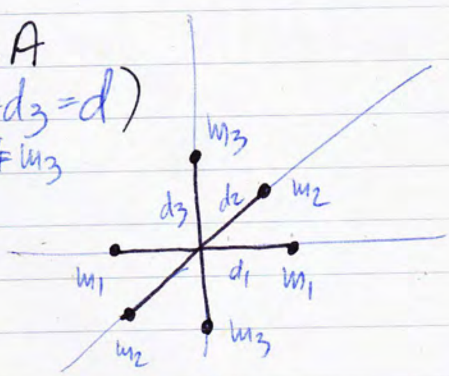
- Symmetric

Canonical Rigid Objects

A) Jack A

($d_1 = d_2 = d_3 = d$)

$m_1 \neq m_2 \neq m_3$



B) Jack B

$$\left(\begin{array}{l} m_1 = m_2 = m_3 = m/6 \\ d_1 \neq d_2 \neq d_3 \end{array}\right)$$

MIQUELRIUS

Rigid object has 7 free parameters:

6 components of I + 1 mass m

Given principal directions $\rightarrow$ 4 free parameters: I_1, I_2, I_3, m

OR

m_1, m_2, m_3, d

OR

d_1, d_2, d_3, m

Examples

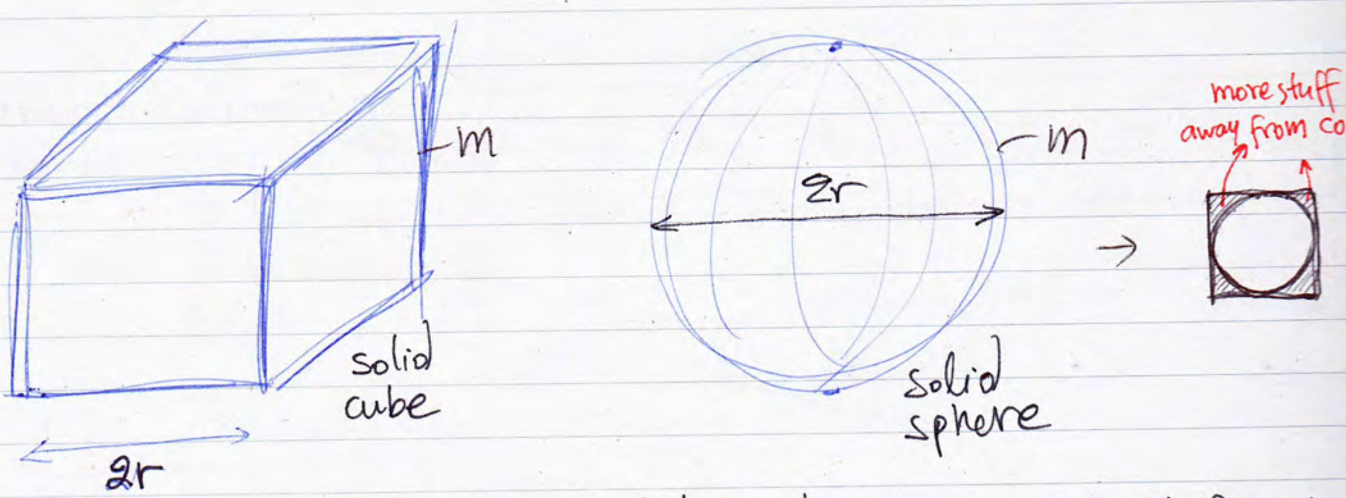
1) Hoop, (r, m)

$$[I]_B = \begin{bmatrix} mr^2/2 & 0 & 0 \\ 0 & mr^2/2 & 0 \\ 0 & 0 & mr^2 \end{bmatrix}$$ → the classic one, I_z

For planar objects $\left(\int z^2 dm = 0 \right)$:

$$\boxed{I_1 + I_2 = I_3}$$

Perpendicular Axis Theorem



* Which one has bigger moment of inertia

$V_c = 8r^3, \quad V_s = \frac{4}{3}\pi r^3$

H/W $\quad I_c = \frac{2}{3}mr^2 \qquad I_s = \frac{2}{5}mr^2$

"Surely you're joking Mr. Feynman" → MUST READ
↪ Cornell alumn?

"Who cares what other people think", Feynman

PUZZLE PROBLEM (p.157)

"Plate in midair spins twice as fast as it wobbles." 2:1

Rigid-object Rotation

$AMB_{/C}$:

$$\Sigma \vec{M}^{ext}_{/C} = \int \vec{r}_{/C} \times \vec{a}_{\mathcal{F}} dm$$

$$\Downarrow$$

$$\Sigma \vec{M}^{ext}_{/G} = \frac{d}{dt}(\vec{H}_{/G}) \quad \vec{H}_{/G} = \int \vec{r}_{/G} \times \vec{v}_{/G} dm$$

For a rigid object: $\vec{H}_{/G} = \underline{\underline{I}} \cdot \vec{\omega}$ ⇒

constant in B

↪ $\vec{\omega}_{B/\mathcal{F}}$

$$\Sigma \vec{M}^{ext}_{/G} = \frac{{}^{\mathcal{F}}d}{dt}(I \cdot \vec{\omega}) = \frac{{}^{B}d}{dt} I\vec{\omega} + \vec{\omega} \times (I\vec{\omega}) \Rightarrow$$

$\dot{Q}$ formula

$$\boxed{\Sigma \vec{M}^{ext}_{/G} = \underline{\underline{I}} \cdot \dot{\vec{\omega}} + \underbrace{\vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega})}}$$

Differential Equation

(*) Euler equation

Was 0 in 2D rotation because it was: $\hat{k} \times \hat{k} = \vec{0}$

• Is $\dot{\vec{\omega}}$ vague notation?

$${}^{\mathcal{F}}\dot{\vec{\omega}} = {}^{B}\dot{\vec{\omega}} + \cancelto{\vec{0}}{\vec{\omega} \times \vec{\omega}}$$

$$\vec{\omega} = \vec{\omega}$$

$$\downarrow$$

$$\omega_i \hat{e}_i = \omega_{i'} \hat{e}_{i'} \rightarrow {}^{\mathcal{F}}\dot{\vec{\omega}} = {}^{B}\dot{\vec{\omega}} \rightarrow \dot{\omega}_i \hat{e}_i = \dot{\omega}_{i'} \hat{e}_{i'}$$

(✱) Solve Euler equation in body-fixed coordinates

Let's assume $\hat{e}_i'$ are aligned with the e-vectors of $\underline{\underline{I}}$:

$$\underline{\underline{I}} = I_1 \hat{e}_1' \hat{e}_1' + I_2 \hat{e}_2' \hat{e}_2' + I_3 \hat{e}_3' \hat{e}_3'$$

$$[\underline{\underline{I}}]_B = \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix}$$

$$[\vec{\omega}]_B = \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix}$$

$$(*) \Rightarrow \dot{\vec{\omega}} = \underline{\underline{I}}^{-1} \cdot \left[\vec{M}_{/G} - \vec{\omega} \times (\underline{\underline{I}} \vec{\omega})\right] \Rightarrow$$

$$[\dot{\vec{\omega}}]_B = [\underline{\underline{I}}]_B^{-1} \cdot \left[[\vec{M}_{/G}]_B ? - [\vec{\omega}]_B \times \left([\underline{\underline{I}}]_B \cdot [\vec{\omega}]_B \right) \right]$$

Make it look simple: "All things in B frame coords"

$$\dot{\omega} = \underbrace{I^{-1}}_{\begin{bmatrix} 1/I_1 & & 0 \\ & 1/I_2 & \\ 0 & & 1/I_3 \end{bmatrix}} (M - \omega \times (I \cdot \omega))$$

→ 3 nonlinear ODEs that can be solved in MATLAB

If $\cancel{M} \to 0$, we have the plate spinning and wobbling in mid-air.

$$\underset{3\times1}{\dot{\omega}} = I^{-1} \cdot \left[\underset{3\times1}{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} \times \underset{3\times1}{\begin{bmatrix} \omega_1 I_1 \\ \omega_2 I_2 \\ \omega_3 I_3 \end{bmatrix}} \right] = \underset{3\times3}{\begin{bmatrix} 1/I_1 & & 0 \\ & 1/I_2 & \\ 0 & & 1/I_3 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} \omega_2 \omega_3 I_3 - \omega_3 \omega_2 I_2 \\ \omega_3 \omega_1 I_1 - \omega_1 \omega_3 I_3 \\ \omega_1 \omega_2 I_2 - \omega_2 \omega_1 I_1 \end{bmatrix}}$$

Famous Problem #1

A) Can you find **ANY** non-zero solution?

$\vec{\omega} = \text{constant} \quad \left([\vec{\omega}]_B\right)$

Check: $0 = I^{-1} \cdot (\underbrace{0 - \omega \times (I\omega)}_{\text{null space of } I^{-1}}) \Rightarrow$

non-singular matrix

null space of $I^{-1} \Longrightarrow$

$$\vec{\omega} \times (I\vec{\omega}) = \vec{0} \Rightarrow \vec{\omega} \parallel \text{ to } \underline{\underline{I}}\vec{\omega} \Rightarrow$$

$c\,\vec{\omega} = (I\vec{\omega}) \Rightarrow \vec{\omega}$ is an e-vector of $\underline{\underline{I}}$!!!

$\vec{\omega} = \omega_1 \hat{e}_1'$ or $\vec{\omega} = \omega_2 \hat{e}_2'$ or $\vec{\omega}_3 = \omega_3 \hat{e}_3'$

$\boxed{\vec{\omega} = \text{const.} \Longrightarrow \text{spin about principal axis}}$ OR there are torques.

B) Const. $\vec{\omega}$ stable? → as in, subset of robustness

without loss of generality: $[\vec{\omega}] = \begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix}$, perturbation $\boxed{\hat{\omega}_i \ll \omega}$

$$\dot{\omega} = 0 \rightarrow \begin{bmatrix} \dot{\omega}_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 1/I_1 & & 0 \\ & 1/I_2 & \\ 0 & & 1/I_3 \end{bmatrix} \left(\begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix} \times \begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix} \begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix} \right) \xrightarrow{\text{let's check}} \text{e-values}$$

$$\dot{\hat{\omega}}_1 = \frac{1}{I_1}\left[-\underbrace{(\hat{\omega}_2 \hat{\omega}_3 I_3 - \hat{\omega}_3 \hat{\omega}_2 I_2)}_{\text{2nd order in small perturbations}} \right]$$

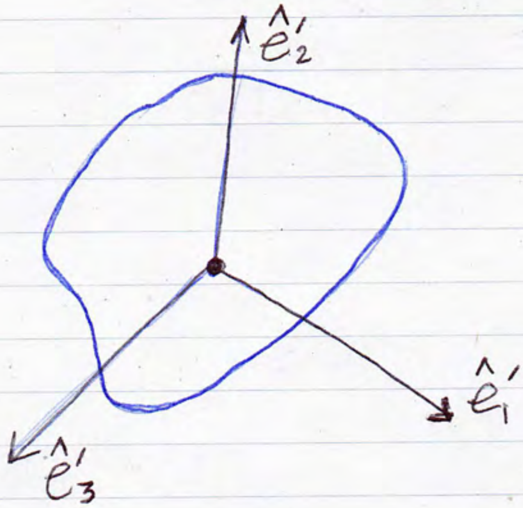
2nd order in small perturbations $\Rightarrow \phi$

MIQUELRIUS

$$\dot{\hat{\omega}}_2 = \frac{1}{I_2}\left[-\left(\hat{\omega}_3(\hat{\omega}_1+\omega)I_1 - (\omega+\hat{\omega}_1)\omega_3 I_3\right)\right]$$

$$\dot{\hat{\omega}}_3 = \frac{1}{I_3}\left[-\left((\omega+\hat{\omega}_1)\hat{\omega}_2 I_2 - \hat{\omega}_2(\omega+\hat{\omega}_1)I_1\right)\right]$$

Rigid Object (cont'd) 2/21



ω_i : components in $\mathcal{F}$

ω_i' -- " -- -- " -- B

$\underline{\underline{I}}$: tensor (matrix)

$$\vec{M}_{/G} = \underline{\underline{I}}\vec{\omega} + \vec{\omega}\times(\underline{\underline{I}}\vec{\omega}) \xrightarrow[\text{coordinates (drop ')}]{\text{Body-fixed}} \xrightarrow{\vec{M}_{/G}=\vec{0}}$$

$$\begin{cases} \dot{\omega}_1 = \omega_2\omega_3\dfrac{(I_2-I_3)}{I_1} \\ \dot{\omega}_2 = \omega_3\omega_1\dfrac{(I_3-I_1)}{I_2} \\ \dot{\omega}_3 = \omega_1\omega_2\dfrac{(I_1-I_2)}{I_3} \end{cases} \longrightarrow$$

For $\dot{\vec{\omega}}=\vec{0}$, only one of ω_i can be nonzero! ¿:

$\vec{M}_{/G}=\vec{0}$ $\quad i=\{1,2,3\}$

Look at small perturbations:

$$\left[\vec{\omega}\right] = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix}$$

$$\dot{\hat{\omega}}_1 = \cancel{\hat{\omega}_2 \hat{\omega}_3 \cdot (I_2 - I_3)}^{\simeq 0} / I_1$$

$$\dot{\hat{\omega}}_2 = \hat{\omega}_3 (\omega + \underbrace{\hat{\omega}_1}_{\sim 0}) \cdot (I_3 - I_1)/I_2 \cong \omega . \frac{\hat{\omega}_3 (I_3 - I_1)}{I_2} \Rightarrow$$

$$\dot{\hat{\omega}}_3 = (\omega + \underbrace{\hat{\omega}_1}_{\sim 0}) \hat{\omega}_2 \cdot (I_1 - I_2)/I_3 \cong \omega . \frac{\hat{\omega}_2 (I_1 - I_2)}{I_3}$$

$$\left.\begin{aligned} \dot{\hat{\omega}}_2 &= \overbrace{\left[\omega \frac{(I_3 - I_1)}{I_2}\right]}^{\text{constant}} \cdot \hat{\omega}_3 \\ \dot{\hat{\omega}}_3 &= \overbrace{\left[\omega \frac{(I_1 - I_2)}{I_3}\right]}^{\text{constant}} \cdot \hat{\omega}_2 \end{aligned}\right\} \Longrightarrow$$

1 2nd order d.e.

$$\boxed{\ddot{\hat{\omega}}_2 = \left[\omega \frac{(I_3 - I_1)}{I_2}\right] \cdot \dot{\hat{\omega}}_3 = \underbrace{\left[\omega \frac{(I_1 - I_2)}{I_2}\right] \cdot \left[\omega \frac{(I_1 - I_2)}{I_3}\right]}_{[\dots]\text{ sign decides stability}} \hat{\omega}_2}$$

- exp. growth ~~[ ] > 0~~
- exp decay
- cos + sin

$\longrightarrow$

solutions are linear combinations of these two.

- If $[\,]>0$,

$\Rightarrow \hat{\omega}_2 = e^{\sqrt{[\,]}t}, \; e^{-\sqrt{[\,]}t}$ → UNSTABLE (will only be stable for VERY SPECIAL ICs)

(or) $\sinh\sqrt{[\,]}t, \; \cosh\sqrt{[\,]}t$

- If $\boxed{[\,]<0}$, STABLE

$\Rightarrow \hat{\omega}_2 = \sin\sqrt{[\,]}t$ or $\cos\sqrt{[\,]}t$

"STABLE" iff $[\,]<0 \Leftrightarrow$

$$(I_3-I_1)(I_1-I_2)<0$$

Cases:

- $I_1<I_2$ & $I_1<I_3 \Rightarrow [\,]<0$
- $I_1>I_2$ & $I_1>I_3 \rightarrow [\,]<0$

"I_1 either smallest, or largest." ~ Tunc Ertan

- else $\rightarrow [\,]>0$

Spin about biggest I axis, or about smallest I axis is STABLE.

UNSTABLE cases:

$I_2<I_1<I_3$ or $I_2>I_1>I_3$

Objects of interest: American football, pencil, plate

Example: Axi-symmetric object, and symm axis I_1 axis

$$I_1, \quad I_2 = I_3$$

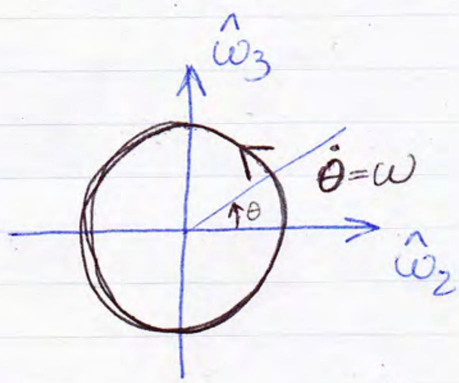
$$\boxed{\text{For plate: } \quad I_1 = 2I_2 = 2I_3} \rightarrow \perp \text{axis thm}$$

(let all mass in $X=0$ plane)

Perturbation equations:

$$\dot{\hat{\omega}}_2 = \omega(-1)\cdot\hat{\omega}_3$$

$$\dot{\hat{\omega}}_3 = \omega(+1)\cdot\hat{\omega}_2 \quad \Rightarrow$$



"Axis of rotation ($\vec{\omega}$ direction) precesses around $\hat{e}_1'$ direction at rate ω."

What about body orientation?

$$\begin{bmatrix} \hat{e}_1'(t) \\ \hat{e}_2'(t) \\ e_3'(t) \end{bmatrix} \rightarrow ? \quad \text{(in } \mathcal{F} \text{ frame)} \qquad \text{Given } [\vec{\omega}(t)]_B \text{ and } \hat{e}_i'(0)$$

$[\vec{\omega}(t)]_B$ ← solution to "Euler eqs".

$$^{\mathcal{F}}\dot{\hat{e}}_1 = \vec{\omega} \times \hat{e}_1' \quad \Rightarrow [\dot{\hat{e}}_1]_{\mathcal{F}} = [\vec{\omega}]_{\mathcal{F}} \times [\hat{e}_1']_{\mathcal{F}}$$

We need those in $\mathcal{F}$ frame/system.

1st column of $^{\mathcal{F}}R_B(\cdot)$

$$[\vec{\omega}]_{\mathcal{F}} = [^{\mathcal{F}}R_B]\cdot[\vec{\omega}]_B$$

Recall:

$$[R] = \begin{bmatrix} [\hat{e}_1']_{\mathcal{F}} & [\hat{e}_2']_{\mathcal{F}} & [\hat{e}_3']_{\mathcal{F}} \end{bmatrix}$$

and $R_{ij} = \hat{e}_i \cdot \hat{e}_j' = \hat{e}_j' \cdot \hat{e}_i$

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Likewise with $\left[\dot{\hat{e}}_2'\right]_{\mathcal{F}}$ and $\left[\dot{\hat{e}}_3'\right]_{\mathcal{F}}$ $\longrightarrow$

$$[\dot{R}] = \underbrace{\left[[R][\vec{\omega}]_B\right]}_{S(R[\omega]_B)} \text{"}\times\text{"} R \quad \Rightarrow \quad \boxed{[\dot{R}] = \underbrace{S(R\cdot[\omega]_B)}\cdot R}$$

(any) use skew symmetric function with that argumen

9 ODEs that you solve with Euler equs.

$\hookrightarrow$ too much information (there are constraints)

Integration error $\Rightarrow$ $R \notin SO(3)$ "R will fall apart"

$\hookrightarrow$ there are a couple of ways to tackle this. (some kind of projection)

What to do?

- Go to Euler angles
- — " — Quaternions / Euler parameters (renormalize 1 number
- ~~Find~~ a nearby matrix with $R^TR = 1$ (projection)

Calculate

$\hookrightarrow$ How? $\rightarrow$ 2 methods

a) Graham-Schmidt orthogonalization on 3 columns.

$$\hat{e}_1' = \hat{e}_1'^{\,bad} / \|\hat{e}_1'^{\,bad}\|, \quad \hat{e}_2' = \left(\hat{e}_2'^{\,bad} - (\hat{e}_2'^{\,bad}\cdot\hat{e}_1')\hat{e}_1'\right) / \|\text{its magnitude}\|,$$

$$\hat{e}_3 = \hat{e}_1' \times \hat{e}_2'$$

$\uparrow$ shortcut in 3D

Polar decomposition

b) Any matrix: $A = RU$ $\Rightarrow$ MATLAB to extra

R: rotation, $R^TR = 1$

U: a symmetric positive semidefinite matrix

relatively new discovery

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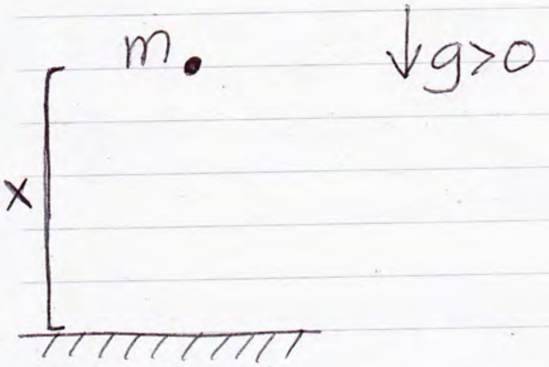
2/26

$$\underline{\underline{R}} = R_{ij}^{\mathcal{F}}\, \hat{e}_i \hat{e}_j = R_{ij}^{B}\, \hat{e}_i' \hat{e}_j'$$

$$R_{ij}^{B} = R_{ij}^{\mathcal{F}}$$

$$R_{BB} = R_{FF} \quad \left([\underline{R}]_{\mathcal{FF}} = [\underline{R}]_{BB} \right)$$

NW's Todd Murphey (guest speaker)



Ans: $\ddot{x} = -g$ → we'll get to this with variational methods.

E-L eqs comes from a minimization of a path integral.
(⟺ equivalent to taking the derivative and setting = 0.)

Gateaux → directional derivative
or
~~Frechet~~?

$f(x^*)$ is an extremum if $\underbrace{\frac{\partial f}{\partial x}}_{\text{i.e. we want this to be 0!}} \cdot v = 0 \;\; \forall v$ (every direction)

Action integral:

$$\overset{A(x(t))}{\overset{\shortparallel}{A}} = \int_0^T L(x, \dot{x})\, dt$$

→ perturbation on x(t)

To take directional derivative of $A(x(t))$ in direction v, v must be a curve $v(t)$.
← differentiable wrt time

derivative wrt argument

Aside: $$Df(x)\cdot v = \frac{d}{d\varepsilon} f(x+\varepsilon v)\Big|_{\varepsilon=0}$$ (from directions to a scalar)

for particle in gravity: $L = KE - PE =$
$= \frac{1}{2}m\dot{x}^2 - mgx$

$$\frac{d}{d\varepsilon}\int_0^T \frac{1}{2}m\left[\frac{d}{dt}(x+\varepsilon v)\right]^2 - mg(x+\varepsilon v)\,dt\,\Big|_{\varepsilon=0} =$$

Assumption 1: v(t) has to be differentiable (wrt to time)

$$= \frac{d}{d\varepsilon}\int_0^T \frac{1}{2}m(\dot{x}^2 + 2\varepsilon\dot{v}\dot{x} + \varepsilon^2\dot{v}^2) - mg(x+\varepsilon v)\,dt\,\Big|_{\varepsilon=0} =$$

$$= \int \frac{d}{d\varepsilon}\left[\frac{1}{2}m(\dot{x}^2 + 2\varepsilon\dot{v}\dot{x} + \varepsilon^2\dot{v}^2) - mg(x+\varepsilon v)\right]dt\,\Big|_{\varepsilon=0} =$$

$$= \int \frac{1}{2}m\,2\dot{v}\dot{x} + 2\varepsilon\dot{v}^2 - mgv\,dt\,\Big|_{\varepsilon=0} =$$

$$= \underbrace{\int m\dot{v}\dot{x}}_{\text{integration by parts!}} - mgv\,dt$$ → this is the directional derivative

integration by parts! $\left(\int u\,dv = uv - \int v\,du\right)$

$$\hookrightarrow \int_0^T m\dot{v}\dot{x}\,dt = m\left(\dot{x}v\Big|_0^T - \int_0^T \ddot{x}v\,dt\right)$$

$\Rightarrow$

$$\Rightarrow m\dot{x}v\Big|_0^T + \int -m\ddot{x}v - mgv\,dt \;\left(= 0 \quad , \forall v \;(v \text{ diff in } t)\right)$$

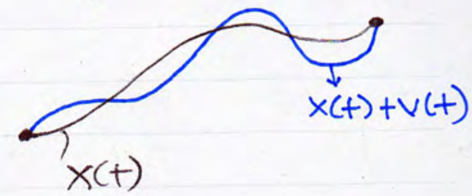
$$\Rightarrow \underbrace{m\dot{x}v\Big|_0^T}_{\text{get rid of this}} - \int m(\ddot{x}+g)v\,dt = 0 \quad , \forall v$$

→

Assumption 2:

Also require that $\boxed{v(0) = v(T) = 0}$ $\Longrightarrow$

graphically



$\{\text{second term} = 0\} \Longrightarrow \ddot{x} + g = 0 \Rightarrow$

$$\boxed{\ddot{x} = -g}$$

✓

Moment of Inertia, Simple 3D Solns 2/28

<u>Sphere:</u> $I_1 = I_2 = I_3 = \frac{2}{3}\int r^2 dm = \cancel{\frac{2}{3}\int_0^R \rho \, \pi 4 \, r'^2 \, dr'}$

$\int dm = \iiint \rho \, dV$ ← dxdydz

$I_{zz} = \int (x^2+y^2)\,dm$

$I_{xx} = \int (y^2+z^2)\,dm$

$I_{yy} = \int (x^2+z^2)\,dm$

$m = \rho \frac{4}{3}\pi R^3$

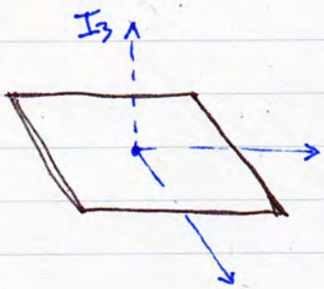
$$= \frac{2}{3}\int_0^R r'^2 \underbrace{\rho \, \underbrace{\overbrace{4\pi r'^2}^{\text{Area}} dr'}_{dV}}_{dm} = \frac{8}{3}\pi\rho\frac{R^5}{5} = \boxed{\frac{2}{5} m R^2}$$

<u>Spherical shell:</u> $I_1 = I_2 = I_3 = \frac{2}{3}\int r^2 dm = \boxed{\frac{2R^2}{3} m} \Rightarrow$

$$[\underline{\underline{I}}]_B = \frac{2}{3} m R^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\longrightarrow$

Cube: $(a=2R)$



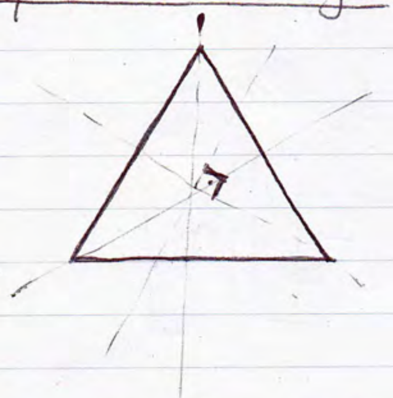
$$I_3 = \int x^2+y^2\,dm$$ (same for a square and a cube)

$$\overset{\perp\text{axis theorem}}{=} 2\int_{-R}^{R} y^2\,dm \Rightarrow I_3 = 2\cdot I_{\text{line segment}} \Rightarrow$$

$\uparrow$ for a line — line segment of length $2R$ and mass m.

$$I_3 = 2\int_{-R}^{R} y^2 \underset{= m/2R}{\rho}\,dy = \frac{2m}{2R}\cdot y^3/3\Big|_{-R}^{+R} = \boxed{\frac{2}{3}mR^2} = I_2 = I_1$$

Equilateral triangle:



$$\left.\begin{aligned} \underline{\underline{I}}\vec{v}_1 &= \lambda_1\vec{v}_1 \\ \underline{\underline{I}}\vec{v}_2 &= \lambda_1\vec{v}_2 \\ \underline{\underline{I}}(a\vec{v}_1+b\vec{v}_2) &= \lambda_1(a\vec{v}_1+b\vec{v}_2) \end{aligned}\right\} \Rightarrow$$

The eq. triangle is a circle! The whole plane of the triangle is eigenvectors.

☆ $\underline{\underline{I}}$ takes velocities $\vec{\omega}$, and outputs $\vec{H}$ vector!

In 3D, if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ have the same e-value $\lambda \Rightarrow$ object = sphere as far as dynamics are concerned.

e.g. All regular polyhedra!

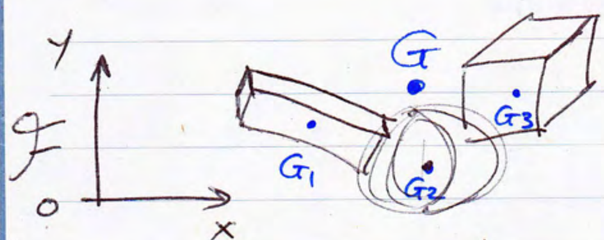
Applied Problem

Given: a bunch of objects welded together

Find: $\underline{\underline{I}}_{total}$

For each object, we know $\underline{\underline{I}}^{i}$, m_i, $\vec{r}_{G_i/O}$

Has taken into account that objects are crooked (tilted).



Step 1: Find G: $\vec{r}_G = \dfrac{\sum m_i \vec{r}_i}{\sum m_i}$

Step 2:

$$\left[\underline{\underline{I}}^{G}\right] = \sum_i \left[\left[\underline{\underline{I}}^{G_i}\right] - \left[\vec{r}_{G_i/G}\right]\left[\vec{r}_{G_i/G}\right]^T + \left[\vec{r}_{G_i/G}\right]^T \cdot \left[\vec{r}_{G_i/G}\right]\left[\underline{\underline{1}}\right] \right] m_i$$

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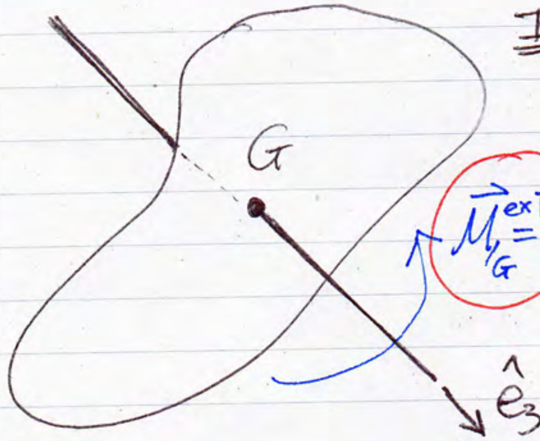
Spin about a fixed axis :



$\vec{\mathcal{M}}^{ext}_{/G} = ?$ $\quad \vec{\omega} = \omega \hat{e}_3 = \text{constant.}$

$$\frac{d^B}{dt}\left[\underline{\underline{I}} \cdot \vec{\omega}\right]$$

constant in B-frame

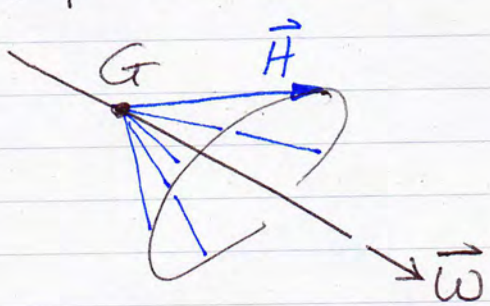
constant in Bframe (for this problem)

$$\vec{M} = \dot{\vec{H}} = \vec{\omega} \times \vec{H} + \cancel{\frac{d^B}{dt}\vec{H}} \rightarrow 0$$

$$\dot{\vec{H}} = \frac{d^F}{dt}(\vec{H})$$

$$\Rightarrow \quad \vec{M} = \vec{\omega} \times \underbrace{\left[\underline{\underline{I}} \cdot \vec{\omega}\right]}_{\not\parallel \text{ to } \vec{\omega} \text{ generally}}$$

$\vec{H}$ spins around:



Careful:

$$I_{xy} = \begin{cases} \int xy\, dm \\ \text{or} \\ -\int xy\, dm \end{cases}$$ depending on the book

$$\left[\vec{M}_{/G}\right]_F = \omega \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left[\begin{bmatrix} & & I_{xz} \\ & & I_{yz} \\ & & I_{zz} \end{bmatrix} \cdot \overset{\omega}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right] =$$

$$= \omega^2 \cdot \begin{bmatrix} I_{yz} \\ I_{xz} \\ 0 \end{bmatrix}$$

} non-zero in other two axes!!!

→ ZERO in axis of revolution

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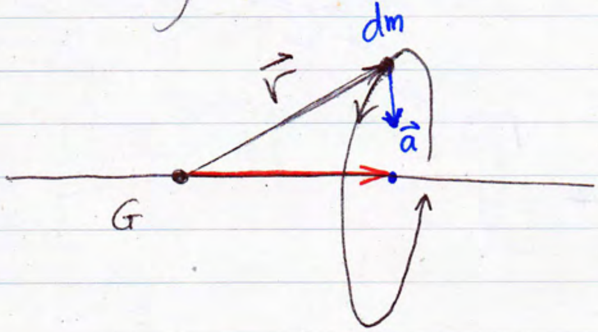
Another way to calculate:

$$\vec{M}_{/G} = \int \vec{r}_{/G} \times \vec{a}\, dm =$$

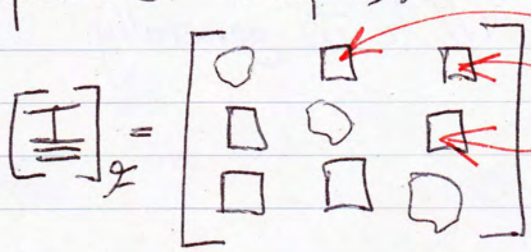
$$\hookrightarrow \vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \int \vec{r} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}))\, dm$$



$$\Rightarrow \int \omega^2 d \cdot d^{\perp} dm$$

Wabbling of unbalanced spinning is imbalance of centripetal torques.



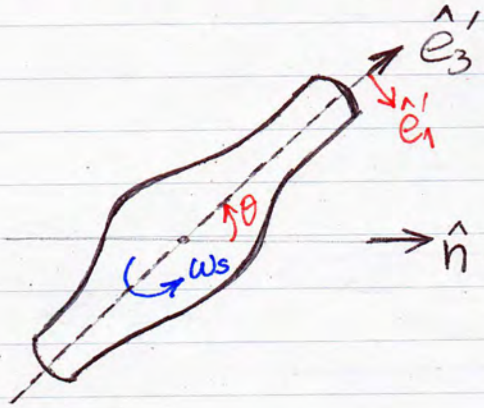
centrifugal terms of $\underline{\underline{I}}$.

Special Motions of Axisymmetric Objects

- Constant spin in a precessing frame, ie,

$$\underline{\underline{I}}_B = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \qquad (I_1 = I_2 \rightarrow \text{axisymmetric}) \text{ eg. football.}$$

Forced or not forced.



$\rightarrow \hat{n}$ = precession axis

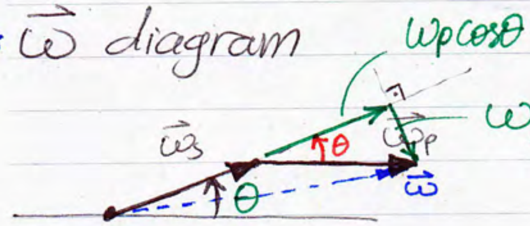
<u>Constants:</u>
$\hat{n}, \omega_s, \omega_p$
and: $\dot{\hat{e}}_3 = (\omega_p \hat{n}) \times \hat{e}_3'$
$\hat{e}_3'$ = constant <u>in the precessing frame</u>

- <u>Assume:</u> $\vec{\omega} = \underbrace{\omega_s \hat{e}_3'}_{\text{spin}} + \underbrace{\omega_p \hat{n}}_{\text{precession}}$

$$\vec{H} = \underline{\underline{I}} \cdot \vec{\omega} \quad \Rightarrow \quad {}^{\mathcal{F}}\dot{\vec{H}} = \cancel{{}^{P}\dot{\vec{H}}}^{\,\vec{0}} + \vec{\omega}_{P/\mathcal{F}} \times \vec{H} = \underset{\uparrow \atop \omega_p \hat{n}}{\vec{\omega}_{P/\mathcal{F}}} \times \vec{H}$$

↳ ang. mom relative to com

* $\vec{\omega}$ diagram

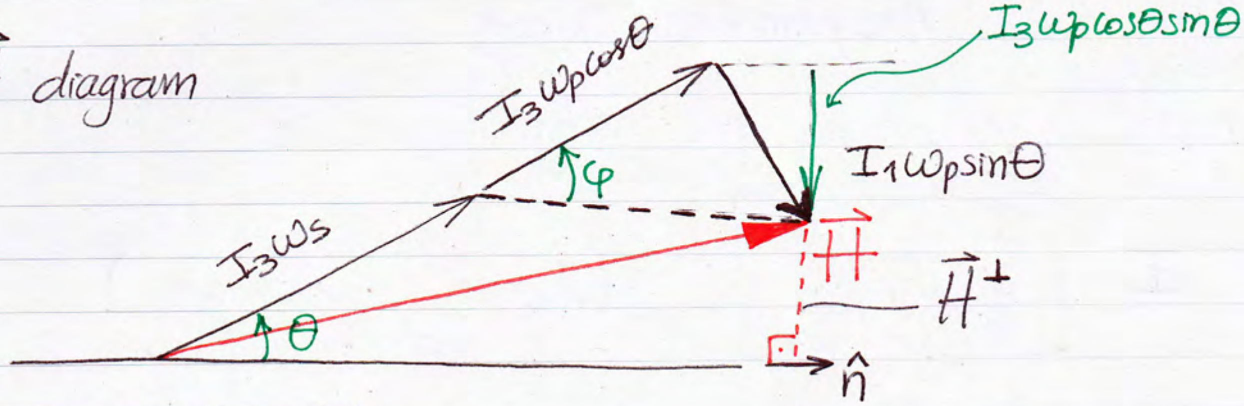


$\omega_p \sin\theta$ = proj of precession $\perp$ to spin axis in the $\hat{n}$-$\hat{e}_3'$ plane.

$$\vec{H} = \underline{\underline{I}} \cdot \vec{\omega} = \underline{\underline{I}} \cdot \left[\omega_s \hat{e}_3' + \omega_p \cos\theta\, \hat{e}_3' + \omega_p \sin\theta\, \hat{e}_1' \right]$$

$\longrightarrow$

★ $\vec{H}$ diagram

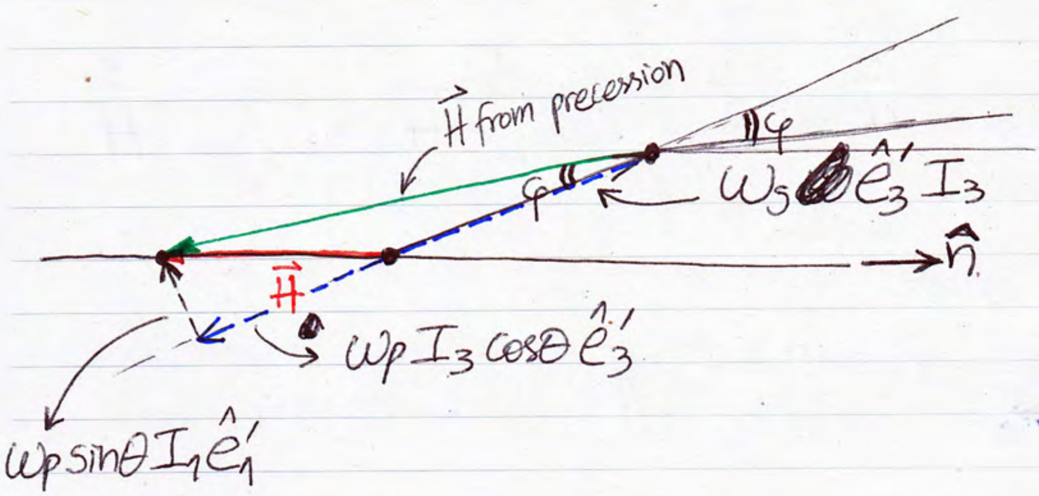


$\boxed{\tan\varphi = \frac{I_1}{I_3} \tan\theta}$, because $I_1 \neq I_3 \Rightarrow \varphi \neq \theta$

$\vec{M} = \vec{\omega}_p \times \vec{H}$

$\boxed{\text{Free precession, iff, } \vec{M} = \vec{0} \Rightarrow \vec{H}^{\perp} = \vec{0}}$ $\longrightarrow$

e.g. $I_3 = 2I_1$ (Flat object, like a plate)



$$\boxed{\vec{H}^{\perp} = \vec{O}} \Rightarrow 0 = I_3 \omega_s \cancel{\sin\theta} + I_3 \underbrace{\omega_p \cos\theta}_{\text{proj. } \omega_p \text{ onto spin direction}} \cancel{\sin\theta} - \omega_p \cancel{\sin\theta} \cos\theta\, I$$

$$\Rightarrow \boxed{\omega_s = \omega_p \left(\frac{I_1}{I_3} - 1 \right) \cos\theta}$$

PLATE

e.g. If, $I_3 = 2I_1$, $\theta \ll 1$,

then, $\omega_s = -\frac{1}{2}\omega_p$

$$\vec{\omega} = \omega_p \hat{n} - \frac{1}{2}\omega_p \hat{e}_3' \Rightarrow \boxed{\|\vec{\omega}\| \simeq \frac{1}{2}\omega_p}$$

$\omega_p \approx 2\|\vec{\omega}\|$

This is what the eye would perceive! ~~(wobbling)~~

3/14

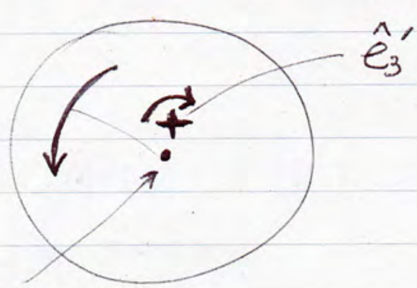
e.g. Earth $\left(1 - \frac{I_1}{I_3}\right) \ll 1 \Rightarrow \omega_s = -\epsilon\, \omega_p$

very small

revolution/many many years

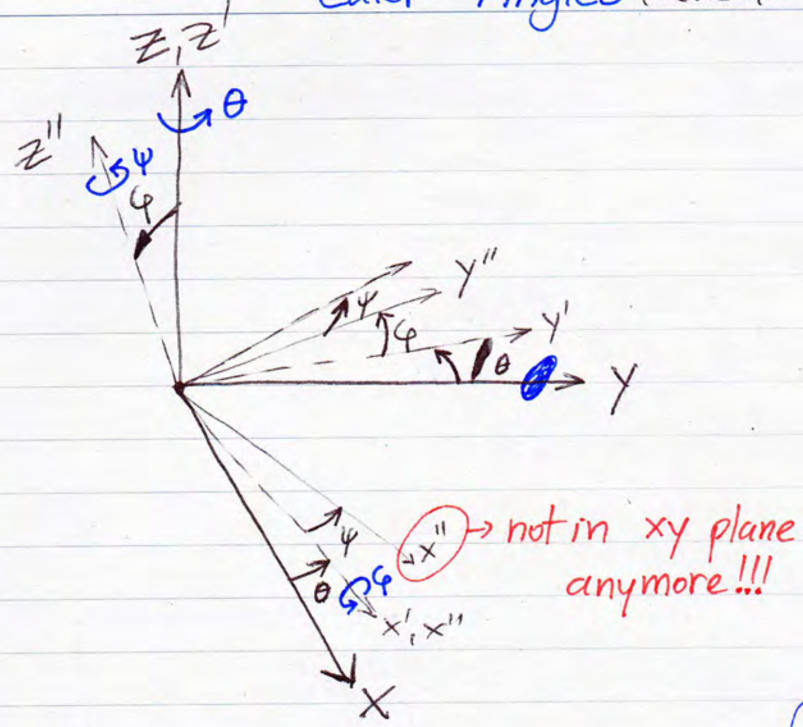
1 revolution/day

View above North Pole



precession axis $\hat{n}$

Euler Angles (revisited)



3-1-3 Eulerangles

1) Rotate θ about z-axis
2) Rotate φ -"- x'-axis
3) -"- ψ -"- z''-axis

Which is the same end result as:
1) ψ about z
2) φ -"- x
3) θ -"- z (original)

Net Rotation:

$$\underline{\underline{R}} = \underline{\underline{R}}(\hat{e}_3,\theta)\,\underline{\underline{R}}(\hat{e}_1,\varphi)\,\underline{\underline{R}}(\hat{e}_3,\psi)$$

$$\left(= \underline{\underline{R}}(\hat{e}_3'',\psi)\,\underline{\underline{R}}(\hat{e}_1',\varphi)\,\underline{\underline{R}}(\hat{e}_3,\theta)\right)$$

first we need to rotate $\hat{e}_1$:)

same for this unit vector

Recall: $\underline{\underline{R}}(\hat{n},\beta) = \cos\beta\,\underline{\underline{1}} + (1-\cos\beta)\,\hat{n}\hat{n} + \sin\beta\,\underline{\underline{S}}(\hat{n})$

Recall dynamics: $\vec{M}_{/G} = \underbrace{\underline{\underline{I}}\vec{\alpha} + \vec{\omega}\times(\underline{\underline{I}}\cdot\vec{\omega})}_{\dot{\vec{H}}_{/G}} \Rightarrow$

$$\vec{\alpha} = \underline{\underline{I}}^{-1}(\vec{M}_{/G} - \vec{\omega}\times\underline{\underline{I}}\vec{\omega})$$

$$\underbrace{\dot{\vec{\omega}} = \vec{\alpha}, \quad \dot{\underline{\underline{R}}} = \underline{\underline{S}}(\vec{\omega})\,\underline{\underline{R}}}_{12\text{ ODEs}}$$

Closed set of ODEs for evolution of ~~pose!~~ attitude

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where $\dot{\underline{\underline{R}}} = \dot{R}_{\mathcal{F}ij}\,\hat{e}_i\hat{e}_j$

or

$$= \dot{R}_{Bij}\,\hat{e}'_i\hat{e}'_j + R_{Bij}\,\dot{\hat{e}}'_i\hat{e}'_j + R_{Bij}\,\hat{e}'_i\dot{\hat{e}}'_j$$

Let's replace $\dot{\underline{\underline{R}}}$ d.e. with:

$$\dot{\Phi} = \begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \\ \dot{\psi} \end{pmatrix} = \text{sth that depends on } \vec{\omega}$$

→ Then we'll have 6 ODEs

Motion of particle on a plane in polar coordinates is also singular when it goes through the origin!

$$\vec{\omega}_{B/\mathcal{F}} \overset{=\vec{\omega}}{=} \dot{\theta}\,\hat{e}_3 + \dot{\varphi}\,\hat{e}'_1 + \dot{\psi}\,\hat{e}''_3 ,$$

where $\hat{e}'_1 = \underline{\underline{R}}(\hat{e}_3, \theta)\,\hat{e}_1$

$\hat{e}''_3 = \underline{\underline{R}}(\hat{e}'_1, \varphi)\,\hat{e}_3$

$$\omega_{\mathcal{F}} = \left[[\hat{e}_3]_{\mathcal{F}} \;\middle|\; [\hat{e}'_1]_{\mathcal{F}} \;\middle|\; [\hat{e}''_3]_{\mathcal{F}} \right] \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \\ \dot{\psi} \end{bmatrix} \Longrightarrow$$

$$\omega_{\mathcal{F}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = A \cdot \dot{\Phi}$$

← if we know θ, φ, ψ, we can calculate matrix A

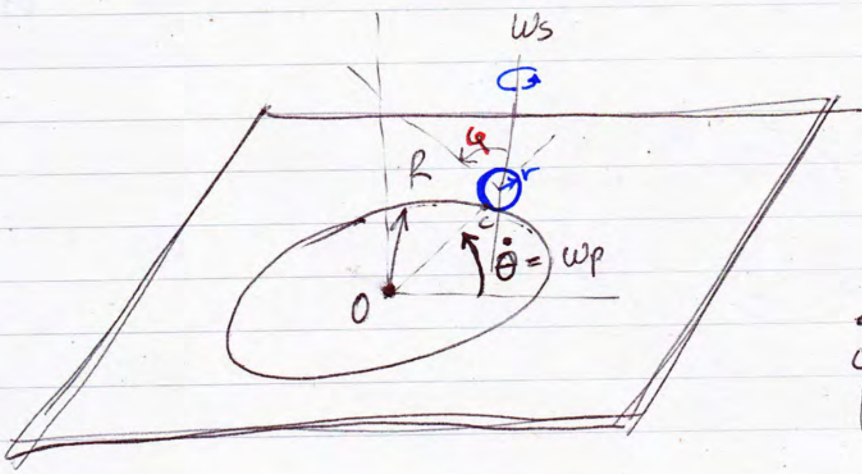
$$\Rightarrow \boxed{\dot{\Phi} = \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \\ \dot{\psi} \end{bmatrix} = A^{-1} \cdot \omega_F}$$

Now we got 6 des: $\left\{ \begin{array}{l} \overset{\circ}{\vec{\omega}} = \underline{\underline{I}}^{-1} \cdot (\cdots) \\ \dot{\Phi} = A^{-1} \cdot \omega \end{array} \right\}$ closed set of 6 des.

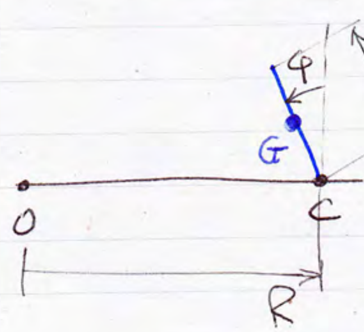
$A = A(\theta, \varphi, \psi)$

Gimbal lock when $\varphi = 0 \rightarrow A$ not invertible.

(Coin) Disk on a plane



Side view



Given: r = radius of disk
$[I] = \mathrm{diag}([I\ I\ 2I])$
m = mass , gravity = g

What are the restrictions on $R, \omega_s, \omega_p, \varphi$ = "lean angle" ~~lean angle~~

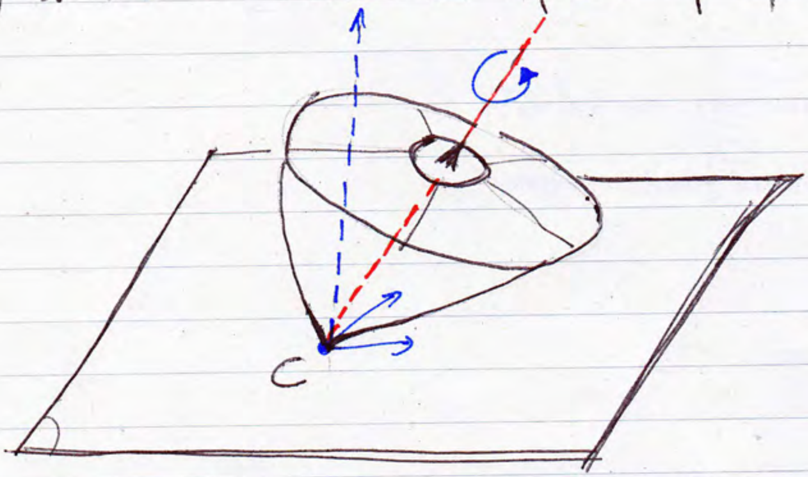
a) No slip 2D?
b) No friction 2D?
c) Both 1D?

$\boxed{AMB/c}$ to tackle unknown reaction forces!

Full des → Algebraic → closed form → constraints on 4 parameters. (how many free?)

New Problem:

How fast do we have to spin a top for it to stay up?



Full DEs
↓
Linearize about spinning upright
↓
Stable? for which parameters

Can we answer without writing DEs that we would "solve"?
↳ OK to ~~do~~ algebraic equations, but don't solve DEs.
↳ find nice simple (circular?) solutions when it's spinning upright.

NEW CHAPTER:
ANALYTICAL DYNAMICS

$\vec{F}=m\vec{a} \rightarrow$ Variational principle(s) $\rightarrow$ Lagrange equations
(for particles)
↳ assume all things are made of particles.

Notation: N vector eqs $\longleftrightarrow$ 3N scalar eqs. (3D space)

Forces on system:
- Constraint forces (often internal)
- External forces

$$\forall i, \quad \vec{F}_i = m_i\vec{a}_i \Rightarrow \vec{F}_i - m_i\vec{a}_i = \vec{0}$$

MUST BE THAT: $(\vec{F}_i - m_i\vec{a}_i)\cdot\underbrace{\delta\vec{v}_i}_{} = 0 \quad , \forall \delta\vec{v}_i$

virtual variation in velocity

or

any vector valued function in time

$$\Downarrow$$

$$\sum(\vec{F}_i - m_i\vec{a}_i)\cdot\delta\vec{v}_i = 0$$

$\quad \hookrightarrow = \vec{F}_i^{constr} + \vec{F}_i^{ext}$

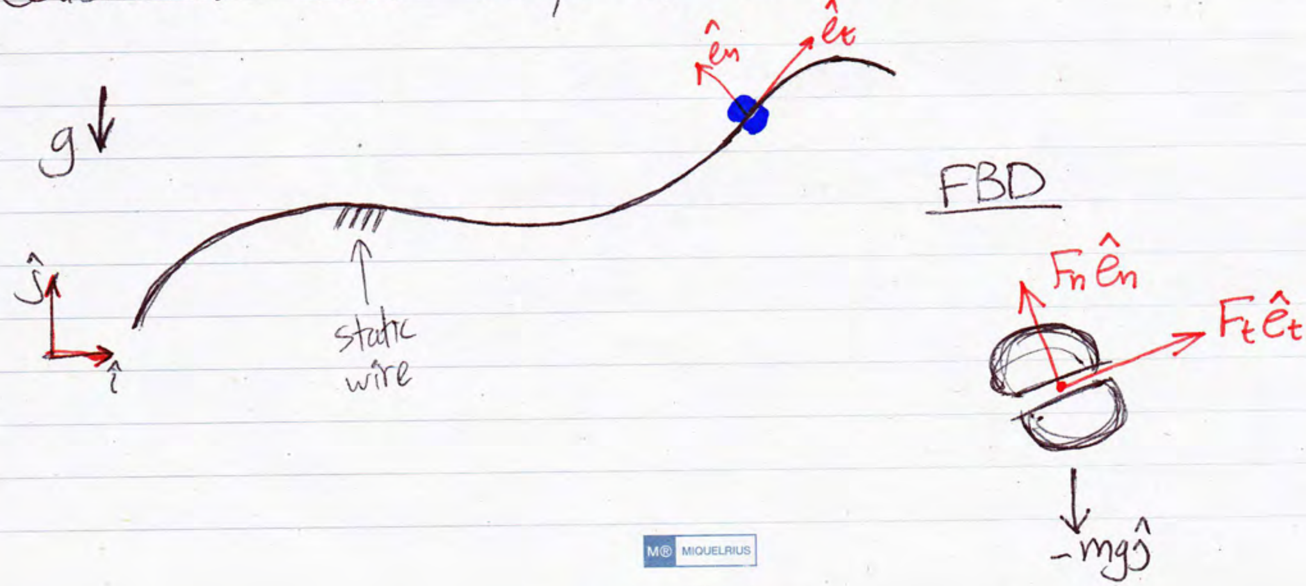
Postulate A (assumption): $\boxed{\sum \vec{F}^{constr.}\cdot\delta\vec{v}_i = 0}, \forall\delta\vec{v}_i$

that satisfy the constraints to 1st order.

($\Rightarrow$ virtual work of constraint forces is zero for virtual displacements/velocities that satisfy the constraints)

ex) bead on rigid wire!

Constraint: bead stays on the wire:

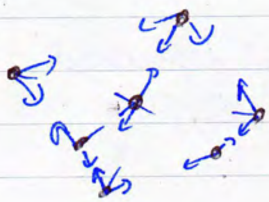


$\vec F = F_t \hat e_t + F_n \hat e_n$ (sth that keep bead on wire)

$\delta W = 0 \Rightarrow \vec F^{constr} \cdot \delta \vec v = 0 \Rightarrow$

$$(F_t \hat e_t + F_n \hat e_n) \cdot \underbrace{v \hat e_t}_{\text{satisfies constraint!}} = 0 \Rightarrow \boxed{F_t = 0}$$

ex) Rigid object



Constraint forces keep all d_{ij} = consta

virtual velocities that satisfy ...

$$\delta W = 0 \Rightarrow \sum \vec F_i^{con} \cdot \delta \vec v_i = 0 \Rightarrow$$

$$\sum \vec F_i^{con} \cdot \delta \Big[\underbrace{\vec V_G + \vec \omega \times \vec r_{i/G}}_{\text{6 dof variation}} \Big]$$ → has to hold for all such variatio

Consider: $\delta \vec \omega = 0$, $\delta \vec V_G = \hat e_1$ $\Rightarrow \sum F_1^{con} = 0$ ...

Likewise for $\vec V_G = \hat e_2, \hat e_3$ $\Bigg\} \Rightarrow$

$$\boxed{\sum \vec F_i^{con} = \vec 0}$$

$(\vec A \times \vec B) \cdot \vec C = \vec A \cdot (\vec B \times \vec C)$

Now consider: $\delta \vec V_G = \vec 0$

$$\sum \vec F_i^{con} \cdot (\vec \omega \times \vec r_{i/G}) = 0 \Rightarrow \sum (\vec \omega \times \vec r_{i/G}) \cdot \vec F_i^{con} = 0 \Rightarrow$$

$$\sum \vec \omega \cdot \vec r_{i/G} \times \vec F_i^{con} = 0$$

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Consider $\delta\vec{\omega} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \Rightarrow$

$$\sum \vec{r}_{i/G} \times \vec{F}^{con} = \vec{0} \Rightarrow \text{net internal moment} = \vec{0}$$

(cont'd) 3/28

$$\vec{F} = m\vec{a} \longrightarrow \sum_i [F_i^{ext} - m_i a_i] \underbrace{\delta x_i}_{\text{or } \delta v_i} = 0 \quad \text{(last lecture)}$$

$\hookrightarrow$ virtual displacements.

$$\Downarrow$$

$$\sum F_i^{ext} \delta x_i - \sum m_i \ddot{x}_i \delta x_i = 0$$

*Assume all forces are conservative $\Rightarrow \bar{F}_i = \bar{F}_i(x_1, x_2, \dots)$

*Some facts: $\int_{x_A}^{x_B} \sum F_i \, dx_i$ is path independent (path AB)

$$\oint \sum F_i \, dx_i = 0$$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{everywhere}$$

$$\exists V, \text{ s.t. } V(\vec{x}) = -\int_0^{\bar{x}} \sum F_i \, dx_i$$

$$\boxed{F_i = -\frac{\partial V_i}{\partial x_i}}$$

$$\Longrightarrow$$

scalar

1) $\sum F_i^{ext}\delta x_i = \sum -\frac{\partial V}{\partial x_i}\overbrace{\delta x_i}^{\epsilon\cdot\eta_i(t)} = -\delta V$,

$\left(\delta V = \frac{d}{d\epsilon}V(x_i + \epsilon\eta_i) \quad , \forall \eta_i\right)$

2) $\sum m_i \ddot{x}_i \delta x_i = \sum \frac{d}{dt}(\dot{x}_i\delta x_i)m_i - \underbrace{\sum \dot{x}_i \delta \dot{x}_i m_i}$ $\quad (d/dt(f\cdot g))$

like a differential

2b) $m_i \dot{x}_i \delta \dot{x}_i = \frac{1}{2} m_i \delta[(\dot{x}_i)^2]$

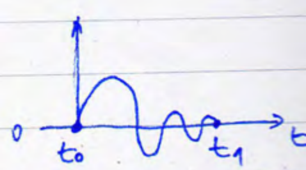
$$\sum m_i \ddot{x}_i \delta x_i = \sum \frac{d}{dt}[m_i \dot{x}_i \delta x_i] - \underbrace{\sum m_i \frac{\delta(\dot{x}_i)^2}{2}}_{\delta T\text{ : varration in kinetic energy E}}$$

Back to start and take it's integral:

$$\int_{t_0}^{t_1} (*)\,dt = \int\left[\sum F_i \delta x_i - \sum m_i \ddot{x}_i \delta x_i\right]dt = 0 \Rightarrow$$

$$\int_{t_0}^{t_1} -\delta V - \Big[\underbrace{d/dt \sum m_i \dot{x}_i \delta x_i}_{\text{We want this zero... (3)}} - \delta T\Big]\, dt = 0$$

3) $\int_{t_0}^{t_1} \frac{d}{dt}\sum m_i \dot{x}_i \delta x_i \, dt = \cancel{\sum m_i \dot{x}_i \delta x_i}\Big|_{t_0}^{t_1} \to 0$



*Assume $\delta x_i = 0$ at t_0, t_1 , i.e, $\delta x_i(t_0) = \delta x_i(t_1) = 0$

$\Longrightarrow$

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$$\delta \int_{t_0}^{t_1} \underbrace{\overbrace{(T-V)}^{\text{Lagrangian :)}}\,dt}_{\text{Action integral, } A} = 0$$

Principle of Least (stationary) action or "Hamilton's Principle"

*Assumptions so far:
- Conservative forces
- Differentiable δx_i
- $\delta x_i(t_0) = \delta x_i(t_1) = 0$

} that satisfy the constraints too !!!

Statement:

"For a solution of $\vec{F} = m\vec{a}$, if a solution goes through X_0, X_1, then the solution has the property that:

$$\frac{d}{dt} A = 0, \quad \forall \eta_i(t) \text{ that satisfy kinematic constraint} + \text{assumption } 2"$$

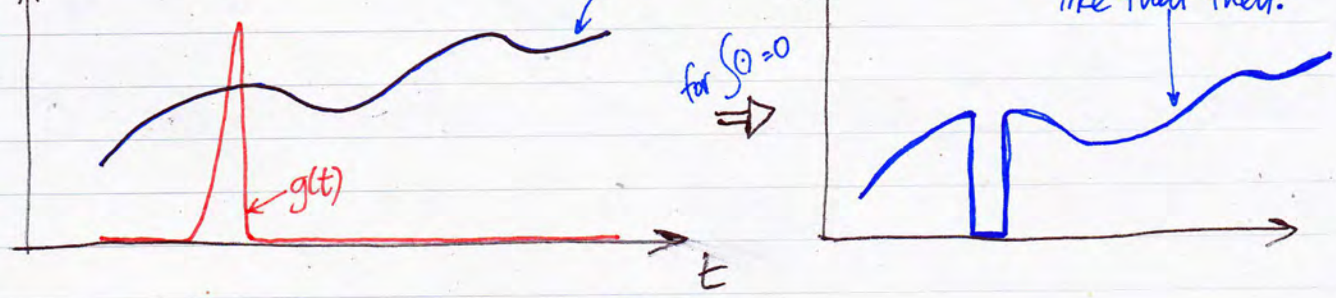
$$\longrightarrow \int_{t_2}^{t_1} \mathcal{L}(x_1 + \epsilon\eta_1, x_2 + \epsilon\eta_2, \dots \dot{x}_1 + \epsilon\dot{\eta}_1, \dot{x}_2 + \epsilon\dot{\eta}_2 \dots)\,dt$$

— 2 Asides —

1) Given $\vec{V} \cdot \vec{b} = 0$, $\forall \vec{b} \Rightarrow \vec{V} = \vec{0}$

2) $\int_0^1 f(t)g(t)\,dt = 0, \forall g(t) \Rightarrow f(t) = 0$

Look at g(t) like this:



But we can ~~use~~ use any of those $g(t)$'s $\Longrightarrow$ $\boxed{f(t)=0}$

Derive Lagrange eqs from Hamilton's Principle

Particles at positions $x_i(t)$

Constraints on x_i

parametrization of kinematically allowed configura

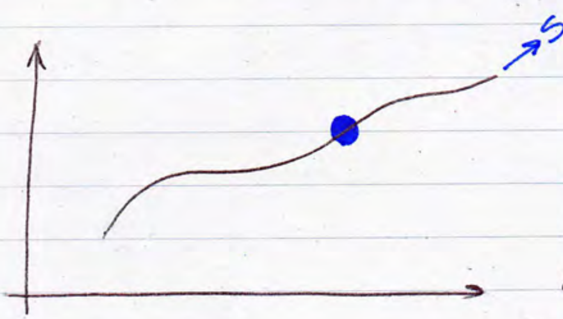
Assume we have minimal/generalized coordinates q_i s.t.

$$\left.\begin{array}{l} x_1 = x_1(q_1, q_2 \dots q_n) \\ x_2 = x_2(\quad - '' - \quad) \\ \vdots \end{array}\right\}$$

All positions and velocities can be found from q_i, $\dot{q}_i$.

The q_i are independent (no constraints on them $\rightarrow$ minimal)

ex) Bead on wire



$q = s$

or $q = x$

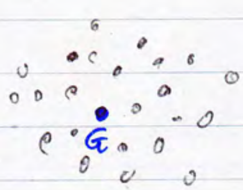
or $q = y$

or $q = x^2$

Then, $x = f(s)$ and $y = f(s)$

ex) N particles

$d_{ij} = \text{constant} \ \forall i,j \in N$



6 free dimensions:

$q_1 = x_G$

$q_2 = y_G$

$q_3 = z_G$

$\left.\begin{array}{l} q_4 = \varphi \\ q_5 = \psi \\ q_6 = \theta \end{array}\right\}$ We are stuck with Euler angles b/c we want no constraints on q_i's (for now).

Start from: $\delta A = 0$, $A = \int_{t_1}^{t_2} \mathcal{L}\,dt$, $\mathcal{L} = T - V = \mathcal{L}(q_i, \dot{q}_i)$

$T(q_1, q_2, \dots \dot{q}_1, \dot{q}_2, \dots)$ $V(q_1, q_2, \dots q_n)$

$$\Downarrow$$

$$0 = \int_{t_0}^{t_1} \delta\mathcal{L}(q_i, \dot{q}_i)\,dt = \int_{t_0}^{t_1} \sum \frac{\partial \mathcal{L}}{\partial q_i}\delta q_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i}\,dt = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i$$

$\Longrightarrow$

$$0 = \int_{t_0}^{t_1} \left\{ \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right]\delta q_i + \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right)} \right\} dt \Longrightarrow$$

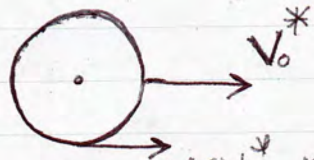
$$\cancel{\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right]_{t_0}^{t_1}} \rightarrow 0 \text{, b/c of assumptions!}$$

$$0 = \int_{t_0}^{t_1} \sum \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right]\delta q_i\,dt \Rightarrow$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] = 0 \quad , \forall i}$$

■

Escape Velocity



$V < V_0^*$ no escape

$V > V_0^*$ are we sure it's gonna escape? → Show that $\frac{1}{2}mV_0^2$ goes to zero

Euler disk

$\varphi \to 90°$ (tip angle) → $\omega_p \to \infty$ (sound frequency gets very high)

Derive Lagrange Eqs (not just conservative forces)

without Hamilton's principle

N particles $(1, 2, \ldots, j, \ldots N)$

n minimal/generalized coordinates $(1, 2, \ldots i, \ldots n)$. q_i

→ physical location/~~orientation~~ of each particle

$$\vec{r}_j = \vec{r}_j(q_1, q_2, \ldots q_n, t)$$

Think of these as 3 independent functi for the purposes of partia derivatives.

$$\dot{\vec{r}}_j = \sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_j}{\partial t} \quad \left(= \dot{\vec{r}}_j(q, \dot{q}, t) \right)$$

"Jacobian"

$$\boxed{\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_k} = \boxed{\frac{\partial \vec{r}_j}{\partial q_k}}} \quad (1)$$

$$\frac{\partial \dot{\vec{r}}_j}{\partial q_k} = \sum_i \frac{\partial^2 \vec{r}_j}{\partial q_k \partial q_i}\dot{q}_i + \frac{\partial^2 \vec{r}_j}{\partial q_k \partial t} \quad (A)$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) = \sum_i \frac{\partial\left(\frac{\partial \vec{r}_j}{\partial q_k}\right)}{\partial q_i}\dot{q}_i + \frac{\partial^2 \vec{r}_j}{\partial t \partial q_k} \quad (B)$$

$A = B$!

because $\frac{d}{dt} f(x,t) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) = \frac{\partial \dot{\vec{r}}_j}{\partial q_k}}$$

2

Principle of Virtual Work
or
D'Alembert's Principle

forces other than constr. forces

virtual displacement in physical space. (positions of particles)

$$\vec{F}_i = m_i \vec{a}_i \quad \Rightarrow \sum_{j=1}^{N} \left(\underset{1}{\vec{F}_j^*} - \underset{2}{m_j \vec{a}_j}\right) \cdot \delta \vec{r}_j = \vec{0}$$

variations that satisfy the constr.

variations: $\delta \vec{r}_j = \sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i$

Look at (1): $\sum_{j=1}^{N} \vec{F}_j^* \cdot \delta \vec{r}_j = \sum_{j=1}^{N} \sum_{i=1}^{n} \vec{F}_j^* \cdot \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i =$

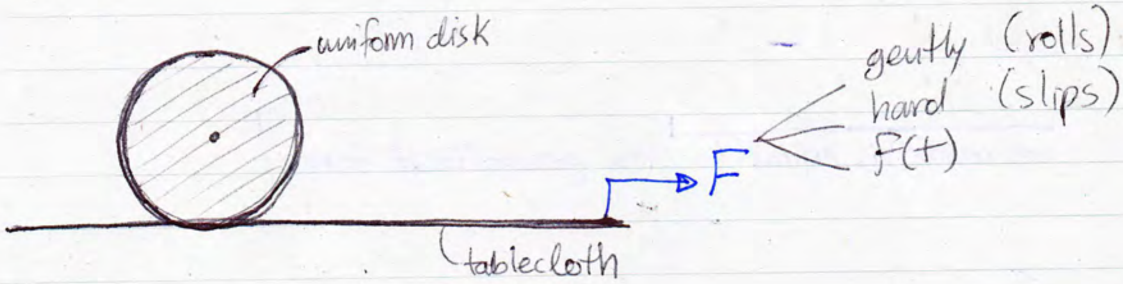
$$= \sum_{i=1}^{n} \underbrace{\left[\sum_{j=1}^{N} \vec{F}_j^* \cdot \frac{\partial \vec{r}_j}{\partial q_i}\right]}_{} \delta q_i = \overbrace{\sum_{i=1}^{n} Q_i \, \delta q_i}^{\delta W\text{: virtual work}}$$

δq_i: generalized displacement

Q_i = i^{th} generalized force. (projection of forces in displacements)

$$\boxed{Q_i \equiv \sum_{j=1}^{N} \vec{F}_j^* \cdot \frac{\partial \vec{r}_j}{\partial q_i}}$$

Q-exam Problem:



? Afterwards it will skid, and then roll. What is the "rolling velocity"? (SPEED)

Problem has 3 phases: (i) on tablecloth, (ii) on table skidding, (iii) on table rolling

---

Lagrange Eqs (cont'd)

Look at various terms in final answer (that we "know"):

$$T = E_K \ , \ \frac{\partial T}{\partial q_k} \ , \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)$$

(T) $T = \sum_{j=1}^{N} \frac{1}{2} m_j \vec{v}_j \cdot \vec{v}_j \quad \vec{v}_j = \dot{\vec{r}}_j \quad = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\vec{r}}_j \cdot \dot{\vec{r}}_j$

$\left(\frac{\partial T}{\partial q_k}\right)$ $\frac{\partial T}{\partial q_k} = \sum_{j=1}^{N} m_j \frac{\partial \dot{\vec{r}}_j}{\partial q_k} \cdot \dot{\vec{r}}_j \quad \overset{(2)}{\Longrightarrow} \frac{\partial T}{\partial q_k} = \sum m_j \frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) \dot{\vec{r}}_j \quad (3)$

(3^{rd}) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \left(\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_k}\right) \cdot \dot{\vec{r}}_j\right) = \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k} \cdot \dot{\vec{r}}_j\right) =$

$$= \sum_{j=1}^{N} m_j \frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) \dot{\vec{r}}_j + \sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k} \ddot{\vec{r}}_j \quad \Longrightarrow$$

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$\cancel{d/dt\left(\frac{\partial T}{\partial \dot{q}_k}\right)}$

term cancelation!

Notice that: $\left\{ \underbrace{d/dt\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}}_{\text{we want this equal to the generalized force!}} = \sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k} \cdot \ddot{\vec{r}}_j \right\}$

$$\{\cdots = \cdots\}\delta q_i \Rightarrow \sum_{i=1}^{n} \{\cdots\}\delta q_i \overset{\text{"}k \to i\text{"}}{\Longrightarrow}$$

* only when summed

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right)\delta q_i = \underbrace{\sum_i^n \sum_j^N m_j \frac{\partial \vec{r}_j}{\partial q_i}}\ddot{\vec{r}}_j\, \delta q_i \quad \xrightarrow{\vec{F}_j = m_j \ddot{\vec{r}}_j}$$

this is "Jacobian^T".

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right)\delta q_i = \sum_{i=1}^{n} Q_i\, \delta q_i$$

ex) $\delta q_1 \neq 0$, $\delta q_2 = \delta q_3 = \cdots = 0$, likewise for all $i = 1, \ldots, n$

$$\Downarrow$$

$$\boxed{d/dt\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i}$$

Lagrange Eqs for non-conservative forces.

Next time:

"Styrofoam" Forces $\xrightarrow{\text{assumptions}}$ LMB/AMB

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$$Q_i = \sum_{j=1}^{3N \to 3D} \frac{\partial r_j}{\partial q_i} F_j^*$$

Assume conservative forces:

$$F_j = -\frac{\partial V^{\leftarrow E_p}}{\partial r_j} \quad , \text{"} V = V(r_1, r_2, \dots r_{3N}) = V(q_1, q_2, \dots q_n) \text{"}$$

$$\Rightarrow Q_i = -\sum_{j=1}^{3N} \frac{\partial r_j}{\partial q_i} \frac{\partial V}{\partial r_j} = -\frac{\partial V}{\partial q_i} \quad \ddot{\smile}$$

V: doesn't depend on $\dot{q}_i$ (conservative force!!!)

Lagrange $\Rightarrow$ Eqs

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \Rightarrow$$

$$\frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_i} - \frac{\partial (T-V)}{\partial q_i} = 0 \Rightarrow$$

$$\boxed{\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad \mathcal{L} = T - V}$$

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4/9

Axioms of Mechanics

Basics ("Facts")

← "true" to $1/10^{-8}$ for terrestrial mechanics

$$10^{-9}\,m < d < 10^{9}\,m$$
$$V < 10^{5}\, m/s$$

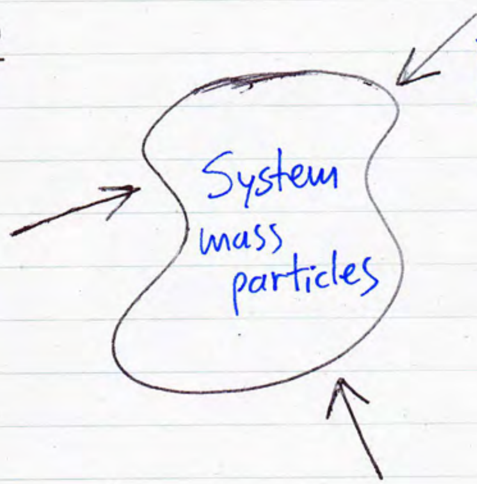
- Space & time are flat and follow all laws you know and like (Euclidean, continuity of time).
- Mass is immutable. ⇒ Doesn't come or go, is local, you can identify it / label it etc.
- A Newtonian frame exists, where the laws hold.

Ruina recommends:

Forces and Moments are <u>the</u> means of system interaction.

FBD

$\vec{F}$ ext. forces & moments (influence of outside world on the systems)



0) Action & Reaction

vectors are frame independent

"If system A causes $\vec{F}$ and $\vec{M}$ on B
then —"— B —"— $-\vec{F}$ —"— $-\vec{M}$ on A."

1) For any system

a) LMB: $\sum \vec{F}^{ext} = \begin{cases} \sum m_i \vec{a}_i \\ \int \vec{a}\, dm \end{cases}$

$\vec{a} \equiv \vec{a}_{/F}$

b) $\sum \vec{M}^{ext}_{/C} = \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{a}_i \\ \int \vec{r}_{/C} \times \vec{a}\, dm \end{cases}$, for ANY point C

Already, these are not independent!

ex) $\boxed{AMB/_{C_1} + AMB/_{C_2} + AMB/_{C_3} \Rightarrow LMB}$ C_1, C_2, C_3 NOT colinear!

$AMB/_{C_1} \Rightarrow \sum \left(\vec{r}_{i/C_1} \times \left(\vec{F}_i - m_i \vec{a}_i \right) \right) = \vec{0}$ (*)

↑net external force on particle i

$AMB/_{C_2} \Rightarrow$ (**)

$AMB/_{C_3} \Rightarrow$ (***)

$\vec{r}_{C_2} - \vec{r}_{C_1} = \vec{d}_1$, $\vec{r}_{C_3} - \vec{r}_{C_1} = \vec{d}_2$

Subtract eqs.

$\vec{d}_1 \times \underbrace{\sum (\vec{F}^{ext}_i - m_i \vec{a}_i)}_{\vec{A}} = \vec{0} \Rightarrow \vec{d}_1 \times \vec{A} = \vec{0} \Rightarrow \vec{A} \parallel \vec{d}_1$

$\vec{d}_2 \times \sum (\quad\cdots\quad) = \vec{0} \Rightarrow \vec{d}_2 \times \vec{A} = \vec{0} \Rightarrow \vec{A} \parallel \vec{d}_2$

→ independent of $\sum_i$

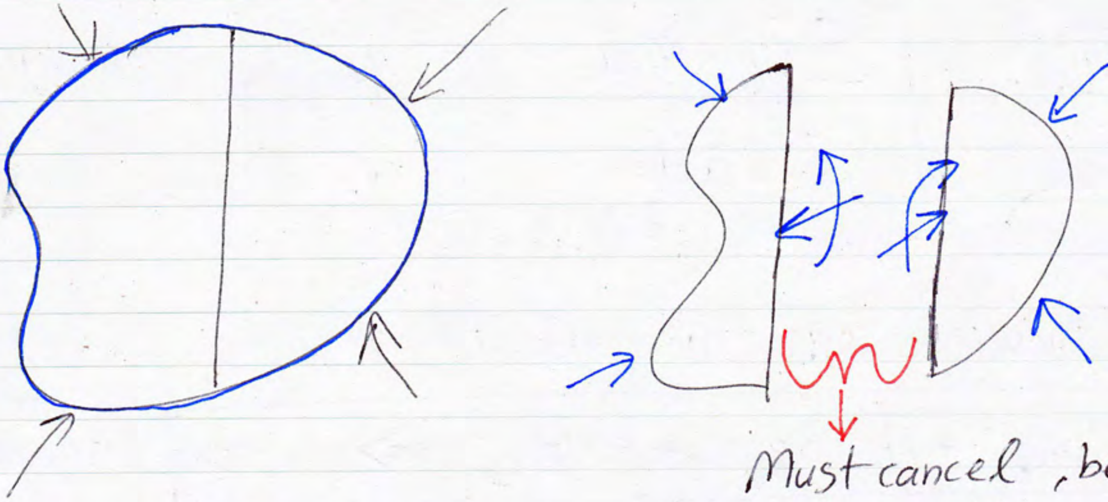
But C_1, C_2, C_3 are not colinear $\Rightarrow \vec{d}_1 \neg\parallel \vec{d}_2 \Rightarrow$

$\vec{A} = 0$, which is LMB! ✓

Applying LMB & AMB to arbitrary systems → Action & Reaction

→ How? →

FBD



Must cancel, because RHS add.

Usual (BAD) assumptions:

Start $\vec{F} = m\vec{a}$ for particles

Use all internal forces are pairwise & equal forces (*)

Then $\sum(\vec{F}_i^{ext} + \vec{F}_i^{int}) = \sum m_i \vec{a}_i$

(*) $\sum \vec{F}^{int} = \vec{0} \Rightarrow \boxed{LMB}$

or $\sum \vec{r}_{i/c} \times (\vec{F}_i^{ext} + \vec{F}_i^{int}) = \sum \vec{r}_{i/c} \times m_i \vec{a}_i$

(*) $\sum \vec{r}_{i/c} \times \vec{F}_i^{int} = 0 \Rightarrow \boxed{AMB}$

Why bad? $\longrightarrow$

*Why bad?

2) Doesn't agree with microscopic physics (more to physics than electrostatics and big G)

1) It's not consistent to make mechanics rest on physics which you don't know about (internal stuff)

3) Bad assumption implies restrictions on modulii of material, called the relations.

↳ For isotropic materials $\nu = 1/4$ (or $1/3$?)

↑ Poisson ratio

Bad not all materials have this ratio! (Bad macroscopic prediction)

Better way:

Start with $\vec{F} = m\vec{a}$ for particles and add:

a) Internal forces add to zero and have no net moment

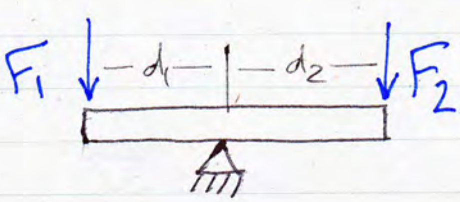
OR ⇕ (similar to previous, but no pairwise equal assumption).

b) Internal forces do no work in virtual translations and rotations.

---

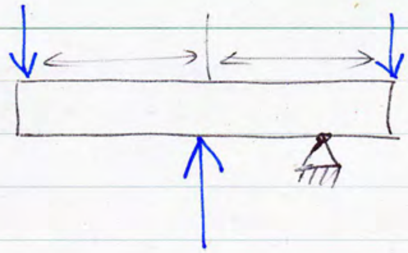
6 Important quantities: $\vec{L}, \vec{H}, E_K, \dot{\vec{L}}, \dot{\vec{H}}, P = \dot{E}_K$

The approach: Start with Principle of the lever.



⇓

$$d_1 F_1 = d_2 F_2$$



"Move hinge anywhere after putting a force at its initial location"

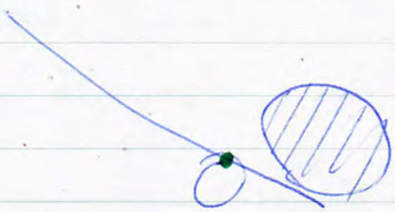
Forces // to hinge have no effect.
-"- intersect -"- -"- -"- -"- $\Bigg\} \Rightarrow$

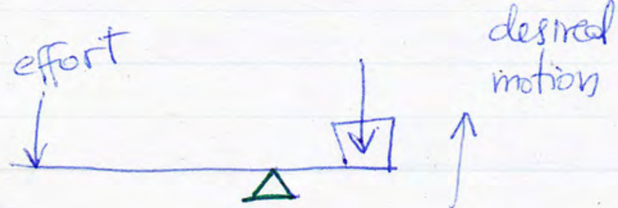
$\underbrace{\hat{\lambda}\cdot\sum \vec{M}_{/C} = 0}_{\text{moments about all axes} = 0}$, C point on the axis, $\hat{\lambda}$ along the axis. , $\forall C$ & $\underbrace{\hat{\lambda}}$ coming out of the page.

$\Rightarrow \sum \vec{M}_{/C} = \vec{0}$

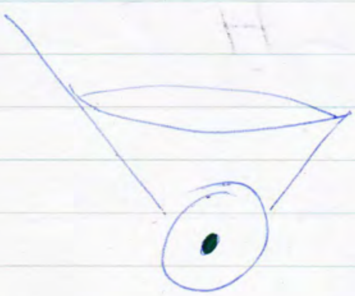
Class

1.

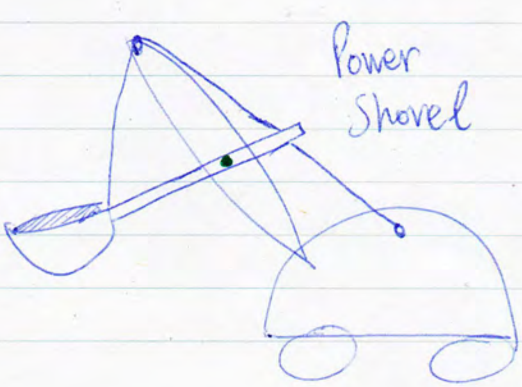




2.

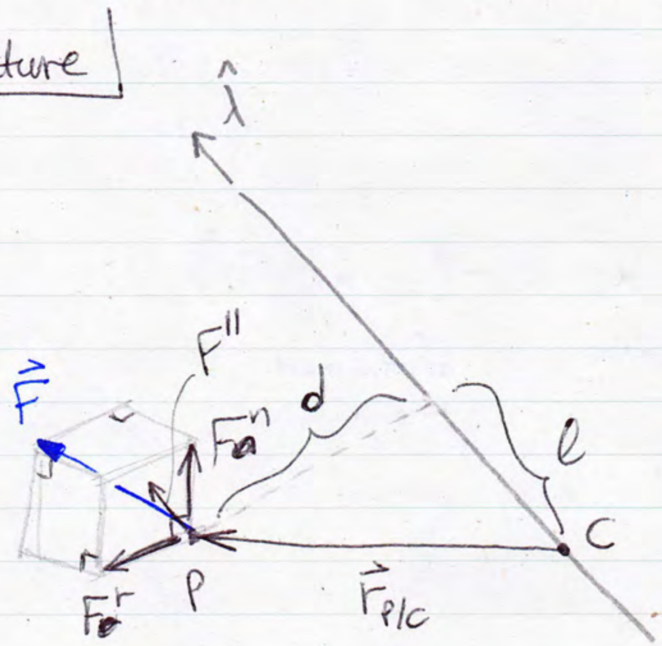


3.



Axioms of Mechanics (cont'd)

3D picture



$$\vec{F} = F^r \hat{e}_r + F^n \hat{e}_n + F^{\parallel} \hat{\lambda}$$

$$\vec{r}_{P/C} = \ell \hat{\lambda} + d \hat{e}_r$$

(because $F^{\parallel}$ is $\parallel \hat{\lambda}$ and F^r intersects axis.)

<u>Intuition:</u> Net Turning effect of force $\vec{F}$ is $F^n d$
(Principle of the lever)

STATIC EQUILIBRIUM eq

For any axis: $\sum_i F_i^n d_i = 0$ (for body in equilibrium, <u>STATICS</u>)

Claim: $F^n d \overset{\checkmark}{=} \hat{\lambda} \cdot (\vec{r}_{P/C} \times \vec{F})$ → <u>prove</u> by doing cross and dot products.

Lever Principle ⇒ $\sum M_{axis} = \vec{0} \Rightarrow \hat{\lambda} \cdot \sum \vec{M}_{/C} = 0 \;, \forall \hat{\lambda} \Rightarrow$

$$\boxed{\sum \vec{M}_{/C} = \vec{0}}$$

$\overset{\text{also}}{\Longrightarrow} \sum \vec{F} = \vec{0}$ (by picking 3 non-colinear C_1, C_2, C_3)

All objects are in static equilibrium, but have to push around bits of mass, which obey $\vec{F} = m\vec{a}$ and action & reaction. They thus push back with $-m\vec{a}$

$$\Sigma \vec{M}_{/C} = \vec{O} \Rightarrow \Sigma(\underbrace{\vec{r}_{i/C}}_{\text{all ext forces}} \times \vec{F}_i) - \Sigma(\underbrace{\vec{r}_{j/C}}_{\text{all pts of mass}} \times m_j \vec{a}_j) = \vec{O}$$

Think of it as: Matter is made of structure (massless) and obeys statics, and has external loads and inertial reactions.

Styrofoam in equilibrium loaded by forces and B-Bs pushing back.

$\Updownarrow$

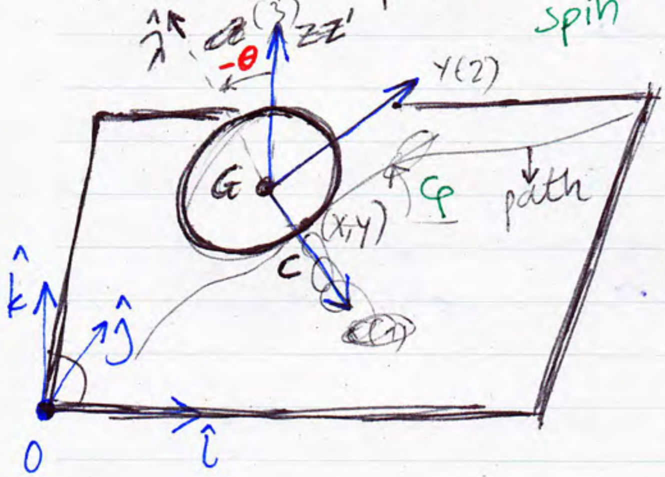
Internal forces have no net moment.
(instead of pairwise equal forces assumption)

$\cancel{2\ell\cos\theta\cos\theta - 2\ell\sin\theta\sin\theta = 0}$

3D Problems with Constraints

* Rolling disk w/ Euler angles (no Rot matrices)

Steer φ(3) – tip θ(2) – ~~roll~~ ψ (1) spin → 321 Euler angles

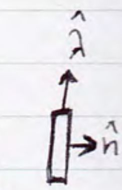


q: x, y, (steer angle) φ, (lean angle) ϑ + ψ angle

ν: steer rate, ~~roll~~ spin rate

Disk: $r, m, I = I^G$

φ = steer, yaw (rotation about z-axis)
ϑ = lean, "roll" (RPY) (–"– –"– y'-axis)
ψ = spin, "pitch" sense → but rolling in English
(rotation about the x''-axis) ($x'' = \hat{n}$)



Starting configuration: $\hat{n} = \hat{e}_1$ ($\hat{n}$ is normal to plane of disk)
$\hat{\lambda} = \hat{k} = \hat{e}_3$

accessible config. space

$\dim(\mathbf{Q}) = 5 \quad (x, y, \varphi, \vartheta, \psi)$

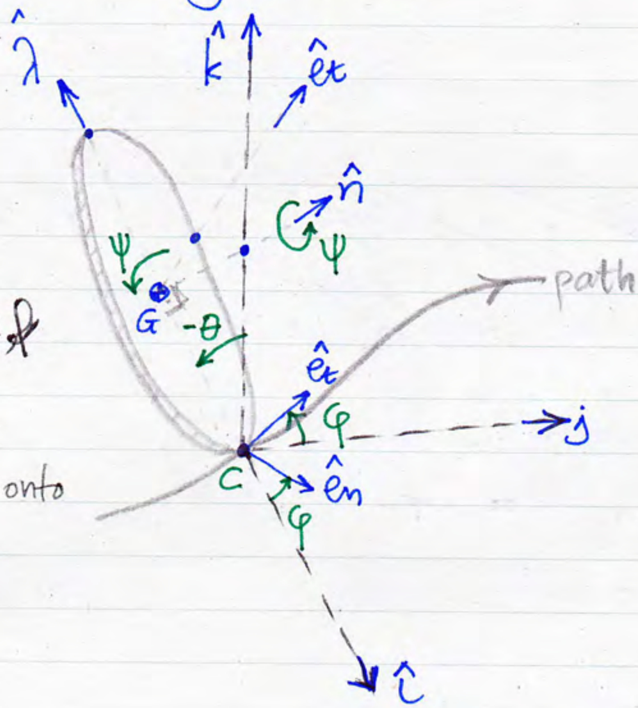
$\dim(\mathbf{V}) = 3$ (stuff that instantaniously satisfy constraint)

space of admissible velocities.

! Nonholonomic because "5 > 3" → 2 nonholonomic constraints
(velocity constraints are not integrable)

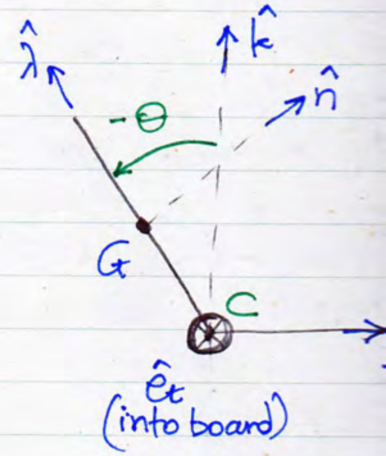
Equations ~~for~~ with: $\theta, \dot{\theta}, \dot{\psi}, \dot{\varphi}$ →

Rolling Disk



$\hat{n}$: normal to disk
$\hat{e}_t$: tangent to path, (in the disk plane & in xy plane)
$\hat{e}_n = -\hat{k} \times \hat{e}_t$ (projection of $\hat{n}$ onto xy plane)
$\hat{\lambda} = \vec{r}_{G/C} / \|\vec{r}_{G/C}\|$

Steer/yaw = φ
Lean = θ
Spin/Pitch = ψ



At rest configuration, $\begin{cases} \theta = \psi = \varphi = 0 \\ \hat{e}_n = \hat{n} = \hat{\imath}, \ \hat{\lambda} = \hat{k} \\ \hat{e}_t = \hat{j} \end{cases}$

We don't really care about ψ, so no need for rotation matric

4 reference frames

$\mathcal{F}$: $\hat{\imath}, \hat{j}, \hat{k}$

$\mathcal{P}$ Precessing : $\hat{e}_n, \hat{e}_t, \hat{k}$

$\mathcal{T}$ Tipped precessing : $\hat{n}, \hat{e}_t, \hat{\lambda}$

} ~~Same frame, but two different~~ OR ~~coordinate systems.~~ These differ by θ.

$\mathcal{B}$ Body-frame : $\hat{n}$, ?, ? (axi-symmetric object)

$\boxed{\text{We will do AMB/C}}$

Geometry:

$$\hat{k} = \cos\theta\,\hat{\lambda} - \sin\theta\,\hat{n}$$

$$\dot{\hat{k}} = \cancel{-\sin\theta\,\dot{\theta}\,\hat{\lambda} + \cos\theta\,\dot{\hat{\lambda}} + \cos\theta\,\dot{\theta}\,\hat{n} + \sin\theta\,\dot{\hat{n}}} = \vec{0}$$

$$\dot{\hat{e}}_t = \overbrace{-\dot{\varphi}\,\hat{e}_n}^{=\dot{\phi}\hat{k}\times\hat{e}_t}, \quad \hat{e}_n = \cos\theta\,\hat{n} + \sin\theta\,\hat{\lambda} \implies$$

$$\dot{\hat{e}}_t = -\dot{\varphi}(\cos\theta\,\hat{n} + \sin\theta\,\hat{\lambda})$$

$$\dot{\hat{n}} = \overbrace{-\dot{\theta}\,\hat{\lambda} + \dot{\varphi}\cos\theta\,\hat{e}_t}^{= \vec{\omega}_{T/F}\times\hat{n}}$$

(changes due to leaning and steering)

$$\vec{r}_{G/C} = R\,\hat{\lambda}$$

Euler angles — $\underline{\underline{I}}$ is nice in this frame

$$\vec{\omega}_{B/F} = \dot{\varphi}\,\hat{k} + \dot{\theta}\,\hat{e}_t + \dot{\psi}\,\hat{n} \overset{\hat{k}}{=} \vec{\omega}(\dot{\varphi}, \dot{\theta}, \dot{\psi}, \underline{\hat{n}}, \underline{\hat{\lambda}}, \underline{\hat{e}_t}) \quad (1)$$

Goal: AMB/C $\Rightarrow$ $\sum \vec{M}^{ext}_{/C} = \dot{\vec{H}}_{/C} = \vec{r}_{G/C}\times m\vec{a}_G + \underline{\underline{I}}\cdot\dot{\vec{\omega}} + \vec{\omega}\times(\underline{\underline{I}}\cdot\vec{\omega})$

$\hookrightarrow$ in terms of Euler angles and their rates of change.

differentiate expression WITH $\hat{k}$

$$\vec{\alpha} = \dot{\vec{\omega}} = {}^{F}\!\frac{d}{dt}(\vec{\omega}) \overset{\text{Q-dot}}{=} \ldots \text{ mess } \ldots =$$

$$= \ddot{\varphi}\,\hat{k} + \ddot{\theta}\,\hat{e}_t + \dot{\theta}\,\dot{\hat{e}}_t + \ddot{\psi}\,\hat{n} + \dot{\psi}\,\dot{\hat{n}}$$

$\longrightarrow$

$$\vec{r}_{G/G}$$

Rolling constraint : $\boxed{\vec{V}_c = \vec{0}} \Rightarrow \vec{0} = \vec{V}_G + \vec{\omega}_{B/\mathcal{F}} \times (-R\hat{\lambda}) \xrightarrow{(1)}$

$\vec{V}_G = \vec{V}_G$ (Euler angles , $\mathcal{T}$ base vectors)

$d/dt\{\ \} \Rightarrow \vec{a}_G = \vec{a}_G($ —"— , —"— $)$

$AMB_{/C} \Rightarrow \sum \vec{M}^{ext}_{/C} = \dot{\vec{H}}_{/C} \Rightarrow$

diagonal

$$\vec{r}_{G/C} \times (-mg\hat{k}) = \vec{r}_{G/C} \times m\vec{a}_G + \underline{\underline{I}}\dot{\vec{\omega}} + \vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega})$$

$\Rightarrow$ 3 equations to solve for $\underline{\ddot{\theta}, \ddot{\varphi}, \ddot{\psi}}$ in terms of $\underline{(\theta, \dot{\theta}, \dot{\varphi}, \dot{\psi})}$

Do I need $\hat{i}, \hat{j}, \hat{k}$? {
- Derive equations explicitly (isolate $\ddot{\theta}, \ddot{\varphi}, \ddot{\theta}$)
- Symbolic solve in MATLAB. ↳ is that possible?

$\vec{\omega}$ will be in B $\xrightarrow{?}$ use R to plot/animate in $\mathcal{F}$.

$$\vec{V}_G = \vec{\omega}_{B/\mathcal{F}} \times R\hat{\lambda}$$
$$= (\dot{\varphi}\cos\theta\,\hat{\lambda} + \dot{\theta}\,\hat{e}_t + (\dot{\psi} - \dot{\varphi}\sin\theta)\hat{n}) \times R\hat{\lambda}$$
$$= R\dot{\theta}\,\hat{n} - R(\dot{\psi} - \dot{\varphi}\sin\theta)\,\hat{e}_t$$

$$V_G^2 = R^2(\dot{\theta}^2 + (\dot{\psi} - \dot{\phi}\sin\theta)^2)$$

H/W 4/18

Problem: Spinning Top

This time, it's not standing upright!

There is a simplification we should discover. (there are missing dependences on angles and rates of change)

Problems that can be solved without solving d.e.s

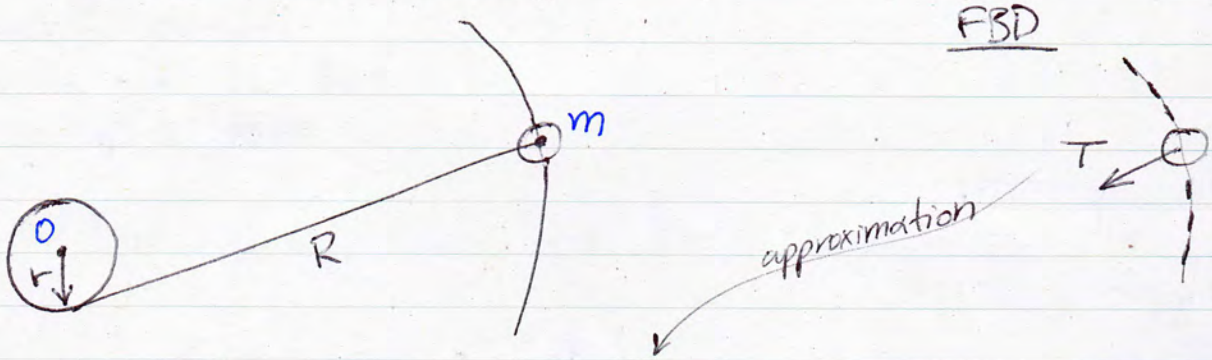
- Fixed-axis rotation constrained
- Axi-symmetric objects spinning w constant rate about axis of symm.
  e.g. plate, coin, football, rolling disk?

$3\text{-}2\text{-}1$ = yaw-pitch-roll

$3\text{-}1\text{-}2$ = ~~ϕ~~ ~~$\phi\,\theta\,\psi$~~ ψ θ φ

spin (pitch) tip (roll) steer/yaw

- Bicycle pictures
- Ice rink velocity quiz.
  - → speed cannot increase because it would result in a perpetual motion machine
  - → Quick calculation



$$\sum \vec{M}_{/O}^{ext} = \dot{\vec{H}}_{/O} \Rightarrow -r\left(\frac{mv^2}{R}\right) = \frac{d}{dt}(mvR)$$

non-zero

$$-\frac{rmv^2}{R} = m\dot{v}R + mv\dot{R}$$

$$\dot{R} = -\frac{v}{R}r$$

$$\Rightarrow$$

$$-\frac{r\cancel{m}v^2}{\cancel{R}} = \cancel{m}\dot{v}R - \frac{\cancel{m}v^2}{\cancel{R}}r \Rightarrow \dot{v}R = -\frac{v^2}{\cancel{R}}r + \frac{v^2}{\cancel{R}}r \Rightarrow$$

$$\boxed{\dot{v} = R}$$

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- Adding kinematic Constraints to Lagrange Equations.

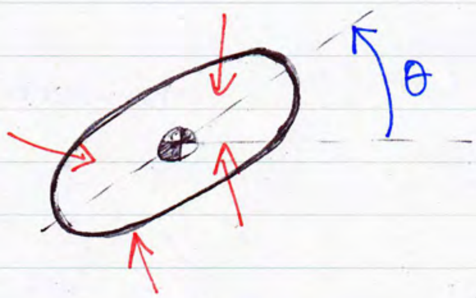
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i = \overbrace{\sum_{j=1}^{3N} \frac{\partial r_i}{\partial q_i}}^{\text{Work done per unit change in } q_i \text{, by force system}} F_j^*$$

$\mathcal{L} = E_K - E_p$ (or $T - V$)

F_j^*: Forces other than those of constraint by the holonomic constraints. & not yet taken account of other $V = E_p$ forces

Example system:

Chaplygin sleigh



$$\begin{pmatrix} q_1 = x_G \\ q_2 = y_G \\ q_3 = \theta \end{pmatrix}$$

$E_p = 0$

$\sum_{j=1}^{2} \dots$

Assume no skate/wheel and write Lagrange eqs:

$$E_K = \frac{1}{2} m \left(\underset{\dot{x}_G}{V_x}^2 + \underset{\dot{y}_G}{V_y}^2 \right) + \frac{1}{2} I \dot{\theta}^2$$

use $N = 1 \Rightarrow j = 1, 2, 3$ (2D)

i) $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_G} - \frac{\partial L}{\partial x_G} = Q_1 = \Sigma F_x^* \Rightarrow$

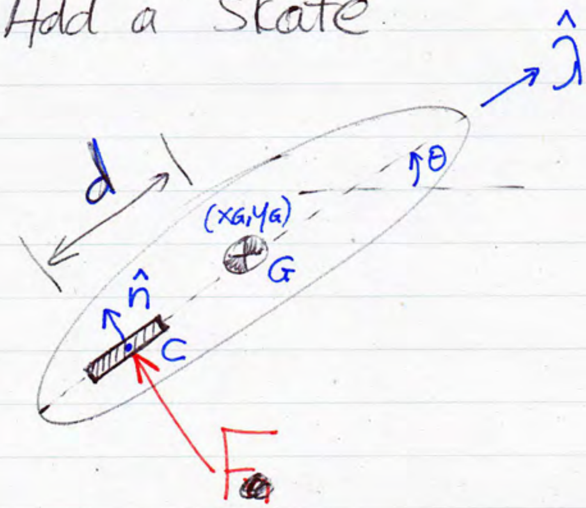
$m\ddot{x}_G = \Sigma F_x^*$

ii) $m\ddot{y}_G = \Sigma F_y^*$

iii) $I\ddot{\theta} = \Sigma M^*_{/G}$

$\longrightarrow$

Add a skate



If we had an expression for $F_a(t)$, we could write $\mathcal{L}$ eqs:

$$\left.\begin{aligned}
&1)\quad m\ddot{x}_G = -F_a \sin\theta && = Q_x \\
&2)\quad m\ddot{y}_G = +F_a \cos\theta && = Q_y \\
&3)\quad I\ddot{\theta} = -F_a d && = Q_\theta
\end{aligned}\right\} \text{generalized forces.}$$

3 equation, 4 stuff: x, y, θ, F!!

add kinematic equation : $\boxed{\vec{V}_c \cdot \hat{n} = 0}$ $\Rightarrow$

$$\left(\dot{x}_G \hat{\imath} + \dot{y}_G \hat{\jmath} + \dot{\theta} \hat{k} \times (-d\hat{\lambda})\right) \cdot \hat{n} = 0$$

$$\Big\Downarrow {}^{d}/_{dt}$$

... 4th equation

Eqs (1→4) → OK to solve now

(Force F will be in $\hat{n}$-direction, because ~~they do~~ it does no work.)

Constraints and Lagrange Equations April 24, 2013

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

$$\mathcal{L} = T - V$$

$$Q_i = \sum_{j=1}^{3N} \frac{\partial r_j}{\partial q_i} F_j^*$$ ← forces not included in Lag. Eq.

$Q_i \, \delta q_i = \delta W$ of the F_j^*

$q_1 = x_G$
$q_2 = y_G$
$q_3 = \Theta$

Method 1

Know direction of constraint forces by physical reasoning

$$\left.\begin{aligned} Q_1 &= -N \sin\Theta \\ Q_2 &= N \cos\Theta \\ Q_3 &= -N d \end{aligned}\right\}$$ and solve constraint equations

$\Rightarrow$ DAE

Method 2

Method of Lagrange Multipliers

Assume constraints of this form

$$\sum_i a_i \dot{q}_i + a_t(t) = 0$$

$$a_i(q_1, q_2 \ldots, t)$$

$$\vec{V}_c \cdot \hat{n} = 0$$

$$\hat{n} = \hat{k} \times \hat{\lambda} = \hat{k} \times (\cos\theta \hat{i} + \sin\theta \hat{j})$$

$$\vec{V}_c = \vec{V}_G + \vec{V}_{c/G} = \vec{V}_G + (\dot{\theta}\hat{k}) \times (-d\hat{\lambda})$$

$$\Rightarrow -\dot{x}_G \sin\theta + \dot{y} \cos\theta - d\dot{\theta} = 0$$

$$\Rightarrow \begin{bmatrix} -\sin\theta & \cos\theta & -d \end{bmatrix} \begin{bmatrix} \dot{x}_G \\ \dot{y}_G \\ \dot{\theta} \end{bmatrix}$$

$a_1 = -\sin\theta$

$a_2 = \cos\theta$

$a_3 = -d$

of form $\sum a_i \dot{q}_i = 0$

Put together 2 ideas

1) Set of allowed variations q_i are all those which satisfy $\sum_i a_i \delta q_i = 0$

2) Constraint forces Q_i do no work in allowed motions

$$\sum_i Q_i \delta q_i = 0 \text{ for all allowed motions}$$

Before we had the constraints

q_i were a vector in $\mathbb{R}^n$

So set of allowed variations is $\perp$ to $\vec{a}$ in $\mathbb{R}^n$

Q_i ~~vec~~ $\perp$ to all allowed δq_i

$$Q_i = \lambda a_i$$

ex)

$$M_G \ddot{x}_G = -\lambda \sin\theta$$

$$M_G \ddot{y}_G = \lambda \cos\theta$$

$$I^G \ddot{\theta} = \lambda(-d)$$

add constraint equations $\Rightarrow$ 4 eqs for $\ddot{\theta}, \ddot{x}_G, \ddot{y}_G, \lambda$

experimental fact

Lagrange multiplier has a simple physical meaning

Homework: rolling disk w/ Lagrange Equations

What if $n=2$ (2 dim configuration space before "non-holonomic" constraints

$a_1\dot{q}_1 + a_2\dot{q}_2 = 0$

$[a_1, a_2]$

direction field $\vec{a}$

$\dot{q} \perp \vec{a}$ everywhere

curves can't cross

level lines of some function $F(q_1, q_2)$

constraint equation is "integrated"

All first order differential equations are integrable

Vibrations of Continuous Systems

In principle $\Rightarrow \infty$ DOFs

But approach is pretend you only have 1-3 that matter.

Find normal modes would be good, but if that's too hard, guess mode ~~string~~ shapes

Ex: String   mode shape = $\frac{1}{2}$ sine wave

What if we guess a parabola

$$\left(\frac{l}{2} - x\right)\left(\frac{l}{2} + x\right) = \phi(x)$$

$$u(x,t) = q\,\phi(x)$$

Lagrange equations $q(t)$

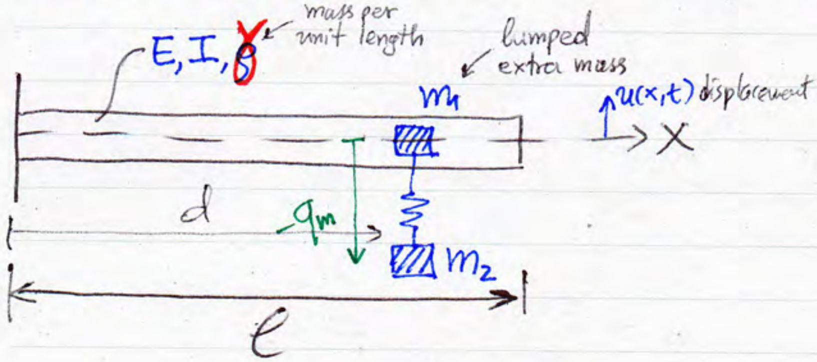
T, V from q and $\dot{q}$

$\Rightarrow$ equations of motion

$\Rightarrow$ approximate calculate of frequency

$\Rightarrow$ ½ % error in frequency

New Homework: (Continuous body vibration)



How does this vibrate?
(approximate solution)

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$q_i = 0 \rightarrow$ potential energy minimum

# of shapes · assumed "mode" shapes

Guess: $\cancel{q}\,u?(x,t) = \sum_{i=1} \underbrace{q_i(t)}_{\text{generalized coordinates}} \phi_i(x) + \underbrace{\sum_{n+1}^{n+?} q_i(t)}_{\text{discrete modes}}$

Write Lagrange equations:

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \xrightarrow[\text{get}]{\text{we wanna}} M\ddot{q} + Kq = \underline{0}$

$L = T^{?} - U^{?}$ ← strain energy of beam + potential energy of spring

Solution must respect boundary conditions:

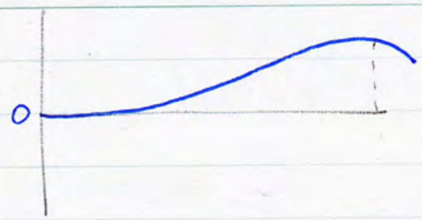
$u(0,t) = 0$
$u'(0,t) = 0$

ex) $\phi_1(x) =$ 

x^2, $e^{cx} - 1 - cx$, $\cosh(cx) - 1$, $1 - \cos(cx)$

3rd o. cubic beam with point load at end
4th o. cubic beam + uniform load.

$x = \ell \Rightarrow cx = \pi/2 \Rightarrow \boxed{c_1 = \frac{\pi}{2\ell}}$

$\phi_2(x)$ 

$\|$

$1-\cos(c_2 x)$, $c_2 = \pi/\ell \;\left(= 2\cdot\frac{\pi}{2\ell}\right)$

$\phi_3(x) = 1-\cos(c_3 x)$, $c_3 = \frac{3\pi}{2}\;\left(= 3\cdot\frac{\pi}{2\ell}\right)$

q_4 = deflection of m_2

$$T = E_K = \sum_{\substack{\text{all}\\\text{particles}}} \frac{1}{2} m_i \dot{u}_i^2 = \overbrace{\int \frac{1}{2}\dot{u}^2\,dm}^{\text{continuous stuff}} + \overbrace{\sum \frac{1}{2}\dot{u}_i^2 m_i}^{\text{discrete stuff.}}$$

Point mass $E_K = \frac{1}{2}\dot{q}_m^2 m_2$

Beam $E_K = \frac{1}{2}\int_0^\ell \dot{u}^2 \cancel{\gamma}\,dx + \frac{1}{2}\dot{u}(d)^2 m_1$

But $\dot{u} = \sum \dot{q}_i \phi_i(x)$ $\Rightarrow$ $\int_0^\ell (\quad)^2 \cancel{\gamma}\,dx = [\dot{q}_1\ \dot{q}_2 \dots][M]\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \vdots\end{bmatrix}$

For this matrix: $M_{ij}^{cont.} = \int_0^\ell \phi_i(x)\phi_j(x)\cancel{\gamma}\,dx$ ← the time consuming calculation (numerical integration)

$M_{ij}^{m_1} = \phi_i(d)\phi_j(d)$

$$\boxed{M_{ij}^{beam} = M_{ij}^{cont} + M_{ij}^{m_1}}$$

→

Inman p. 459

$$E_k^{m_2} = \frac{1}{2} m_2 \dot{q}_m^2$$

$$\boxed{E_k^{tot}} = [\dot{q}_1\ \dot{q}_2 \cdots \dot{q}_m] \cdot \left[\begin{array}{c|c} M^{beam} & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ \hline 0\ \ 0\ \cdots & m_2 \end{array}\right] \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_m \end{bmatrix}$$

$$V^{spring} = \frac{1}{2} k \left(\sum q_i \phi_i(d) - q_m \right)^2$$ → potential energy of spring

$$U^{beam} = \int_0^l \frac{1}{2} EI \underbrace{(u'')^2}_{\frac{\partial^2 u}{\partial x^2}} dx$$

$$= \frac{1}{2} EI \int_0^l \sum \left[(q_i \phi_i''(x)) \right]^2 dx$$

M, I, E, $1/\rho = \kappa$ (curvature of beam)

M ↳ bending moment

$$M = \frac{EI}{\rho} = EI\kappa = EI \frac{\partial^2 u}{\partial x^2}$$

"Bending moments-given by E-I-over-ρ" song mnemonic

E: Young's modulus (material stiffness) $= \frac{F/A}{d/l}$

I: Area moment of inertia

$$U = [q_1\ q_2 \cdots q_m] \cdot \left[K^{beam} + K^{spring} \right] \cdot \begin{bmatrix} \vdots \\ q_2 \\ \vdots \\ q_m \end{bmatrix},$$

last row $\left[-k\phi(d)^T \mid +k \right] \begin{bmatrix} q_1 \\ q_2 \\ q_m \end{bmatrix}$

where $K_{ij}^{beam} = \cancel{\frac{1}{2}} EI \int_0^l \phi_i'' \phi_j'' dx$

→

$\frac{1}{2} m \dot{x}^2 \Rightarrow m\ddot{x}$

$M\ddot{x}$

$\frac{1}{2} x^T M x$

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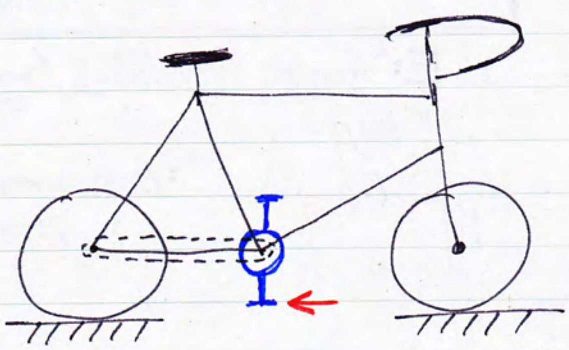
finally
$\Longrightarrow$ $M\ddot{\vec{q}} + K\vec{q} = \vec{0}$ $\rightarrow$ Find normal modes, frequencies etc.

To compare answers, at the end, substitute:

$$\boxed{\begin{array}{l} l = 1, \quad m_1 = 1, \quad m_2 = 1 \\ E = 1, \quad I = 1, \quad d = 1/2, \quad \gamma = 1 \end{array}}$$

13-14 May: 30' Homework review with Fuinx
(animations for all H/W that ask for one)

Another H/W Problem:



Which way will it go?

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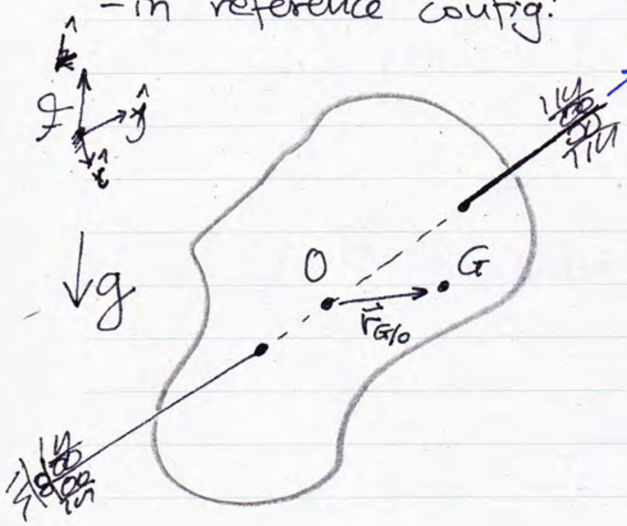
5/2

<u>Goal</u>: Double pendulum in 3D
1) fixed axis of rotation (single pend)
2) variable -"- -"- (-"-)
3) 2 axes of rotation (double pend)

- Simple pendulum in 3D
  - arbitrary rigid object.
  - in reference config: $\vec{r}_{G/O}^{\,ref}$, $\underline{\underline{I}}^{ref}$, $\underline{\underline{R}} = \underline{\underline{1}}$, m

not rotated at reference.



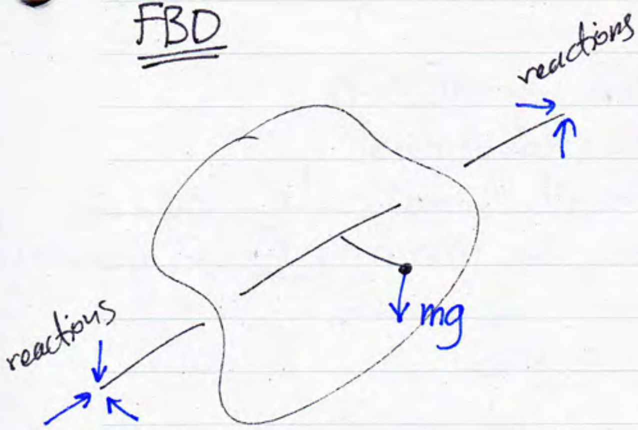
$\boxed{1 \text{ DoF}} \rightarrow \theta$

This will have simple harmonic oscillator equations, but we need to find the effective length, mass, inertia etc:

$$\boxed{\ddot{\theta} + c \cdot \sin\theta = 0}$$

$\hookrightarrow c(\hat{\lambda}\cdot\hat{k},\ \hat{\lambda}\cdot\underline{\underline{I}}\hat{\lambda},\ d^{\perp})$

<u>FBD</u>



LMB & AMB should give 6 eqs for 5 reaction forces & $\ddot{\theta}$.

But, take $\{AMB/_O\} \cdot \hat{\lambda} \Rightarrow \ddot{\theta}$ directly

$$\left\{ \Sigma \vec{M}_{/0} = \dot{\vec{H}}_{/0} \right\} \cdot \hat{\lambda} \Rightarrow$$

$$\vec{r}_{G/0} \times (-mg\hat{k}) \cdot \hat{\lambda} = \left\{ \vec{r}_{G/0} \times m\vec{a}_G + \underline{\underline{I}}\dot{\vec{\omega}} + \vec{\omega} \times (\underline{\underline{I}}\vec{\omega}) \right\} \cdot \hat{\lambda} \Rightarrow$$

Calculate relevant terms:

$$\underline{\underline{R}} = \underline{\underline{R}}(\theta) = (1-\cos\theta) \cdot \hat{\lambda}\hat{\lambda} + \cos\theta \cdot \underline{\underline{1}} + \sin\theta \cdot \underline{\underline{S}}(\hat{\lambda})$$

$$\vec{r}_{G/0} = \underline{\underline{R}} \cdot \vec{r}_{G/0}^{\,ref.}$$

$$\vec{\omega} = \dot{\theta}\hat{\lambda}$$

Only unknown is $\ddot{\theta}$ DONE

$$\dot{\vec{\omega}} = \ddot{\theta}\hat{\lambda}$$

$$\boxed{\underline{\underline{I}} = \underline{\underline{R}} \cdot \underline{\underline{I}}^{ref} \cdot \underline{\underline{R}}^T} \rightarrow {}^{curr}\underline{\underline{I}} = {}^{curr}\underline{\underline{R}}_{ref} \cdot {}^{ref}\underline{\underline{I}} \cdot \underline{\underline{R}}^T$$

$$\underline{\underline{R}}^T = {}^{ref}\underline{\underline{R}}_{curr}$$

How to extract $\ddot{\theta}$? in practice (3 ways)

1) symbolic solve in MATLAB
2) Isolate $\ddot{\theta}$ in equations.
3) some "on the fly" method: use that eqs are linear in $\ddot{\theta}$ (true for all dynamics)

(3) Dumb, on the fly, method. (sits inside .rhs file)

(a) Put in numbers for all constants ($\underline{\underline{I}}^{ref}$, m, $\vec{r}_{G/0}^{\,ref}$, $\hat{\lambda}$ ...) and for present value of θ & $\dot{\theta}$.

Set $\ddot{\theta} = 1 \Rightarrow$ evaluate $\left(-\Sigma \vec{M}_{/0}^{ext} + \dot{\vec{H}}_{/0}\right) \cdot \hat{\lambda} = \underline{-f(\text{params}, \theta, \dot{\theta}) + M_{\theta\theta}\ddot{\theta}}$

$A_1 =$

function of params & θ.

(b) Set $\ddot{\theta} = 0 \Rightarrow A_0 = \cdots$

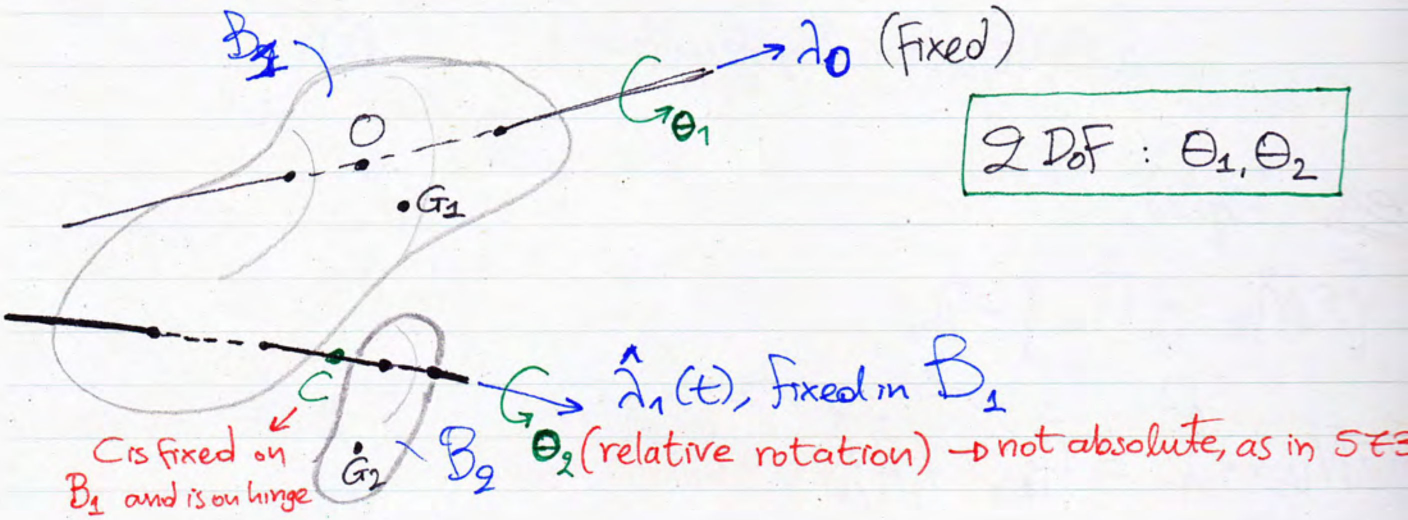
(c) $A_1 - A_0 = M_{\theta\theta}$, $A_0 = -f(\text{params}, \theta, \dot{\theta}) \Rightarrow$

(d) $\left\{ \begin{array}{l} A_1 - A_0 = M_{\theta\theta} \\ \qquad A_0 = -F(\cdot) \end{array} \right\}$

(e) $\boxed{M_{\theta\theta} \cdot \ddot{\theta} = f}$

now known   now known

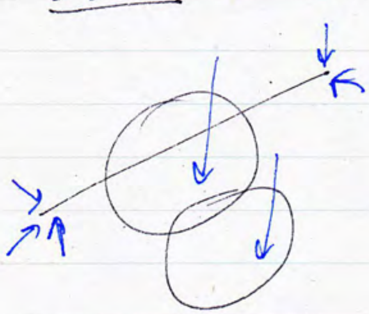
- Double Pendulum in 3D



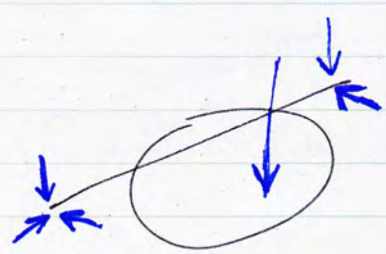
$\boxed{2\ \text{DoF}: \theta_1, \theta_2}$

FBDs:

SYSTEM



B_2



- Given parameters: $\left\{ \begin{array}{l} m_1, m_2, g, \hat{\lambda}_0, \hat{\lambda}_1^{ref} \\ \underline{\underline{I}}_1^{ref}, \underline{\underline{I}}_2^{ref}, \vec{r}_{G/O}^{\ ref}, \vec{r}_{C/O}^{\ ref}, \vec{r}_{G_2/C}^{\ ref} \end{array} \right\} \underline{P}$

- Goal: Given : $P, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2$

  Find : $\ddot{\theta}_1, \ddot{\theta}_2$

- Equations: $\{AMB_{/C} \text{ for } B_2\} \cdot \hat{\lambda}_1$ (1)

  $\{AMB_{/O} \text{ for System}\} \cdot \hat{\lambda}_0$ (2)

<u>e.g.</u> Eq.(2)

$$\{\Sigma \vec{M}_{/O} = \dot{\vec{H}}_{/O}\} \cdot \hat{\lambda}_0$$

$$\{\Sigma \vec{M}_{/O}\} \cdot \hat{\lambda}_0 = \underbrace{\dot{\vec{H}}^{1}_{/O}}_{B_1} + \underbrace{\dot{\vec{H}}^{2}_{/O}}_{B_2}$$ → hardest term

$$\dot{\vec{H}}^{(2)}_{/O} \cdot \hat{\lambda}_0 = \left\{ \vec{r}_{G_2/O} \times m_2 \vec{a}_{G_2} + \underline{\underline{I}}_2 \dot{\vec{\omega}}_2 + \vec{\omega}_2 \times \left(\underline{\underline{I}}_2 \cdot \vec{\omega}_2 \right) \right\} \cdot \hat{\lambda}_0$$

- Calculate all terms:

$$\underline{\underline{R}}_{B_1/\mathcal{I}} = \underline{\underline{R}}_{B_1/\mathcal{I}}(\theta_1) = \dots \ \hat{\lambda}_0 \hat{\lambda}_0 \dots \ \dots$$

$$\cancel{\underline{\underline{R}}_{B_2/B_1} = \underline{\underline{R}}_{B_2/B_1}}$$

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$$\hat{\lambda}_1 = \underline{\underline{R}}_{B_1/\mathcal{F}} \cdot \hat{\lambda}_1^{ref}$$

$$\vec{r}_{G_1/O} = \underline{\underline{R}}_{B_1/\mathcal{F}} \cdot \vec{r}_{G_1/O}^{\,ref}$$

$$\vec{r}_{C/O} = \underline{\underline{R}}_{B_1/\mathcal{F}} \cdot \vec{r}_{G_1 O}^{\,ref}$$

$$\underline{\underline{R}}_{B_2/B_1} = \underline{\underline{R}}_{B_2/B_1}(\theta_2, \hat{\lambda}_1) \rightarrow (\theta_2, \theta_1, \hat{\lambda}_1^{ref})$$

$$\underline{\underline{R}}_{B_2/\mathcal{F}} = \underline{\underline{R}}_{B_1/\mathcal{F}} \underline{\underline{R}}_{B_2/B_1} \cdot \cancel{\underline{\underline{R}}_{B_1/\mathcal{F}}} \qquad \cancel{B} \qquad {}^{\mathcal{F}}R_{B_2} = {}^{\mathcal{F}}R_{B_1} \cdot {}^{B_1}R_{B_2}$$

$$\vec{r}_{G_2/C} = \underline{\underline{R}}_{B_2/\mathcal{F}} \cdot \vec{r}_{G_2/C}^{\,ref}$$

$$\vec{r}_{G_2/O} = \vec{r}_{G_2/C} + \vec{r}_{C/O}$$

$$\vec{\omega}_{B_2/B_1} = \dot{\theta}_2 \hat{\lambda}_1$$

$$\vec{\omega}_{B_1/\mathcal{F}} = \dot{\theta}_1 \hat{\lambda}_0$$

$$\vec{\omega}_{B_2/\mathcal{F}} = \vec{\omega}_{B_1/\mathcal{F}} + \underline{\vec{\omega}_{B_2/B_1}}$$

Q-dot formula

$$\dot{\vec{\omega}}_{B_2/\mathcal{F}} = \dot{\vec{\omega}}_{B_1/\mathcal{F}} + \underline{\vec{\omega}_{B_1/\mathcal{F}} \times \vec{\omega}_{B_2/B_1} + \ddot{\theta} \hat{\lambda}_1}$$

$$\underline{\underline{I}}_2 = \underline{\underline{R}}_{B_2/\mathcal{F}} \cdot \underline{\underline{I}}_2^{ref} \cdot \underline{\underline{R}}_{B_2/\mathcal{F}}^T$$

$$\vec{a}_C = \dot{\vec{\omega}}_{B_1/\mathcal{F}} \times \vec{r}_{C/O} + \vec{\omega}_{B_1/\mathcal{F}} \times (\vec{\omega}_{B_1/\mathcal{F}} \times \vec{r}_{C/O})$$

$$\vec{a}_{G_2/C} = \dot{\vec{\omega}}_{B_2/\mathcal{F}} \times \vec{r}_{G_2/C} + \vec{\omega}_{B_2/\mathcal{F}} \times \vec{\omega}_{B_2/\mathcal{F}} \times \vec{r}_{G_2/C}$$

Done with the hard piece of $\dot{\vec{H}}_{/O}$.

Similar but easier calculations for other terms

$\Downarrow$

2 eqs in $\ddot{\theta}_1, \ddot{\theta}_2$