MAE 6700 - Advanced Dynamics
- Andy Ruina

$\boxed{1/23/2014}$

Course Outline

I. 3D dynamics of particles & rigid objects
   $\vec{\omega}$, $\underline{\underline{R}}$, $\underline{\underline{I}}$, $\vec{H}_{/c}$

II. Lagrange Eqns.
   - Derive 2 ways:
     1. Newton Laws
     2. Princ. of least action
   - Forcing & with extra constraints

III. Vibrations of strings & beams

IV. Misc. Topics: Friction, collisions, non-holonomic constraints

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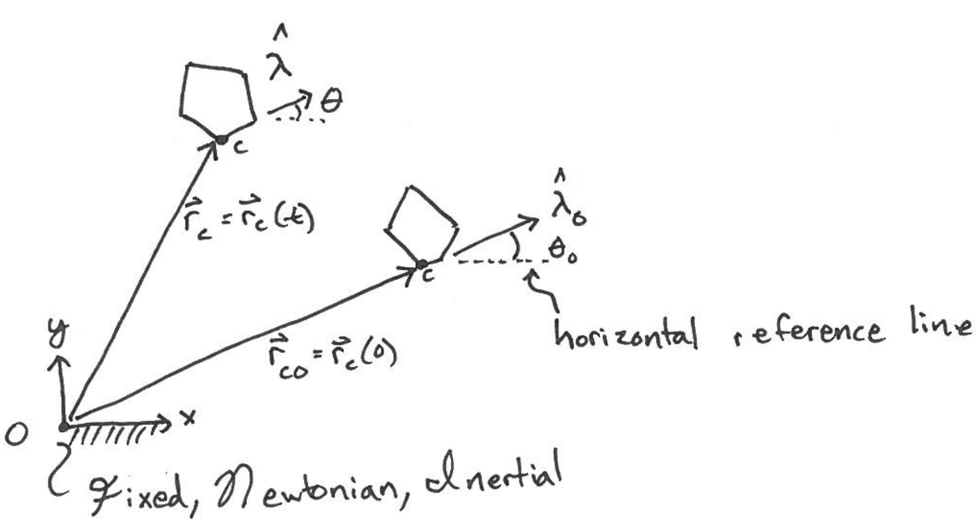
Course Organization

- HW ≈ weekly
- FINAL EXAM
- FINAL PROJECT

2D Rigid Object Geometry



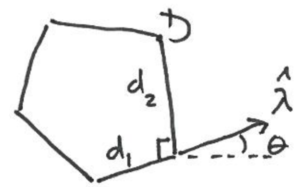
Claim:

If know position of all points at $t=0$

2) $\hat{\lambda}(t)$

3) $\theta(t)$

$\Rightarrow$ position of all pts for all time

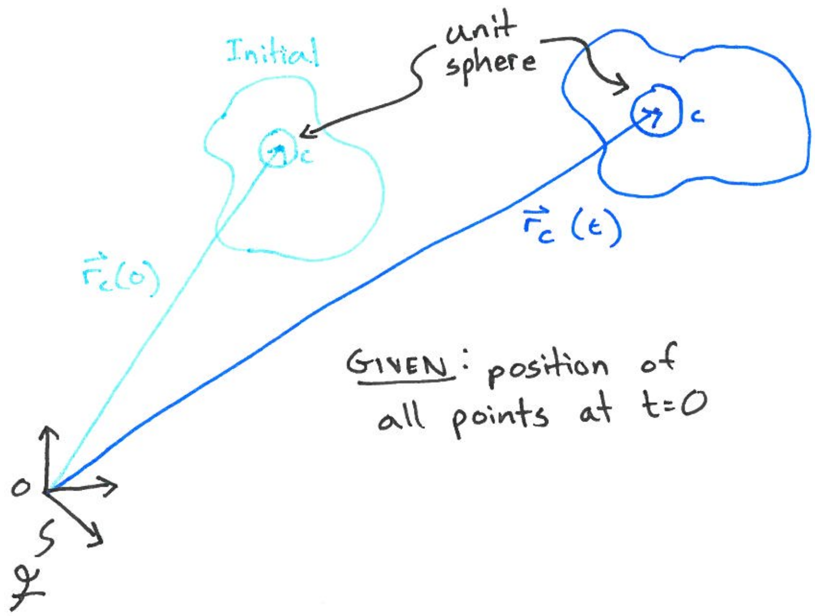


ex) Pt D is d_1 in $\hat{\lambda}$ dir + d_2 $\perp$ to left of $\hat{\lambda}$

$\Rightarrow$ In 2D motion is characterized by 3 #'s: $\vec{r}_c$ + θ

$2 + 1 = 3$

3D



GIVEN: position of all points at $t=0$

where D is point on a unit sphere

$$\vec{r}_{D/c}(0) = r_{D/c}\,\hat{\lambda}(0)$$

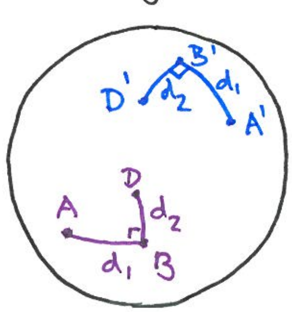
$$\vec{r}_{D/c}(t) = r_{D/c}\,\hat{\lambda}(t)$$

Claim: All we need is $\vec{r}_c(t)$ + position of all points on unit sphere

e.g.) $\hat{\lambda}$ is a unit vector with tip on unit sphere

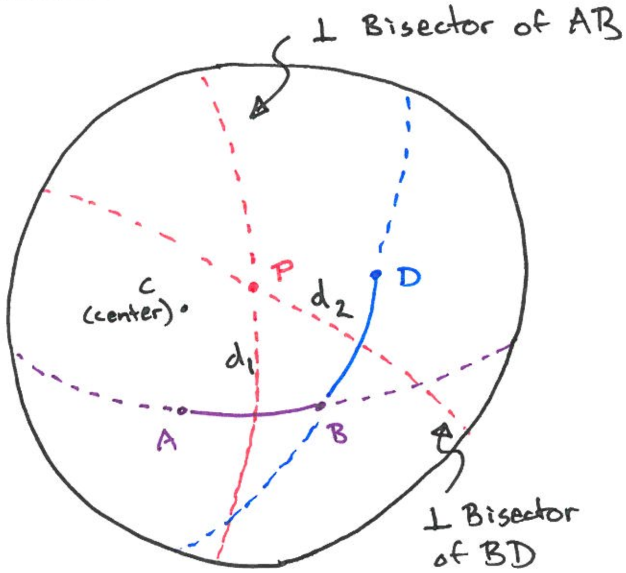
$\Rightarrow$ $\vec{r} = r\hat{\lambda}$

Claim: Only need to know position of two points before + after



Point D is d_1 along great circle AB + then d_2 to the left

Euler's Thm



$$\boxed{\begin{aligned}\theta &= m_1 P m_2 \\ &= APB \\ &= BPD\end{aligned}}$$

B is any point before motion

D is where B gets to in motion

A is the point that ends up at B after the motion

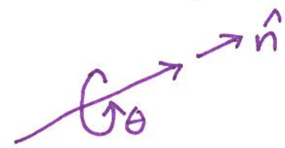
$\Rightarrow$ AB $\rightarrow$ BD

After motion, draw ⊥ bisectors and their intersection has not moved

P is a fixed point

CP is the axis of rotation, $\hat{n}$

θ is the angle of rotation



<u>FACT</u>: $\hat{n}$ is independent of choice of C for given motion

First Parameterization of rotations:

4 #'s: $\hat{n}$, θ

$3 + 1 = 4$

<u>Not independent</u>: $n_x^2 + n_y^2 + n_z^2 = 1$

<u>Not unique</u>: a) $\hat{n}, \theta$ "=" $-\hat{n}, -\theta$

b) $\hat{n}, \theta$ "=" $\hat{n}, \theta + 16\pi$

<u>Representation #2</u>

$\vec{N} = \theta \hat{n}$

= the rotation "vector"

NOTE: not unique

$\theta \hat{n} = (\theta + 2\pi)\hat{n}$

<u>Not a vector</u>

$\vec{N} = 3\vec{N}_1 + 5\vec{N}_2$

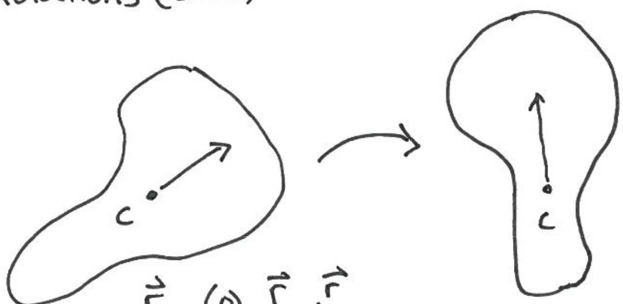
★ Rules of Vector addition have no sensible interpretation in geometry of rotations

e.g.) Rotate $\vec{N}_1$ then $\vec{N}_2$, net rotation is not $\vec{N}_1 + \vec{N}_2$

★ Also, $\vec{N}_1$ then $\vec{N}_2 \neq \vec{N}_2$ then $\vec{N}_1$

1/28/2014

Rotations (con't)



$\vec{r}_{p/c}(0), \vec{r}_0, \vec{r}$

$\underbrace{\vec{r}_{p/c}(t), \vec{r}, \vec{r}\,'}$ Rotated version of original vector

<u>Recall</u>: Rigid motion = translation + "rotation"

Rotation = rotation θ about axis $\vec{N}$

$\vec{N}$ doesn't change in rotation

Axis angle representation of rotation

$\vec{N}, \theta$

Rotation vector $\theta \hat{n}$ where $\hat{n} = \vec{N}/|\vec{N}|$

Or use unit vector

$\hat{n}, \theta$

Not unique $\hat{n}, \theta$ "=" $-\hat{n}, -\theta$

and $\hat{n}, \theta$ "=" $\hat{n}, \theta + 2n\pi$

$n = 1, 2, 3$

Can be made unique: $\boxed{\sin\theta\,\hat{n}, \cos\theta}$ or $\boxed{\sin(\theta/2)\,\hat{n}, \cos(\theta/2)}$

<u>TODAY'S Question</u>

<u>given</u> $\hat{n}, \theta, \vec{r}_0$

<u>find</u> $\vec{r}$

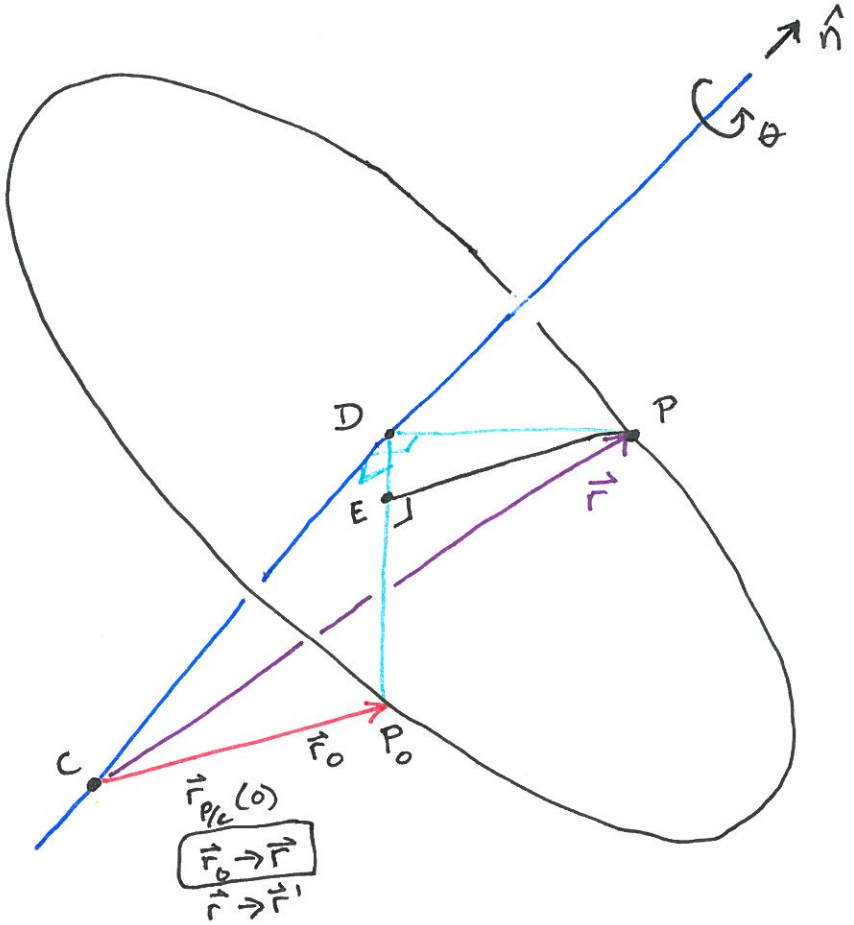
$\vec{r}_0 \rightarrow \vec{r}$

$\vec{r}_{p/c}(0) \rightarrow \vec{r}_{p/c}(t)$

$\vec{r} \rightarrow \vec{r}\,'$

<u>GIVEN:</u> $\hat{n}, \theta, \vec{r}_0$

<u>FIND:</u> $\vec{r}$



NOTE: $DP \perp CD$

$DP_0 \perp CD$

$DP_0 \perp EP$

$$\vec{r}_{P/C} = \vec{r}_{CD} + \vec{r}_{DE} + \vec{r}_{EP}$$

where

$$\vec{r}_{CD} = (\vec{r}_0 \cdot \hat{n})\,\hat{n}$$

$$\vec{r}_{DP_0} = \vec{r}_0 - (\vec{r}_0 \cdot \hat{n})\hat{n}$$

$$\vec{r}_{DE} = \cos\theta\, \vec{r}_{DP_0} = \cos\theta\left[\vec{r}_0 - (\vec{r}_0 \cdot \hat{n})\hat{n}\right]$$

NOTE: $\hat{n} \times \vec{r}_{DP_0} = \hat{n} \times \vec{r}_{CP_0}$

$$\vec{r}_{EP} = \hat{n} \times \vec{r}_0 \sin\theta$$

$$\Rightarrow \boxed{\vec{r} = \hat{n}\,\hat{n} \cdot \vec{r}_0 + \cos\theta\left(\vec{r}_0 - \hat{n}(\hat{n} \cdot \vec{r}_0)\right) + \sin\theta\, \hat{n} \times \vec{r}_0}$$ ☆☆☆

NOTE: Rotation is linear in $\vec{r}_0$

$$\mathrm{Rot}(\underbrace{a\vec{r}_1 + b\vec{r}_2}_{\vec{r}_0}) = a\,\mathrm{Rot}(\vec{r}_1) + b\,\mathrm{Rot}(\vec{r}_2)$$

Two ways to see:

a) geometry of rigid objects

b) Look at formula ★★★

Identity Tensor

$\underline{\underline{I}} \cdot \vec{r}_0 = \vec{r}_0$

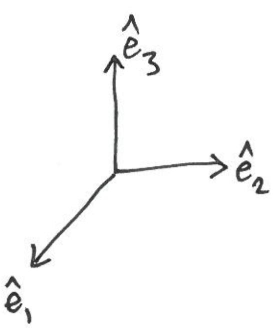
$$= \underbrace{\left[\hat{n}\hat{n}\cdot + \cos\theta\,(\underline{\underline{I}}\cdot - \overbrace{\hat{n}\hat{n}}^{\text{diad}}\cdot) + \sin\theta\,\hat{n}\times\right]}_{\underline{\underline{R}} \text{ rotation tensor}}\vec{r}_0$$

Define diad $\vec{a}\vec{b}$ as vector $\vec{a}$ next to vector $\vec{b}$

$$\vec{a}\,\vec{b}\cdot\vec{v} = \vec{a}(\vec{b}\cdot\vec{v})$$

$$\vec{r} = \left[\hat{n}\hat{n} + \cos\theta\,(\underline{\underline{I}} - \hat{n}\hat{n})\right]\cdot\vec{r}_0 + \underbrace{\sin\theta\,\hat{n}\times\vec{r}_0}_{=\sin\theta\,\underline{\underline{\&}}(\hat{n})\cdot\vec{r}_0}$$

ASIDE: $\vec{a}\times\vec{b} = \underline{\underline{\&}}(\vec{a})\cdot\vec{b}$



$$\vec{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$

$$= \sum_{i=1}^{3} b_i\hat{e}_i = b_i\hat{e}_i \quad \Big\} \text{ Einstein's summation convention}$$

1/30/2014

TODAY
- VECTORS
- TENSORS
- DIADICS
- MATRICES
- ROTATIONS

Vectors (Recommend Ruina/Pratap Chp 2) ☆

Dave Block, book on tensors

Q: What is a vector?

A: A vector is a vector, is a vector...

Vector:
- something with magnitude + direction
- the concept of adding is defined
- the concept of scalar multiplication is defined

& must obey the rules of vector arithmetic:

$$(a+b)\vec{v} = a\vec{v} + b\vec{v}$$

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$$

$$\vec{v} + \vec{0} = \vec{v}$$

$$0\vec{v} = \vec{0}$$

etc.

2 main examples: ① FORCE + ② RELATIVE POSITION

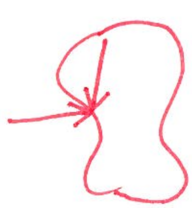
FORCE: some measure of mechanical interaction.

Define in terms of spring

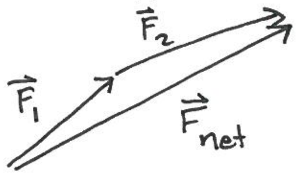
dir is orientation

mag is strength of spring

Definition of addition: Apply 2 forces at same place



$\vec{F}_1 + \vec{F}_2$ is the single spring & stretch with same effect as $\vec{F}_1$ & $\vec{F}_2$ applied together



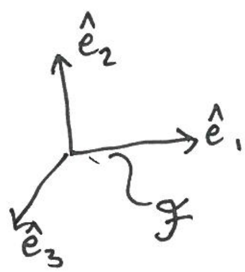
Scalar Multiplication: $c\vec{F}$ means apply c copies of $\vec{F}$

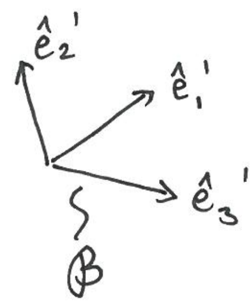
How to represent vectors

① $a \nearrow$ means a * unit vector in $\nearrow$ dir

② add other vectors: $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$

$= v_i\hat{e}_i$ (Einstein





③ $[\vec{v}]_{\mathcal{S}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

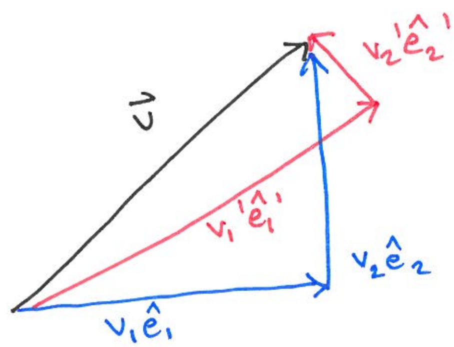
Important $\vec{v} = \vec{v}$

$v_i\hat{e}_i = v_i'\hat{e}_i'$

$v_i \neq v_i'$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$

$\boxed{v_i\hat{e}_i = v_i'\hat{e}_i'}$



Tensors = Linear Functions

ex) vectors $\xrightarrow[\text{order } 2]{}$ vectors

ex) vectors $\xrightarrow[\text{order } 1]{}$ scalars

What is the most general linear function from vectors to scalars?

$$W = \mathcal{L}(\vec{v})$$

vector input

linear function (tensor)

scalar output

$$= \mathcal{L}(\underbrace{v_i \hat{e}_i}_{v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3})$$

Linear function implies

$$\left.\begin{aligned} &\mathcal{L}(\vec{a}+\vec{b}) = \mathcal{L}(\vec{a}) + \mathcal{L}(\vec{b}) \\ &\mathcal{L}(c\vec{a}) = c\,\mathcal{L}(\vec{a}) \end{aligned}\right\} \star\star\star$$

$$= v_i \underbrace{\mathcal{L}(\hat{e}_i)} = v_1\mathcal{L}(\hat{e}_1) + v_2\mathcal{L}(\hat{e}_2) + v_3\mathcal{L}(\hat{e}_3)$$

$L(\hat{e}_1) = L_1 \quad \ldots \quad L(\hat{e}_i) = L_i$

$$w = v_i L_i$$

$= \vec{v}\cdot\vec{L}$ where $\vec{L} = L_1\hat{e}_1 + L_2\hat{e}_2 + L_3\hat{e}_3 = L_i\hat{e}_i$

$$\boxed{\mathcal{L}(\vec{v}) = \vec{L}\cdot\vec{v}}$$

Now look at order 2 tensors

$$\vec{w} = \mathcal{L}(\vec{v})$$

vector input

linear function (tensor)

vector output

$$\vec{w} = \mathcal{L}(\vec{v})$$

$$= \mathcal{L}(v_i \hat{e}_i)$$

linearity

$$= v_i \mathcal{L}(\hat{e}_i)$$

Define $\vec{L}_i$ as $\mathcal{L}(\hat{e}_i)$ → e.g. $\vec{L}_2 = \mathcal{L}(\hat{e}_2)$

$$= v_i \vec{L}_i$$

$$= \vec{L}_i v_i$$

NOTE $v_2 = \hat{e}_2 \cdot \vec{v}$

$$= \vec{L}_i \hat{e}_i \cdot \vec{v}$$

↳ $\vec{L}_i = L_{ji} \hat{e}_j$

$$= \underbrace{L_{ji} \hat{e}_j}_{\vec{L}_i} \hat{e}_i \cdot \vec{v}$$

NOTE: $\hat{e}_j \cdot \hat{e}_k = \delta_{jk}$

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\vec{w} = \underbrace{L_{ij}\, \hat{e}_i (\hat{e}_j}_{\underline{\underline{L}}} \cdot \vec{v})$$

↳ $\vec{v} = v_k \hat{e}_k$

$$= L_{ij} \hat{e}_i \hat{e}_j \cdot (v_k \hat{e}_k)$$

$$\boxed{\vec{w} = L_{ij} v_j \hat{e}_i}$$

$$[\vec{w}]_{\mathcal{F}} = [L][\vec{v}]_{\mathcal{F}}$$

Various representations of order 2 tensors operating on $\vec{v}$

$$\vec{w} = \mathcal{L}(\vec{v}) \quad (1)$$
$$= \vec{L}_i \hat{e}_j \cdot \vec{v} \quad (2)$$
$$= L_{ij} \hat{e}_i \hat{e}_j \cdot \vec{v} \quad (3)$$

$$w = Lv \quad (4)$$
$$w_i = L_{ij} v_j \quad (5)$$

ex) $\vec{a} \times \vec{v}$ $\quad \vec{a} = \text{given}$

NOTE: $\vec{a} \times (c_1 \vec{v}_1 + c_2 \vec{v}_2) = \underbrace{c_1 \vec{a} \times \vec{v}_1 + c_2 \vec{a} \times \vec{v}_2}_{\text{FACT (see Ruina/Pratrap)}}$

$\Rightarrow \vec{w} = \vec{a} \times \vec{v} = \mathcal{L}(\vec{v})$

$\uparrow \mathcal{L}$ is a tensor (order 2)

How to find $\underline{\underline{L}}$?

FIND $\vec{L}_i = \vec{a} \times \hat{e}_i$

$$[\vec{L}_1] = [\vec{a} \times \hat{e}_1] = [\ ]$$
$$[\vec{L}_2] = [\vec{a} \times \hat{e}_2] = [\ ]$$
$$[\vec{L}_3] = [\vec{a} \times \hat{e}_3] = [\ ]$$

$$a \times \hat{e}_1 = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \times (\hat{e}_1 + 0 + 0)$$
$$= 0\hat{e}_1 + a_3 \hat{e}_2 - a_2 \hat{e}_3$$

$$\Rightarrow [\vec{L}_1] = \begin{bmatrix} 0 \\ a_3 \\ -a_2 \end{bmatrix} \qquad [\vec{L}_2] = \begin{bmatrix} -a_3 \\ 0 \\ a_1 \end{bmatrix} \qquad [\vec{L}_3] = \begin{bmatrix} a_2 \\ -a_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow [\underline{\underline{L}}] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$\underline{\underline{\mathcal{S}}}(\vec{a})$ = skew matrix that is like $\vec{a}$ for cross products

$$\vec{a} \times \vec{v} = \underline{\underline{\mathcal{S}}}(\vec{a}) \cdot \vec{v}$$

$$[\vec{a} \times \vec{v}]_{\underline{\underline{\mathcal{S}}}} = [\underline{\underline{\mathcal{S}}}(\vec{a})][\vec{v}]$$

Back to Rotations

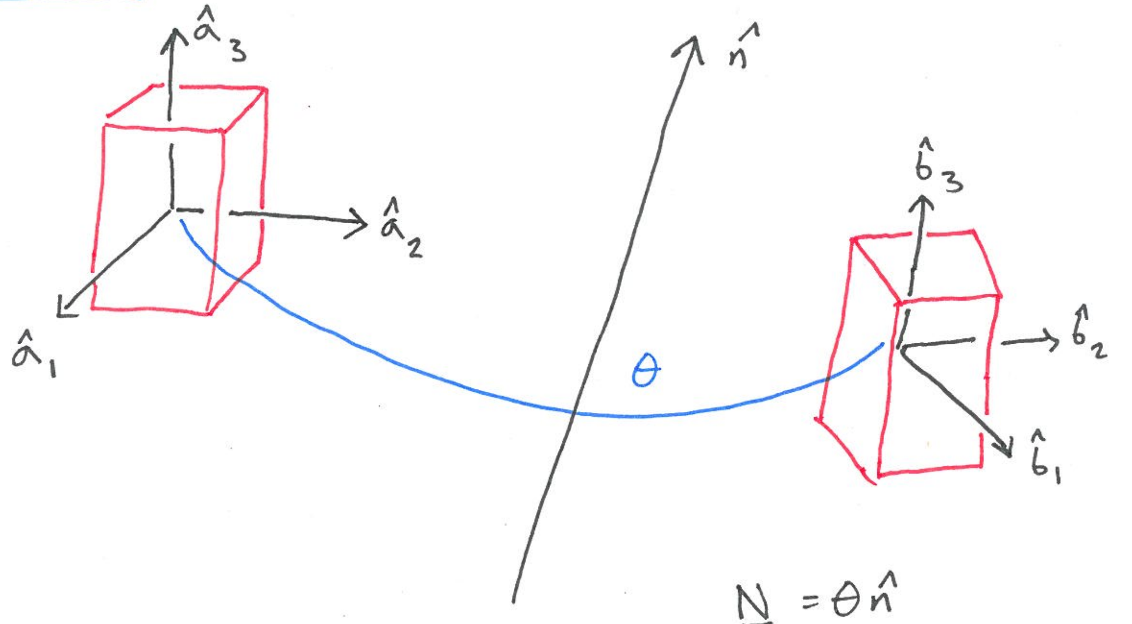
Rot. of $\vec{v} = \hat{n}\hat{n} \cdot \vec{v} + \cos\theta(\underline{\underline{I}} - \hat{n}\hat{n}) \cdot \vec{v} + \sin\theta\, \hat{n} \times \vec{v}$

Linear func. of $\vec{v}$

$$\boxed{\mathrm{Rot}(\vec{v}) = \underbrace{\left[\hat{n}\hat{n} + \cos\theta\,(\underline{\underline{I}} - \hat{n}\hat{n}) + \sin\theta\, \underline{\underline{\mathcal{S}}}(\hat{n})\right]}_{\underline{\underline{R}}} \cdot \vec{v}}$$ ☆☆☆

$$[\underline{\underline{R}}] = [\mathrm{Rot}(\hat{e}_1) \mid \mathrm{Rot}(\hat{e}_2) \mid \mathrm{Rot}(\hat{e}_3)]$$

2/4/2014



$$\underline{N} = \theta \hat{n}$$

Recall

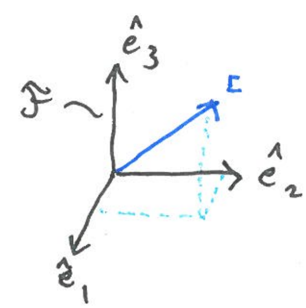
$$\underline{r} = \hat{n}\hat{n} \cdot \underline{r}_0 + \cos(\theta)\left(\underline{r}_0 - \hat{n}(\hat{n} \cdot \underline{r}_0)\right) + \sin(\theta)\hat{n} \times \underline{r}_0$$

$$\mathrm{Rot}(\underline{r}, \theta) = \underline{\underline{R}} \cdot \underline{r} \Rightarrow \underline{\underline{R}} \triangleq \hat{n}\hat{n} + \cos(\theta)(\underline{\underline{I}} - \hat{n}\hat{n}) + \sin(\theta)\,\&(\hat{n})$$

$$\underline{b} = \underline{\underline{R}} \cdot \underline{a}$$

※ ASIDE:

$$\underline{r} = a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3 \Leftrightarrow [\underline{r}]_{\mathcal{F}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



$$\underline{\underline{T}} = \underline{a} \otimes \underline{b}$$

$$\Rightarrow (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) \otimes (b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)$$

$$= T_{ij}(\hat{e}_i \otimes \hat{e}_j)$$

$$[\underline{a} \cdot \underline{b}]_{\mathcal{F}} = [\underline{a}]_{\mathcal{F}}^T [\underline{b}]_{\mathcal{F}}$$

$$[\underline{a} \otimes \underline{b}]_{\mathcal{F}} = [\underline{a}]_{\mathcal{F}} [\underline{b}]_{\mathcal{F}}^T$$

※ END ASIDE

$$\mathcal{A} = (A, \hat{a}_1, \hat{a}_2, \hat{a}_3)$$

$$\mathcal{B} = (B, \hat{b}_1, \hat{b}_2, \hat{b}_3)$$

Define 9 values: $c_{ij} \triangleq \hat{b}_j \cdot \hat{a}_j$

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \underbrace{{}^{\mathcal{B}}C^{\mathcal{A}}}_{\substack{[c_{ij}] \\ 3\times 3}} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} \quad ; \quad \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = {}^{\mathcal{A}}C^{\mathcal{B}} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

$$\Rightarrow {}^{\mathcal{A}}C^{\mathcal{B}} = \underbrace{\left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{-1}}_{\text{Linear algebra definition}} = \underbrace{\left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^{T}}_{\text{To be proven}}$$

$$\underline{r} = \sum_i (\underline{r} \cdot \hat{\underline{e}}_i)\hat{\underline{e}}_i$$

$$\underline{r} = (\underline{r} \cdot \hat{a}_i)\hat{a}_i = (\underline{r} \cdot \hat{b}_i)\hat{b}_i \quad ①$$

$${}^{\mathcal{A}}C^{\mathcal{B}} : \{\underline{r} \cdot \hat{a}_i\} \Leftarrow \{\underline{r} \cdot \hat{b}_i\}$$

$$[\underline{r}]_{\mathcal{A}} = {}^{\mathcal{A}}C^{\mathcal{B}} [\underline{r}]_{\mathcal{B}} \quad ②$$

<u>Proof</u>:

Consider $\underline{r} \equiv \hat{b}_i$

$$① \Rightarrow \underline{r} \equiv \hat{b}_i = (\hat{b}_i \cdot \hat{b}_j)\hat{b}_j = \underbrace{(\hat{b}_i \cdot \hat{a}_j)}_{c_{ij}}\hat{a}_j$$

$$\left.\begin{aligned} \hat{b}_i &= c_{ij}\hat{a}_j \\ \hat{a}_i &= c_{ji}\hat{b}_j \end{aligned}\right\} {}^{\mathcal{A}}C^{\mathcal{B}} = \left({}^{\mathcal{B}}C^{\mathcal{A}}\right)^T$$

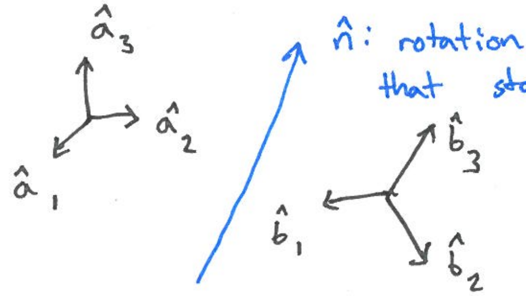
$$[\underline{r}_i]_{\mathcal{B}} = \underline{r} \cdot {}^{\mathcal{B}}C^{\mathcal{A}}_{ij}\hat{a}_j = \underbrace{(\underline{r} \cdot \hat{a}_1)}_{[\underline{r}_1]_{\mathcal{A}}} {}^{\mathcal{B}}C^{\mathcal{A}}_{i1} + \underbrace{(\underline{r} \cdot \hat{a}_2)}_{[\underline{r}_2]_{\mathcal{A}}} {}^{\mathcal{B}}C^{\mathcal{A}}_{i2} + \underbrace{(\underline{r} \cdot \hat{a}_3)}_{[\underline{r}_3]_{\mathcal{A}}} {}^{\mathcal{B}}C^{\mathcal{A}}_{i3}$$

$$\Rightarrow ② \quad [\underline{r}]_{\mathcal{A}} = {}^{\mathcal{A}}C^{\mathcal{B}} [\underline{r}]_{\mathcal{B}}$$

$$\underline{\underline{D}} \qquad [D_{ij}]_A \triangleq \hat{a}_i \cdot \underline{\underline{D}} \cdot \hat{a}_j$$

$$[D_{ij}]_B \triangleq \hat{b}_i \cdot \underline{\underline{D}} \cdot \hat{b}_j$$

$$[D]_B = {}^B C^A \, [D]_A \, {}^A C^B$$



$\hat{n}$: rotation axis, "only thing that stays the same"

$$\hat{n} \cdot \hat{a}_i = \hat{n} \cdot b_i$$

$\underline{b} = \underline{\underline{R}} \cdot \underline{a}$ for $\underline{b} \equiv \underline{a}$ before rotation

$$\hat{b}_i = \underline{\underline{R}} \cdot \hat{a}_i$$

$$[c_{ij}] = \hat{b}_i \cdot \hat{a}_j = (\underline{\underline{R}} \cdot \hat{a}_i) \cdot \hat{a}_j$$

$$(\underline{\underline{R}} \cdot \hat{a}_i) \cdot \hat{a}_i = \left[\left(\cos(\theta)\, \underline{\underline{I}} + \hat{n}\hat{n}(1-\cos(\theta)) + (\hat{n} \times \underline{\underline{I}}) \sin(\theta) \right) \cdot \hat{a}_i \right] \cdot \hat{a}_j$$

$$({}^B C^A)_{ij} = \hat{a}_i \cdot \hat{a}_j \cos(\theta) + \hat{a}_i \cdot \hat{n}\hat{n} \cdot a_j (1-\cos(\theta)) + \hat{n} \times \hat{a}_i \cdot \hat{a}_j \sin(\theta)$$

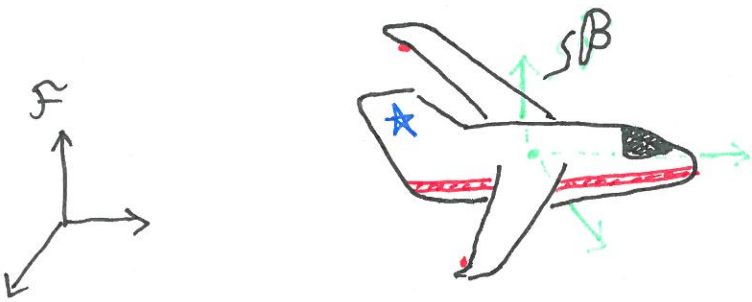
NOTE

$$\hat{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Example

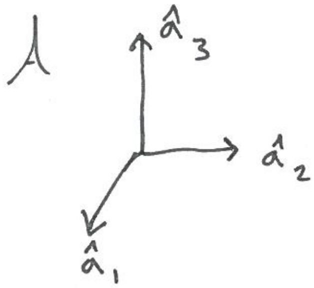
$$\begin{bmatrix} \cos(\theta) + n_1^2(1-\cos(\theta)) & n_1 n_2 (1-\cos(\theta)) + n_3 \sin(\theta) & \\ -n_3 \sin(\theta) + n_1 n_2 (1-\cos(\theta)) & \cos(\theta) + n_2^2 (1-\cos(\theta)) & \\ & & \cos(\theta) + n_3^2 (1-\cos(\theta)) \end{bmatrix}$$

skew symmetric matrix



$$[r]_{\mathcal{F}} = {}^{\mathcal{F}} C^{\mathcal{B}} [r]_{\mathcal{B}}$$

Let $\hat{n} = \hat{a}_1$



$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\underline{r} = (\underline{r} \cdot a_i)\, a_i$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_A$$

Rotation matrix is greatly simplified

$${}^{B}C^{A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$\hat{n} \equiv \hat{a}_1$

Example : $\hat{n} \equiv \hat{a}_2$

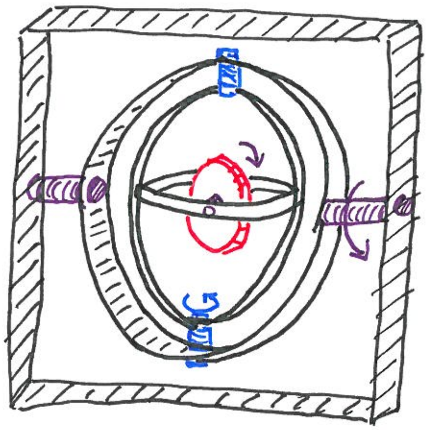
$${}^{B}C^{A}(\theta) = \begin{bmatrix} c_\theta & 0 & -s_\theta \\ 0 & 1 & 0 \\ s_\theta & 0 & c_\theta \end{bmatrix}$$

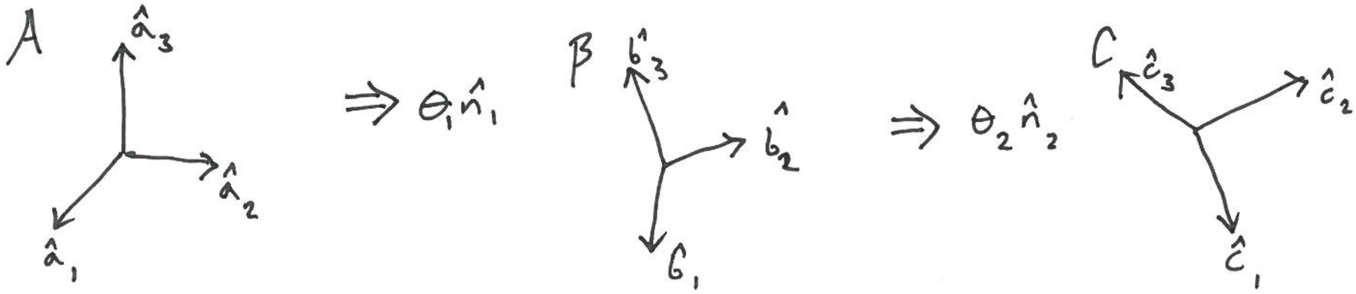
NOTE : $s_\theta = \sin(\theta)$
$c_\theta = \cos(\theta)$

Example : $\hat{n} \equiv \hat{a}_3$

$${}^{B}C^{A}(\theta) = \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2/6/2014





$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix} = {}^{B}C^{A}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{bmatrix}$$

$$\begin{bmatrix}\hat{c}_1\\ \hat{c}_2\\ \hat{c}_3\end{bmatrix} = {}^{C}C^{B}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix}$$

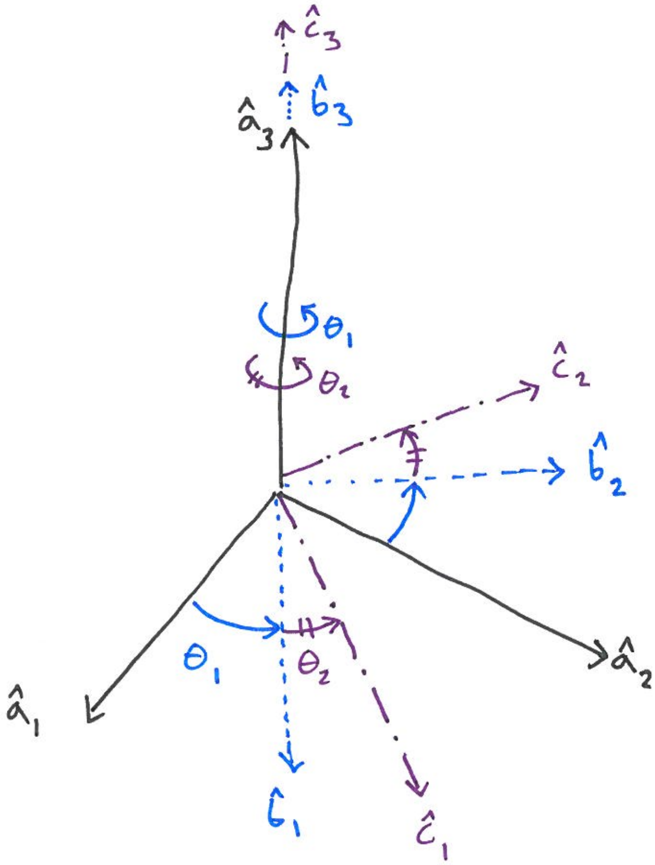
$$\begin{bmatrix}\hat{c}_1\\ \hat{c}_2\\ \hat{c}_3\end{bmatrix} = \underbrace{{}^{C}C^{B}\;{}^{B}C^{A}}_{{}^{C}C^{A}}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{bmatrix}$$

$${}^{A}C^{C} = \left({}^{C}C^{A}\right)^T = \left({}^{C}C^{B}\,{}^{B}C^{A}\right)^T = {}^{A}C^{B}\,{}^{B}C^{C}$$

${}^{B}C^{A}: \hat{n} \equiv \hat{a}_i$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & c_\theta & s_\theta\\ 0 & -s_\theta & c_\theta\end{bmatrix} \quad \begin{bmatrix}c_\theta & 0 & -s_\theta\\ 0 & 1 & 0\\ s_\theta & 0 & c_\theta\end{bmatrix} \quad \begin{bmatrix}c_\theta & s_\theta & 0\\ -s_\theta & c_\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

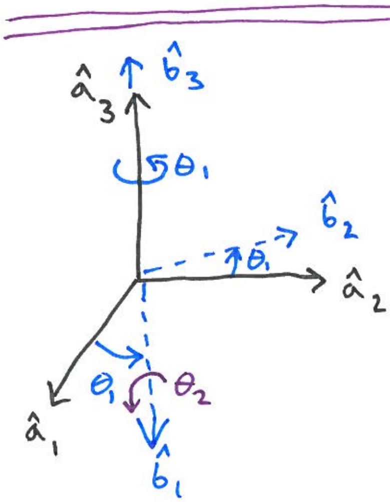
NOTE: $c_\theta \equiv \cos(\theta)$ $\quad s_\theta \equiv \sin(\theta)$

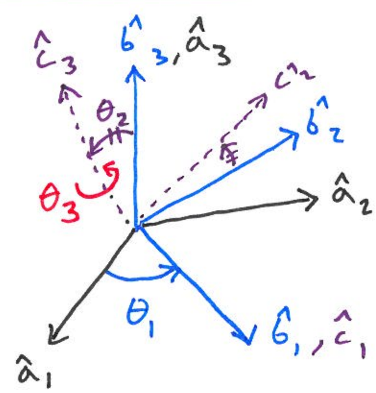


Rotate about the same fixed axis twice

$$\left.\begin{matrix} {}^{B}C^{A} = \begin{bmatrix} c_{\theta_1} & s_{\theta_1} & 0 \\ -s_{\theta_1} & c_{\theta_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\[1em] {}^{C}C^{B} = \begin{bmatrix} c_{\theta_2} & s_{\theta_2} & 0 \\ -s_{\theta_2} & c_{\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}\right\} \begin{bmatrix} c_{\theta_1+\theta_2} & s_{\theta_1+\theta_2} & 0 \\ -s_{\theta_1+\theta_2} & c_{\theta_1+\theta_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

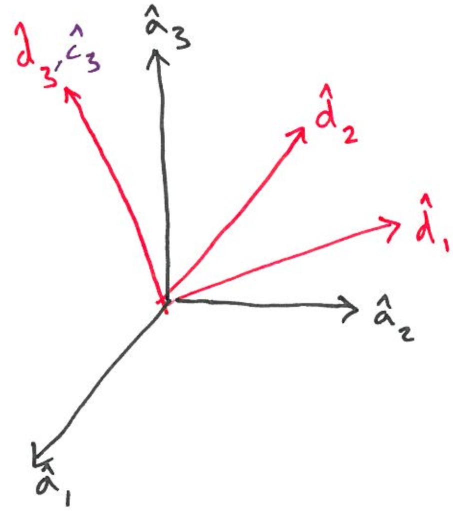
<u>NOTE</u>: same thing as rotating once by angle $\theta = \theta_1 + \theta_2$





$${}^{C}C^{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{\theta_2} & s_{\theta_2} \\ 0 & -s_{\theta_2} & c_{\theta_2} \end{bmatrix}$$

$${}^{C}C^{B}\,{}^{B}C^{A} = \begin{bmatrix} c_{\theta_1} & s_{\theta_1} & 0 \\ -s_{\theta_1}c_{\theta_2} & c_{\theta_1}c_{\theta_2} & s_{\theta_1} \\ s_{\theta_1}s_{\theta_2} & -s_{\theta_2}c_{\theta} & c_{\theta_2} \end{bmatrix}$$

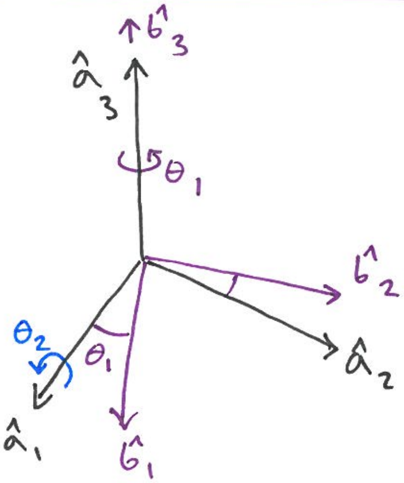


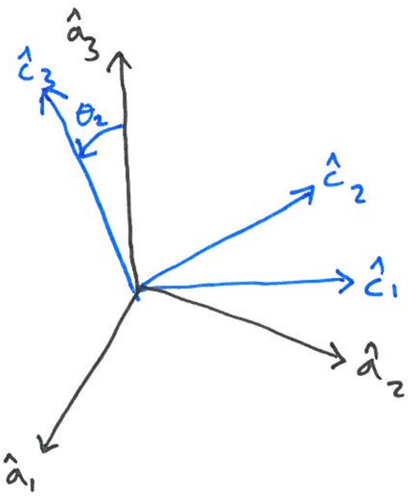
$$
{}^{D}C^{C} = \begin{bmatrix} c_{\theta_3} & s_{\theta_3} & 0 \\ -s_{\theta_3} & c_{\theta_3} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{D}C^{A} = {}^{D}C^{C}\,{}^{C}C^{B}\,{}^{B}C^{A} = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}}_{\text{Full matrix, therefore can represent any rotation}}
$$

Conclusions:

you need ≥ 2 non-repeating axes & 3 rotations to reach any arbitrary rotation (orientation) in space





${}^{B}C^{A}$ is simple form, what about ${}^{C}C^{B}$? Not as simple, must go back to first principles

$$
c_{ij} = \hat{b}_i \cdot \hat{a}_j
$$
$$
= \hat{c}_i \cdot \hat{b}_j
$$

$$
c_{ij} = \delta_{ij} c_\theta + \underbrace{\epsilon_{ijk}}_{\substack{\text{when } ijk \text{ is:} \\ \text{an even permutation, } = 1 \\ \text{an odd permutation, } = -1}} n_k s_\theta + n_i n_j (1 - c_\theta)
$$

$\hat{n} \equiv \hat{a}_1$

$$[\hat{n}]_B = {}^B C^A \underbrace{[\hat{n}]_A}_{\begin{bmatrix}1\\0\\0\end{bmatrix}} = \begin{bmatrix} c_{\theta_1} \\ -s_{\theta_1} \\ 0 \end{bmatrix}_B$$

$${}^C C^B = \begin{bmatrix} s_{\theta_1}^2 c_{\theta_2} + c_{\theta_1}^2 & -s_{\theta_1} c_{\theta_1}(1-c_{\theta_2}) & s_{\theta_2} s_{\theta_1} \\ -(1-c_{\theta_2}) s_{\theta_1} c_{\theta_1} & s_{\theta_1}^2 + c_{\theta_1}^2 c_{\theta_2} & s_{\theta_2} c_{\theta_1} \\ -s_{\theta_2} s_{\theta_1} & -s_{\theta_2} c_{\theta_1} & c_{\theta_2} \end{bmatrix}$$

$${}^C C^B \, {}^B C^A = \begin{bmatrix} c_{\theta_1} & s_{\theta_1} c_{\theta_2} & s_{\theta_1} s_{\theta_2} \\ -s_{\theta_1} & c_{\theta_1} c_{\theta_2} & c_{\theta_1} s_{\theta_2} \\ 0 & -s_{\theta_2} & c_{\theta_2} \end{bmatrix}$$

Common nomenclature for rotations

space/body - 2/3

$\theta_1, \theta_2, \theta_3$

NOTE: 24 different possible conventions

e.g.) Body-2 3-1-3

Body-3 1-2-3

Small angles

What if $\theta^2 \ll 1$

$\sin(\theta) \approx \theta$

$\cos(\theta) \approx 1$

Recall: $\underline{r} = (\cancel{\hat{n}\hat{n}} + \overset{1}{\cancel{\cos(\theta)}}(\underline{\underline{I}} - \cancel{\hat{n}\hat{n}}) + \mathcal{S}(\hat{n})\sin\theta) \cdot \underline{r}_0$

$$= \underbrace{\left(\underline{\underline{I}} + \mathcal{S}(\hat{n})\theta\right)}_{\underline{\underline{R}}} \cdot \underline{r}_0$$

$${}^B C^A = I + \theta \begin{bmatrix} 0 & -n_2 & \\ \end{bmatrix}$$

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$${}^{B}C^{A} \approx I + S(\theta)$$

2/11/2014

Rotations Continued

Axis-angle: $\hat{n}, \theta$

or $\cos\frac{\theta}{2}, \sin\left(\frac{\theta}{2}\right)\hat{n}$

Tensor matrix: $\underline{\underline{R}} = R_{ij}\,\hat{e}_i\hat{e}_j$

Today review, small rotations

$$\vec{r} = \underbrace{\left[\hat{n}\hat{n} + \cos\theta\left(\underline{\underline{I}} - \hat{n}\hat{n}\right) + \sin\theta\,\underline{\underline{\mathcal{E}}}(\hat{n})\right]}_{\underline{\underline{R}}} \cdot \vec{r}_0$$

$$\vec{r} = \underline{\underline{R}} \cdot \vec{r}_0$$

$$R_{ij} = \hat{e}_i\left[R_{kl}\,\hat{e}_k\hat{e}_l\right]\cdot\hat{e}_j$$

$$= n_i n_j + \cos\theta\left[\delta_{ij} - n_i n_j\right] + \sin\theta\,\epsilon_{ijk}\,n_k$$

δ_{ij} = kronicker delta

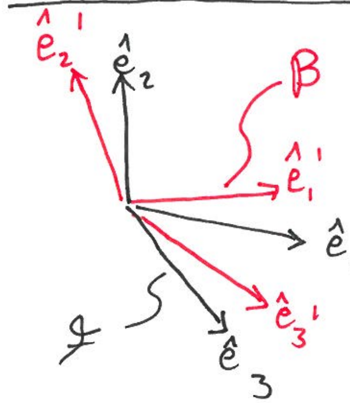
$$= \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$= \left[\underline{\underline{I}}\right]$$

ϵ_{ijk} = alternating epsilon symbol

$$= \begin{cases} 0 & \text{if } i=j \text{ or } i=k \text{ or } j=k \\ 1 & \text{for } ijk = 123 \text{ or } 231 \text{ or } 312 \text{ (even)} \\ -1 & \text{for } ijk = 213 \text{ or } 132 \text{ or } 321 \text{ (odd)} \end{cases}$$

Another representation of $\underline{\underline{R}}$



$$\boxed{\underline{\underline{R}} \cdot \hat{e}_i = \hat{e}_i'}$$

Rotation:

$$\hat{e}_1 \rightarrow \hat{e}_1{}'$$

$$\hat{e}_2 \rightarrow \hat{e}_2{}'$$

$$\hat{e}_3 \rightarrow \hat{e}_3{}'$$

$$\underline{\underline{R}} = \hat{e}_1{}'\hat{e}_1 + \hat{e}_2{}'\hat{e}_2 + \hat{e}_3{}'\hat{e}_3$$

$$\boxed{\underline{\underline{R}} = \hat{e}_i{}'\hat{e}_i} \quad ★★★$$

ASIDE: $$[\underline{\underline{R}}]_{\underbrace{\beta g}_{\text{mixed base}}} = \underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Check</u>: $$\underline{\underline{R}} \cdot \hat{e}_1 = \hat{e}_i{}'\hat{e}_i \cdot \hat{e}_1 = \hat{e}_i{}'\delta_{i1} = \hat{e}_1{}'$$
$$= \hat{e}_1{}'$$

$$[\underline{\underline{R}}]_{gg} = \begin{bmatrix} | & | & | \\ | & | & | \end{bmatrix}$$

$\hat{e}_3{}'(g)$

$[\hat{e}_2{}']_g$

$[\hat{e}_1{}']_g$

$$\underline{\underline{R}}^T \equiv \hat{e}_i \hat{e}_i{}'$$

$$\underline{\underline{R}}^T \cdot \underline{\underline{R}} = \hat{e}_i\hat{e}_i{}' \cdot \hat{e}_j{}'\hat{e}_j = \delta_{ij}\,\hat{e}_i\,\hat{e}_j = \underline{\underline{I}}$$

$$\Rightarrow \underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$$

★ **Puzzle:** Use geometry to find net $\overbrace{\theta \ \& \ \hat{n}}^{\underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1}$ given successive rotations $\underbrace{\theta_1 , \hat{n}_1}_{\underline{\underline{R}}_1}$ then $\underbrace{\theta_2 , \hat{n}_2}_{\underline{\underline{R}}_2}$ ★

$$R_{ij} = n_i n_j + \cos\theta \left(\delta_{ij} - n_i n_j\right) + \epsilon_{ijk} n_k \sin\theta$$

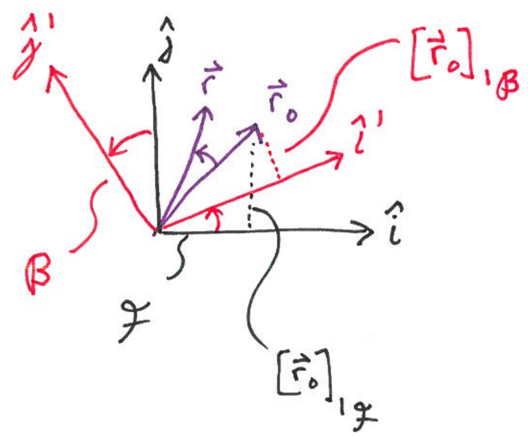
$$\text{trace}\left[\underline{\underline{R}}\right]_{\mathcal{F}} = R_{ii} = \underbrace{n_i n_i}_{1} + \cos\theta \,(\underbrace{\delta_{ii}}_{3 = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1} - \underbrace{n_{ii}}_{1}) + \underbrace{\epsilon_{ii}}_{0+0+0=0} n_k \sin\theta$$

$$R_{ii} = 1 + 2\cos\theta$$

$$\Rightarrow \boxed{\theta = \arccos\left(\frac{\text{trace}(R) - 1}{2}\right)} \qquad \hat{n} = \text{evector of } \underline{\underline{R}}$$

World of Confusion

Given ______ find ______



Given	Find
$\vec{r}_0$	$\vec{r} = \underline{\underline{R}} \cdot \vec{r}_0$
$[\vec{r}_0]_{\mathcal{F}}$	$[\vec{r}_0]_{\beta}$
$[\vec{r}]_{\beta}$	$[\vec{r}]_{\mathcal{F}}$
$\hat{e}_i$	$\hat{e}_i{}'$
$\vdots$	$\vdots$

etc.

Dimitry said: $\hat{e}_i{}' = \underbrace{c_{ij}}_{R_{ji}} \hat{e}_j$

$$\hat{e}_i' = \underline{\underline{R}} \cdot \hat{e}_i$$

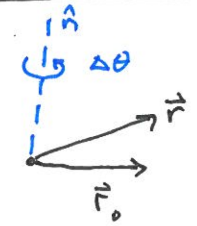
$$\hat{e}_i' = R_{kl}\, \hat{e}_k \hat{e}_l \cdot \hat{e}_i$$

$$\hat{e}_i' = \underbrace{R_{ki}}_{c_{ik}} \hat{e}_k$$

$\boxed{2/13/2014}$

TODAY
1) Small rotations
2) Angular velocity
3) Angular momentum of rigid object

Small rotations



$$\vec{r} = \underline{\underline{R}} \cdot \vec{r}_0$$

$$= \left[\hat{n}\hat{n} + \cancelto{1}{\cos\theta}\left(\underline{\underline{I}} - \hat{n}\hat{n}\right) + \cancelto{\theta}{\sin\theta}\, \underline{\underline{\varepsilon}}(\hat{n})\right] \cdot \vec{r}_0$$

$\theta \ll 1$
$\cos\theta \approx 1$
$\sin\theta \approx \theta$

$$= \underbrace{\left[\underline{\underline{I}} + \theta\, \underline{\underline{\varepsilon}}(\hat{n})\right]}_{\text{small rotation}} \cdot \vec{r}_0$$

$\theta \ll 1$

$$\left[\underline{\underline{R}}\right]_{\mathcal{F}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} + \theta \underbrace{\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}}_{\left[\underline{\underline{\varepsilon}}(\hat{n})\right]}$$

$$\vec{r} = \vec{r}_0 + \theta\, \hat{n} \times \vec{r}$$

Compose two small rotations

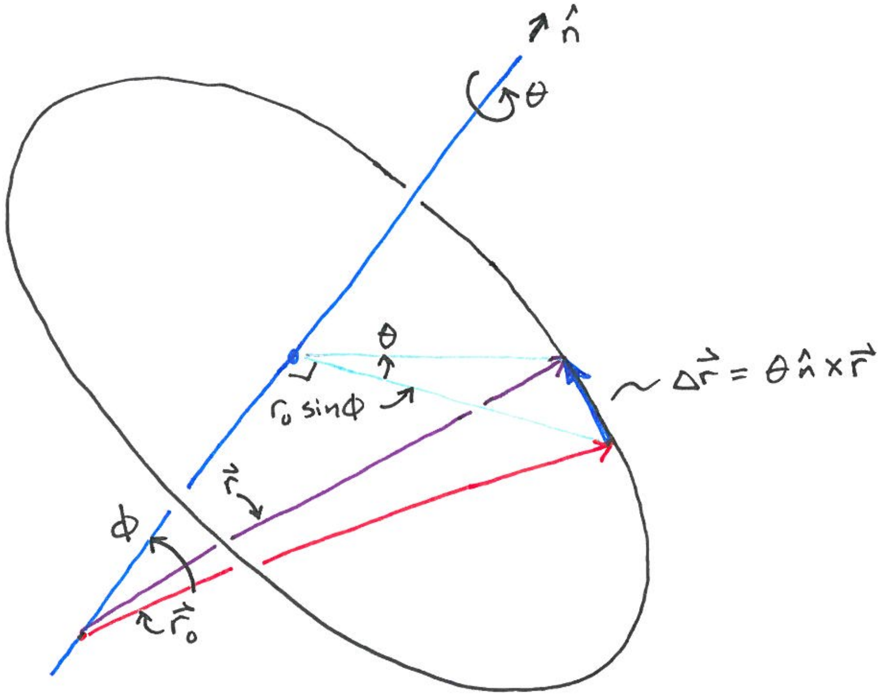
$$\underline{\underline{R}} = \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1$$

$$\vec{r} = \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \vec{r}_0$$

$$\underline{\underline{R}} = \left[\underline{\underline{I}} + \theta_1 \underline{\underline{d}}(\hat{n}_1)\right] \cdot \left[\underline{\underline{I}} + \theta_2 \underline{\underline{d}}(\hat{n}_2)\right]$$

$$= \underline{\underline{I}} + \theta_1 \underline{\underline{d}}(\hat{n}_1) + \theta_2 \underline{\underline{d}}(\hat{n}_2)$$

$$\vec{r} = \vec{r}_0 + \underbrace{\theta_1 \hat{n}_1 \times \vec{r}_0 + \theta_2 \hat{n}_2 \times \vec{r}_0}_{\text{"small rotations add"}}$$



Angular velocity

$\vec{r}$ is fixed in a rigid object

$$\dot{\vec{r}} = \frac{(\vec{r} + \Delta\vec{r}) - \vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}$$

$$= \frac{\Delta\theta}{\Delta t} \underline{\underline{d}}(\hat{n}) \cdot \vec{r} = \frac{\Delta\theta}{\Delta t} \hat{n} \times \vec{r}$$

small rotations

$$\underset{\Delta t \to 0}{=} \underbrace{w\, \underline{\underline{d}}(\hat{n})}_{\underline{\underline{\omega}}} \cdot \vec{r} = \vec{\omega} \times \vec{r}$$

$$\boxed{\vec{\omega} = \dot{\theta}\,\hat{n}}$$

Angular velocity vector

$$\boxed{\boxed{\begin{aligned}\dot{\vec{r}} &= \vec{\omega} \times \vec{r} \\ &= \underline{\underline{\omega}} \cdot \vec{r}\end{aligned}}}$$

$\underline{\underline{\omega}} = \theta \, \underline{\underline{\mathcal{S}}}(\hat{n})$

$= $ ang. vel. tensor

$\vec{r} = \underline{\underline{R}} \cdot \vec{r}_0$

$\dot{\vec{r}} = \dot{\underline{\underline{R}}} \cdot \underbrace{\vec{r}_0}_{\underline{\underline{R}}^{-1} \cdot \vec{r}}$

$\underline{\underline{R}}^{-1} = \underline{\underline{R}}^{T}$

$$= \underbrace{\dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^{T}}_{\underline{\underline{\mathcal{S}}}(\vec{\omega})} \cdot \vec{r}$$

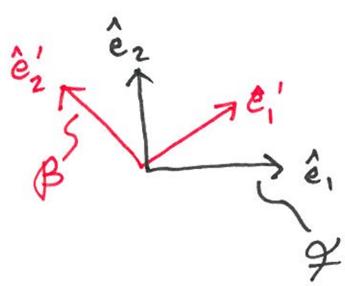
$$\boxed{\boxed{\underline{\underline{\mathcal{S}}}(\vec{\omega}) = \underline{\underline{\omega}} = \dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^{T}}}$$

The Transport Thm / The Q dot formula

consider $\vec{Q}(t)$ ← some vector

$\vec{Q} = \vec{Q}$

$Q_i \hat{e}_i = Q_i' \hat{e}_i'$



$$\boxed{\hat{e}_i' = \underline{\underline{R}} \cdot \hat{e}_i}$$

$$[\vec{Q}]_{\mathcal{E}} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}, \quad [\vec{Q}]_{\beta} = \begin{bmatrix} Q_1' \\ Q_2' \\ Q_3' \end{bmatrix}$$

Time derivative in a frame:

$$^{\mathcal{F}}\dot{\vec{Q}} \equiv \dot{Q}_1\hat{e}_1 + \dot{Q}_2\hat{e}_2 + \dot{Q}_3\hat{e}_3$$

$$^{\beta}\dot{\vec{Q}} \equiv \dot{Q}_1'\hat{e}_1' + \dot{Q}_2'\hat{e}_2' + \dot{Q}_3'\hat{e}_3'$$

$$^{\mathcal{F}}\dot{\vec{Q}} = \frac{d}{dt}\left(Q_i'\hat{e}_i'\right) = \dot{Q}_i'\hat{e}_i' + Q_i'\dot{\hat{e}}_i'$$

$$\dot{\hat{e}}_i' = \vec{\omega}_{\beta/\mathcal{F}} \times \hat{e}_i'$$

$$= \dot{Q}_i'\hat{e}_i' + \vec{\omega} \times Q_i'\hat{e}_i'$$

$$\boxed{^{\mathcal{F}}\dot{\vec{Q}} = {}^{\beta}\dot{\vec{Q}} + \vec{\omega} \times \vec{Q}}$$

The transport thm
= Q dot formula

NOTE: $\vec{\omega}$ is coordinate system independent

Main Examples

$\vec{Q} = \vec{r}$

$\vec{Q} = \vec{v}$

$\vec{Q} = \vec{\omega}$

$\vec{Q} = \vec{H}_{/G}$

Mechanics

Angular Momentum Balance (for any system)

pt. fixed in space C

$$\underset{\text{ext. forces}}{\sum} \vec{M}_{/C} = \frac{d}{dt}\vec{H}_{/C}$$

$$\vec{H}_{/C} = \sum \vec{r}_{i/C} \times (m_i \vec{v}_i)$$

$$\vec{H}_{/C} = \underbrace{\vec{r}_{G/C} \times m_{tot}\vec{v}_G}_{\vec{H}_{G/C}} + \underbrace{\sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}}_{\vec{H}_{/G}}$$

$$\dot{\vec{H}}_{/c} = \vec{r}_{G/c} \times (m_{tot}\vec{a}_G) + \sum \vec{r}_{i/G} \times m_i \vec{a}_{i/G}$$

$$\left\{ \begin{array}{l} \dot{\vec{r}}_{G/c} \times \vec{v}_{G/c} = \vec{0} \\ \dot{\vec{r}}_{i/G} \times \vec{v}_{i/G} = \vec{0} \end{array} \right\}$$

$\dot{\vec{H}}_{/c}$ time derivative in fixed frame unless otherwise stated

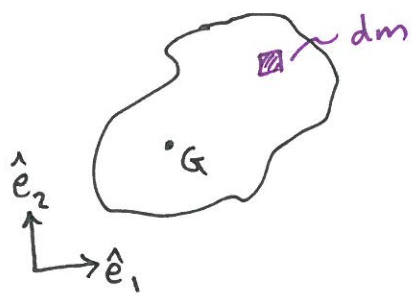
A special pt is $C = G$

$$\vec{r}_{G/G} = \vec{0}$$

$$\dot{\vec{H}}_{/G} = \sum \vec{r}_{i/G} \times m_i \vec{a}_{i/G}$$

$$= \frac{d}{dt} \underbrace{\sum \vec{r}_{i/G} \times m_i \vec{v}_{i/G}}_{\vec{H}_{/G}}$$

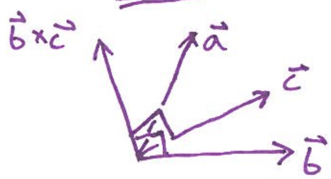
What is the Ang Mom of a Rigid Object w.r.t. G?



$$\vec{H}_{/G} = \underbrace{\int \vec{r}_{/G} \times \vec{v}_{/G} \, dm}_{\sum \vec{r}_{i/G} \times \vec{v}_{i/G} m_i}$$

$$= \int \vec{r}_{/G} \times (\vec{\omega} \times \vec{r}_{/G}) \, dm$$

ASIDE: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ ← FACT



$\vec{b} \times \vec{c} \perp \vec{b}$ & $\vec{c}$

$\vec{a} \times (\vec{b} \times \vec{c}) \perp \vec{b} \times \vec{c} \Rightarrow$ in plane of $\vec{b}$ & $\vec{c}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = d_1 \vec{b}(\vec{a} \cdot \vec{c}) + d_2 \vec{c}\,(\vec{a} \cdot \vec{b}) \quad \star$$

cross
product is linear

$$\vec{g} \times (\vec{l} + \vec{k})$$
$$= \vec{g} \times \vec{l} + \vec{g} \times \vec{k}$$

$\star$ demanded by linearity

ex)

$$\hat{\imath} \times (\hat{\jmath} \times \hat{\imath}) = \left[\begin{array}{c} \hat{\jmath} \\ d_1 \hat{\jmath}(1) + \cancel{d_2 \hat{\imath} \cdot 0} \end{array} \right] \quad d_1 = 1$$

Similarly $d_2 = -1$

<u>END ASIDE</u>

$$\vec{H}_{/G} = \int \vec{r}_{/G} \times (\vec{\omega} \times \vec{r}_{/G})\, dm$$

$$= \int \vec{\omega}\,(\vec{r}_{/G} \cdot \vec{r}_{/G}) - \vec{r}_{/G}(\vec{r}_{/G} \cdot \vec{\omega})\, dm$$

$$= \left[\underline{\underline{1}} \int \vec{r}_{/G} \cdot \vec{r}_{/G}\, dm - \int \vec{r}_{/G}\, \vec{r}_{/G}\, dm \right] \cdot \vec{\omega}$$

<u>Define</u>

$$\underbrace{\underline{\underline{I}}^G \equiv \int \vec{r}_{/G} \cdot \vec{r}_{/G}\, dm\ \underline{\underline{1}} - \int \vec{r}_{/G}\, \vec{r}_{/G}\, dm}_{\text{moment of inertia tensor}}$$

no dot

$$[\underline{\underline{I}}]_{\mathcal{F}} = \int (x^2 + y^2 + z^2)\, dm \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\underline{\underline{1}}]_{\mathcal{F}}} - \int \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ xz & zy & z^2 \end{bmatrix} dm$$

$$\boxed{[\underline{\underline{I}}^G] = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm} \quad \star$$

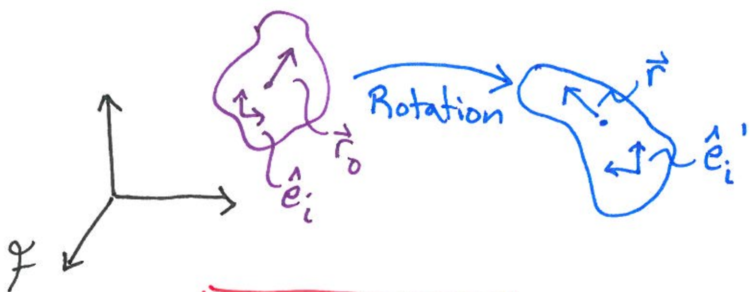
$\star\star\star$

$$\boxed{\vec{H}_{/G} = \underline{\underline{I}}^G \cdot \vec{\omega}}$$

$$\dot{\vec{H}}_{/G} = \underline{\underline{I}}^G \cdot \dot{\vec{\omega}} + \dot{\underline{\underline{I}}}^G \cdot \vec{\omega}$$

2/25/2014

TODAY
1) Recap
2) Motion of one rigid body



$$\boxed{\begin{aligned}\hat e_i\,' &= \underline{\underline{R}} \cdot \hat e_i \\ \vec r &= \underline{\underline{R}} \cdot \vec r_0\end{aligned}}$$

$$\dot{\vec r} = \dot{\vec r}$$

$$\dot{\underline{\underline{R}}} \cdot \vec r_0 = \underbrace{\vec\omega \times \vec r}_{\underline{\underline{\mathcal{A}}}(\vec\omega)\cdot\vec r}$$

$$= \underline{\underline{\mathcal{A}}}(\vec\omega) \cdot \underline{\underline{R}} \cdot \vec r_0$$

$$\Rightarrow \boxed{\begin{aligned}\dot{\underline{\underline{R}}} &= \underline{\underline{\mathcal{A}}}(\vec\omega) \cdot \underline{\underline{R}} \\ \underline{\underline{\mathcal{A}}}(\vec\omega) &= \dot{\underline{\underline{R}}} \cdot \underline{\underline{R}}^{-1}\end{aligned}}$$

$$^{\mathcal{B}}\dot{\vec Q} \equiv \dot Q_{x'} \hat e_x' + \dot Q_{y'} \hat e_y' + \dot Q_{z'} \hat e_z'$$

The "Q-dot formula" = The transport thm.

$$\boxed{^{\mathcal{F}}\dot{\vec Q} = \vec\omega_{\mathcal{B}/\mathcal{F}} \times \vec Q + {}^{\mathcal{B}}\dot{\vec Q}}$$ for any vector $\vec Q$

ex) $^{\mathcal{F}}\dot{\vec\omega} = {}^{\mathcal{B}}\dot{\vec\omega}$

$$\dot\omega_x \hat e_x + \dot\omega_y \hat e_y + \dot\omega_z \hat e_z = \dot\omega_{x'} \hat e_x' + \dot\omega_{y'} \hat e_y' + \dot\omega_{z'} \hat e_z'$$

Back to Mechanics

Recall:

For a rigid object,

$$\underline{H}_{/G} \equiv \int \vec{r}_{/G} \times \vec{V}_{/G}\, dm$$

$$= \underline{\underline{I}} \cdot \vec{\omega}$$

$$\underline{\underline{I}} = \int \vec{r} \cdot \vec{r} \cdot dm\, \underline{\underline{1}} - \int \vec{r}\,\vec{r}\, dm$$

$$\left[\underline{\underline{I}}\right]_{\mathcal{G}} = \int r_k r_k\, dm\, \delta_{ij} - \int r_i r_j\, dm$$

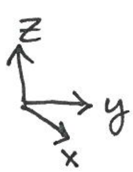
$$I = \int \begin{bmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2+z^2 \end{bmatrix} dm - \int \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} dm$$

$$\boxed{I = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm}$$

About $\underline{\underline{I}}$

Examples:

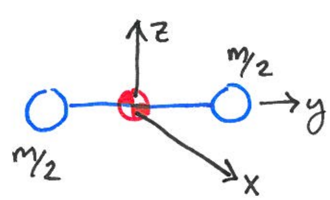
pt. mass:



$$\Rightarrow \underline{\underline{I}} = \underline{\underline{0}}$$

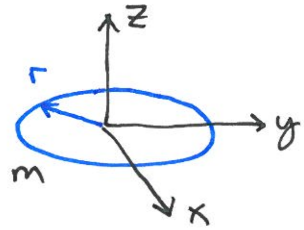
$$I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

dumbell:



$$I = \begin{bmatrix} mr^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & mr^2 \end{bmatrix}$$

hoop:



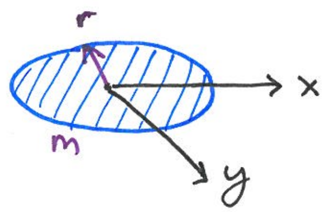
$$I = \begin{bmatrix} \frac{mr^2}{2} & 0 & 0 \\ 0 & \frac{mr^2}{2} & 0 \\ 0 & 0 & mr^2 \end{bmatrix}$$

any planar object:
(on xy plane)

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} = I_{xx} + I_{yy} \end{bmatrix}$$

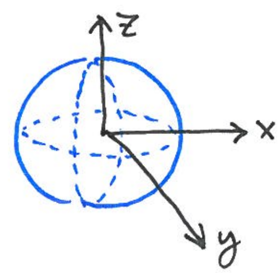
$$\boxed{I_{zz} = I_{xx} + I_{yy}}$$ The perpendicular axis Thm. (for planar objects)

disk:



$$I = \begin{bmatrix} mr^2/4 & 0 & 0 \\ 0 & mr^2/4 & 0 \\ 0 & 0 & mr^2/4 \end{bmatrix}$$

sphere:



$$I = \frac{2}{5} mr^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetry $\Rightarrow I_{xx} = I_{yy} = I_{zz}$

$$\Rightarrow I_{xx} = \frac{1}{3}\left(I_{xx} + I_{yy} + I_{zz}\right)$$

$$= \frac{2}{3}\int (x^2 + y^2 + z^2)\, dm$$

$$I_{xx} = \frac{2}{3}\int_0^r \underbrace{r'^2}_{x^2+y^2+z^2} \underbrace{dm}_{dm = \rho(4\pi r'^2)dr'}$$

$\rho = \frac{m}{[illegible]} = \frac{m}{4\pi r^3 [illegible]}$

$$I_{xx} = \frac{2}{3}\int_0^r (r'^2)\underbrace{(\rho\, 4\pi r'^2)\,dr'}_{dm}$$

$$I_{xx} = \frac{2}{3}\cdot 4\pi \underbrace{\left(\frac{m}{\frac{4}{3}\pi r^3}\right)}_{\rho} \int_0^r r'^4\,dr'$$

$$= \frac{2}{\cancel{3}}\cdot\frac{\cancel{4\pi}}{(\cancel{4}/3)\cancel{\pi}}\, m\frac{r^5}{5}$$

$$\Rightarrow \boxed{I_{xx} = \frac{2}{5}mr^2 = I_{yy} = I_{zz}}$$

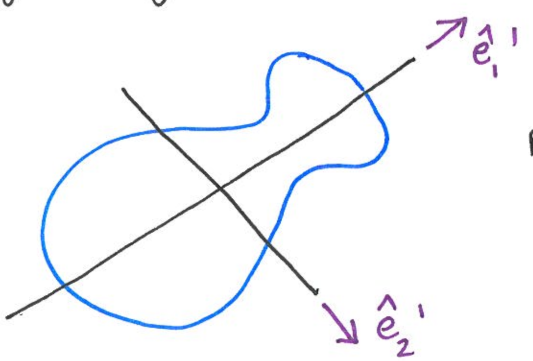
$$\boxed{\text{Aside: } \int_0^\infty e^{-x^2}dx = ?}$$

More facts about $\underline{\underline{I}}$:

Symmetric $\Rightarrow$ diagonalizable

$$\Rightarrow I_\beta = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

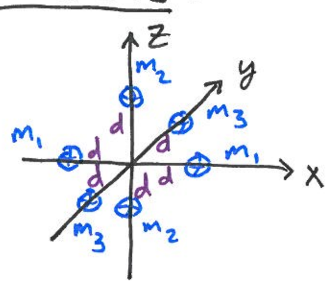
symmetry of objects shows in $\underline{\underline{I}}$



princ. dir. align w/ symmetry axes

Minimal representations:

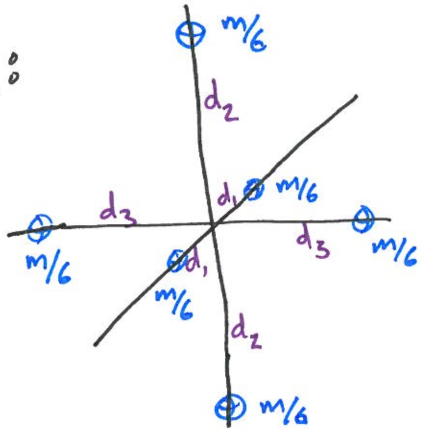
jacks 1:



$$d^2\begin{bmatrix} 2(m_3+m_2) & & \\ & 2(m_1+m_2) & \\ & & 2(m_1+m_3) \end{bmatrix}$$

Pick m_1, m_2, m_3 to get desired $I_1\ I_2\ I_3$

jacks 2:



Pick d_1, d_2, d_3 to give desired $\underbrace{I_1, I_2, I_3}$

☆ e-values of moment of inertia matrix ☆

$$\underline{\underline{I}} = I_1 \hat{e}_1' \hat{e}_1' + I_2 \hat{e}_2' \hat{e}_2' + I_3 \hat{e}_3' \hat{e}_3'$$

☆ $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ are e-vectors of $\underline{\underline{I}}$ ☆

Restrictions on I_1, I_2, I_3

$$I_1 \geq 0, \; I_2 \geq 0, \; I_3 \geq 0$$

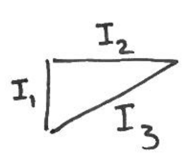
Look at $I_1 + I_2 = \int (y'^2 + z'^2) + (x'^2 + z'^2) \, dm$

$$= \int y'^2 + x'^2 + 2z'^2 \, dm$$

$$= I_3 + 2 \int z'^2 \, dm$$

$$\Rightarrow \quad I_1 + I_2 \geq I_3$$

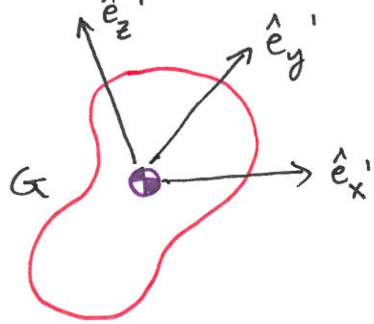
Triangle inequality



$$I_2 + I_3 \geq I_1$$
$$I_1 + I_3 \geq I_2$$
$$I_1 + I_2 \geq I_3$$

$\boxed{2/27/2014}$

Dynamics of Rigid Objects



β axes are aligned with princ. dir. of $\underline{\underline{I}}$

$$\underline{\underline{I}} = \underline{\underline{I}}$$

$$I_{ij}\hat{e}_i\hat{e}_j = I_1\hat{e}_1'\hat{e}_1' + I_2\hat{e}_2'\hat{e}_2' + I_3\hat{e}_3'\hat{e}_3'$$

$$\left[\underline{\underline{I}}\right]_\beta = \text{const} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$\vec{H}_{/G} = \underline{\underline{I}} \cdot \vec{\omega}$$

$\vec{\omega}_{\beta/\mathcal{F}}$

AMB$_{/G}$

$$\vec{M}_{/G} = {}^{\mathcal{F}}\dot{\vec{H}}_{/G}$$

$$= {}^{\mathcal{F}}\frac{d}{dt}\vec{H}_{/G}$$

$$= {}^{\mathcal{F}}\frac{d}{dt}\left(\underline{\underline{I}}\cdot\vec{\omega}\right)$$

$$= {}^{\beta}\frac{d}{dt}\left(\vec{H}_{/G}\right) + \vec{\omega}\times\vec{H}_{/G}$$

$$\vec{M}_{/G} = {}^{\beta}\frac{d}{dt}\left(\underline{\underline{I}}\cdot\vec{\omega}\right) + \vec{\omega}\times\left(\underline{\underline{I}}\cdot\vec{\omega}\right)$$

$$= \underline{\underline{I}}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\left(\underline{\underline{I}}\cdot\vec{\omega}\right)$$

${}^{\beta}\dot{\vec{\omega}} = {}^{\mathcal{F}}\dot{\vec{\omega}}$

$$\boxed{\begin{aligned}\dot{\vec{\omega}} &= \underline{\underline{I}}^{-1}\cdot\left[\vec{M}_{/G} - \vec{\omega}\times\underline{\underline{I}}\cdot\vec{\omega}\right]\\ \dot{\underline{\underline{R}}} &= \underline{\underline{\mathcal{S}}}(\vec{\omega})\cdot\underline{\underline{R}}\end{aligned}} \; *$$

The Euler Equations

ODEs we can solve for motion of a rigid about given initial conditions (ICs) and $\vec{M}_{/G}(t)$

12 ODEs (1st order)

6 ind. ODEs

Special Solutions

Write * in body coordinates

$$[\dot{\vec{\omega}}]_\beta = [\underline{\underline{I}}^{-1}]_\beta\cdot\left[[\vec{M}_{/G}]_\beta - [\vec{\omega}]_\beta \times [\underline{\underline{I}}]_\beta[\vec{\omega}]_\beta\right]$$

$$[\dot{\vec{\omega}}]_\beta = \begin{bmatrix}\dot{\omega}_x'\\ \dot{\omega}_y'\\ \dot{\omega}_z'\end{bmatrix} \qquad [\underline{\underline{I}}^{-1}]_\beta = \begin{bmatrix}1/I_1 & 0 & 0\\ 0 & 1/I_2 & 0\\ 0 & 0 & 1/I_3\end{bmatrix} \qquad [\underline{\underline{I}}]_\beta = \begin{bmatrix}I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{bmatrix} \qquad [\vec{\omega}]_\beta = \begin{bmatrix}\omega_x'\\ \omega_y'\\ \omega_z'\end{bmatrix}$$

$$\begin{bmatrix}\dot{\omega}_x'\\ \dot{\omega}_y'\\ \dot{\omega}_z'\end{bmatrix} = \begin{bmatrix}M_{Gx}'/I_1\\ M_{Gy}'/I_2\\ M_{Gz}'/I_3\end{bmatrix} - \begin{bmatrix}1/I_1 & 0 & 0\\ 0 & 1/I_2 & 0\\ 0 & 0 & 1/I_3\end{bmatrix}\begin{bmatrix}\omega_x'\\ \omega_y'\\ \omega_z'\end{bmatrix}\times\begin{bmatrix}\omega_x' I_1\\ \omega_y' I_2\\ \omega_z' I_3\end{bmatrix}$$

$$\begin{bmatrix}\dot{\omega}_x'\\ \dot{\omega}_y'\\ \dot{\omega}_z'\end{bmatrix} = \begin{bmatrix}M_{Gx}'/I_1\\ M_{Gy}'/I_2\\ M_{Gz}'/I_3\end{bmatrix} - \begin{bmatrix}1/I_1 & 0 & 0\\ 0 & 1/I_2 & 0\\ 0 & 0 & 1/I_3\end{bmatrix}\begin{bmatrix}(I_3 - I_2)\,\omega_y\omega_z\\ (I_1 - I_3)\,\omega_z\omega_x\\ (I_2 - I_1)\,\omega_x\omega_y\end{bmatrix}$$

$$** \; \boxed{\begin{bmatrix}\dot{\omega}_x'\\ \dot{\omega}_y'\\ \dot{\omega}_z'\end{bmatrix} = \begin{bmatrix}M_{Gx}'/I_1\\ M_{Gy}'/I_2\\ M_{Gz}'/I_3\end{bmatrix} + \begin{bmatrix}\frac{(I_2 - I_3)}{I_1}\,\omega_y'\omega_z'\\ \frac{(I_3 - I_1)}{I_2}\,\omega_x'\omega_z'\\ \frac{(I_1 - I_2)}{I_3}\,\omega_x'\omega_y'\end{bmatrix}}$$

The Euler Equations

Aside: Recall in 2D

$\dot{\theta} = \omega$

$\dot{\omega} = M_{/G}/I^G$

Find any soln's

1) $\vec{M}_{/G} = \vec{0}$

$\vec{\omega} = \vec{0}$

2) $\vec{\omega} = \text{const.}, \quad \overbrace{\vec{M}_{/G} = 0}^{\text{torque free}}$

$\Rightarrow \dot{\vec{\omega}} = \vec{0}$

$$\vec{0} = \underset{\text{non-singular}}{\underline{\underline{I}}^{-1}} \underbrace{\left[\vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega})\right]}_{\Rightarrow \; = \vec{0}}$$

$\Rightarrow \vec{\omega} \parallel \text{ to } \underline{\underline{I}}\,\vec{\omega}$

$\Rightarrow \vec{\omega}$ is an e-vector of $\underline{\underline{I}}$

$\Rightarrow \vec{\omega} = \underbrace{\omega \hat{e}_1' \text{ or } \omega \hat{e}_2' \text{ or } \omega \hat{e}_3'}_{\text{Rot. about a princ. axis}}$

Or from **

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{I_2 - I_3}{I_1} \omega_y' \omega_z' \\ \frac{I_3 - I_1}{I_2} \omega_x' \omega_z' \\ \frac{I_1 - I_2}{I_3} \omega_x' \omega_y' \end{bmatrix} \Rightarrow \left.\begin{array}{c} \omega_x' \neq 0, \text{ others} = 0 \\ \sim\text{or}\sim \\ \omega_y' \neq 0, \text{ others} = 0 \\ \sim\text{or}\sim \\ \omega_z' \neq 0, \text{ others} = 0 \end{array}\right\} \text{Rot. about a princ. axis}$$

Torque-free motion obeys

$$\begin{bmatrix} \dot{\omega}_x' \\ \dot{\omega}_y' \\ \dot{\omega}_z' \end{bmatrix} = \begin{bmatrix} \frac{I_2 - I_3}{I_1} \omega_y' \omega_z' \\ \frac{I_3 - I_1}{I_2} \omega_x' \omega_z' \\ \frac{I_1 - I_2}{I_3} \omega_x' \omega_y' \end{bmatrix}$$

A solution is: $[\vec{\omega}]_\beta = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} = \text{const.}$

$$[\vec{\omega}]_\beta = \underset{\text{big, but const}}{\begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}} + \underset{\text{small perturbation fns of time}}{\begin{bmatrix} \hat{\omega}_x' \\ \hat{\omega}_y' \\ \hat{\omega}_z' \end{bmatrix}}$$

$$\begin{bmatrix} \dot{\hat{\omega}}_x{}' \\ \dot{\hat{\omega}}_y{}' \\ \dot{\hat{\omega}}_z{}' \end{bmatrix} = \begin{bmatrix} \frac{I_2 - I_3}{I_1}\left(\hat{\omega}_y{}' \hat{\omega}_z{}'\right) \\ \frac{I_3 - I_1}{I_2}\left(\omega \hat{\omega}_z{}' + \hat{\omega}_x{}' \hat{\omega}_z{}'\right) \\ \frac{I_1 - I_2}{I_3}\left(\omega \hat{\omega}_y{}' + \hat{\omega}_x{}' \hat{\omega}_y{}'\right) \end{bmatrix}$$

Dropping $\hat{\omega}'^2$ terms ("small x small")

$$\dot{\hat{\omega}}_x{}' = 0$$

$$\boxed{\begin{aligned} \dot{\hat{\omega}}_y{}' &= \left[\frac{I_3 - I_1}{I_2}\,\omega\right] \hat{\omega}_z{}' \\ \dot{\hat{\omega}}_z{}' &= \left[\frac{I_1 - I_2}{I_3}\,\omega\right] \hat{\omega}_y{}' \end{aligned}}$$

A pair of linear 1st order ODEs

Aside: form of the ODEs

$$\dot{x} = c_1 y$$
$$\dot{y} = c_2 x$$

$$\ddot{\hat{\omega}}_{y'} = \omega^2 \frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \hat{\omega}'_y$$

$$\ddot{\hat{\omega}}_y{}' = D\, \hat{\omega}_y{}'$$

$D < 0 \Rightarrow$ stable

$D > 0 \Rightarrow$ unstable

$$\left.\begin{matrix} I_1 > I_2 \\ I_1 > I_3 \end{matrix}\right. \Rightarrow D < 0 \text{ stable}$$

$$\left.\begin{matrix} I_1 < I_2 \\ I_1 < I_3 \end{matrix}\right. \Rightarrow D < 0 \text{ stable}$$

$$\underbrace{I_2 < I_1 < I_3}_{\text{Rotate about the intermediate princ. axis}} \Rightarrow D > 0 \text{ unstable}$$

3) Fixed axis rotation, const $\vec{\omega}$

$$\boxed{\vec{M}_{/G} = \vec{\omega} \times \left(\underline{\underline{I}} \cdot \vec{\omega} \right)}$$ $\leftarrow$ $\underline{\vec{\omega} = \text{const}}$

If rotating about x-axis,
Moments are due to off-diagonal terms in $\underline{\underline{I}}$

In old books
I_{xy}, I_{xz}, I_{yz} are called the centrifugal terms in $[\underline{\underline{I}}]$

Today

Lecture 10
March 11th, 2014

Rotations: ① $\underline{\underline{R}}$ falling apart due to numerical error
② Steady precession of axis symmetric objects

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① $\underline{\underline{R}}$ falling apart

We know that $\dot{\underline{\underline{R}}} = \underline{\underline{\delta}}(\vec{\omega}) \cdot \underline{\underline{R}}$

Problem: $\underline{\underline{R}}$ needs to be such that $\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{1}}$.
$\underline{\underline{R}}$ has 9 numbers that may drift due to numerical error. (Analogous to DAEs solved as ODEs where the constraint satisfaction drifts.)

Let $\underline{\underline{R}} = [\vec{R}_1 | \vec{R}_2 | \vec{R}_3]$.

Then
$$\left.\begin{aligned} \vec{R}_1 \cdot \vec{R}_1 &= 1 \\ \vec{R}_2 \cdot \vec{R}_2 &= 1 \\ \vec{R}_3 \cdot \vec{R}_3 &= 1 \\ \vec{R}_1 \cdot \vec{R}_2 &= 0 \\ \vec{R}_2 \cdot \vec{R}_3 &= 0 \\ \vec{R}_3 \cdot \vec{R}_1 &= 0 \end{aligned}\right\} \text{6 constraints on 9 numbers}$$

How to deal with this?

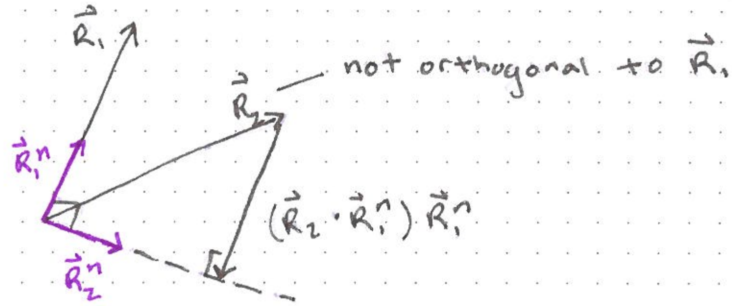
1) Ignore it and check that $\underline{\underline{R}}^T \underline{\underline{R}} - \underline{\underline{1}} \approx \underline{\underline{0}}$
(error is small when integrating over small time steps.)
2) Euler Angles (3 numbers) - requires ODEs with Euler angles...yuck
3) Quaternians (4 numbers)
4) Others ways to parameterize rotations
5) Straighten out $\underline{\underline{R}}$ now & again.
   a) Gram-Schmidt Orthogonalization
   b) Polar Decomposition

a) Gram-Schmidt Orthogonalization

Let $\underline{\underline{R}} = [\vec{R}_1 | \vec{R}_2 | \vec{R}_3]$.

Start by replacing $\vec{R}_1$ with $\frac{\vec{R}_1}{|\vec{R}_1|}$.

$$\vec{R}_1^n = \frac{\vec{R}_1}{|\vec{R}_1|}$$



$$\vec{R}_2^n = \frac{\vec{R}_2 - (\vec{R}_2 \cdot \vec{R}_1^n)\vec{R}_1^n}{|\vec{R}_2 - (\vec{R}_2 \cdot \vec{R}_1^n)\vec{R}_1^n|}$$

$$\vec{R}_3^n = \frac{\vec{R}_3 - (\vec{R}_3 \cdot \vec{R}_1^n)\vec{R}_1^n - (\vec{R}_3 \cdot \vec{R}_2^n)\vec{R}_2^n}{|\vec{R}_3 - (\vec{R}_3 \cdot \vec{R}_1^n)\vec{R}_1^n - (\vec{R}_3 \cdot \vec{R}_2^n)\vec{R}_2^n|}$$

$$\boxed{\underline{\underline{R}}^{new} = [\vec{R}_1^n | \vec{R}_2^n | \vec{R}_3^n]}$$

b) Polar Decomposition

$\underline{\underline{R}} = U S V^T$ $\quad$ ✱ true for all matrices

U, V^T: orthogonal

S: diagonal

$\underline{\underline{R}} = \underbrace{U V^T}_{\text{orthog.}} \underbrace{V S V^T}_{\text{symmetric}}$ $\quad$ → this decomposition is unique iff VSV^T is positive definite

UV^T should be close to $\underline{\underline{R}}$, since $\underline{\underline{R}}$ is supposed to be orthogonal.
Then VSV^T should be close to $\underline{\underline{1}}$.

$$\boxed{\underline{\underline{R}}^{new} = UV^T}$$

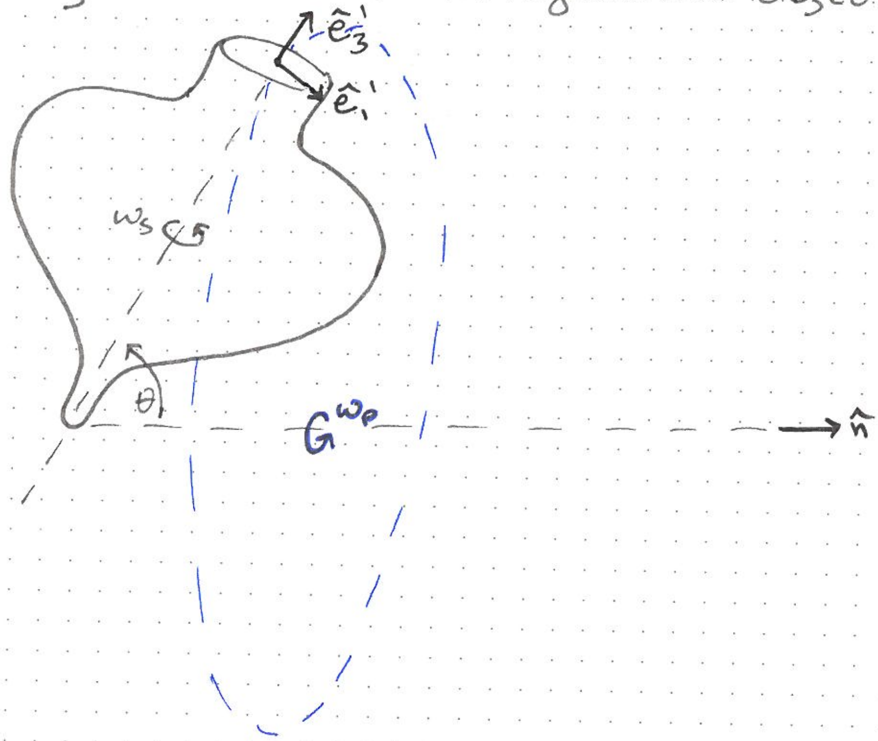
In Matlab:

```
(U, S, V) = svd(R);
R_new = U * V';
```

$\dot{R} = \vec{\omega} \times R + c(R\_new - R)$

c: pick and play with small values

② Steady Precession of Axis Symmetric Objects

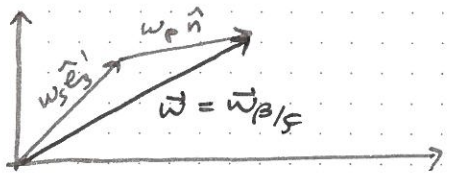


$\vec{\omega}_{B/F} = \vec{\omega} = \omega_s \hat{e}_3' + \omega_p \hat{n}$ $\qquad$ ω_s, ω_p are constant

$\hat{e}_3', \hat{e}_1', \hat{e}_2'$ = precessing frame P
$\hat{e}_3', ?, ?$ = body frame B

$\underbrace{?, ?}$ we don't need to pay attention to these

θ = precession angle



Assume that θ is constant.
Our governing equation: $\Sigma \vec{M}_{/G} = {}^F\dot{\vec{H}}_{/G}$

$= \cancel{{}^P\dot{\vec{H}}_{/G}} + \vec{\omega}_{P/F} \times \vec{H}_{/G}$ $\qquad$ (Qdot formula) lecture 8

$= \vec{0}$ because $\dot{\theta} = \dot{\omega}_p = \dot{\omega}_s = 0$
$\Rightarrow$ only change is rotation about $\hat{n}$.

$$[\underline{\underline{I}}]_P = [\underline{\underline{I}}]_B = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 = I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

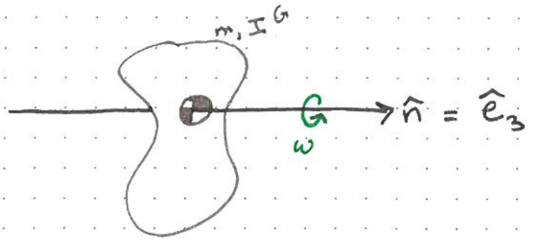
$\vec{H}_{/G} = \underline{\underline{I}} \cdot \vec{\omega}$

$= [I_1 \hat{e}_1' \hat{e}_1' + I_2 \hat{e}_2' \hat{e}_2' + I_3 \hat{e}_3' \hat{e}_3'] \cdot [\omega_s \hat{e}_3 + \omega_p \hat{n}]$

Today

Special Motions of Rigid Objects (con't)

1). Rotation about a fixed axis:



$\vec{\omega}_{B/F} = \underset{\text{const}}{\omega}\ \underset{\text{const}}{\hat{n}}$

$\vec{H}_{/G} = \underline{\underline{I}}^G \vec{\omega}$

${}^F\dot{\vec{H}}_{/G} = {}^B\dot{\vec{H}}_{/G} + \vec{\omega}_{B/F} \times \vec{H}_{/G}$

$= \vec{\omega}_{B/F} \times \vec{H}_{/G}$

$= \vec{\omega}_{B/F} \times \underline{\underline{I}}^G \cdot \vec{\omega}_{B/F}$

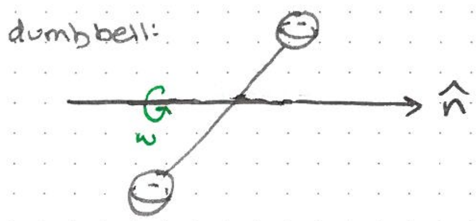
$$[\dot{\vec{H}}] = [\vec{\omega}] \times \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \omega \begin{bmatrix} I_{13} \\ I_{23} \\ I_{33} \end{bmatrix}$$

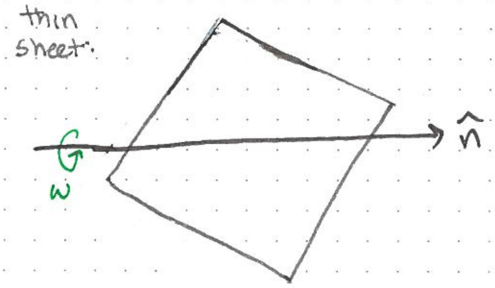
$$[\dot{\vec{H}}] = \begin{bmatrix} -\omega^2 I_{23} \\ \omega^2 I_{13} \\ 0 \end{bmatrix}$$

Since $[\vec{M}] = [\dot{\vec{H}}]$, the torque required to spin @ constant $\vec{\omega} = \omega \hat{e}_3$ only equals 0 if $I_{23} = I_{13} = 0$.

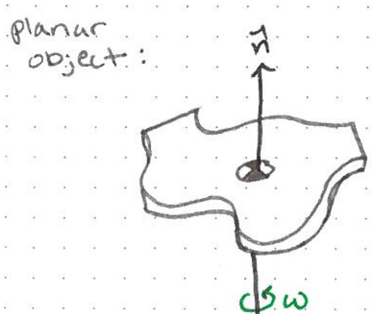
Examples:



needs torque



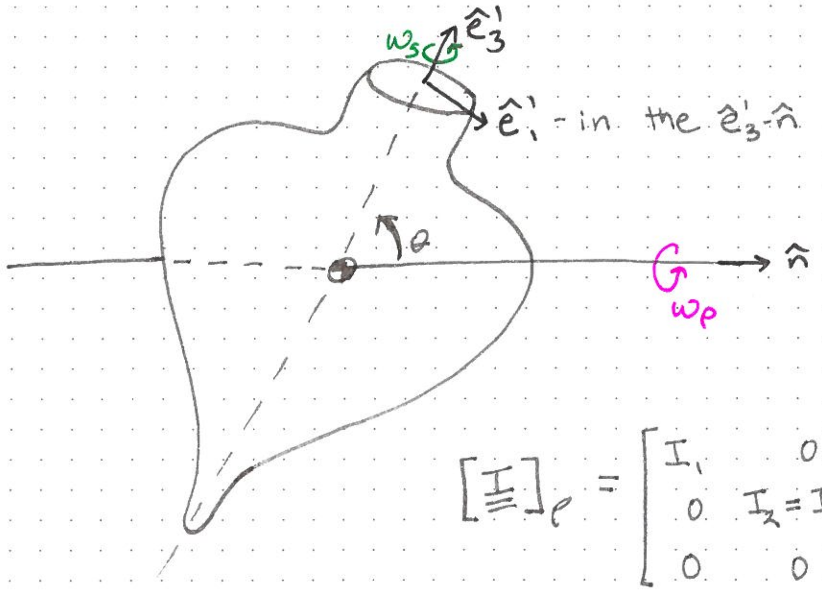
no torque required



no torque needed

2) Rotation about an axis near a principle axis (such as in the flipping board)

3) Steady Precession



$$[\underline{\underline{I}}]_\rho = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 = I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Questions

① What torque is required

② If there is no torque, are there any restrictions on $\theta, I_1, I_3, \omega_s$, or ω_p?

$\vec{\omega} = \omega_s \hat{e}_3' + \omega_p \hat{n}$

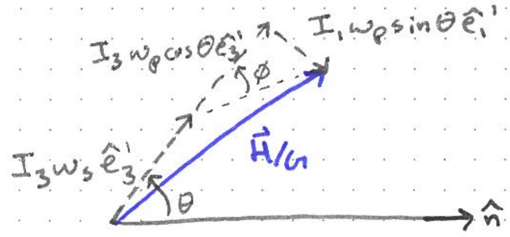
for small θ, $\vec{\omega} \approx (\omega_s + \omega_p)\hat{n}$

$$\vec{M}_{/G} = {}^F\dot{\vec{H}}_{/G} = \underbrace{\cancel{{}^\rho\dot{\vec{H}}_{/G}}}_{0} + \omega_p \hat{n} \times \vec{H}_{/G}$$

$$= \omega_p \hat{n} \times [\underline{\underline{I}} \cdot \vec{\omega}]$$

$$= \omega_p \hat{n} \times [I_1 \hat{e}_1' \hat{e}_1' + I_1 \hat{e}_2' \hat{e}_2' + I_3 \hat{e}_3' \hat{e}_3'] \cdot (\omega_s \hat{e}_3' + \omega_p \hat{n})$$

$$= \omega_p \hat{n} \times \underbrace{(\omega_s I_3 \hat{e}_3' + \omega_p \sin\theta I_1 \hat{e}_1' + \omega_p \cos\theta I_3 \hat{e}_3')}_{\vec{H}_{/G}}$$



* $\phi \neq \theta$ unless $I_1 = I_3$

Question: Is this a torque-free motion?

Answer: For it to be torque free, $\vec{M}_{/G} = \dot{\vec{H}}_{/G} = 0$

$\dot{\vec{H}}_{/G} = 0$ only if $\vec{H}_{/G} \parallel \hat{n}$

Therefore the sum of the components in $\vec{H}_{/G}$ that are perpendicular to $\hat{n}$ must $=0$:

$$\left\{ I_3 \omega_s \cancel{\sin\theta} + I_3 \omega_p \cos\theta \cancel{\sin\theta} - I_1 \omega_p \sin\theta \cancel{\cos\theta} = 0 \right\}$$

$$\frac{1}{\omega_p I_3} \cdot \{\ \} \Rightarrow \frac{\omega_s}{\omega_p} + \cos\theta - \frac{I_1}{I_3}\cos\theta = 0$$

$$\rightarrow \boxed{\frac{\omega_s}{\omega_p} = \left(\frac{I_1}{I_3} - 1\right)\cos\theta}$$

Wobbling Plate: $\theta \ll 1$

$I_1 = \frac{I_3}{2}$

$$\frac{\omega_s}{\omega_p} = \left(\frac{1}{2} - 1\right) \cdot 1$$

$\hookrightarrow \omega_p = -2\omega_s$

recall that: $\vec{\omega} = \omega_s \hat{e}_3' + \omega_p \hat{n}$

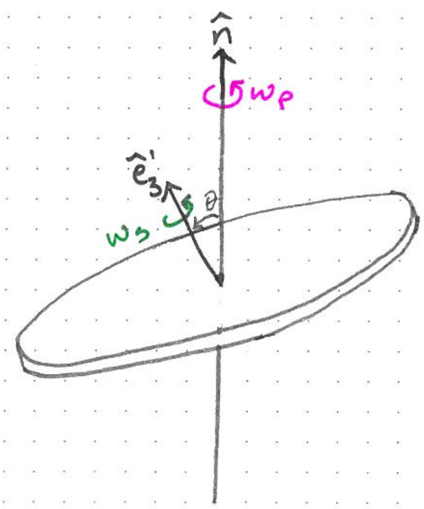
and for small θ: $\vec{\omega} \approx (\omega_s + \omega_p)\hat{n}$

$$\rightarrow \omega \approx \omega_s + \omega_p$$

$$\approx -\frac{\omega_p}{2} + \omega_p$$

$$\Rightarrow \boxed{\omega \approx \frac{\omega_p}{2}}$$

* true for any planar object



$\boxed{3/18/2014}$

Analytical Dynamics

1) Derive Princ. of least action for $F=ma$
2) Derive Lag. Eqn. from Princ. of least action
3) Derive Lag. Eqn. from $\vec{F}=m\vec{a}$.

---

1. Start w/ $\vec{F}=m\vec{a} \Rightarrow$ Princ. of least action

General system is system of particles

For each,

$$\vec{F}_i = \underbrace{m_i \vec{a}_i}_{\text{no sum}}$$

all forces on particle

$$\vec{F}_i - m_i\vec{a}_i = \vec{0}$$

$$(\vec{F}_i - m_i\vec{a}_i)\cdot \underbrace{\delta\vec{v}_i}_{\text{any vector fn of time}} = 0$$

Add up over all particles in system:

$$\sum (\vec{F}_i - m_i\vec{a}_i)\cdot\delta\vec{v}_i = 0$$

$$\vec{F}_i = \vec{F}_i^{\text{ constraint}} + \vec{F}_i^{\text{ non-constraint}}$$

constraint = kinematic constraint

e.g.: * hinges
* distances between pts are const
* rigid object

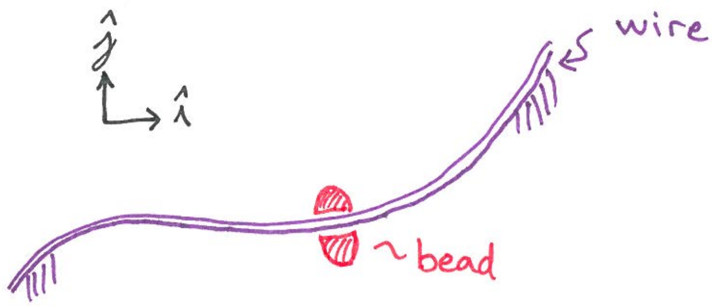
Postulate A from Variational Principles of Mechanics by Lanczos (sp)

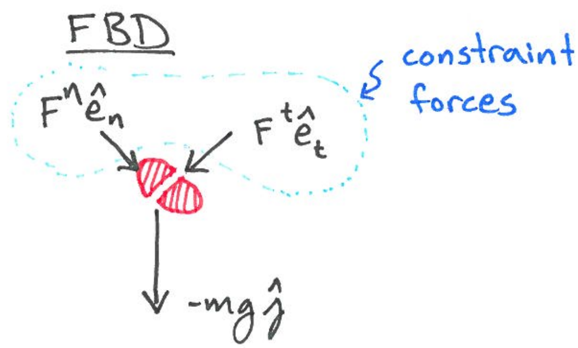
$$\sum \vec{F}^{\,const} \cdot \delta \vec{v}_i = 0$$

for all $\delta \vec{v}_i$ that respect the constraints

$\Rightarrow$ The work of constraint forces is zero for all real or imagined displacements that satisfy the constraints

ex) bead on rigid wire





Postulate A

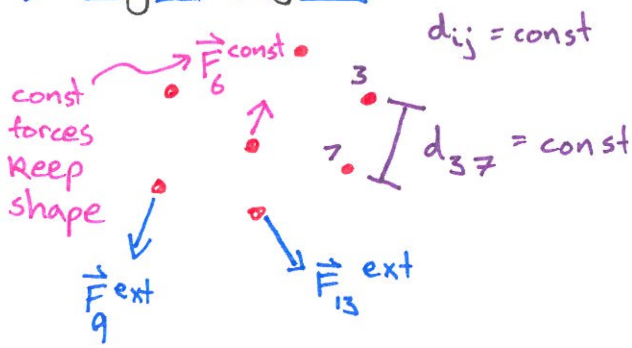
$$\sum \vec{F}^{const} \cdot \delta \vec{v} = 0$$

$$(F^n \hat{e}_n + F^t \hat{e}_t) \cdot \delta v \hat{e}_t = 0$$

$$\Rightarrow \boxed{F^t = 0}$$

Postulate A $\Rightarrow$ constraint has no friction

ex) Rigid object



For this object (collection of particles),

Postulate A: $\sum \vec{F}^{const} \cdot \underbrace{\delta \vec{v}_i}_{\text{that respect constraints}} = 0$

The set of $\delta \vec{v}_i$ that correspond to rigid motions are

$$\delta \vec{v}_i = \delta \vec{v}_G + \delta \vec{\omega} \times \vec{r}_{i/G}$$

$$\Rightarrow \sum \vec{F}_i^{\,const} \cdot \left[\delta \vec{v}_G + \delta \vec{\omega} \times \vec{r}_{i/G}\right] = 0$$

for all $\delta \vec{v}_G$ & $\delta \vec{\omega}$

Set $\delta \vec{V}_G$ "=" $\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\hat{e}_1 \text{ or } \hat{i}}$, $\delta \vec{\omega}$ "=" $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\sum \vec{F}^{\,const} \cdot \hat{e}_1 = 0$$

again, $\delta \vec{v}_G = \hat{e}_2$, $\delta \vec{\omega} = \vec{0}$

$$\sum \vec{F}^{\,const} \cdot \hat{e}_2 = 0$$

again, $\delta \vec{v}_G = \hat{e}_3$, $\delta \vec{\omega} = \vec{0}$

$$\sum \vec{F}^{\,const} \cdot \hat{e}_3 = 0$$

$$\Rightarrow \sum \vec{F}^{\,const} = \vec{0}$$

Aside:
"The Culture of Force"
~Wilczek
Mentions sum of internal forces

Postulate A $\Rightarrow$ internal forces in rigid objects have no net forces

Now set $\delta \vec{v}_G = 0$:

$$\sum \vec{F}^{\,const} \cdot \left(\delta \vec{\omega} \times \vec{r}_{i/G}\right) = 0$$

$$\Rightarrow \sum_i \delta \vec{\omega} \cdot \vec{r}_{i/G} \times \vec{F}_i^{\,const} = 0$$

$$\underbrace{\delta \vec{\omega}} \cdot \left[\sum \vec{r}_{i/G} \times \vec{F}_i^{\,const}\right] = 0$$

set $= \hat{e}_1$
set $= \hat{e}_2$
set $= \hat{e}_3$

$$\Rightarrow \sum \vec{r}_{i/G} \times \vec{F}_i^{\,const} = \vec{0}$$

Postulate A: internal const forces have no net moment

Back to derivation,

$$\sum \left(\vec{F}_i^{\,\text{non-const}} - m_i \vec{a}_i\right) \cdot \overbrace{\delta \vec{v}_i}^{\text{for virtual motions that satisfy constraints}} = 0$$

no const forces in eqn

Add assumption $\vec{F}_i^{\,\text{non-const}}$ are conservative

Notation: F_i is list of 3n forces

a_i are 3n acceleration components

$$\begin{array}{ll}
3\left\{\begin{matrix} F_{x1} \\ F_{y1} \\ F_{z1} \end{matrix}\right. & (1) \\
3\left\{\begin{matrix} F_{x2} \\ F_{y2} \\ F_{z2} \end{matrix}\right. & (2) \\
3\left\{\begin{matrix} \vdots \end{matrix}\right. & (3) \\
\quad\ \vdots & (n)
\end{array}$$

$$\text{Conservative} \Leftrightarrow \oint \sum F_i \, dx_i = 0 \Leftrightarrow \int_{\vec{a}}^{\vec{b}} \sum F_i \, dx_i \text{ is ind. of path}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$F_i = \frac{-\delta E}{\partial x_i} \Leftrightarrow \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

3/20/2014

Recall (last lecture):

$$\sum (\vec{F}_i^{\text{non-const}} - m\vec{a}_i) \cdot \delta\vec{v}_i = 0$$

where i is particle number

$$\vec{F} = \Big[\quad\Big]^{3n} \qquad \int_{t_1}^{t_2} \sum \Big(\overset{\text{potential energy}}{\frac{-\partial V}{\partial x_i}} - ma_i \Big) \underset{\text{satisfy constraints}}{\delta x_i}\, dt = 0 \quad ☆$$

i = force number

$i = 5 \Rightarrow$

$$\sum \frac{-\partial V}{\partial x_i} \delta x_i = -\delta V$$

ASIDE

$$a_i \delta x_i = \ddot{x}_i \, \delta x_i$$
$$= \frac{d}{dt}(\dot{x}_i \delta x_i) - \dot{x}_i \delta \dot{x}_i$$
$$= \frac{d}{\partial t}(\dot{x}_i \delta x_i) - \delta\left(\frac{\dot{x}_i^2}{2}\right)$$

$$☆ \Rightarrow \int_{t_1}^{t_2} -\delta V + \Big[\underbrace{m_i \delta\left(\frac{\dot{x}_i^2}{2}\right)} - \frac{d}{dt}(\dot{x}_i \delta x_i) m_i\Big] \, dt = 0$$

$$\delta \int_{t_1}^{t_2} (\overset{\sum \frac{1}{2}\dot{x}_i^2 m_i}{T} - V)\, dt - \underbrace{\int_{t_1}^{t_2} \frac{d}{dt}(\dot{x}_i \delta x_i) m_i \, dt}_{m_i x_i \delta x_i \big|_{t_1}^{t_2} = 0} = 0$$

$\delta x_i = 0$ at t_1 and t_2

$$\Rightarrow \boxed{\delta \int_{t_1}^{t_2} (T - V) = 0}$$

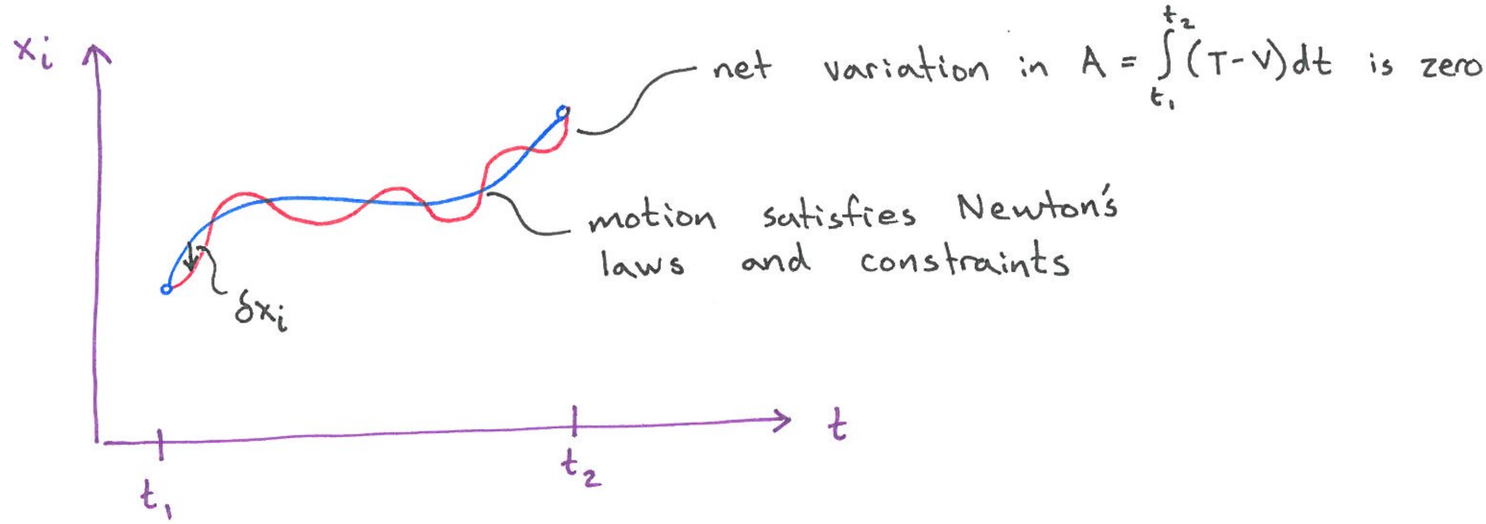
Princ. of Stationary Action

Given a collection of particles,

* each moving with $\vec{F} = m\vec{a}$
* constraint forces that do no work for real or imagined motions that satisfy constraints
* all other forces are conservative

The nature of motion is such that

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$ for all variations in motion that satisfy constraints and have $\delta x_i = 0$ at t_1 and t_2



Start w/ Principle of "Least" Action
($\uparrow$ stationary)

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

Assume we have generalized (minimal) coordinates q_i (respect constraints & give all possible motions that do respect the constraints)

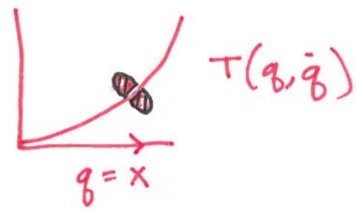
Can calculate

$V(\vec{q})$

$T(\vec{q}, \dot{\vec{q}})$

for known system

ASIDE



$$0 = \int_{t_1}^{t_2} \sum \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \sum \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i \, dt$$

ASIDE $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i\right)$

$$= \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \delta q_i$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \delta q_i + \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i\right)$$

$$\Rightarrow 0 = \int_{t_1}^{t_2} \sum \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right) \delta q_i \, dt + \underbrace{\int_{t_1}^{t_2} \frac{d}{dt} \sum \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i\right) dt}_{\left.\sum \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} \to 0}$$

consider variation at t_1 + t_2 = 0

$$\delta A = 0 \Rightarrow \int_{t_1}^{t_2} \sum \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right) \delta q_i = 0$$

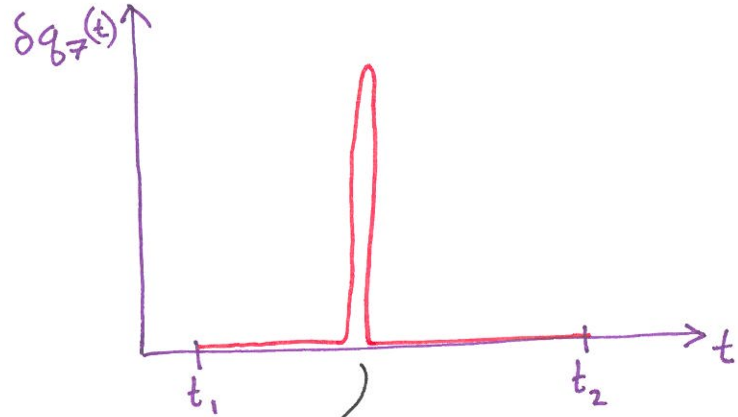
for all $\delta q_i = 0$ at t_1 + t_2

Assume all $\delta q_i(t) = 0$ for all t except δq_7

$$0 = \int_{t_1}^{t_2} \underbrace{\left(\frac{\partial \mathcal{L}}{\partial q_7} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_7}\right)\right)}_{g(t)} \underbrace{\delta q_7}_{f(t)} \, dt$$

ASIDE

$$\int_{t_1}^{t_2} f(t)\, g(t) = 0 \quad \text{for all } g(t)$$
$$\Rightarrow f(t) = 0 \quad \text{for all } t$$



$$\Rightarrow \underbrace{(\qquad)}_{g(t)} = 0 \quad \text{in that interval}$$

True at all times for q_7, q_8, q_5, etc.

$\Rightarrow$ For all q:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

Lagrange eqns.

assuming: conservative forces, holonomic constraints

3/25/2014

Lagrange Eqns

① So far: $\left.\begin{array}{c}\vec{F} = m\vec{a} \\ + \\ \text{postulate A}\end{array}\right\} \Rightarrow \delta(\text{Action}) = 0$ "Hamilton's Principle"

that constrain end conditions

$$\text{Action} = \int_{t_1}^{t_2} \underbrace{T - V}_{\mathcal{L}}\, dt$$

② $\left.\begin{array}{c}\delta\ \text{Action} = 0 \\ + \\ \text{Cons. Forces}\end{array}\right\} \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$

$\mathcal{L} = T - V$

③ Today: $\vec{F} = m\vec{a} \Rightarrow \boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i}$

Lag Eqn. w/ non-conservative forces

Derive Lagrange Eqns

N particles $(1, \dots N)$

\# of particles

n $\underbrace{\text{generalized coordinates}}_{\text{"minimal"}}$ q_i $(1, \dots n)$

Constrained system with all positions of all particles determined by $q_1, q_2, \cdots$

$$\vec{r}_j = \vec{r}_j(\vec{q}, t) = \vec{r}_j(q_1, q_2 \cdots q_n, t)$$

Look at $\vec{r}_j$

$$\vec{r}_j = \vec{r}_j(\vec{q}, t)$$

$$\dot{\vec{r}}_j = \underbrace{\sum_{K=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i} \dot{q}_i + \frac{\partial \vec{r}_j}{\partial t}}_{\dot{\vec{r}}_j = \dot{\vec{r}}_j(\vec{q}, \dot{\vec{q}}, t)}$$

$$\boxed{\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_K} = \frac{\partial \vec{r}_j}{\partial q_K}} \quad (1)$$

Look at $\dfrac{\partial \dot{\vec{r}}_j}{\partial q_K}$

(A)

$$\frac{\partial \dot{\vec{r}}_j}{\partial q_K} = \sum_{i=1}^{n} \frac{\partial^2 \vec{r}_j}{\partial q_K \partial \vec{q}_i} \dot{q}_i + \frac{\partial^2}{\partial q_K \partial t}(\vec{r}_j)$$

Look at $\dfrac{d}{dt}\left(\dfrac{\partial \vec{r}_j(\vec{q}, t)}{\partial q_K}\right)$ (B)

$$= \sum_{i=1}^{n} \frac{\partial^2 \vec{r}_j}{\partial q_i \partial q_K} \dot{q}_i + \frac{\partial^2 \vec{r}_j}{\partial t \, \partial q_K}$$

NOTE: (A) = (B)

$$\boxed{\frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_K}\right) = \frac{\partial \dot{\vec{r}}_j}{\partial q_K}} \quad (2)$$

(1) + (2) will be used in the following derivation

Derivation from Frank Dimaggio of Columbia, ≈ 1960

For each particle:

$$\vec{F}_j = m_j \vec{a}_j$$

$$\vec{F}_j - m_j \vec{a}_j = \vec{0}$$

$$(\vec{F}_j - m_j \vec{a}_j) \cdot \underbrace{\delta \vec{r}_j}_{\text{any variation in position}} = 0$$

$$\Rightarrow \sum_{j=1}^{N} (\vec{F}_j - m_j \vec{a}_j) \cdot \delta \vec{r}_j = 0$$

non-constraint forces

$$\left\{ \sum_{j=1}^{N} \left(\underbrace{\vec{F}_j^{\,*}}_{**} - \underbrace{m_j \vec{a}_j}_{***} \right) \cdot \underbrace{\delta \vec{r}_j}_{\text{variations consistent with constraints}} = 0 \right\}$$

Fundamental eqn of analytical mechanics

NOTE: Using Postulate A from last class or define constraint forces to be the forces with

$$\sum_{j=1}^{N} \vec{F}^{\,const} \cdot \underbrace{\delta r}_{\text{satisfy constraints}} = 0$$

$\vec{q}$ already incorporates constraints

$$\delta \vec{r}_j = \sum_{i=1}^{N} \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i$$

Look at ** in {Fund eqn}

$$\underbrace{\sum_{j=1}^{N} \vec{F}_j^{\,*} \cdot \delta \vec{r}_j}_{** \;\; \delta W = \text{virtual work}} = \sum_{j=1}^{N} \sum_{i=1}^{n} \vec{F}_j^{\,*} \cdot \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i$$

$$= \sum_{i=1}^{n} \underbrace{\left[\sum_{j=1}^{N} \vec{F}_j^{\,*} \cdot \frac{\partial \vec{r}_j}{\partial q_i} \right]}_{Q_i \equiv \sum \vec{F}_j^{\,*} \cdot \frac{\partial \vec{r}}{\partial q_i} \,=\, i^{th} \text{ generalized force}} \delta q_i = \overbrace{\sum_{i=1}^{n} Q_i \delta q_i}^{\delta W = \text{virtual work}}$$

Look at $T = E_K$ + $\dfrac{\partial T}{\partial q_K}$ + $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_K}\right)$

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right)$$

Now:

$$\frac{\partial T}{\partial q_K} = \sum_{i=1}^{N} m_i \underbrace{\frac{\partial \dot{\vec{r}}_i}{\partial q_K}}_{② \Rightarrow \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_K}\right)} \cdot \dot{\vec{r}}_i$$

$$\star = \sum_{i=1}^{N} m_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_K}\right) \cdot \dot{\vec{r}}_i$$

Now:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) = \frac{d}{dt} \frac{\partial\left(\sum \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right)}{\partial \dot{q}_K}$$

$$= \frac{d}{dt} \sum_{i=1}^{N} m_i \underbrace{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_K}}_{①} \cdot \dot{\vec{r}}_i$$

$$= \frac{d}{dt} \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_K}\right) \cdot \dot{\vec{r}}_i$$

$$\star\star = \sum_{i=1}^{N} m_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_K}\right) \cdot \dot{\vec{r}}_i + \sum_{i=1}^{N} m_i \underbrace{\frac{\partial \vec{r}_i}{\partial q_K}}_{\text{The Jacobian}} \cdot \ddot{\vec{r}}_i$$

Look at difference between two previous expressions $(\star)$

$$\overbrace{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_K}}^{\star\star} - \overbrace{\frac{\partial T}{\partial q_K}}^{\star} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_K} \cdot \ddot{\vec{r}}_i$$

$$= \sum_{i=1}^{N} \underbrace{m_i \ddot{\vec{r}}_i}_{= \vec{F}} \cdot \frac{\partial \vec{r}_i}{\partial q_K}$$

$$\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_K} - \frac{\partial T}{\partial q_K}\right] = \left\{\sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_K}\right\}$$

$$[\quad]\,\delta q_K = \{\quad\}\,\delta q_K$$

$$\sum_{k=1}^{n} [\quad]\,\delta q_K = \sum \{\quad\}\,\delta q_K$$

NOTE: i switched to k, k switched to i

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right)\delta q_i = \sum_{i=1}^{n} \underbrace{\sum_{k=1}^{N} \vec{F}_K \cdot \frac{\partial \vec{r}_K}{\partial q_i}}_{Q_i} \delta q_i$$

$$= \sum_{i=1}^{n} Q_i\,\delta q_i$$

set $\delta q_1 \neq 0$, all other $\delta q_i = 0$

then for 2, 3, etc $\Rightarrow$ $$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i}$$ ☆ ☆ ☆

Lagrange Eqns

$\boxed{3/27/2014}$

1) Lagrange Eqns. (cont'd)

2) Axioms of Mechanics (styrofoam)

Recall

$$Q_i = \sum_{j=1}^{N} \frac{\partial \vec{x}_j}{\partial q_i} \cdot \vec{F}_j^{*}$$

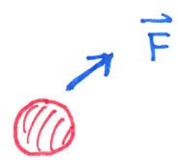
$\vec{F}_j^{*}$: total non-constraint force on particle j

$$\underbrace{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}}_{\text{proj. of } m\vec{a} \text{ in } q_i \text{ direction}} = \underbrace{Q_i}_{\text{projection of force in } q_i \text{ direction}}$$

---

ex ①

a single particle



$q_1 = x$

$q_2 = y$

$q_3 = z$

$$T = \frac{1}{2}\dot{q}_1^2 m + \frac{1}{2}\dot{q}_2^2 m + \frac{1}{2}\dot{q}_3^2 m$$

$$\frac{\partial T}{\partial \dot{q}_1} = \dot{q}_1 m$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} = \ddot{q}_1 m$$

$$\frac{\partial T}{\partial q_1} = 0$$

Lag Eqn #1: $m\ddot{q}_1 = Q_1$

where $\vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$Q_1 = \frac{\partial \vec{x}}{\partial q_1} \cdot \vec{F}$$

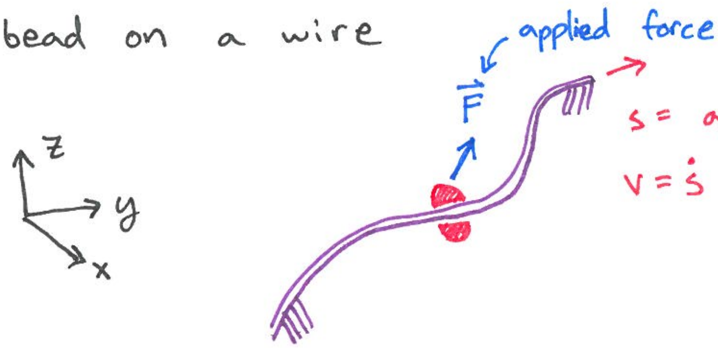
$$= \hat{\imath} \cdot \vec{F}$$

$$= F_x$$

$\Rightarrow$ Lag Eqn #1: $\boxed{m\ddot{x} = F_x}$

ex ②

bead on a wire



$s =$ arc length

$v = \dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

wire: $\vec{x}(s)$

$\quad \llcorner s = q$

> **ASIDE:**
>
> $d\vec{x} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$
>
> $ds = \sqrt{dx^2 + dy^2 + dz^2}$

$$T = \frac{1}{2} m \dot{s}^2$$

$$Q = \underbrace{\frac{\partial \vec{x}}{\partial s}}_{\text{Jacobian}} \cdot \vec{F} = \hat{e}_t \cdot \vec{F}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{s}} - \frac{\partial T}{\partial s} = Q_s$$

$$\boxed{m\ddot{s} = \vec{F} \cdot \hat{e}_t}$$

Literally equivalent to:

$$\left\{ \vec{F} = m\vec{a} \right\} \cdot \hat{e}_t$$

What if $\vec{F}^*$ is conservative?

Simplify notation: $\quad i = 1, \ldots 3N$

$$F_i^* = \frac{-\partial V}{\partial x_i} \qquad \begin{aligned} V &= V(x_i) \\ &= V(x_i(q_j)) \\ &= V(q_j) \end{aligned}$$

Look at Q_i

$$Q_i = \sum_{j=1}^{3N} \frac{\partial x_j}{\partial q_i} F_j$$

$$F_j = \frac{-\partial V}{\partial x_j}$$

$$= \sum \frac{-\partial V}{\partial x_j} \frac{\partial x_j}{\partial q_i}$$

$$Q_i = \frac{-\partial V}{\partial q_i}$$

Lag. Eq:

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \frac{\partial V}{\partial q_i}$$

NOTE: $V = V(q_i, \cancel{\dot{q}_i})$

V is only a function of q_i, not $\dot{q}_i$

$$\frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_i} - \frac{\partial (T-V)}{\partial q_i} = 0$$

$$\Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \begin{cases} 0 & \text{if all forces are conservative} \\ Q_i & \text{if forces not conserved} \end{cases}$$

where $\mathcal{L} = T - V = E_K - E_V$

$$\boxed{\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i}$$ ☆☆☆

1) Internal forces cancel

a) $\sum \vec{F}^{int} = \vec{0}$
b) $\sum \vec{M}^{int}_{/c} = \vec{0}$

Internal forces have no net force or moment

2) LMB: $\underbrace{\sum \vec{F}^{ext}}_{\text{external forces}} = \sum m_i \vec{a}_i$

AMB: $\underbrace{\sum \vec{r}_{i/c} \times \vec{F}_i}_{\text{external moments}} = \sum \vec{r}_{i/c} \times m_i \vec{a}_i$

3) Even for non-rigid systems work of internal forces is zero for imagined rigid motions.

HW: Show 3 $\Rightarrow$ 1

4) Think of matter as made of massless styrofoam that carries all loads & interactions.

a) Foam obeys statics
b) Is embedded with lead particles, each of which obeys $\vec{F} = m\vec{a}$ and causes reaction $\vec{F}$ on foam.

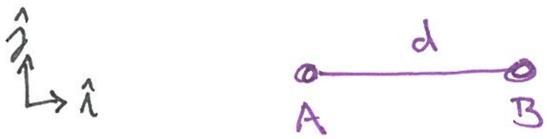
e.g. LMB $\sum_{\substack{\text{ext}\\\text{forces}}} \vec{F}_i \underbrace{- m_i \vec{a}_i}_{\text{D'Alambert reaction forces}} = \vec{0}$

$\sum_{\substack{\text{ext +}\\\text{D'Alambert}}} \vec{F} = \vec{0}$

NOTE: Moment balance $\Rightarrow$ force balance

Assume $\sum \vec{M}_{/A} \cdot \hat{k} = 0$ & $\sum \vec{M}_{/B} \cdot \hat{k} = 0$

& $\vec{r}_{B/O} = \vec{r}_{A/O} + d\,\hat{\imath}$



$$\boxed{\sum \vec{M}_{/B} = \sum \vec{M}_{/A} + \vec{r}_{A/B} \times \sum \vec{F}}$$

where $\sum \vec{M}_{/B} = 0$ and $\sum \vec{M}_A = 0$,

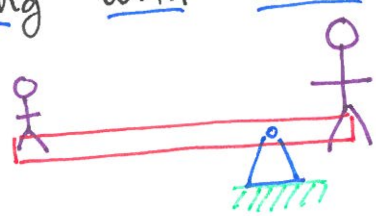
$$\left\{ \underbrace{\vec{r}_{A/B}}_{d\hat{\imath}} \times \sum \vec{F} = \vec{0} \right\} \cdot \hat{k}$$

$$d\,\hat{\imath} \times \left[\sum F_x \hat{\imath} + F_y \hat{\jmath} + F_z \hat{k} \right] \cdot \hat{k} = 0$$

$$\Rightarrow \boxed{\sum F_y = 0}$$ likewise for other components

---

Starting with Teeter Totter



4/8/2014

Lagrange Eqs. with Constraints

1) Informal
2) Lagrange Multipliers

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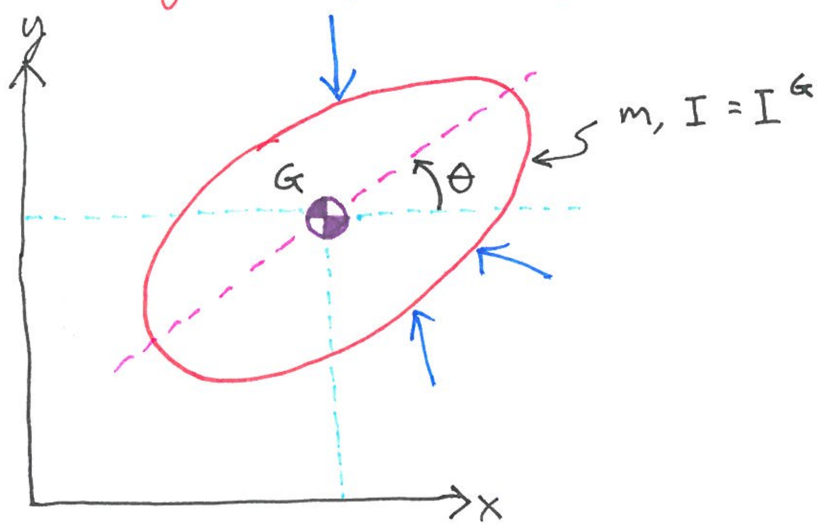
ex) Chaplygih Sleigh (sp?)

Rigid object on plane



$q_1 = x = x_G$

$q_2 = y = y_G$

$q_3 = \theta$

$$\left.\begin{aligned} E_K = T &= \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} m \dot{y}^2 + \tfrac{1}{2} I \dot{\theta}^2 \\ E_p &= 0 = V \end{aligned}\right\} \Rightarrow \boxed{\mathcal{L} = E_K}$$

Lag Eqns

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = Q_x$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = Q_y$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = Q_\theta$$

$$m\ddot{x} = \sum_{\substack{\text{all}\\\text{atoms}}}^{3N} \frac{\partial x_i}{\partial x} F_i = \sum F_x$$

$$m\ddot{y} = \sum F_y$$

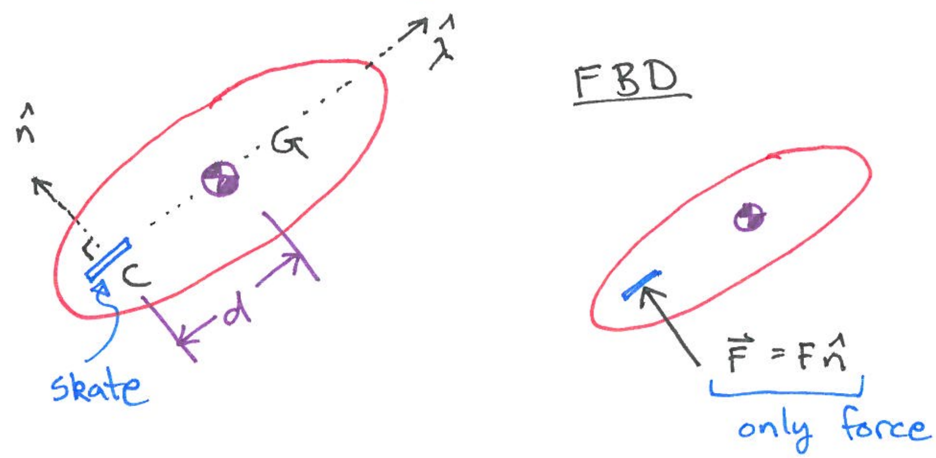
$$I\ddot{\theta} = \sum M$$

Consider the case where forces come from a constraint:

Add a skate

└ kinematic constraint

$$\vec{V}_c \cdot \hat{n} = 0$$



Need to:

① Write constraint eqns in terms of q_1, q_2, q_3

② Find Q_x, Q_y, Q_θ

Constraint Eqns

$$\vec{V}_c \cdot \hat{n} = 0$$

$(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$

$$\dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + (\dot{\theta}\,\hat{k}) \times \vec{r}_{c/G}$$

└ $-d\hat{\lambda}$

└ $\hat{\lambda} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$$f(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = 0$$

$$\boxed{-\sin\theta\,\dot{x} + \cos\theta\,\dot{y} - \dot{\theta} d = 0} \quad ①$$

$$m\ddot{x} = Q_x = -F\sin\theta \quad ②$$

$$m\ddot{y} = Q_y = F\cos\theta \quad ③$$

$$I\ddot{\theta} = M = -Fd \quad ④$$

$\frac{d}{dt}$① $\Rightarrow$ ①* & 2, 3, 4 are 4 eqns for $\ddot{x}, \ddot{y}, \ddot{\theta}, F$ at every instant in time

---

Method 2: Method of Lagrange Multipliers

have system, know Lag Eqns

Add a constraint:

$$\sum a_i \dot{q}_i + a_t(t) = 0 \quad ①$$

$a_i = a_i(q_1, q_2 \dots)$

ex) $$\underbrace{(-\sin\theta)}_{a_x}\dot{x} + \underbrace{\cos\theta}_{a_y}\dot{y} + \underbrace{-d}_{a_\theta}\dot{\theta} + \underbrace{0}_{a_t} = 0$$

$\frac{d}{dt}$① = ①* eqn in terms of gen. accelerations

Need a const force to make const satisfied

Assume "workless constraints"

$$\sum F_i^{const} \underbrace{\delta q_i}_{\text{motions consistent with constraints}} = 0$$

Constraint Equation

$$\sum a_i \underbrace{\delta q_i}_{} = 0$$

all motions consistent with constraints

n-dimensional space with D.O.F of the original problem

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{bmatrix}$$ some vector in $\mathbb{R}^n$

allowed variations in motion $\perp$ to $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}$

Constraint forces $\perp$ allowed motions

$$\Rightarrow \quad F_i = \lambda a_i$$

Lagrange multiplier

const force $\parallel$ to a vector

1* Kinematic Eqn.

2 $m\ddot{x} = \lambda \underbrace{(-\sin\theta)}_{a_x}$

3 $m\ddot{y} = \lambda \underbrace{(\cos\theta)}_{a_y}$

4 $I\ddot{\theta} = \lambda \underbrace{(-d)}_{a_\theta}$

1-4 are 4 eqns to solve for

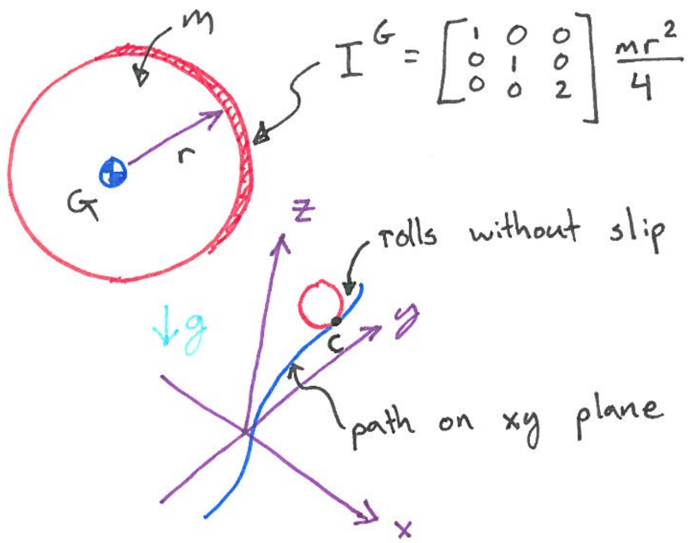
$\ddot{x}, \ddot{y}, \ddot{\theta}, \lambda$

F

Experimental fact:
Lagrange multipliers always have physical meaning
$\lambda = F$

4/10/2014

Rolling Disc



$$I^G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{mr^2}{4}$$

of DoFs

$$\underbrace{6 \ (3 \text{ translation} + 3 \text{ rotation})}$$

6 - constraints

└ constraints: holonomic = no ground penetration

$$\boxed{z_G - r\cos\theta = \text{const}}$$

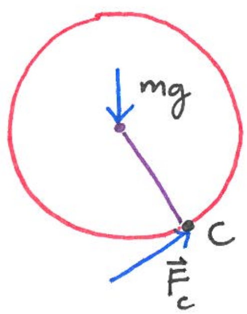
⇒ 5 configuration space variables

5 dimensional, accessible configuration space: x, y + 3 rotation angles

3 velocity degrees of freedom

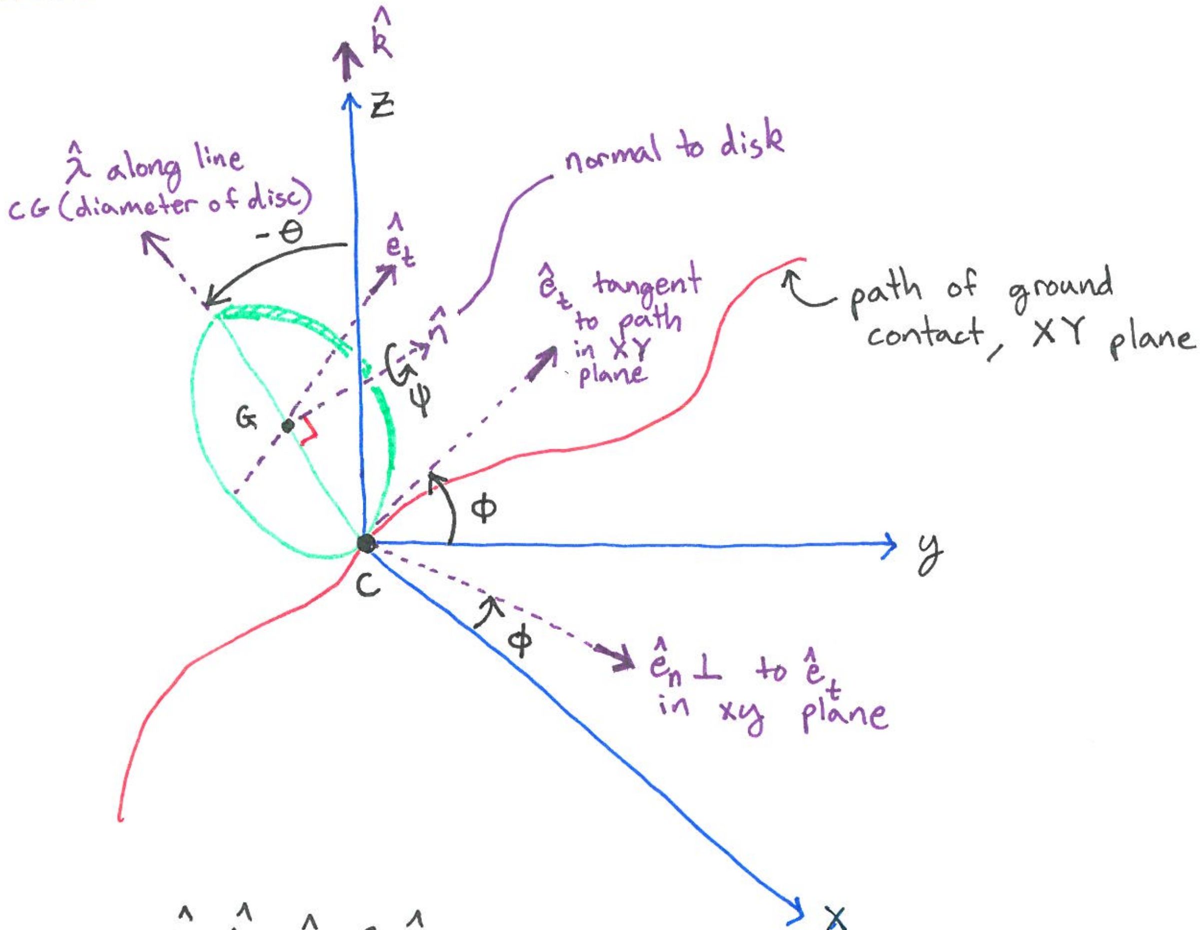
e.g. Euler angles

FBD



AMB/c

$$\boxed{\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}} \quad *$$



$\hat{\lambda}, \hat{k}, \hat{n}$ & $\hat{e}_n$ are coplanar

X, Y, Z local coordinates instantaneously coincident with C

Goal: Evaluate left & right sides of * in terms of

$\theta, \phi, \psi, \quad \dot\theta, \dot\phi, \dot\psi, \quad \ddot\theta, \ddot\phi, \ddot\psi$

4 reference frames

$\mathcal{F}$ = fixed = $\hat\imath, \hat\jmath, \hat k$

$\mathcal{P}$ = precessing steering frame = rot $\hat\imath, \hat\jmath, \hat k$ angle ϕ about $\hat k$ axis

$\hat\imath \to \hat e_n = \hat\imath'$

$\hat\jmath \to \hat e_t = \hat\jmath'$

$\hat k \to \hat k = \hat k'$

$\mathcal{T}$ = tipped / leaned / rolled Frame

$\hat e_n, \hat e_t, \hat k$ rotate about $\hat e_t$ axis angle θ

$\hat e_n \to \hat n = \hat\imath''$

$\hat e_t \to \hat e_t = \hat\jmath''$

$\hat k \to \hat\lambda = \hat k''$

$\mathcal{B}$ = body frame

$\hat n, \hat e_t, \hat\lambda$ and rotate about $\hat n$ by ψ

$\hat n \to \hat n$

$\left.\begin{array}{l}\hat e_t \to ? \\ \hat\lambda \to ?\end{array}\right\}$ not needed because of axis symmetry

ϕ, θ, ψ called 3 2 1 Euler angles

3 – z

2 – new y axis

1 – newest x axis

$$\boxed{\vec\omega_{\mathcal{B}/\mathcal{F}} = \dot\phi \hat k + \dot\theta \hat e_t + \dot\psi \hat n}$$

Nice frame for calculations $\underbrace{\hat{n}, \hat{e}_t, \hat{\lambda}}_{\text{tipped frame}}$

$$\vec{\omega}_{\beta/\mathcal{F}} = \dot{\phi}\,\hat{k} + \dot{\theta}\,\hat{e}_t + \dot{\psi}\,\hat{n}$$

$\hat{k} = \hat{\lambda}\cos\theta - \sin\theta\,\hat{n}$

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega}_{\beta/\mathcal{F}} = \ddot{\phi}\left(\hat{\lambda}\cos\theta - \sin\theta\,\hat{n}\right) + \dot{\phi}\left[\left(\dot{\hat{\lambda}}\cos\theta - \hat{\lambda}\sin\theta\,\dot{\theta}\right) \ldots\ldots\right.$$

messy trig

Easy way:

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega}_{\beta/\mathcal{F}} = \ddot{\phi}\,\hat{k} + \dot{\phi}\,\cancel{\dot{\hat{k}}}^{\,0} + \ddot{\theta}\,\hat{e}_t + \dot{\theta}\,\dot{\hat{e}}_t + \ddot{\psi}\,\hat{n} + \dot{\psi}\,\dot{\hat{n}}$$

$$\dot{\hat{e}}_t = \dot{\phi}\,\hat{k} \times \hat{e}_t = -\dot{\phi}\,\hat{e}_n = -\dot{\phi}\left(\cos\theta\,\hat{n} + \sin\theta\,\hat{\lambda}\right)$$

$\hat{e}_n = \cos\theta\,\hat{n} + \sin\theta\,\hat{\lambda}$

$$\dot{\hat{n}} = \vec{\omega}_{\tau/\mathcal{F}} \times \hat{n}$$

$\vec{\omega}_{\tau/\mathcal{F}} = \dot{\theta}\,\hat{e}_t + \dot{\phi}\,\hat{k}$

$$= -\dot{\theta}\,\hat{\lambda} + \dot{\phi}\cos\theta\,\hat{e}_t$$

$$\Rightarrow \vec{\alpha} = \vec{\alpha}_{\beta/\mathcal{F}}\left(\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}, \hat{\lambda}, \hat{n}, \hat{e}_t\right)$$

AMB/c

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

$$\vec{r}_{G/c} \times (-mg\hat{k}) = \vec{r}_{G/c} \times m\,\vec{a}_G + \underline{\underline{I}} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \left(\underline{\underline{I}} \cdot \vec{\omega}\right)$$

$\vec{r}_{G/c} = r\hat{\lambda}$

$\underline{\underline{I}} = I_t\,\hat{e}_t\hat{e}_t + I_n\,\hat{e}_n\hat{e}_n + I_\lambda\,\hat{e}_\lambda\hat{e}_\lambda$

where $I_n = \frac{1}{2}mr^2$

$I_\lambda = I_t = \frac{1}{4}mr^2$

NEED TO FIND

Method: Laurie Anderson

"let x = x"

Rolling without slip

$$\vec{v}_c = \vec{v}_c$$

$$\vec{0} = \vec{v}_G + v_{c/G}$$

$$\vec{v}_G = -\vec{v}_{c/G}$$

$$= \vec{v}_{G/c}$$

$$= \vec{\omega} \times \vec{r}_{G/c}$$

$r\hat{\lambda}$

$\vec{\omega}_{\beta/\mathcal{F}}$ ✓

$$\vec{v}_G = f(\underbrace{\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}}, \ \& \ \underbrace{\text{base vectors}})$$

$$\vec{a}_G = \frac{d}{dt}(\vec{v}_G) = f(\downarrow, \ddot{\theta}, \ddot{\phi}, \ddot{\psi}, \downarrow)$$

$AMB_{/c}$ $\Rightarrow$ vector eqn with all terms expressed in terms of Euler angles, rates and 2nd derivatives

$\Rightarrow$ can solve for $\ddot{\theta}, \ddot{\phi}, \ddot{\psi}$ in terms of state

ODEs $\rightarrow$ Solve!

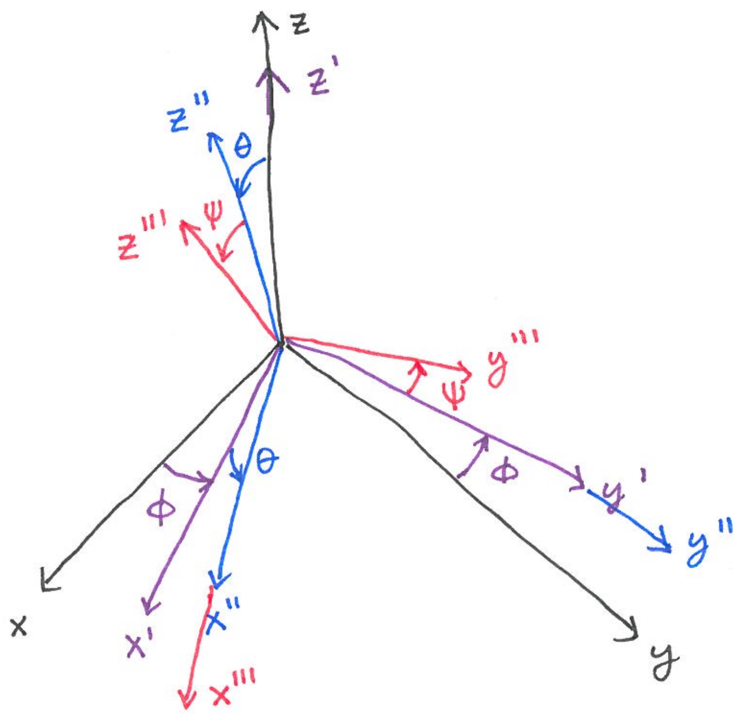
$\boxed{4/15/2014}$

Euler Angles (with matrices)

Consider 3 2 1 Euler angles

- 1 → ψ then newest x-axis, $\hat{e}_1''$
- 2 → θ then new y-axis, $\hat{e}_2'$
- 3 → ϕ rotate about z-axis, $\hat{e}_3$

[**ASIDE**: most common Euler angles 3-1-3]



$$\underline{\underline{R}} = \underbrace{\overbrace{\underline{\underline{R}}(\hat{e}_1'', \psi)}^{\underline{\underline{R}}_3}}_{\text{rotation about newest x-axis}} \cdot \underbrace{\overbrace{\underline{\underline{R}}(\hat{e}_2', \theta)}^{\underline{\underline{R}}_2}}_{\text{rotation about new y-axis}} \cdot \underbrace{\overbrace{\underline{\underline{R}}(\hat{e}_3, \phi)}^{\underline{\underline{R}}_1}}_{\text{rotation about z-axis}}$$

$\hat{e}_1'' = \underline{\underline{R}}_2 \cdot \underline{\underline{R}}_1 \hat{e}_1$

$\hat{e}_2' = \underline{\underline{R}}_1 \hat{e}_2$

[**Recall**

$$\underline{\underline{R}}(\hat{n}, \beta) = \cos\beta \, \underline{\underline{1}} + (1-\cos\beta)\hat{n}\hat{n} + \sin\beta \, \underline{\underline{\epsilon}}(\hat{n})$$
]

$$\Rightarrow \underline{\underline{R}} = \underline{\underline{R}}(\phi, \theta, \psi)$$

Alternative Approach:

Use gimbals to see

$$\underline{\underline{R}} = \underline{\underline{R}}(\hat{e}_3, \phi) \cdot \underline{\underline{R}}(\hat{e}_2, \theta) \cdot \underline{\underline{R}}(\hat{e}_1, \psi)$$

no primes, easier to calculate

ASIDE

$$\left[\underline{\underline{R}}(\hat{e}_3, \phi)\right]_{\mathcal{F}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What about dynamics?

AMB/G

$$\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

$$= \underline{\underline{I}}\,\underset{\dot{\vec{\omega}}}{\vec{\alpha}} + \vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega})$$

$$\vec{\alpha} = \underline{\underline{I}}^{-1}\left[\sum \vec{M}_{/G} \; \vec{\omega} \times \underline{\underline{I}} \cdot \vec{\omega}\right] \quad ** $$

Knowing $\underline{\underline{R}} \Rightarrow \underline{\underline{I}} = \underbrace{\underline{\underline{R}}\,\underline{\underline{I}}^{ref}\,\underline{\underline{R}}^T}_{?}$

$$\vec{\omega} = \dot{\phi}\,\hat{e}_3 + \dot{\theta}\,\hat{e}_2{}' + \dot{\psi}\,\hat{e}_1{}''$$

$\hat{e}_1{}'' = \underline{\underline{R}}(\hat{e}_3, \phi) \cdot \underline{\underline{R}}(\hat{e}_2, \theta)\,\hat{e}_1$

$\hat{e}_2{}' = \underline{\underline{R}}(\hat{e}_3, \phi) \cdot \hat{e}_2$

$$[\vec{\omega}]_{\mathcal{F}} = \underbrace{\Big[\,[\hat{e}_3]_{\mathcal{F}} \;\Big|\; [\hat{e}_2{}']_{\mathcal{F}} \;\Big|\; [\hat{e}_1{}'']_{\mathcal{F}}\,\Big]}_{A(\phi,\, \theta,\, \psi)} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad *$$

How to get ODEs for $\ddot{\phi}, \ddot{\theta}, \ddot{\psi}$

ASIDE $[\;]$ indicates $[\;]_{\mathcal{F}}$

Method ①

Differentiate *

$$\Rightarrow \left[\dot{\vec{\omega}}\right] = \dot{A}\,\dot{\Phi} + A\,\ddot{\Phi}$$

** (AMB)

$$\Phi = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

$$\Rightarrow \ddot{\Phi} = A^{-1}\left[\vec{\alpha} - \underbrace{\dot{A}\,\dot{\Phi}}_{\text{quadratic in angular rates}}\right]$$

AMB

Method ②

Lagrange Eqns

$* \Rightarrow \underline{\omega} \cdot \underline{\underline{I}} \cdot \vec{\omega}$

$$\Rightarrow E_K = E_K(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})$$

$\Rightarrow \mathcal{L} = E_K$ (free motion)

$\Rightarrow$ Lagrange eqns $\Rightarrow \ddot{\phi}, \ddot{\theta}, \ddot{\psi}$

H.W.:

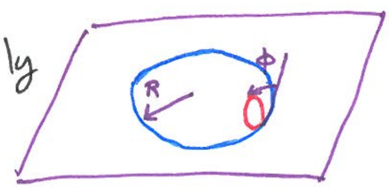
① Do free motion again the 2 ways from today's (4/15/2014) lecture. Check that all 3 give same motion. (Animation, plots, etc.)

Estimated work: 8-16 hrs

Final Project:

① Rolling disk by elementary methods (circles on circles, simple precession)

2 parameter family of solutions (R, Φ)



② Same for frictionless disk

2 parameter family of solutions

(tip angle, spin rate, precession rate) ← pick 2, solve for third

③ Find the common subset of ① + ②

④ Find general motion of the rolling disc.

As many test cases as possible.

e.g. check against ① above

check conservation of energy

roll in straight line will small perturbation

(Critical speed to not fall down

⋮

etc

⑤ Extra Credit Derive ④ a different way

e.g. Rot matrix + constraints, Lag eqns + constraints,

Euler angles with matrices

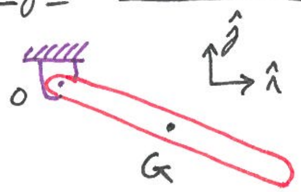
4/29/2014

Today

Single pendulum

Double pendulum

Single Pendulum: 2D Review



$AMB_{/O}$

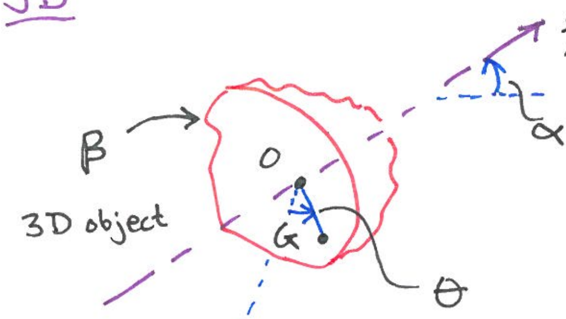
$$\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}$$

$$\vec{r}_{G/O} \times -mg\hat{\jmath} = \vec{r}_{G/O} \times m\vec{a}_G + I\ddot{\theta}\hat{k}$$

$$\ddot{\theta}\hat{k} \times \vec{r}_{G/O} + -\omega^2 \vec{r}_{G/O}$$

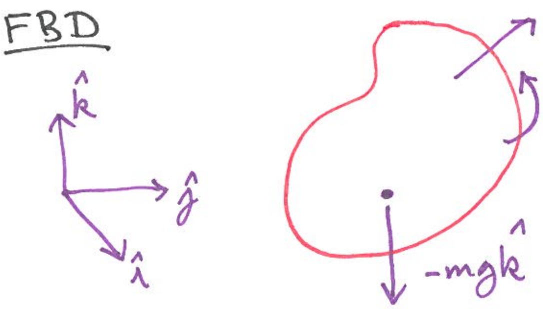
$$\Rightarrow \boxed{\ddot{\theta} + \frac{mgd}{I + md^2}\sin\theta = 0}$$

3D



$\hat{\lambda}$ = arbitrary fixed axis direction

FBD



$\vec{F}$ = from hinge on object

$\vec{M}$ = moment of hinge on object with $\vec{M} \cdot \hat{\lambda} = 0$

AMB & LMB $\Rightarrow$ 6 scalar eqns for 5 components of $\vec{F}$ & $\vec{M}$ & $\ddot{\theta}$

AMB about axis $\hat{\lambda}$

$$\left\{ \Sigma \vec{M}_{/O} = \dot{\vec{H}}_{/O} \right\} \cdot \hat{\lambda} \; *$$

<u>Given</u>: $\overbrace{\underline{\underline{I}}^{ref}, m, \vec{r}_{G/O}^{\;ref}}^{P}, \hat{\lambda}$

pretend to know $(\theta, \dot{\theta})$

$$① \quad \left\{ \vec{r}_{G/O} \times (-mg\,\hat{k}) = \vec{r}_{G/O} \times m\vec{a}_G + \vec{\omega} \times \underline{\underline{I}} \cdot \vec{\omega} + \underline{\underline{I}} \cdot \dot{\vec{\omega}} \right\} \cdot \hat{\lambda}$$

Evaluate all of above in terms of $P, \theta, \dot{\theta}, \ddot{\theta}$ $\Rightarrow$ Eqns of Motion

<u>Figure out terms</u>:

$$\vec{r}_{G/O} = \underline{\underline{R}} \cdot \vec{r}_{G/O}^{\;ref}$$

L $\underline{\underline{R}}\,(\hat{\lambda}, \theta) = \ldots$

$$\vec{\omega} = \dot{\theta}\,\hat{\lambda}, \quad \dot{\vec{\omega}} = \ddot{\theta}\,\hat{\lambda}$$

↑ $\vec{\omega}_{B/\mathcal{F}}$

$$\vec{a}_G = \dot{\vec{\omega}} \times \vec{r}_{G/O} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{G/O})$$

$$\underline{\underline{I}} = \underline{\underline{R}}\, \underline{\underline{I}}^{ref}\, \underline{\underline{R}}^T$$

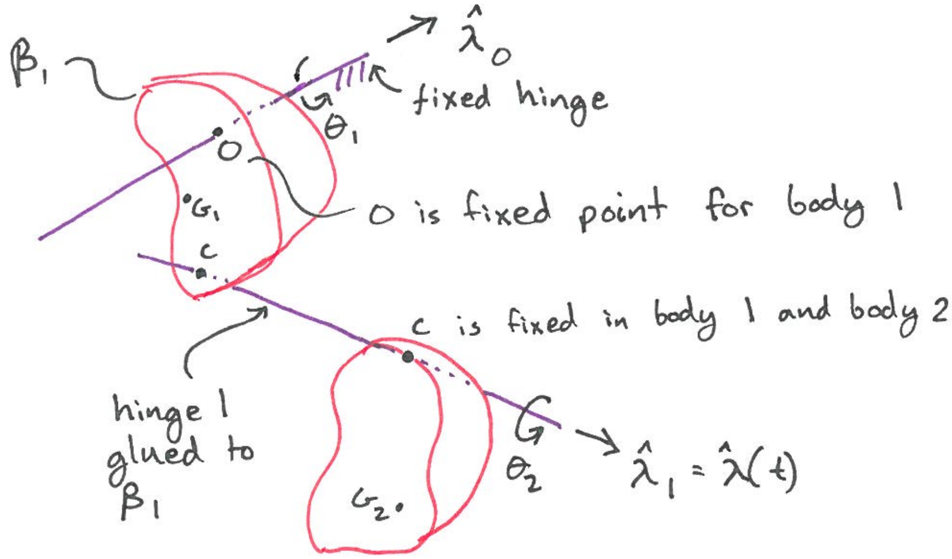
① is one scalar eqn for $\ddot{\theta}$

<u>Shortcut</u>:

$$\left.\begin{aligned} \vec{M}_{/O} \cdot \hat{\lambda} &= -\cos\alpha \; mgd \sin\theta \\ \dot{\vec{H}}_{/O} \cdot \hat{\lambda} &= (I_{\lambda\lambda} + d^2 m)\,\ddot{\theta} \end{aligned}\right\} \Rightarrow \boxed{\ddot{\theta} + \frac{mgd\cos\alpha}{I_{\lambda\lambda} + md^2} \sin\theta = 0}$$

L $I_{\lambda\lambda} = \hat{\lambda} \cdot \underline{\underline{I}} \cdot \hat{\lambda}$

Double Pendulum in 3D

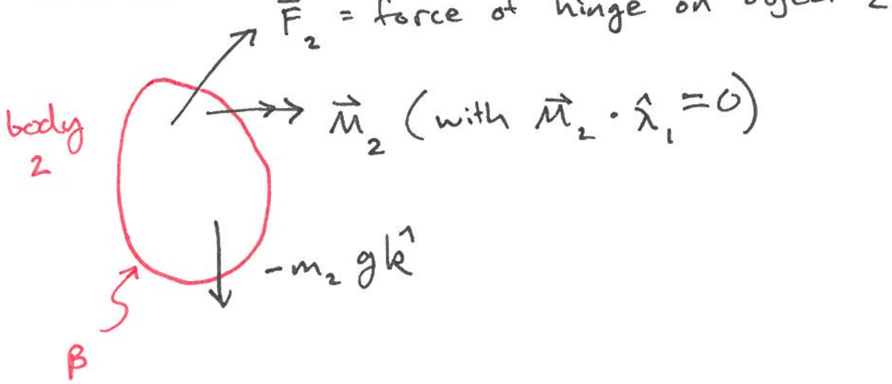


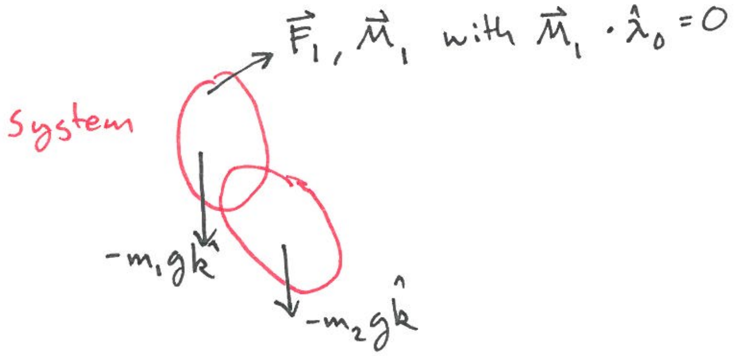
Minimal coordinates θ_1, θ_2

<u>Given</u>: $\underline{\underline{I}}_1^{ref}$, $\underline{\underline{I}}_2^{ref}$, $\vec{r}_{G/O}^{\,ref}$, m_1, m_2, $\vec{r}_{C/O}^{\,ref}$, $\vec{r}_{G_2/C}^{\,ref}$, $\hat{\lambda}_0$, $\hat{\lambda}_1^{ref}$ $\Big\}$ P = parameters

given: $P, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2$ want: $\ddot{\theta}_1, \ddot{\theta}_2$

<u>FBDs</u>





System: $\left\{ \sum \vec{M}_{/0} = \dot{\vec{H}}_{/0} \right\} \cdot \hat{\lambda}_0$ (2)

Object 2: $\left\{ \sum \vec{M}_{/c} = \dot{\vec{H}}_{/c} \right\} \cdot \hat{\lambda}_1(t)$ (1)

(1) & (2) are 2 eqns for $\ddot{\theta}_1$, $\ddot{\theta}_2$

For Body 2:

$$\sum \vec{M}_{/c} = \vec{r}_{G2/c} \times (-mg\hat{k})$$

$$\underline{\underline{R}}_2 \cdot \vec{r}^{\,ref}_{G2/c}$$

$$\underline{\underline{R}}_2 = \underline{\underline{R}}(\hat{\lambda}_1, \theta_2)$$

$$\underline{\underline{R}}_1 \cdot \hat{\lambda}_1^{ref}$$

$$\underline{\underline{R}}_1 = \underline{\underline{R}}(\hat{\lambda}_0, \theta_1)$$

For System:

$$\sum \vec{M}_{/0} = \vec{r}_{G1/0} \times -m_1 g\hat{k} + \vec{r}_{G2/0} \times -m_2 g\hat{k}$$

$$\vec{r}_{c/0} + \vec{r}_{G2/c}$$

$$\underline{\underline{R}}_1 \cdot \vec{r}^{\,ref}_{c/0}$$

Body 2:

$$\dot{\vec{H}}_{/c} = \vec{r}_{G2/c} \times m_2 \vec{a}_{G2} + \underline{\underline{I}}_2 \cdot \vec{\alpha}_2 + \vec{\omega}_2 \times \left(\underline{\underline{I}}_2 \cdot \vec{\omega}_2\right)$$

$$\vec{\omega}_2 = \vec{\omega}_{\beta_2/\mathcal{F}} = \vec{\omega}_{\beta_1/\mathcal{F}} + \vec{\omega}_{\beta_2/\beta_1} = \underbrace{\dot{\theta}_1 \hat{\lambda}_0}_{\vec{\omega}_1} + \dot{\theta}_2 \hat{\lambda}_1$$

$$\vec{\alpha}_2 = \frac{d^{\mathcal{F}}}{dt}\left(\vec{\omega}_{\beta_2/\mathcal{F}}\right) = \ddot{\theta}_1 \hat{\lambda}_0 + \underbrace{\frac{d^{\beta_1}}{dt}\left(\vec{\omega}_{\beta_2/\beta_1}\right) + \vec{\omega}_{\beta_1/\mathcal{F}} \times \vec{\omega}_{\beta_2/\beta_1}}_{\text{"}\dot{Q}\text{"}}$$

$$\vec{\alpha}_2 = \ddot{\theta}_1 \hat{\lambda}_0 + \ddot{\theta}_2 \hat{\lambda}_1 + \dot{\theta}_1 \dot{\theta}_2 \hat{\lambda}_0 \times \hat{\lambda}_1$$

$$\underline{\underline{I}}_2 = \underline{\underline{R}}_2 \cdot \underline{\underline{I}}_2^{ref} \cdot \underline{\underline{R}}_2^T$$

$$\underline{\underline{I}}_1 = \underline{\underline{R}}_1 \cdot \underline{\underline{I}}_1^{ref} \cdot \underline{\underline{R}}_1^T$$

System:

$$\dot{\vec{H}}_{/0} = \vec{r}_{G1/0} \times m_1 \vec{a}_{G1} + \underline{\underline{I}}_1 \cdot \vec{\alpha}_1 + \vec{\omega}_1 \times (\underline{\underline{I}}_1 \cdot \vec{\omega}_1)$$

$$+ \vec{r}_{G2/0} \times m_2 \vec{a}_{G2} + \underline{\underline{I}}_2 \cdot \vec{\alpha}_2 + \vec{\omega}_2 \times (\underline{\underline{I}}_2 \cdot \vec{\omega}_2)$$

$$\vec{a}_{G1} = \vec{\alpha}_1 \times \vec{r}_{G1/0} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{G1/0})$$

$$\vec{a}_{G2} = \vec{a}_c + \vec{a}_{G2/c}$$

$$\vec{a}_{G2} = (\vec{\alpha}_1 \times \underbrace{\vec{r}_{c/0}}_{\underline{\underline{R}}_1 \cdot \vec{r}_{c/0}^{\,ref}} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_{c/0})) + (\vec{\alpha}_2 \times \vec{r}_{G2/c} + \underbrace{\vec{\omega}_2}_{\vec{\omega}_{\beta_2/\beta_1} + \vec{\omega}_{\beta_1/\mathcal{F}}} \times (\vec{\omega}_2 \times \vec{r}_{G2/c}))$$

Now have 2 eqns that are linear in $\ddot{\theta}_1$, $\ddot{\theta}_2$, $m_1 g$, $m_2 g$
& have complicated non-linear terms with θ_1, θ_2, $\dot{\theta}_1^2$, $\dot{\theta}_1 \dot{\theta}_2$, $\dot{\theta}_2^2$

$$\underset{2\times2}{[M]} \underset{2\times1}{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}} = \underset{2\times1}{[F]} + \underset{2\times1}{[\dot{\theta}^2 \text{ stuff}]}$$

How to find M?

1. Use Jacobian command
2. Be smarter: Books by Craig + Featherstone about arranging terms to solve for M directly
3. Do all calculations with numbers, not symbols, 3 times

"Probe method" "Andy's dumb method"

once: $\ddot{\theta}_1 = \ddot{\theta}_2 = 0$ ①
once: $\ddot{\theta}_1 = 1, \ddot{\theta}_2 = 0$ ② $\Rightarrow$ ② − ① $\Rightarrow$ 1^{st} col of M
once: $\ddot{\theta}_1 = 0, \ddot{\theta}_2 = 1$ ③ $\Rightarrow$ ③ − ① $\Rightarrow$ 2^{nd} col of M

5/8/2014

Angular momentum

Start with $\vec{F} = m\vec{a}$

$$\Rightarrow \vec{F}_i = m_i \vec{a}_i$$

$$\sum \vec{F}_i = \sum m_i \vec{a}_i$$

$\vec{0}$ assumption ⓐ

$$\sum \vec{F}_i^{ext} + \cancel{\sum \vec{F}_i^{int}} = \sum m_i \vec{a}_i$$

$$\boxed{\sum \vec{F}^{ext} = \frac{d}{dt}\sum (m_i \vec{v}_i) = m_{tot}\vec{a}_i}$$

LMB ⓑ

Linear Momentum

Postulates ⓐ or ⓑ

ⓒ or ⓓ

Ang Mom.

$$\sum \vec{r}_{i/c} \times \vec{F}_i = \sum \vec{r}_{i/c} \times m_i \vec{a}$$

$\vec{F}_i^{int} + \vec{F}_i^{ext}$

$\vec{0}$ by assumption ⓒ

$$\sum \vec{r}_{i/c} \times \vec{F}_i^{ext} + \cancel{\sum \vec{r}_{i/c} \times \vec{F}_k^{int}} = \sum \vec{r}_{j/c} \times m_j \vec{a}_j$$

$$\boxed{\sum \vec{M}_{/c} = \underbrace{\sum \vec{r}_{j/c} \times m_j \vec{a}_j}_{J}}$$

AMB ⓓ

Candidate definitions of $\vec{H}_{/c}$

A. $\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{v}_{i/c}$

B. $\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{v}_{i/\mathcal{F}}$

Definition A

$$\frac{d}{dt}\left(\vec{H}_{/c}\right) \overset{?}{=} J$$

NOTE: o is a fixed point

$$\frac{d}{dt}\{A\} \Rightarrow \frac{d\vec{H}_{/c}}{dt} = \frac{d}{dt}\sum\left(\vec{r}_{i/o} - \vec{r}_{c/o}\right) \times m_i\left(\vec{v}_{i/o} - \vec{v}_{c/o}\right)$$

$$= \sum\left[\underbrace{(\vec{v}_{i/o} - \vec{v}_{c/o}) \times m_i(\vec{v}_{i/o} - \vec{v}_{c/o})}_{\vec{0}} + \vec{r}_{i/c} \times m_i(\vec{a}_i - \vec{a}_c)\right]$$

$$= \sum\left(\vec{r}_{i/c} \times m_i\vec{a}_i + \vec{r}_{i/c} \times m_i\vec{a}_c\right)$$

$$= \underbrace{\sum \vec{r}_{i/c} \times m_i\vec{a}_i}_{J} - \underbrace{\vec{r}_{G/c} \times m_{TOT}\vec{a}_c}_{=\vec{0}\ ?}$$

$$\boxed{\begin{array}{ll} \text{def A} & \dfrac{d\vec{H}_{/c}}{dt} = J \quad \text{if } \vec{a}_c = \vec{0} \\ & \qquad \text{or if } c = G \\ & \qquad \text{or if } \vec{a}_c \parallel \vec{r}_{G/c} \end{array}}$$

Extra Results

If $G = C$ (C.O.M.)

$$J = \sum \vec{r}_{i/G} \times m_i\vec{a}_i$$

$\vec{a}_i$: $\vec{a}_G + \vec{a}_{i/G}$

$\vec{r}_{i/G}$: $\vec{r}_{i/G}$

$$= \underbrace{\sum \vec{r}_{i/G} \times m_i\vec{a}_G}_{=\left(\underbrace{\sum \vec{r}_{i/G}m_i}_{\vec{0}}\right) \times \vec{a}_G} + \sum \vec{r}_{i/o} \times m_i\vec{a}_{i/G}$$

$c=G$

$$\boxed{J = \sum \vec{r}_{i/G} \times m_i\vec{a}_{i/G}}$$

COM equation

Side Story

def A: $\vec{H}_{/c} = \sum \vec{r}_{i/c} \times m_i \vec{v}_{i/c}$

$\vec{r}_{i/c} = \vec{r}_{i/G} + \vec{r}_{G/c}$, $\quad \vec{v}_{i/c} = \vec{v}_{i/G} + \vec{v}_{G/c}$

$$= \sum \vec{r}_{i/G} \times \vec{v}_{i/G} m_i + \cancelto{\vec{0}}{\sum \vec{r}_{i/G} \times \vec{v}_{G/c} m_i} + \cancelto{\vec{0}}{\sum \vec{r}_{G/c} \times \vec{v}_{i/G} m_i} + \sum \vec{r}_{G/c} \times \vec{v}_{G/c} m_i$$

$$\vec{H}_{/c} = \sum \vec{r}_{i/G} \times (m_i \vec{v}_i) + m_{TOT} \vec{r}_{G/i} \times \vec{v}_{G/c}$$

$$= \vec{H}_{/G} + \text{"} \vec{H}_{G/c} \text{"}$$

$$\boxed{\text{ASIDE: for rigid objects} \quad \vec{H}_{/G} = \underline{\underline{I}}^G \vec{\omega}}$$

For rigid object:

def A $\Rightarrow$ $\vec{H}_{/c} = \underline{\underline{I}}^G \cdot \vec{\omega} + \vec{r}_{G/c} \times m_{TOT} \vec{v}_{G/c}$

$$\Rightarrow J = \frac{d}{dt}\left(\underline{\underline{I}}^G \vec{\omega}\right) + \vec{0} + \vec{r}_{G/c} \times m_{TOT} \vec{a}_{G/c}$$

$$= \underline{\underline{I}} \, \underbrace{{}^{\beta}\dot{\vec{\omega}}} + \vec{\omega} \times (\underline{\underline{I}} \cdot \vec{\omega}) + \vec{r}_{G/c} \times m_{TOT} \vec{a}_G$$

$${}^{F}\dot{\vec{\omega}} = {}^{\beta}\dot{\vec{\omega}} + \cancelto{\vec{0}}{\vec{\omega} \times \vec{\omega}} = {}^{\beta}\dot{\vec{\omega}}$$

$$= \underline{\underline{I}} \, {}^{F}\dot{\vec{\omega}} + \vec{\omega} \times \underline{\underline{I}} \vec{\omega} + \vec{r}_{G/c} \times m \vec{a}_G = \sum \vec{r}_{i/c} \times m_i \vec{a}_i$$

$$= \frac{d\vec{H}_{/c}}{dt} \quad \text{if} \quad c \text{ is fixed or}$$

$\vec{a}_c = 0$ or

$c = G$ or

$\vec{a}_G \parallel \vec{r}_{G/c}$

$\boxed{5/13/2014}$

Vibes

Emphasis on Continuous Systems

Review of discrete systems

A. Equations of motion & linearize, or

B. Lagrange Equations

$$E_p = \frac{1}{2} q_i \underbrace{k_{ij}}_{\text{const}} q_j$$

* 0^{th} order term irrelevant
* 1^{st} order term = 0 because equilibrium at $q = 0$ (by assumption) $\vec{q} = \vec{0}$ solutions
* K is positive semi-definite, symmetric (stability)
* small motions so keep higher order terms $\Rightarrow q^3, q^4$ etc

$$E_K = \frac{1}{2} \dot{q}_i \overset{\text{const}}{M_{ij}} \dot{q}_j$$

M is positive definite

Symmetric

$$\mathcal{L} = E_K - E_P$$

Lag. Eqn.: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_K} - \frac{\partial \mathcal{L}}{\partial q_K} = 0$

Look at

$$\frac{\partial}{\partial q_K} \frac{1}{2}(q_i K_{ij} q_j) = \frac{1}{2}\left(\frac{\partial q_i}{\partial q_K} K_{ij} q_i\right) + \frac{1}{2} q_i K_{ij} \frac{\partial q_i}{\partial q_K}$$

$$= \frac{1}{2}\left[\delta_{iK} K_{ij} q_j + q_i K_{ij} \delta_{jk}\right]$$

$$= \frac{1}{2}\left[K_{Kj} q_j + q_i K_{iK}\right]$$

$$= \frac{1}{2}\left[K_{Kj} q_j + K_{Ki} q_i\right]$$

$= K_{kj}\, q_j$

Likewise with $\dot{q}_i M_{ij} \dot{q}_j$

$\Rightarrow$ Lag. Eqns $\Rightarrow$ $M_{kj}\ddot{q}_j + K_{kj} q_j = 0$

$$\boxed{[M]\ddot{\vec{q}} + [K]\vec{q} = \vec{0}}$$

canonical undamped governing eqns. Given * can calculate normal modes etc

---

Continuous Systems

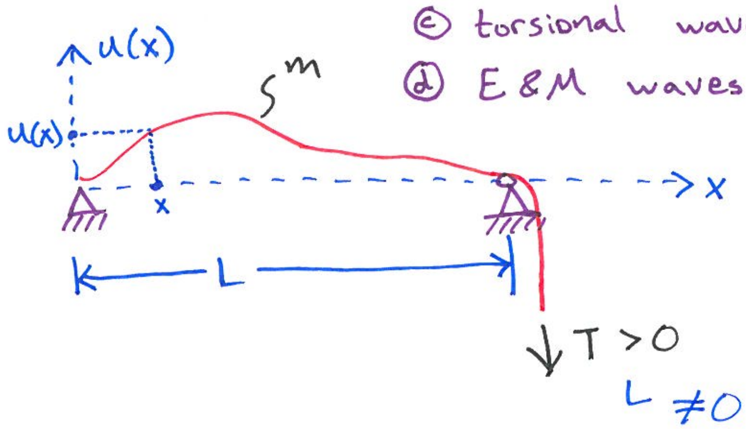
A. PDEs & their solutions
1. D'Alambert (only for the pure wave eqn)
2. Separation of variables (mostly used in vibes)

B. Discretize
1. Finite elements / Finite difference
2. Assumed (modal) shapes

PDEs: follow from LMB, AMB

ex) String Eqn

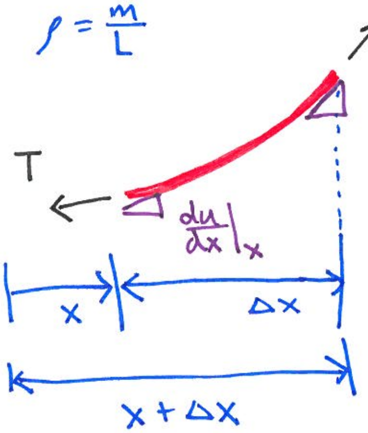
(same eqn as ⓐ rod, gas in tube,
ⓑ shear wave
ⓒ torsional waves (round bars)
ⓓ E&M waves



Method of Δx

$\rho = \frac{m}{L}$

T = const along length



$$\left.\frac{du}{\partial x}\right|_{x+\Delta x} \approx \left.\frac{du}{dx}\right|_{x} + \left.\frac{d}{dx}\frac{du}{dx}\right|_{x} \Delta x$$

NOTE: $\ddot{u}$ is a property of the whole segment

$\{LMB\} \cdot \hat{j}$

$$\sum F_y = \rho \Delta x \ddot{u}$$

$$-T\left.\frac{du}{\partial x}\right|_x + \underbrace{T\left.\frac{du}{\partial x}\right|_{x+\Delta x}}_{T\left.\frac{du}{dx}\right|_x + \left.\frac{\partial^2 u}{\partial x^2}\right|_x \Delta x} = \rho\ddot{u}\,\Delta x$$

$$\frac{d^2 u}{dx^2} T \cancel{\Delta x} = \rho\ddot{u}\,\cancel{\Delta x}$$

$$\boxed{\begin{gathered}\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} \\ \ddot{u} = c^2 u_{xx} \\ c \equiv T/\rho\end{gathered}}$$

The wave equation

The simple wave eqn.

The non-dispersive wave eqn.

ASIDE: dispersive $\equiv$ wave speed depends on frequency

D'Alamdert Solution (REALLY FAMOUS)

consider candidate solution of form $u(x,t) = f(x - ct)$

$$\frac{du}{\partial x} = f', \quad \frac{\partial^2 u}{\partial x^2} = f'', \quad \frac{\partial u}{\partial t} = -cf', \quad \frac{\partial^2 u}{\partial t^2} = c^2 f''$$

Plug into wave equation:

$$u_{tt} = c^2 u_{xx}$$

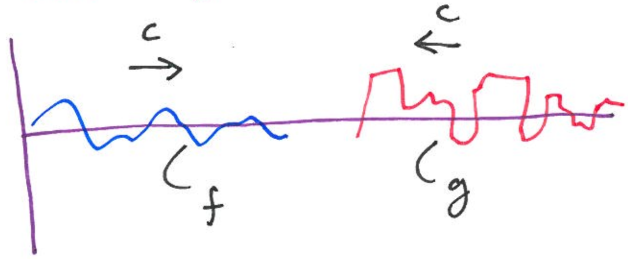
$$c^2 f'' = c^2 f''$$

for any f

Likewise for $g(x+ct)$

General solution of wave eqn is $\boxed{u(x,t) = f(x-ct) + g(x+ct)}$

sum of left-going and right-going waves with arbitrary shapes



② Separation of variables

ASSUME $u = X(x)\,T(t)$

plug in guess $\boxed{u_{tt} = c^2 u_{xx}}$

Intuition: acceleration is proportional to curvature
* pulled up, accelerates up
* pulled down, accelerates down
* higher tension, higher acceleration
* lower mass, higher acceleration

$$\frac{\partial^2}{\partial t^2}\left(X(x)\,T(t)\right) \overset{?}{=} c^2 \frac{\partial^2 X(x)\,T(t)}{\partial x^2}$$

$$X \ddot{T} \overset{?}{=} c^2 X'' T$$

$$\boxed{\frac{X''}{X} = \frac{\ddot{T}}{c^2 T}}$$

define

$$f(x) \equiv \frac{X''}{X} \quad , \quad g(t) = \frac{\ddot{T}}{c^2 T}$$

$f(x) = g(t)$ for all x + t

$$\Rightarrow \quad f(x) = g(t) = C = \text{const} = -\lambda^2$$

[$+\lambda^2$ would give solutions, but we don't like them]

$$X'' + \lambda^2 X = 0 \qquad \ddot{T} + \lambda^2 c^2 T = 0$$

$$X(x) = A \sin(\lambda x) + B \cos(\lambda x) \quad \text{for all } \lambda < 0$$

$$T = C \sin(\lambda c t) + D \cos(\lambda c t) \quad \text{for all } \lambda < 0$$

Solution of $u_{tt} = c^2 u_{xx}$

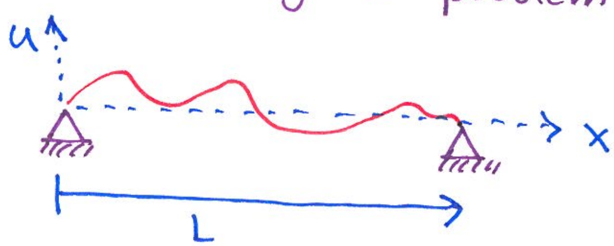
$$u(x,t) = \overbrace{\left(A\cos(\lambda x) + B\sin(\lambda x)\right)}^{X} \cdot \overbrace{\left(C\cos(\lambda c t) + D\sin(\lambda c t)\right)}^{T} \quad *$$

e.g. a) standing wave $\boxed{u = \sin(\lambda x)\sin(\lambda c t)}$

b) traveling wave $u = \cos(\lambda(x - ct))$

main soln used in vibrations

Back to original problem



NOTE: $T > 0$

Boundary Conditions: $u(0) = 0$

$u(L) = 0$

Find all solutions to satisfy boundary conditions

$$* + \text{B.C} \Rightarrow A\cos(\lambda x) + B\sin(\lambda x)\Big|_{0 \text{ and } L} = 0$$

$\Rightarrow \quad A = 0$

$$\Rightarrow \quad \sin\underbrace{(\lambda L)}_{=\,0,\ \pi,\ 2\pi,\ \ldots} = 0$$

$\lambda L = n\pi$, with $n = 1, 2, 3, \ldots$

$\lambda = n\pi/L$

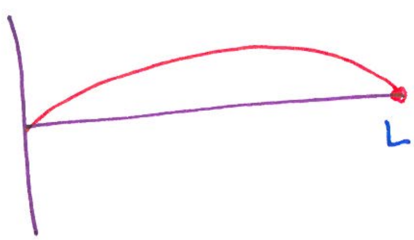
$$\text{Soln} \Rightarrow u(x,t) = \cancel{A}\sin\left(\frac{n\pi x}{L}\right)\left[C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)\right]$$

General solution from this

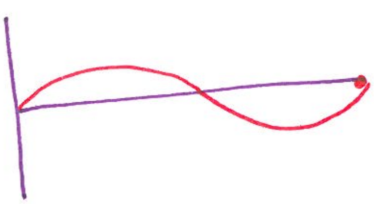
$$u(x,t) = \sum_{n=1}^{\infty}\left(C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Mode shapes n (e-vectors): $\sin\frac{n\pi x}{L} = v_n$

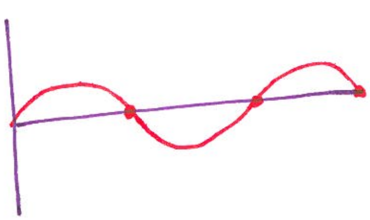
frequencies n: $\omega_n = \frac{n\pi c}{L}$ $\quad\boxed{\infty \text{ \# of modes}}$



mode1 $\quad \omega_1$

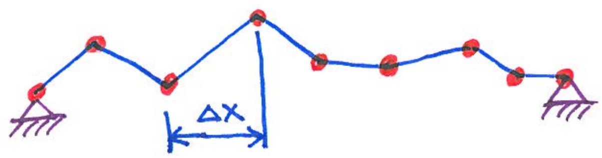


mode 2 $\quad 2\omega_1$

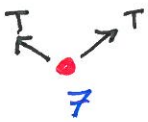


mode 3 $\quad 3\omega_1$

Discrete / Finite Element (Version #1)



Aside: Assume $T > 0$, T const throughout string



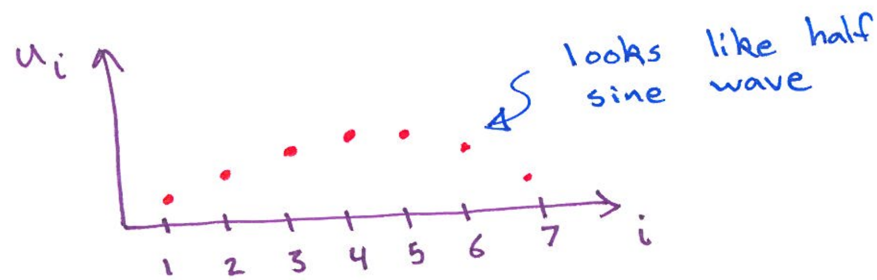
$$\text{LMB:} \quad m_7 \ddot{u}_7 = \frac{u_8 - u_7}{\Delta x} T - \frac{u_7 - u_6}{\Delta x} T$$

$$= \frac{u_8 - 2u_7 + u_6}{\Delta x} T$$

$$\begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & m_3 & \\ 0 & & & \ddots \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \vdots \end{bmatrix} + \frac{T}{\Delta x} \begin{bmatrix} \ddots & & & & & \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \vec{0}$$

Can calculate e-values and vectors

Lowest frequency e-vectors



Discrete Method (Version #2) (Garcia's preferred method)

B. Guess a mode shape for continuous problem

guess: $u(x,t) = g(t)\, f(x)$

can calculate $E_K, E_p = \underbrace{\int \frac{1}{2} T u'^2 \, dx}_{\approx T \cdot \Delta \text{arc length}}$

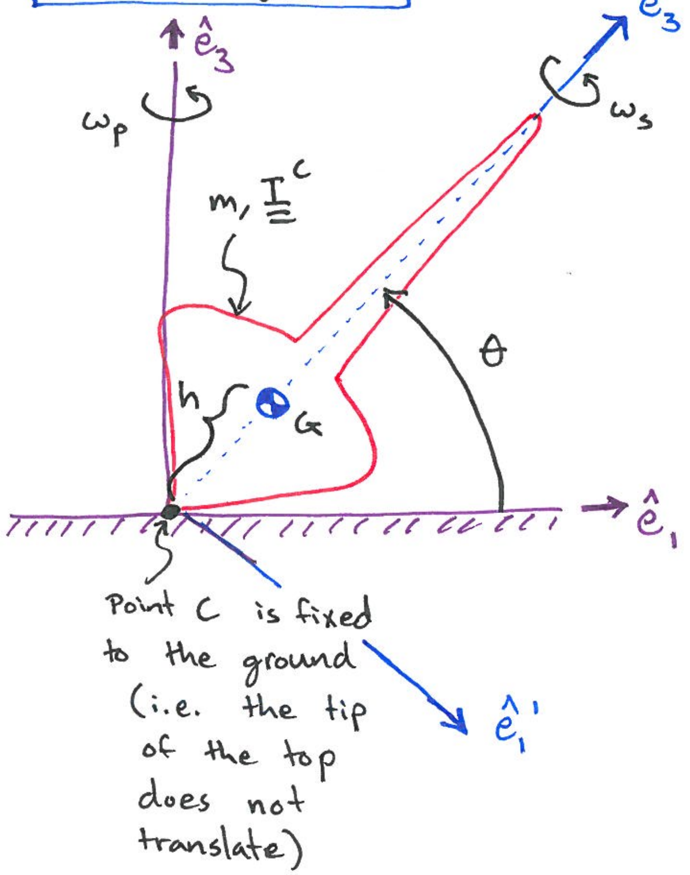
$\Rightarrow$ freq $= \frac{\pi c}{L}$

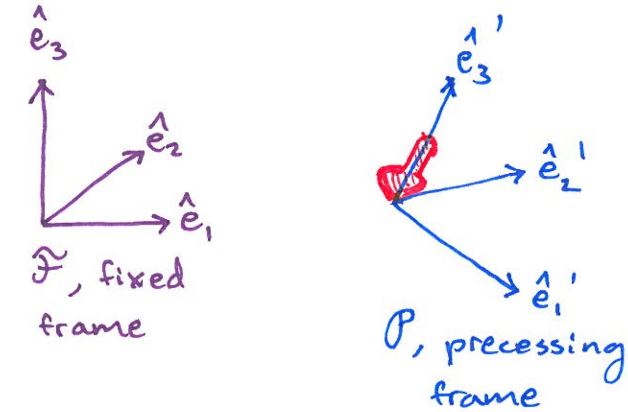
Beams

$$u_{tt} = \frac{1}{\rho} EI \, u_{xxxx}$$

Bryan Peele
MAE 6700
4/22/2014

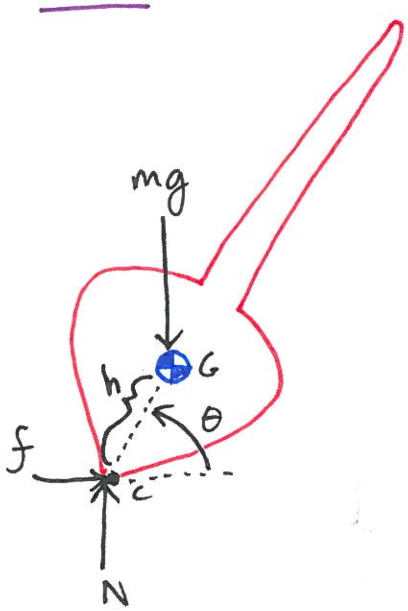
$\boxed{\text{Precessing Top}}$





NOTE: $\hat{e}_1{}'$ is in the plane formed by $\hat{e}_3$ and $\hat{e}_3{}'$

FBD



AMB/c

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

where $\vec{H}_{/c} = \underline{\underline{I}}^c \vec{\omega}$

$$\sum \vec{M}_{/c} = \dot{\vec{H}}_{/c} = \underline{\underline{I}}^c \vec{\alpha} + \vec{\omega} \times \left(\underline{\underline{I}}_c \vec{\omega}\right) \quad (1)$$

where $\underline{\underline{I}}^c = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$ when represented in the body or precessing frame

NOTE: $I_1 = I_2$

Conditions for steady precession:

(kinematic constraints)

$$\left.\begin{array}{l} \theta = \text{const} \\ \omega_p = \text{const} \\ \omega_s = \text{const} \end{array}\right\} \Rightarrow 0 = \dot{\theta} = \ddot{\theta} = \dot{\omega}_p = \dot{\omega}_s$$

$$\Rightarrow \vec{\omega} = \omega_p \hat{e}_3 + \omega_s \hat{e}_3'$$

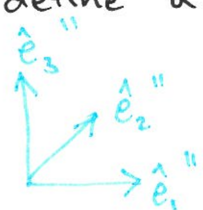
where $\hat{e}_3 = -\cos\theta\, \hat{e}_1' + \sin\theta\, \hat{e}_3'$

$$\vec{\omega} = \omega_p \left(-\cos\theta\, \hat{e}_1' + \sin\theta\, \hat{e}_3'\right) + \omega_s \hat{e}_3'$$

$$\boxed{\vec{\omega} = -\omega_p \cos\theta\, \hat{e}_1' + (\omega_s + \omega_p \sin\theta)\, \hat{e}_3'} \quad 2$$

Now, find $\vec{\alpha} = \dot{\vec{\omega}}$

first, define a new coordinate system in the precessing frame



such that
1. $\hat{e}_3'' = \hat{e}_3$
2. $\hat{e}_1''$ is in the plane formed by $\hat{e}_3$ & $\hat{e}_3'$
3. $\hat{e}_1''$ is in the plane formed by $\hat{e}_1$ & $\hat{e}_2$
4. $\hat{e}_2'' = \hat{e}_2'$

$$\Rightarrow \hat{e}_3' = \sin\theta\, \hat{e}_3 + \cos\theta\, \hat{e}_1''$$

$$\vec{\omega} = \omega_p \hat{e}_3 + \omega_s \hat{e}_3'$$

$$\vec{\omega} = \omega_p \hat{e}_3 + \omega_s \left(\sin\theta\, \hat{e}_3 + \cos\theta\, \hat{e}_1''\right)$$

$$\vec{\omega} = \omega_s \cos\theta\, \hat{e}_1'' + (\omega_p + \omega_s \sin\theta)\, \hat{e}_3$$

$$\Rightarrow \vec{\alpha} = \dot{\vec{\omega}} = \left[\cancelto{0}{\frac{d}{dt}(\omega_s \cos\theta)}\right] \hat{e}_1'' + \omega_s \cos\theta\, \dot{\hat{e}}_1'' + \left[\cancelto{0}{\frac{d}{dt}(\omega_p + \omega_s \sin\theta)}\right] \hat{e}_3$$

$$+ (\omega_p + \omega_s \sin\theta)\, \cancelto{0}{\dot{\hat{e}}_3}$$

$$\vec{\alpha} = \omega_s \cos\theta \, \dot{\hat{e}}_1''$$

where $\dot{\hat{e}}_1'' = \omega_p \hat{e}_2'' = \omega_p \hat{e}_2'$

$$\Rightarrow \boxed{\vec{\alpha} = \omega_s \omega_p \cos\theta \, \hat{e}_2'} \quad (3)$$

Back to Eqn (1)

$$\vec{\Sigma} M_{/c} = \underline{\underline{I}}^c \vec{\alpha} + \vec{\omega} \times (\underline{\underline{I}}^c \vec{\omega})$$

where $\vec{\Sigma} M_{/c} = \vec{r}_{G/c} \times -mg\hat{e}_3 = h\hat{e}_3' \times -mg\hat{e}_3 = mgh\cos\theta \, \hat{e}_2'$

$$mgh\cos\theta \, \hat{e}_2' = \underline{\underline{I}}^c \vec{\alpha} + \vec{\omega} \times (\underline{\underline{I}}^c \vec{\omega})$$

Substituting in (2) and (3)

$$mgh\cos\theta \, \hat{e}_2' = I_2 \omega_s \omega_p \cos\theta \, \hat{e}_2' + \vec{\omega} \times \underbrace{\left[-I_1 \omega_p \cos\theta \, \hat{e}_1' + I_3(\omega_s + \omega_p \sin\theta)\hat{e}_3'\right]}_{\star}$$

$$mgh\cos\theta \, \hat{e}_2' = I_2 \omega_s \omega_p \cos\theta \, \hat{e}_2' + \left[-\omega_p \cos\theta \, \hat{e}_1' + (\omega_s + \omega_p \sin\theta) \, \hat{e}_3'\right] \times [\star]$$

$$\left\{ mgh\cos\theta \, \hat{e}_2' = I_2 \omega_s \omega_p \cos\theta \, \hat{e}_2' + \left[I_3 \omega_p \cos\theta(\omega_s + \omega_p \sin\theta) - I_1 \omega_p \cos\theta (\omega_s + \omega_p \sin\theta)\right]\hat{e}_2' \right\}$$

$$\frac{\{\} \cdot \hat{e}_2'}{\cos\theta} \Rightarrow mgh = I_2 \omega_s \omega_p + I_3(\omega_p \omega_s + \omega_p^2 \sin\theta) - I_1(\omega_p \omega_s + \omega_p^2 \sin\theta)$$

where $I_1 = I_2$

$$mgh = I_1 \omega_s \omega_p + (I_3 - I_1)\omega_s \omega_p + (I_3 - I_1)\omega_p^2 \sin\theta$$

$$\boxed{mgh = I_3 \omega_s \omega_p + (I_3 - I_1)\omega_p^2 \sin\theta}$$