Intro-Rotation,1/22/13
Tuesday, January 22, 2013
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MAE 6700

- INTRO
- ROTATIONS

→ 3-D ROTATIONS

→ Lagrange EQUATIONS
WITH
- NON-CONSERVATIVE FORCES
- EXTRA CONSTRAINTS
→ NON-HOLONOMIC ?

→ CONSTRAINTS: DEAL WITH THEM

→ FRICTION + COLLISIONS

→ MULTI-BODY 3D DYNAMICS
KANE EQ's
FEATHERSTONE
TMT
} NOT TAUGHT BUT, MENTIONED
↳ COOK BOOK TO SET UP EQ's

→ HAMILTON'S PRINCIPLE

NOT HAMILTON'S EQUATIONS
↳ USE IN QUANTUM STUFF

→ FINDING "INTERESTING SOLUTIONS"
* USE ROOT-FINDING + NUMERIC OPTIMIZATION TO FIND OPTIMAL OR PERIODIC OR INTERESTING SOLUTIONS

→ AXIOMS + REASONING

→ NUMERICS THROUGOUT SEMESTER

TENTATIVE GOAL:

Tentative Goal:

Simulate a Bicycle!

Books "None Are Good"!

Syllabus No Syllabus! Start with Rotations

→ will take a month

Q-Like Homework → 6-10 hrs/wk

Exams? Reserve Judgement on Topic

Rotations [in 3D]

"Rigid Object"

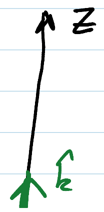
All Distances between all Pairs of Points are constant in time

Point — a material point, not a point in space

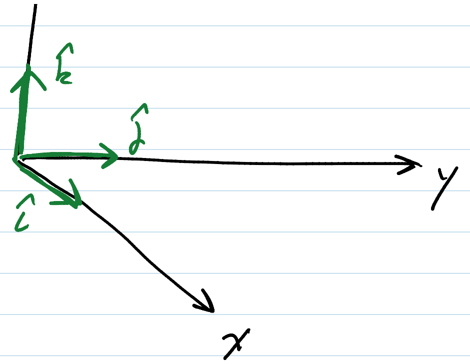
All Angles, lines are constant in time

All shapes are congruent thru time

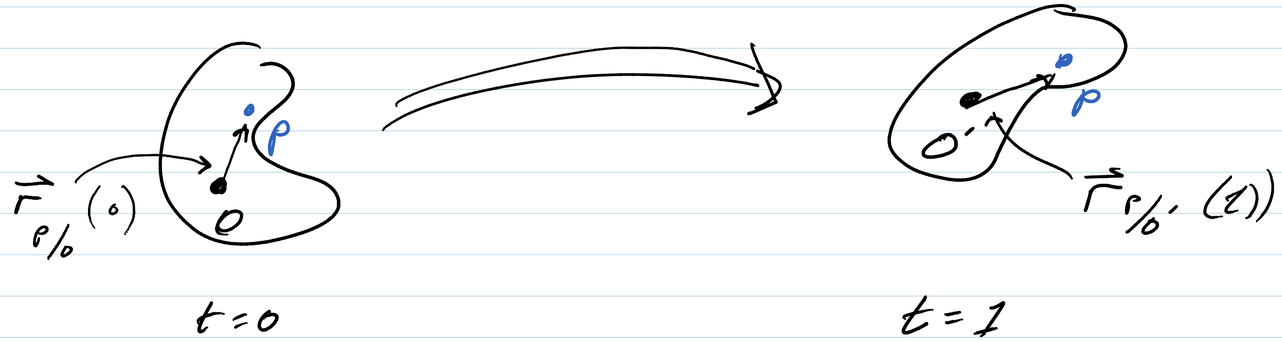
→ No Deformation! ←



We assume we are dealing with a background coordinate system



WE ASSUME WE ARE DEALING WITH A BACKGROUND COORDINATE SYSTEM

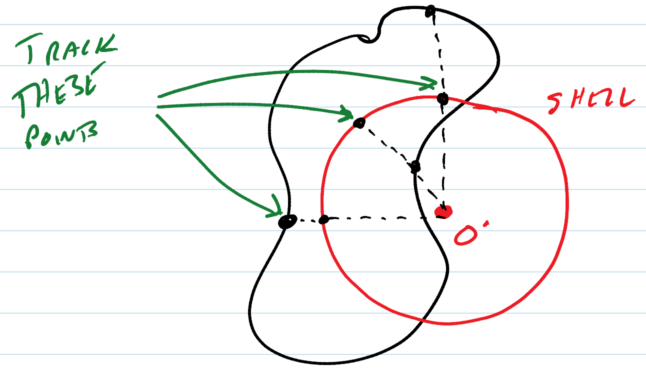


$$\vec{r}_{P/O}(t) = \underbrace{\vec{r}_{O'/O}(t)}_{\text{TRANSLATION}} + \underbrace{\vec{r}_{P/O'}(t)}_{\text{ROTATION}}$$

CONSIDER POINTS ON A SPHERICAL SHELL

INSIDE THE OBJECT

- WE CONSIDER ALL POINTS ATTACHED TO A FRAMEWORK THAT IS ATTACHED TO THE OBJECT

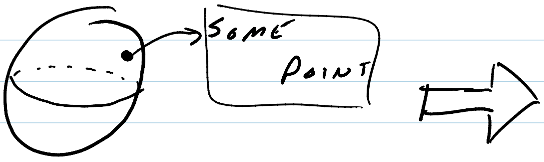


EULER'S THEOREM

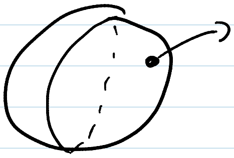
FOR ANY ROTATION AT ANY TIME t THERE IS SOME POINT E SO THAT

$$\vec{r}_{E/O'}(t) = \vec{r}_{E/O'}(0)$$

"SOME POINT ON SPHERE DOESN'T MOVE"

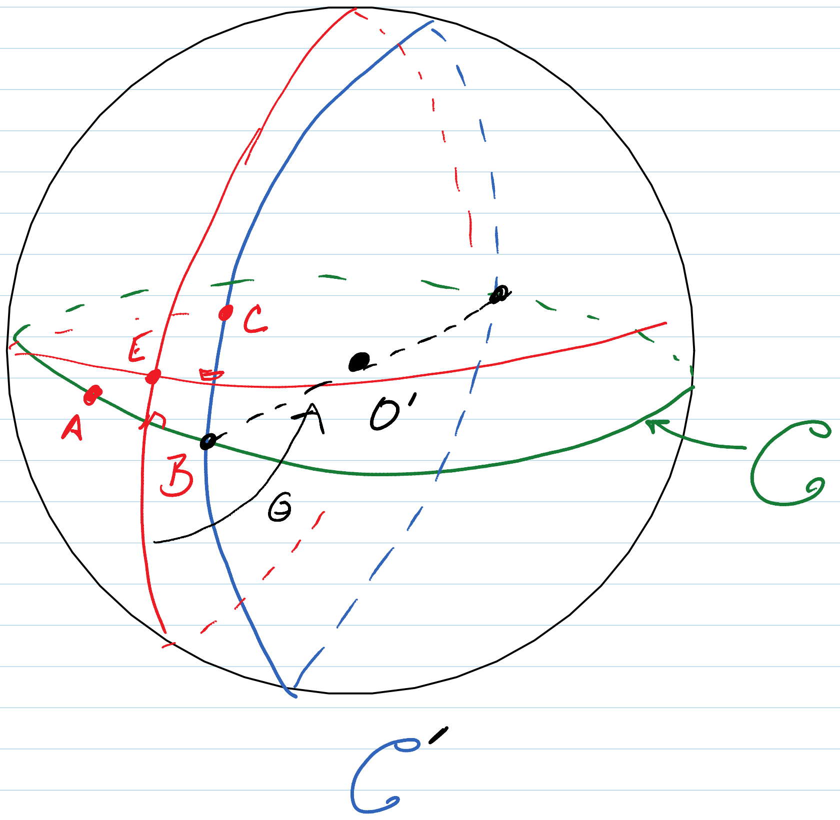


BUNCH OF ROTATIONS



BEFORE ROTATION
DRAW ANY GREAT CIRCLE G

GREAT CIRCLE IS INTERSECTION OF SPHERE WITH PLANE THRU CENTER



Rotation takes

$C \to C'$

B is intersection of C and C'

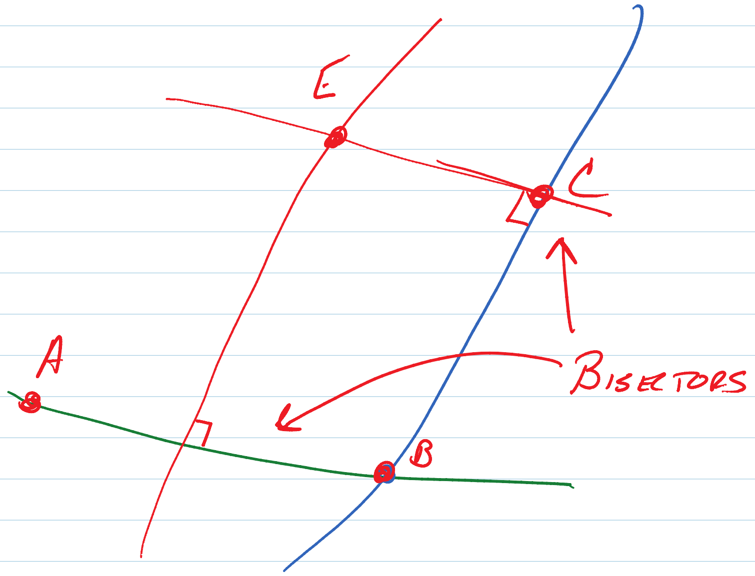
A and B are on C

Rotation

$C \to C'$

$A \to B$

$B \to C$



Point E at intersection

of perpendicular bisectors of AB + BC

DOESN'T MOVE

⇒ ROTATION IS ABOUT AXIS O'E

THAT IS THE BEGINING = END, BUT DOES NOT "STAY STILL"

NOT CONSTANT FOR $0 \leq t \leq t_1$

ONLY $\vec{r}_{E/O'}(0) = \vec{r}_{E/O'}(t)$

HW #1

FIND A BOOK OR WEB SITE

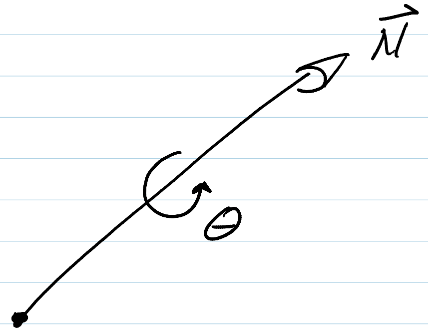
THAT PROVES THIS BETTER THAN THIS EXAMPLE

→ NOT MATRICES

A ROTATION CAN BE DESCRIBED

AS AN AXIS AND A ROTATION ANGLE

AXIS



4 NUMBERS

$n_x \, , \, n_y \, , \, n_z \, , \, \theta$

NOTE:

$n_x^2 + n_y^2 + n_y^2 = 1$

REDUNDANT NOTATION!

$\hat{n}, \theta$ ROTATION
IS SAME AS:

a) $-\hat{n} \, , \, -\theta$

b) $\hat{n} \, , \, \theta + 2m\pi$

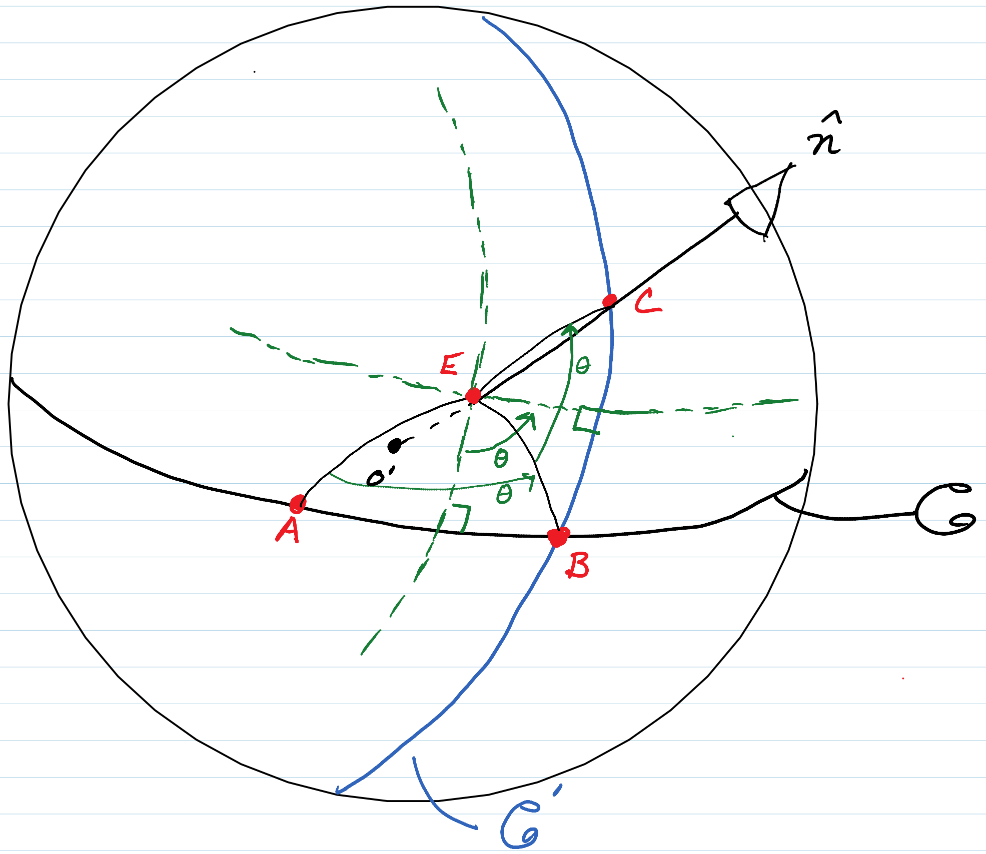
NOT UNIQUE!

ROTATIONS [CONT'D]

WEBSITE UP THIS WEEKEND

→ SYNGE + GRIFFITH   BOOK O/ ROTATIONS

EULERS THEORM [REPRISE]



$\mathcal{G} \to \mathcal{G}'$

$A \to B$
$B \to C$

PERPENDICULAR BISECTORS

$E \to E$

$\angle AEB =$
$\angle BEC =$
$\angle$ PERP BIS $= \theta$

REPRESENTATION #1

$\hat{n}, \theta$ 4 #'s

↳ UNIT VECTOR: $\hat{n} \cdot \hat{n} = 1$

REP #2 | VECTOR OF ROTATION

$\vec{N} = \theta \vec{n}$

↳ 3 #'s: N_x, N_y, N_z

* NOT UNIQUE: HAS THE "2π" PROBLEM OF ROTATION

Aside:

"Rotation is not a vector" $\boxed{?}$

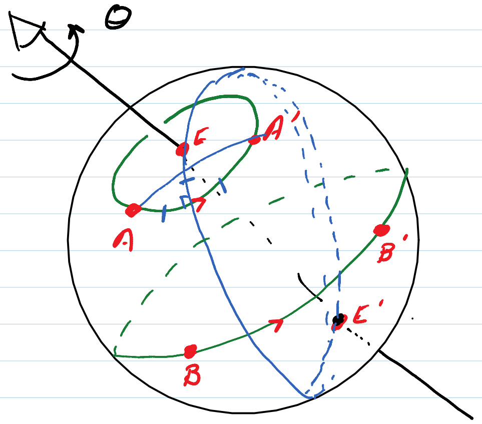
Must obey vector math

- It has magnitude + direction
- Can do vector algebra → But $\boxed{\vec{A}+\vec{B} \neq \vec{B}+\vec{A}}$
- Must have a 'Physical Meaning'

→ There is no meaning to "adding rotations"

Some Geometry of Rotations:

- All points move on latitudinal arcs!



Latitude Arcs
A → A'
B → B'

Longitudinal Bisector Must Go Through E

* ie Longitudinal Arc is a Great Circle

[A → A' circle is Not]

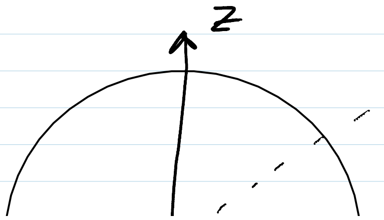
Intersection of longitudinal bisectors of A → A', B → B' is point E

→ You need the motion of 2 points to do this

Rep #3 Look @ Rotated Positions of

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \& \quad \begin{bmatrix}0\\1\\0\end{bmatrix}$$

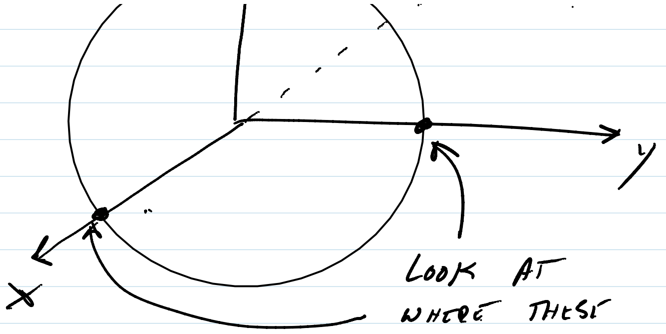
A B



4 #'s

Redundancy

- Distance



LOOK AT
WHERE THESE
POINTS GO

REDUNDANCY

DISTANCE
$A'B' = AB$

$\text{DIST}(A'B') = \text{DIST}(AB)$

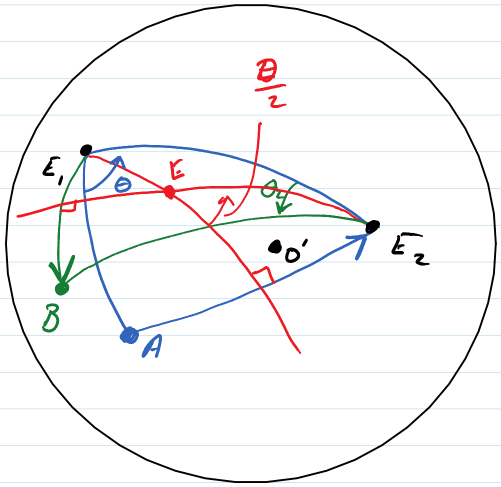
→ ANGLE ON GREAT CIRCLE

SO HOW DO YOU ADD ROTATIONS ?
GIVEN

$\theta_1, \hat{n}_1, E_1$ AND $\theta_2, \hat{n}_2, E_2$

WHAT IS "net ROTATION" $\theta\ \hat{n}$?

ROTATION 1
THEN
ROTATION 2



$A \to E_2$

$E_1 \to B$

E IS NOT ON THE PLANE $O' E_1 E_2$ THEREFORE:

$\vec{N}$ IS NOT $\underbrace{\vec{N}_1 + \vec{N}_2}$

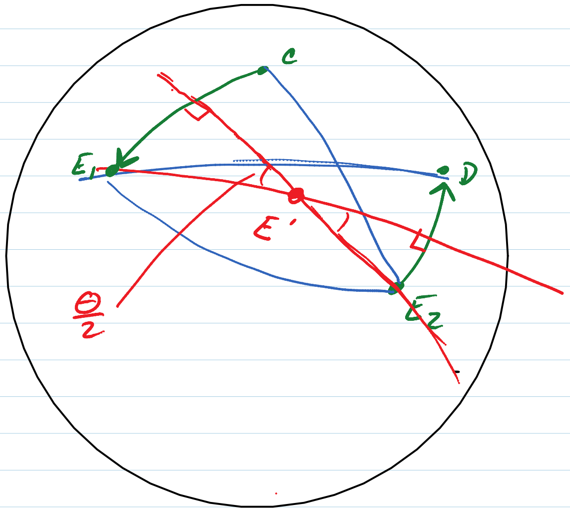
→ IS IN PLANE $O'\ E_1\ E_2$

NOW DO ROTATION 2 THEN ROTATION 1

$E' = E \implies$ ORDER OF ROTATION

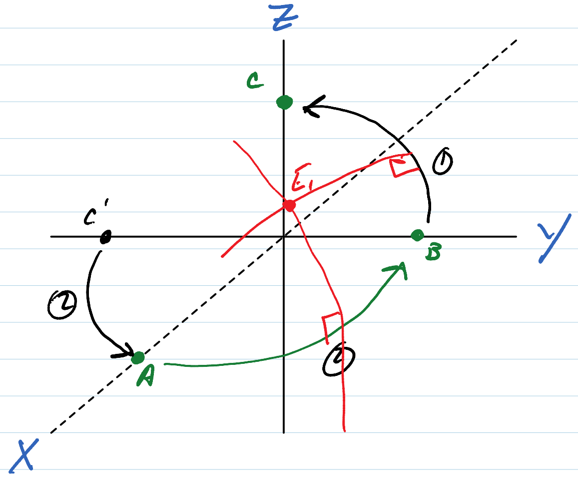
$E' = E \Rightarrow$ ORDER OF ROTATION

CHANGE NET ROTATION



EXAMPLE 1) ROTATE 90° ABOUT X-AXIS

THEN
ROTATE 90° ABOUT Z-AXIS



∀ POINTS IN SPACE
x, y, z

MATERIAL POINTS
A, B, C

① A STAYS AT X
B → Z
C → "−Y"

② A → Y AXIS
B STAYS AT Z
C → X

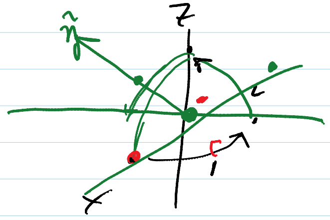
E_1 IS AXIS of ROTATION

E_1 is axis of rotation

$$\hat{n} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\theta = \frac{2\pi}{3} = 120°$$

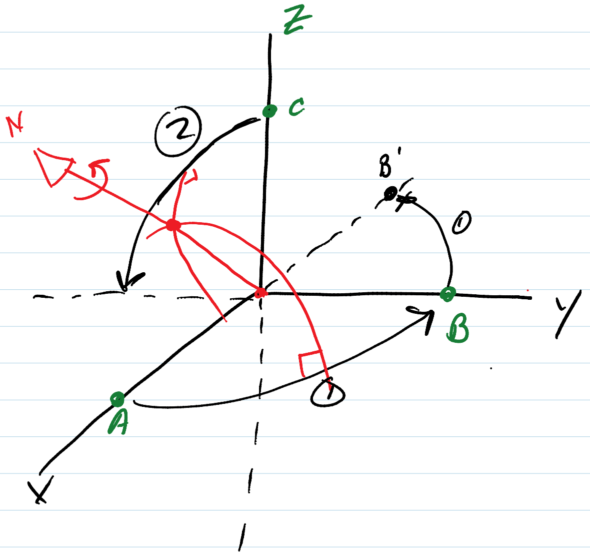
b) Rotate 90° about Z and 90° about X



$$\frac{1}{\sqrt{3}} \begin{bmatrix}-1\\+1\\+1\end{bmatrix} \quad ✗ \ \underline{NG}$$

$\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is <u>Correct</u>

A → B
B → C



Rotation ②
Pick point
that did
not move
in ①

$$\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

<u>We want to compute this!</u>

<u>Goal</u> Find formula that starts with $\hat{n}$, θ, $\vec{r}$

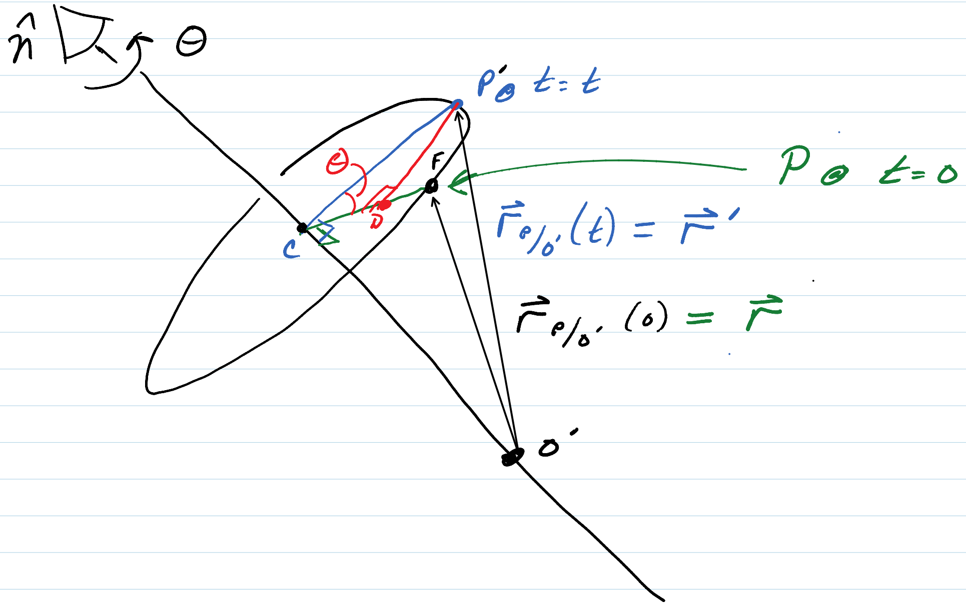
$\vec{r}$ is short for $\vec{r}_{P/O'}(0)$

Find $\vec{r}\,' = \vec{r}_{P/O'}(t)$

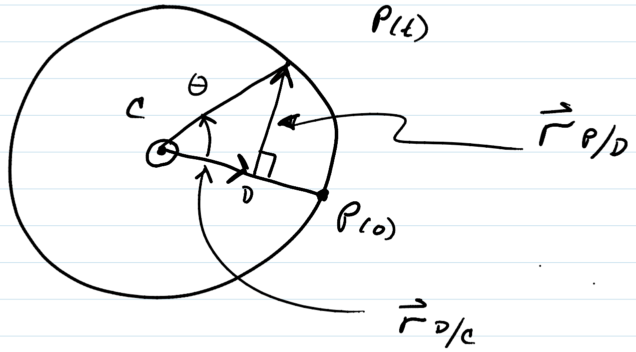
After Rotation

[O' used vs O
since we are
currently ignoring translations]

ROTATION



LOOKING DOWN



$$\vec r\,' = \vec r_{C/O'} + \vec r_{D/C} + \vec r_{P'/D}$$

IN TERMS OF $\tilde n, \theta$.

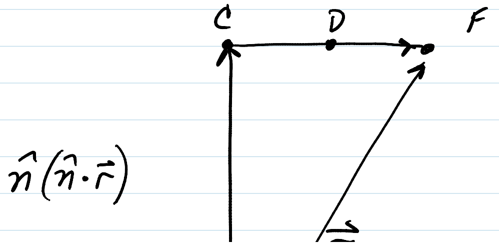
$$\vec r_{C/O'} = \underbrace{\hat n(\vec r \cdot \hat n)} \hookrightarrow = (\vec r \cdot \hat n)\hat n$$

* WE ARE PLACING POSITION VECTOR "ON THE RIGHT" FOR FUTURE CONVENIENCE

$$\vec r_{D/C} = \cos\theta\left[\vec r - \hat n(\hat n \cdot \vec r)\right]$$

$$\vec r_{P'/D} = \sin\theta\left[\hat n \times \vec r\right]$$

HINTS



$\hat{n}(\hat{n}\cdot\vec{r})$

$$\vec{r}_{F/C} = \vec{r} - \hat{n}(\hat{n}\cdot\vec{r})$$

BUT WE WANT $\vec{r}_{D/C}$

SO WE USE $\cos\theta$

WE OBTAIN

$$\vec{r}\,' = \hat{n}(\hat{n}\cdot\vec{r}) + \cos\theta\left[\vec{r} - \hat{n}(\hat{n}\cdot\vec{r})\right] + \sin\theta\left[\hat{n}\times\vec{r}\right]$$

RE WRITE

$$= (1-\cos\theta)\,\hat{n}(\hat{n}\cdot\vec{r}) + \cos\theta\,\vec{r} + (\sin\theta)(n\times\vec{r})$$

NOTE

THIS IS "LINEAR IN $\vec{r}$"

RotationIII,1/29/13
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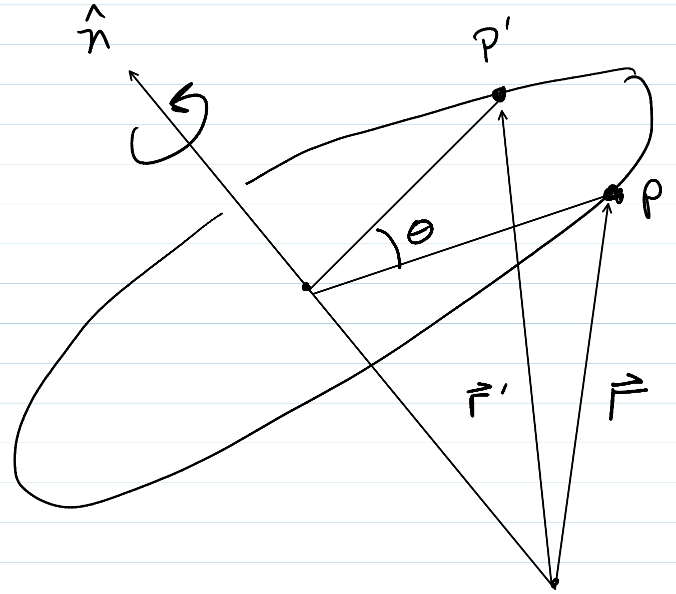
ROTATION (CONT'D)

RECALL



ROTATION

1) $\hat n$, θ

2) $\vec N = \theta\,\hat n$

3) $$\vec r\,' = \hat n(\hat n \times \vec r) + \cos\theta\left(\vec r - \hat n(\hat n\cdot\vec r)\right) \qquad + \sin\theta\, n \times \vec r$$

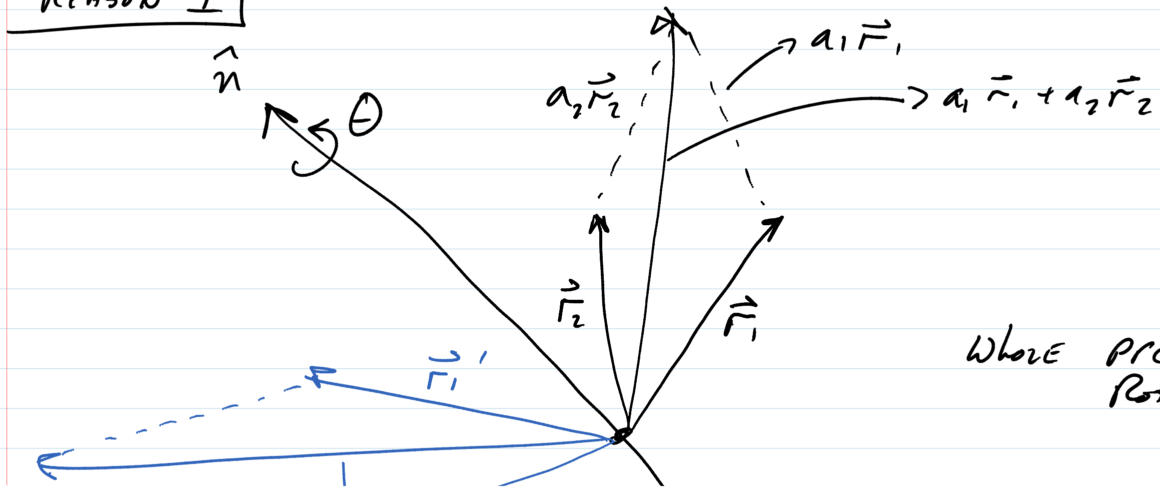
$$= \left(\hat n\hat n\,\cdot \quad + \cos\theta\left(\underline{\underline{I}}\cdot - \hat n\hat n\cdot\right) + \sin\theta\,\hat n \times\right)\vec r$$

OBSERVE: ROTATION IS LINEAR IN $\vec r$

$$\text{ROTATION}(a_1\vec r_1 + a_2\vec r_2) = a_1\,\text{ROT}(\vec r_1) + a_2\,\text{ROT}(\vec r_2)$$

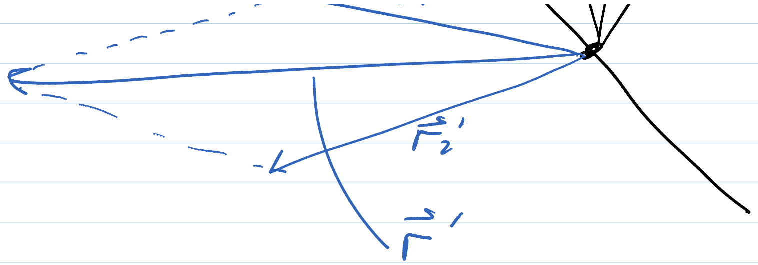
Why?

REASON 1

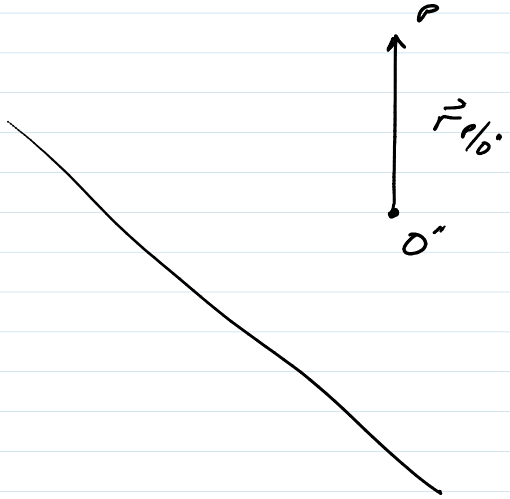


WHOLE PICTURE ROTATES

ROTATED

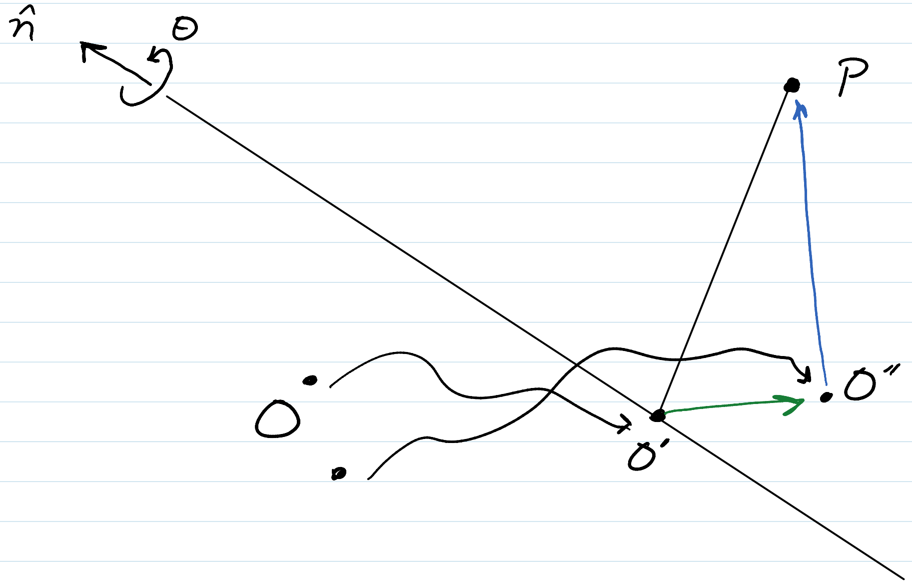


$\Rightarrow$ MEANS THAT $\vec{r}_{P/O''}$ ROTATES ACCORDING TO SOME FORMULA



ROTATION ALGEBRA IS <u>INDEPENDENT OF REFERENCE POINT</u> (O', OR O'')

<u>UNLIKE</u> ROTATION



* REGARDLESS OF REFERENCE POINT

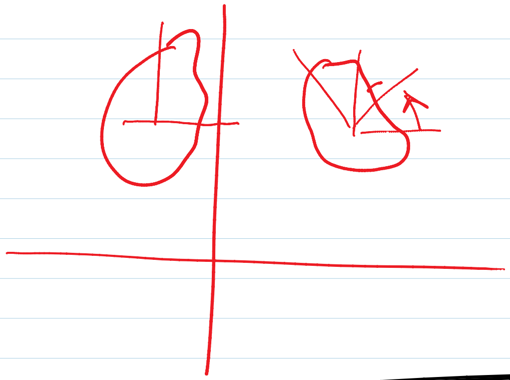
you get THE

SAME $\hat{n}$ AND THE

SAME θ

ROTATION MATRIX DESCRIBES ANGULAR MOTION REGARDLESS OF REFERENCE POINT ON OBJECT

→ WORKS FOR 2D AS WELL:

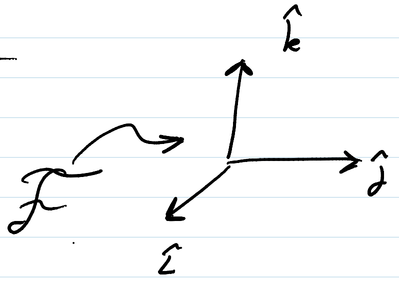


DOES NOT MATTER YOUR REFERENCE POINT, THE OBJECT ROTATES A GIVEN AMOUNT

SEMANTICS of VECTORS

$$[\vec{V}]_{\hat{\imath}\hat{\jmath}\hat{k}} = [\vec{V}]_{\mathcal{F}} = [\vec{V}] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

implicit

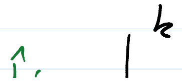


FRAME vs COORDINATE SYSTEM

FRAME IS A SET of ALL COORDINATE SYSTEMS $\hat{\imath}\,\hat{\jmath}\,\hat{k}$ THAT DO NOT ROTATE W/r/T EACH OTHER

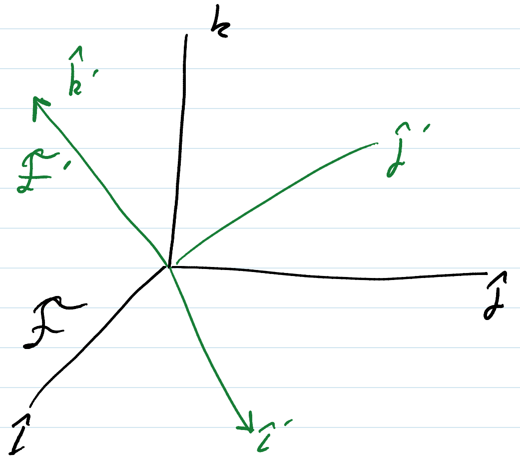
A COORDINATE SYSTEM $\hat{\imath}$, $\hat{\jmath}$ OR $\hat{k}$ DOES NOT MOVE BUT MIGHT HAVE MOVED "BEFORE you DREW THEM"

SOMETIMES → $\begin{bmatrix} V_1 \end{bmatrix}$



SOMETIMES $\vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$

"JUST A LIST OF 3 NUMBERS"



$$\vec{r}' = \left[\underline{\underline{I}}\cdot\vec{r} - \hat{n}(\hat{n}\cdot\vec{r})\right]\cos\theta + \hat{n}\,\hat{n}\cdot\vec{r} + \sin\theta\,\hat{n}\times\vec{r}$$

RECALL DRAWING

HOW YOU GO FROM $\vec{r}$ AND $\vec{r}'$ BY ROTATION

ASIDE: QUATERNIANS / EULER PARAMETERS:

$$\underbrace{\left(\sin\left(\frac{\theta}{2}\right)\hat{n},\ \cos\frac{\theta}{2}\right)}_{4\ \#'s}$$

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① DIADIC NOTATION ② MATRIX NOTATION

2] GIVEN $\vec{r} = r_x\hat{i} + r_y\hat{j} + r_z\hat{k}$

or

$= r_1\hat{e}_1 + \cdots\cdots$

$= \sum_{i=1}^{3} r_i\,\hat{e}_i$

$$= \underbrace{r_i \hat{e}_i}$$

EINSTEIN SUMMATION NOTATION
OR
"INDICIAL NOTATION"

⟹ IMPLIED IS A

$\sum_{i=1}^{3}$ WHERE EVER A SUBSCRIPT APPEARS TWICE

So,
GIVEN $\vec{r}$

GIVEN $\hat{n} = n_i \hat{e}_i$

GIVEN θ

CALCULATE $[\vec{r}\,']$

⟹ GIVEN COMPONENTS of $\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$$\hat{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\theta$$

FIND $\vec{r}\,' = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix}$

THE ANSWER "WILL BE"

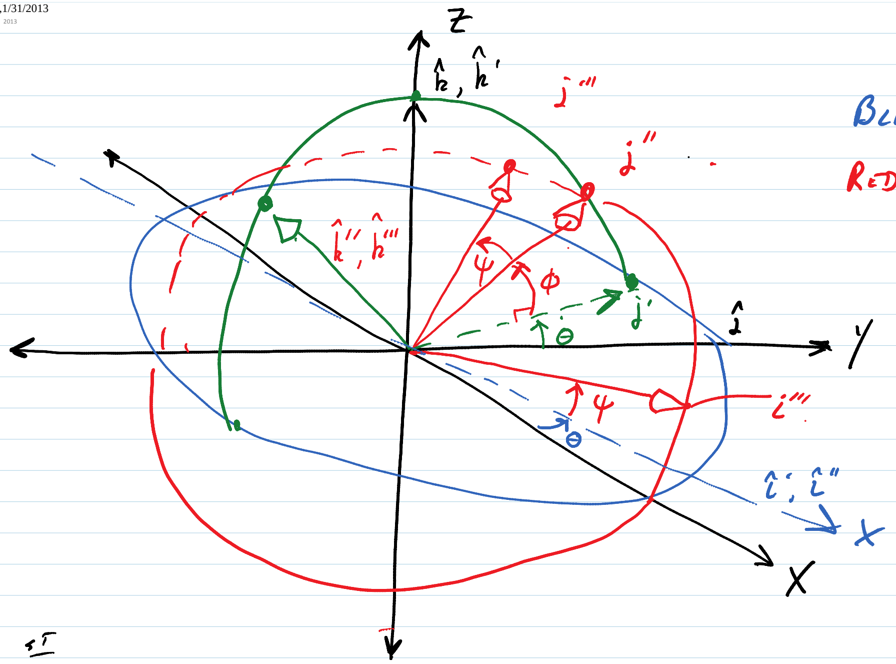
THE ORIGINAL COMPONENTS TIMES A MATRIX

$$\underline{\underline{I}} \cdot \vec{r} = \left[\underline{I}\right] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}$$

$$\left[\hat{n}\,\hat{n} \cdot \vec{r}\right] = \begin{bmatrix} n_1 (n_1 r_1 + n_2 r_2 + n_3 r_3) \\ n_2 \left(\qquad \text{''} \qquad \right) \\ n_3 \left(\qquad \text{''} \qquad \right) \end{bmatrix} = \begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$\left[n \times r\right] = \begin{bmatrix} n_2 r_3 - n_3 r_2 \\ n_3 r_1 - n_1 r_3 \\ n_1 r_2 - n_2 r_1 \end{bmatrix} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\left[\vec{r}\,'\right] = \underbrace{\left[(1 - \cos\theta) \begin{bmatrix} n_1 n_1 & \cdots & \\ & \ddots & \\ & & n_3 n_3 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \right]}_{\left[\underline{R}\right] = R} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$



Blue = x y plane

Red = i', j'' plane

Tilted ϕ about X' axis from x-y plane

"θ, ψ" overlap

1st Rotate about Z axis

$$\left.\begin{array}{l} \hat{i} \to \hat{i}' \\ \hat{j} \to \hat{j}' \\ \hat{k} \to \hat{k}' = \hat{k} \end{array}\right\} \text{ in x-y plane}$$

2nd Rotate about new x axis [called x'] = $\hat{i}'$ axis ϕ

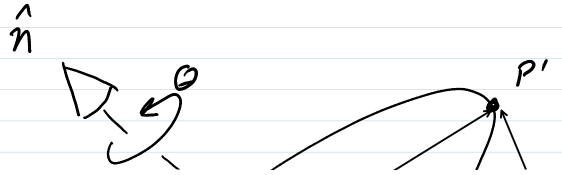
$$\hat{i}' \to \hat{i}'' = \hat{i}'$$
$$\hat{j}' \to \hat{j}''$$
$$\hat{k}' \to \hat{k}''$$

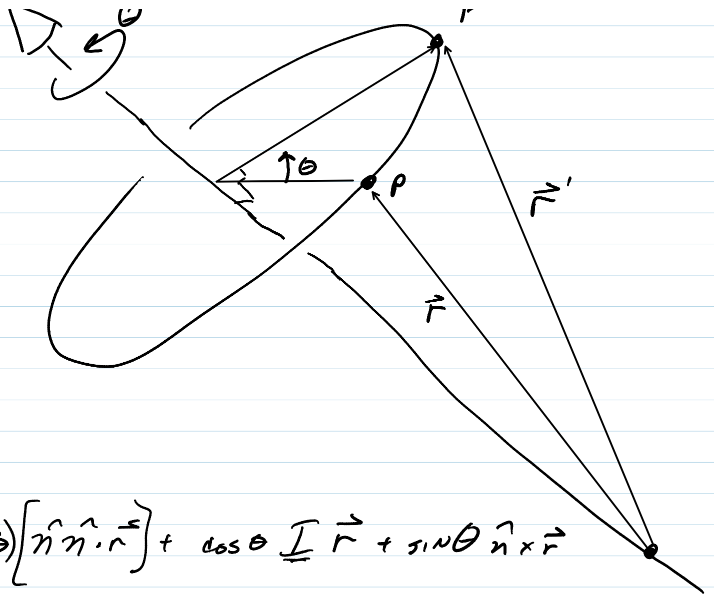
3rd Rotate ψ about new z-axis called z', $\hat{k}''$, $\hat{k}'''$

$$\left.\begin{array}{l} \hat{i}'' \to \hat{i}''' \\ \hat{j}'' \to \hat{j}''' \end{array}\right\} x', \hat{i}', \hat{i}'', \hat{j}'' \text{ plane}$$
$$\hat{k}'' \to \hat{k}''' = \hat{k}''$$

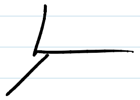
Net

$$\hat{i} \xrightarrow{\theta} \hat{i}' \xrightarrow{\phi} \hat{i}'' = \hat{i}' \xrightarrow{\psi} \hat{i}'''$$
$$\hat{j} \longrightarrow \hat{j}' \xrightarrow{\phi} \hat{j}'' \xrightarrow{\psi} \hat{j}'''$$
$$\hat{k} \longrightarrow \hat{k} = \hat{k}' \longrightarrow \hat{k}'' \xrightarrow{\psi} \hat{k}'' = \hat{k}'''$$





$$\vec r' = (1-\cos\theta)\left[\hat n \hat n \cdot \vec r\right] + \cos\theta\, \underline{I}\, \vec r + \sin\theta\, \hat n \times \vec r$$

$\mathcal{F} =$ 

$$[\vec r]_{\mathcal F} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \overbrace{\left[1-\cos\theta \underbrace{\begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix}}_{\text{"TIMES TABLE"}} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + s \right.}^{R}$$

$[\hat n]\cdot[\hat n]$ ← DIAD $\quad n \cdot n' = \hat n \hat n$

$$\left.\begin{array}{l} [\vec r']_F = [\underline{R}]_{\mathcal F}\,[r]_F \\ \text{"}\vec r' = R \cdot \vec r\text{"} \end{array}\right\}$$ MATRIX REPRESENTATION

<u>COMPONENT REPRESENTATION</u>

$$\vec r_i' = \underbrace{R_{ij}\, r_j}_{}$$

SUMMATION CONVENTION

$$\sin\theta \underbrace{\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}}_{\text{SKEW SYMMETRIC}} \Bigg] \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

CONVENTION

$$R_{ij} = (1-\cos\theta)\, n_i n_j + \cos\theta\, \delta_{ij} - \epsilon_{ijk}\, n_k\, (\sin\theta)$$

DECODER

$$\delta = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = I_{ij}$$ Kroënecker delta

$$\epsilon_{ijk} = \begin{cases} 1 & \text{IF } ijk = 1,2,3 \text{ OR } 2,3,1 \text{ OR } 3,1,2 \\ -1 & \text{IF } ijk = 2\,1\,3 \quad 1\,3\,2 \quad 3\,2\,1 \\ 0 & \text{for the other 21 cases: } i=j \text{ or } j=k \text{ or } k=i \text{ or } i=j=k \end{cases}$$

ROTATE #

TWIDDLE ANY 2,

DIRECT TENSOR NOTATION

$$\vec{r}\,' = \underline{\underline{R}} \cdot \vec{r}$$

$$\underline{\underline{R}} = (1-\cos\theta)\, \hat{n}\hat{n} + \cos\theta\, \underline{\underline{I}} + \underbrace{\underline{\underline{\mathcal{E}}}(\hat{n})}$$

?

$$\underbrace{-\epsilon_{ijk}\, \hat{e}_i\, \hat{e}_j\, n_k}$$

THIS CANNOT BE TRUE

SO HOW DO WE DO THE INVERSE PROBLEM

GIVEN R

CAN WE FIND $\hat{n}, \theta$?

LOOK AT $\text{TRACE}(R) = R_{11} + R_{22} + R_{33}$

$R_{ii} = (1-\cos\theta)(n\,n + n\,n + n\,n) + \cos\theta\,(3) + \sin\theta\,(0)$

Then ROTATE

$$R_{ii} = (1-\cos\theta)\underbrace{(n_1 n_1 + n_2 n_2 + n_3 n_3)}_{n_{ii}} + \cos\theta\underbrace{(3)}_{\delta_{ii}} + \sin\theta(0)$$

$$Tr(R) = (1-\cos\theta)(1) + 3\cos\theta = 1 + 2\cos\theta$$

$$n_i n_i = 1$$

BECAUSE THEY ARE UNIT VECTORS!

⟶

$$\boxed{\cos\theta = \frac{\text{trace}(R) - 1}{2}}$$

Now find $\hat{n}$

TAKE ANTI SYMMETRIC PART of R (LAST PART)

$$\rightarrow \sin\theta\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$\boxed{\frac{R - R'}{2}} \rightarrow \boxed{\begin{aligned} n_3 &= \frac{-(R_{12} - R_{21})}{2\sin\theta} \\ n_2 &= \frac{R_{13} - R_{31}}{2\sin\theta} \\ n_1 &= \frac{-(R_{23} - R_{32})}{2\sin\theta} \end{aligned}}$$

$\underline{R}$

ROTATION MATRIX USING EULER ANGLES

$R = R_z \,,\ R_x \,,\ R_z$ PRODUCT

has sum of 3 MATRIX

$R - R'$ REMOVES SYMETRIC
PART (1ST 2)

AND LEAVES TWICE
THE ASYMMETRIC

$$\underbrace{R_\theta \; R_\phi \; R_\psi}$$

ORIGINAL Z

ORIGINAL X

FIXED Z

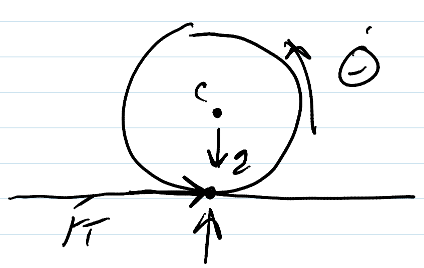
$$\vec{r}\,' = R\vec{r}$$

<u>ex</u> R_ψ = ROTATION of ψ ABOUT "FIXED" Z AXIS

$$= (1-\cos\psi)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \cos\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\psi \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

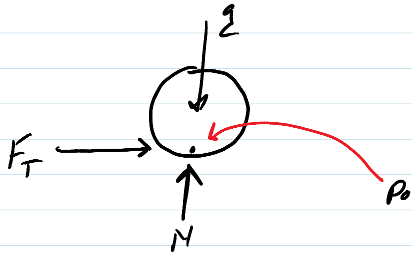
DO OTHERS! SOME SQUARED TERM COME OUT

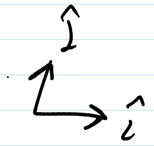


{LMB} ?

or $\tau = ?$

FBD





(D)

POWER IS ACTING ON POINT WITH VELOCITY

3 EQ's $\vec{a}\cdot\hat{j} = 0$

WE CAN FIND $F_T,\ N,\ \ddot{\theta}$

AMB/D

$$0 = [\vec{r}_{G/D} \times m\underbrace{\vec{a}_G}_{-r\ddot{\theta}\hat{i}}] + I\ddot{\theta}\hat{k}$$

$$0 = \ddot{\theta}\,[\text{SOMETHING}]\,\hat{k}$$

BUT $[\text{SOMETHING}] \neq 0$

SO $\ddot{\theta}$ MUST $= 0$

$$\vec{a} = 0$$

{LMB}

$$F_T\,\hat{i} + N\hat{j} - mg\,\hat{j} = m\underset{0}{\cancel{\vec{a}_G}}$$

$$F_T = 0$$

NO ONE ANSWERED HW QUESTION

DO IT AGAIN!

✱ BE SURE OF ANSWER! → THAT BLOCK

might not go further

[PICK A SLOPE AND A μ]

ROTATIONS [CONT'D]

$$\vec{r}\,' = \underline{\underline{R}} \cdot \vec{r}$$

$$\hookrightarrow \underline{\underline{R}} = R_{ij}\, \hat{e}_i \hat{e}_j$$

"ROTATION IN FIXED BASIS"

$$[1-\cos\theta]\,\hat{n}\hat{n} + \cos\theta\, \underset{\hookrightarrow \hat{e}_i \hat{e}_i}{\underline{I}} + \sin\theta\, \underbrace{\mathcal{E}\hat{n}}_{\underbrace{\epsilon_{ijk}\,\hat{e}_i\, n_j\, \hat{e}_k}_{\text{"THIS GIVES THE RIGHT ANSWER"}}}$$

CLAIM CHECK ON LITTLE CROSS PRODUCT (?)

$$(\epsilon_{ijk}\, \hat{e}_i\, n_j\, \hat{e}_k) \cdot \underline{R} = \hat{n} \times \vec{r}$$

CHECK BOTH SIDES

$$= [n_2 r_3 - n_3 r_2]\hat{e}_1 + [n_1 r_3 - n_1 r_3]\hat{e}_2 + [n_1 r_2 - r_2 n_1]\hat{e}_3$$

$$\epsilon_{ijk}\, \hat{e}_i (r_l \hat{e}_l\ \ )$$

$$\hat{e}_k \cdot \hat{e}_l = \delta_{kl}$$

$$\epsilon_{ijk}\, \hat{e}_i\, n_j\, r_k \overset{?}{=}$$

say $i = 1$

you get $(1 \cdot 23) - (1 \cdot 3 \cdot 2)$ ✓

[TRY IT] !

$$\boxed{\underline{\underline{R}} = \hat{e}_i{}' \, \hat{e}_i}$$

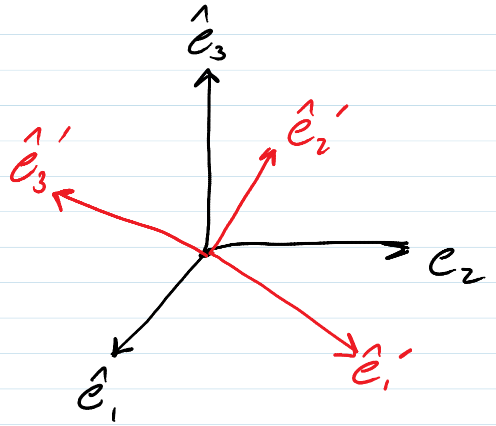
$\hat{e}_3$

$$\boxed{\underline{\underline{R}} = \hat{e}_i' \, \hat{e}_i}$$

check

$$\underline{\underline{R}} \cdot \vec{r} = \hat{e}_i' \, \hat{e}_i \cdot (r_k \hat{e}_k)$$

$$= r_i \hat{e}_i'$$



$$\hat{e}_1' = \text{Rot}(\hat{e}_1)$$
$$\hat{e}_2' = \text{Rot}(\hat{e}_2)$$
$$\hat{e}_3' = \text{Rot}(\hat{e}_3)$$

Components of $\vec{r}$ in $\hat{e}$ is same as components of $\vec{r}\,'$ in $\hat{e}'$

So

$$[\underline{\underline{R}}]_{\mathcal{E}} = \Big[[\hat{e}_1']_{\mathcal{E}} \,\Big|\, [\hat{e}_2']_{\mathcal{E}} \,\Big|\, [\hat{e}_3']_{\mathcal{E}} \Big]$$

$\mathcal{E} = \hat{e}$

↳ components of $\text{Rot}(\hat{e}_1)$ in $\hat{e}$ basis

↳ THE "ROTATION MATRIX"

$$\hat{e}_i' = \underbrace{R_{ji}}_{} \, \hat{e}_j$$

↳ i summing over j → Transpose of R

"SMALL ROTATION"

¥ Ball Rolls on Flat Plane
→ Possible Motions?

→ Infinite Turntable: Possible Motions

$$\underline{\underline{R}}\cdot = (1-\cos\theta)\,\hat{n}\hat{n}\cdot + \cos\theta\,\underline{\underline{I}}\cdot + \sin\theta\,\underline{\underline{\mathcal{E}}}\,\hat{n}$$

say $\theta << 1$ (small)

And we only want 1st order Terms

$$(1-\cos\theta) \approx \theta^2/2 + \theta^4/2 \text{ -----}$$ Goes Away

$$\approx \underline{\underline{I}} + \theta\,\underline{\underline{\mathcal{E}}}(\hat{n})$$

Look at 2 small sequential Rotations R_1, R_2

$$\underline{\underline{R}}_2\cdot\underline{\underline{R}}_1 = \left(\underline{\underline{I}} + \theta_2\,\underline{\underline{\mathcal{E}}}(\hat{n}_2)\right)\cdot\left(\underline{\underline{I}} + \theta_1\,\underline{\underline{\mathcal{E}}}(\hat{n}_1)\right)$$

$$\approx \underline{\underline{I}} + \theta_2\,\underline{\underline{\mathcal{E}}}(\hat{n}_2) + \theta_1\,\underline{\underline{\mathcal{E}}}(\hat{n}_1)$$

But

$\underline{\underline{R}}_1\cdot\underline{\underline{R}}_2 =$ same thing.

→ Troubling since we proved that Rotations are not communicative

!

⇒ SMALL Rotations ARE commutative to the First Order !!

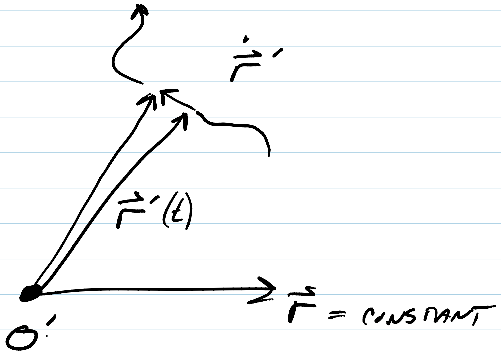
THIS RELATES TO ANGULAR VELOCITY

SMALL ANGLE PER UNIT TIME

$$\vec{r}\,' = \underline{\underline{R}}\cdot\vec{r} \iff \underline{\underline{R}}^{-1}\vec{r}\,'$$

$$\dot{\vec{r}}\,' = \left[\dot{\underline{\underline{R}}}\cdot\vec{r}\right] + \left[\underline{\underline{R}}\cdot\vec{0}\right]$$

PRODUCT RULE



so

$$\underline{\underline{R}} = \underline{\underline{R}}(t): \quad \dot{\vec{r}} = ?$$

$$\dot{\vec{r}}\,' = \dot{\underline{\underline{R}}}\cdot\underline{\underline{R}}^{-1}\,\vec{r}\,'$$

$$\hookrightarrow \underline{\underline{R}}^{T} = R^{-1}$$

$$\dot{\vec{r}}\,' = \underline{\underline{\omega}}\cdot\vec{r}\,'$$

$$\underline{\underline{\omega}} \equiv \dot{\underline{\underline{R}}}\cdot\underline{\underline{R}}^{-1}$$

TOPICS

1) SMALL ROTATIONS & ANGULAR VELOCITY [CONT'D]

2) DERIVATIVES

3) ¿ MECHANICS ?

RECALL

- SMALL ROTATIONS ADD

(I)

$\vec{r}_p$ = VECTOR @ "SOME TIME"

↑ FIXED IN ROTATING RIGID OBJECT

LOOK AT ROTATION RELATIVE TO CONFIGURATION @ t

$$\underline{R}(\Delta t) = \underline{\underline{I}} + \Delta\Theta \, \underline{\underline{\mathcal{S}}}(\hat{n})$$

$$\dot{\vec{r}}_{p_{\Delta t \to 0}} = \frac{\vec{r}_p(t+\Delta t) - \vec{r}_p(t)}{\Delta t} = \frac{\Delta\Theta}{\Delta t}\mathcal{S}(\hat{n}) \cdot \vec{r}_p$$

$$\boxed{\begin{aligned} \vec{r}_p(t) &= \underline{\underline{I}} \cdot \vec{r}_p(t) \\ \vec{r}_p(t+\Delta t) &= \left(\underline{\underline{I}} + \Delta\Theta \, \underline{\underline{\mathcal{S}}}(\hat{n})\right) \vec{r}_p \end{aligned}}$$

DEFINE

$$\underline{\underline{\omega}} = \dot{\theta} \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$-\omega_2 \quad \omega_1 \quad 0 \,]$$

$$\dot{\vec{r}}_p = \underline{\underline{\omega}} \cdot \vec{r}_p = \vec{\omega} \times \vec{r}_p$$

$$\vec{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \dot{\theta} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

☑ NOTE

ANSWER IS NOT DEPENDENT on $\vec{r}_p$

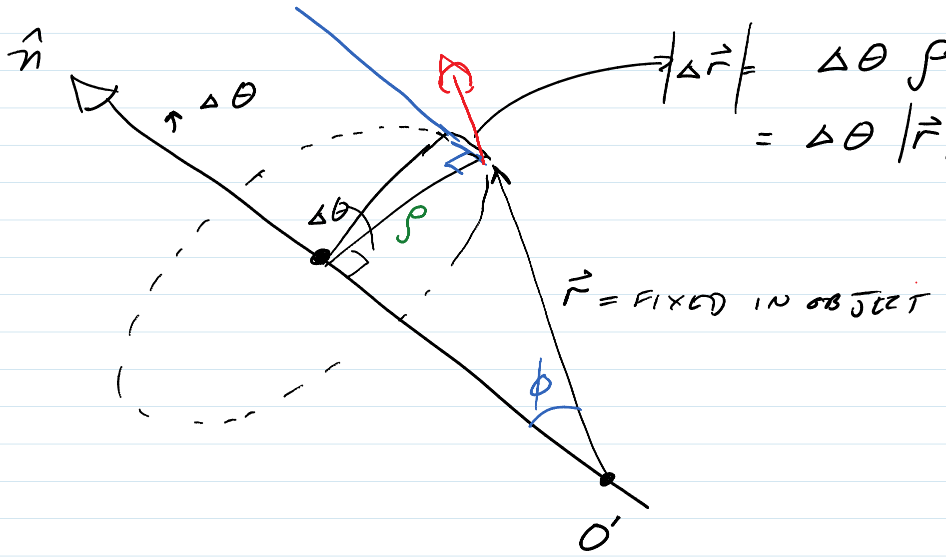
SO IT IS NOT A "PROPERTY" of object

$$\boxed{\dot{\vec{r}}_p = \vec{\omega} \times \vec{r}_p}$$

SHOULD LOOK FAMILIAR!

⇨ STILL WORKS IN 3D!

(II) GEOMETRIC REASONING



$$|\Delta \vec{r}| = \Delta\theta \, \rho$$
$$= \Delta\theta \, |\vec{r}| \sin\phi$$

RECALL $n \times r = |n| \, |r| \sin\phi \, \hat{u} = |\vec{r}| \sin\phi \, \hat{u}$

$\hat{u}$ IS ORTHOGONAL TO $\vec{n}, \vec{r}$ PLANE

$$\Delta\vec{r} = \Delta\theta \, \vec{n} \times \vec{r}$$

$$\dot{\vec{r}} = \dot{\theta}\; \hat{n} \times \vec{r}$$

$$= \vec{\omega} \times \vec{r}$$

$$\vec{\omega} \equiv \dot{\theta}\, \hat{n}$$

III Algebraic Approach

$$\vec{r}\,' = \underline{\underline{R}} \cdot \vec{r}_0$$

$\vec{r}\,' = \vec{r}(t)$

$\vec{r}_0 = \vec{r}(t),\ t=0$

$$\dot{\vec{r}}\,' = \dot{\underline{\underline{R}}} \cdot \vec{r}_0$$

Aside

$$\underline{\underline{R}} = \operatorname{expm}\left(\theta \underbrace{\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}}\right)$$

$$\dot{\vec{r}} = \underline{\underline{\omega}}\; \vec{r}$$

IF $\underline{\underline{\omega}}$ IS CONSTANT

$$\rightarrow \vec{r} = \exp\Big(\underbrace{\underline{\underline{\omega}} t}_{\text{THIS}}\Big)\, \vec{r}_0$$

IS THIS

But Recall

$$\vec{r}\,' = \underline{\underline{R}} \cdot \vec{r}_0$$

$$\vec{r}_0 = R^T \vec{r}\,'$$

$$\boxed{\dot{\vec{r}}\,' = \dot{\underline{\underline{R}}}\, \underline{\underline{R}}^T \cdot \vec{r}\,'}$$

Define $\underline{\underline{\omega}} = \dot{\underline{\underline{R}}}\, \underline{\underline{R}}^T$

$$\dot{\underline{\underline{I}}} = 0$$

$$\dot{\left(\underline{\underline{R}}\, \underline{\underline{R}}^T\right)} = \dot{\underline{\underline{R}}}\, \underline{\underline{R}}^T + \underline{\underline{R}}\, \dot{\underline{\underline{R}}}^T$$

$$\underline{\underline{0}} = \dot{\underline{\underline{R}}}\, \underline{\underline{R}}^T + \left(\dot{R} R^T\right)^T$$

Because $(AB)^T = B^T A^T$

$\Rightarrow$ implies $\underline{\underline{\omega}}$ is skew-symmetric

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

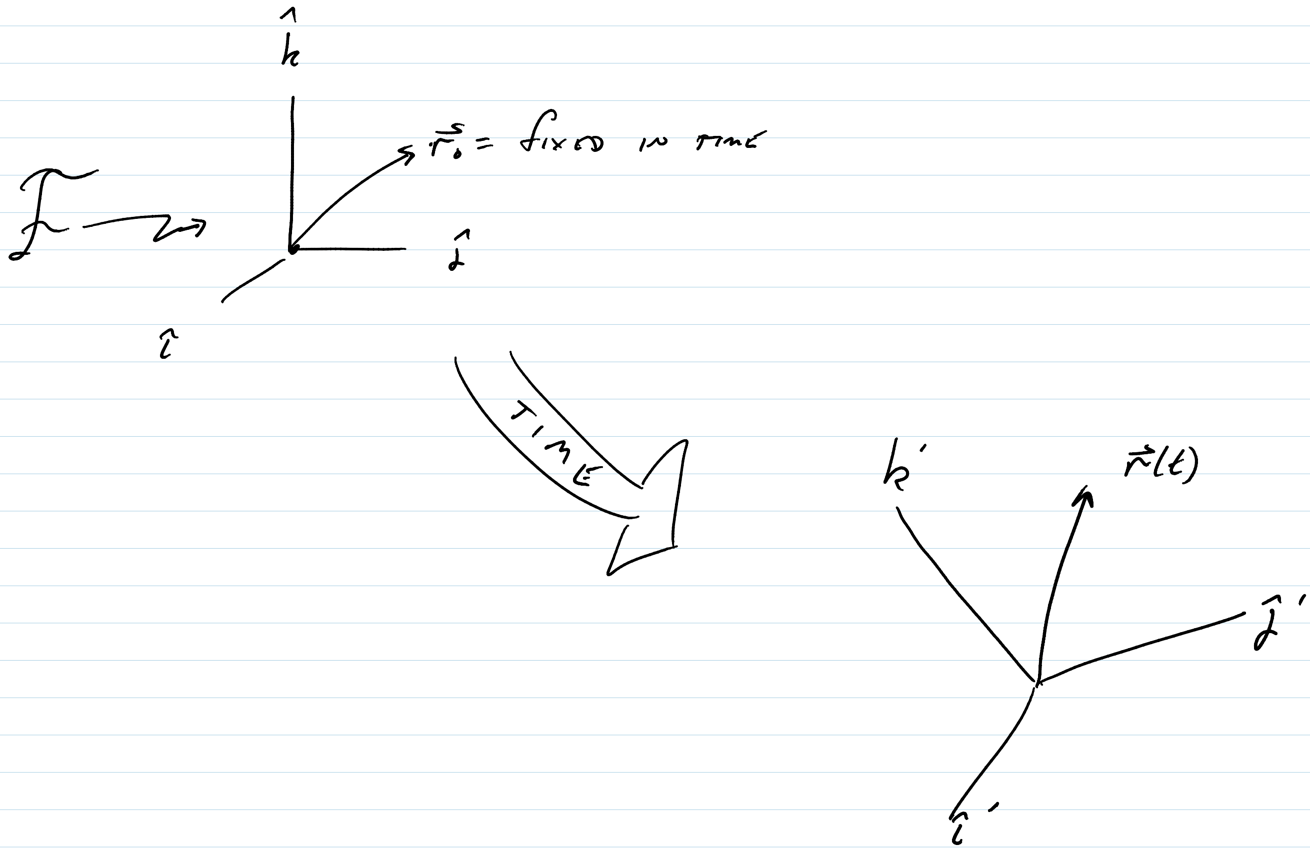
So we know

$$\dot{\vec{r}}\,' = \dot{\underline{\underline{R}}}\ \underline{\underline{R}}^T\ \vec{r}\,'$$

$$= \underline{\underline{\omega}} \cdot \vec{r}\,'$$

$$= \vec{\omega} \times \vec{r}\,'$$

CHANGE of COORDINATES VS ROTATION



$\mathcal{B}$

So take some vector:

$$\vec{r} = \vec{r}$$

$$[\vec{r}]_{\mathcal{F}} \longleftarrow r_i^{\mathcal{F}}\,\hat{e}_i = r_i^{\mathcal{B}}\,\hat{e}_i'$$

$$= \hat{e}_i'\,\vec{r}_i^{\mathcal{B}}$$

$$\vec{r}_i^{\mathcal{F}} = R_{ij}\,\vec{r}_j^{\mathcal{B}}$$

$$[\vec{r}]_{\mathcal{F}} = [\underline{R}][\vec{r}]_{\mathcal{B}}$$

$$[R] = \left[\begin{array}{c|c|c} [\hat{e}_1']_{\mathcal{F}} & [\hat{e}_2']_{\mathcal{F}} & [\hat{e}_3']_{\mathcal{F}} \end{array}\right]$$

R used to change coordinate for a <u>given</u> set of frames $\mathcal{F}$ & $\mathcal{B}$

<u>Recall</u>:

$$\hat{e}_i^{\mathcal{B}} = R_{ij}\,\hat{e}_j^{\mathcal{F}}$$

<u>or</u>

$$\hat{e}_i^{\mathcal{F}} = R_{ji}\,\hat{e}_j^{\mathcal{B}}$$

<u>What is</u> $\dot{\vec{r}}$?

$$\overset{\circ}{\vec{r}} = \dot{\vec{r}}$$

$$\dot{\vec{r}} = \vec{r}$$

$$\dot{r}_i^{\mathcal{F}}\,\hat{e}_i = \dot{r}_i^{\beta}\,\hat{e}_i' + r_i^{\beta}\,\dot{\hat{e}}_i'$$

$\vec{\omega}_{\beta/\mathcal{F}}$

$$\dot{\hat{e}}_i' = \vec{\omega}\times\hat{e}_i'$$

"THE Q FORMULA"!

$$= \dot{r}_i^{\beta}\,\hat{e}_i' + \vec{\omega}\times\vec{r}$$

$$\vec{r} = \vec{r}_i^{\beta}\,\hat{e}_i' = r_i^{\mathcal{F}}\,\hat{e}_i^{\mathcal{F}}$$

$$\boxed{\frac{{}^{\mathcal{F}}d\vec{r}}{dt} = \frac{{}^{\beta}d\vec{r}}{dt} + \left(\vec{\omega}\times\vec{r}\right)}$$

Q-dot formula
or TRANSPORT THEOREM

$$\dot{\vec{Q}}^{\mathcal{F}} = \dot{\vec{Q}}^{\beta} + \left(\vec{\omega}_{\beta/\mathcal{F}}\times\vec{Q}\right)$$

for ANY VECTOR "Q"

DERIVATIVE IN A FRAME DEFINED BY differentiating components ; NOT THE BASE VECTORS!

☆ REMEMBER

r_i^B IS NOT

A VECTOR

IT IS A LIST OF

#'S ; i IS A

SUMMATION

☆ DAVID BLOCK

TAM

DEF OF A VECTOR:

VECTOR IS A VECTOR IS A VECTOR

Mechanics,2/14/13

Tuesday February 12 2013
2 59 PM

MECHANICS ["Ooooooh....."]

"THE PAST 3 WEEKS HAS BEEN AN ASIDE OF KINEMATICS"

3 PILLARS OF MECHANICS [REPRISE]

- MATERIAL PROPERTIES (FORCE LAWS)
- GEOMETRY & KINEMATICS
- LAWS OF MECHANICS

$$\underbrace{\text{LAWS OF MECHANICS}} \hookrightarrow$$

$F = ma = \dot{L}$

$M = \dot{H}$

ACTION / REACTION

$P = \dot{E}_K$

RECALL FROM LAST TERM:

LMB IS A SPECIAL CASE OF AMB

FOR ANY POINT C: [AND ANY $\underbrace{\text{CLOSED}}$ SYSTEM]

$\hookrightarrow$ "FIXED COLLECTION OF ATOMS"

$$\Sigma \vec{M}_{/c} = \dot{\vec{H}}_{/c}$$

EXTERNAL TO THE SYSTEM

$$\hookrightarrow \dot{\vec{H}}_{/c} \equiv \int \vec{r}_{/c} \times \vec{a} \; dm$$

$$\hookrightarrow \vec{a}_{/\mathcal{F}}$$

= HOMEWORK PROBLEM

$$= \frac{d}{dt}\left[\vec{r}_{G/c} \times \vec{V}_{G/\mathcal{F}} \; m_{TOT} + \int \vec{r}_{/G} \times \vec{V}_{/G} \; dm \right.$$

$$\vec{V} = \dot{\vec{r}} = \frac{d}{dt} \vec{r}_{/o}$$

$$\vec{V}_{/G} = \vec{V} - \vec{V}_{/G}$$

$= \vec{V}$ IN A REFERENCE FRAME WHICH DOES NOT ROTATE

$$\vec{H}_{G/c} = \vec{r}_{G/c} \times (m_{TOT} \vec{V}_G)$$

$$\dot{\vec{H}}_{/o} = \frac{d}{dt}\left[\vec{r}_{G/c} \times \vec{V}_{G/c} \left(\sum m_i\right) + \sum_{\substack{\text{ALL} \\ \text{MASS}}} \vec{r}_{i/G} \times V\right.$$

$$\vec{H}_{/G} = \int \vec{r}_{G/c} \times \vec{V}_{G/c} \; dm$$

$$\dot{\vec{H}}_{/c} = \frac{d}{dt}\left(\vec{H}_{/c} \right)$$

$$ {}_{i/G}\; m_i \Big] $$

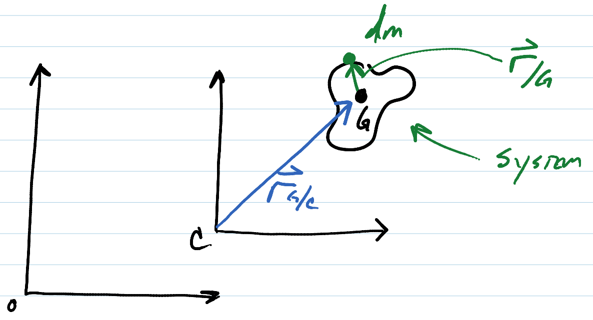
$/C$

$\hookrightarrow H_{/C} = H_{G/C} + H_{/G}$

$\boxed{\text{SPECIAL CASE: } G = C}$

$$\sum \vec{M}_{/G} = \frac{d}{dt}\left(\vec{H}_{/G}\right)$$

$$\hookrightarrow \vec{H}_{/G} = \int \vec{r}_{/G} \times \vec{V}_G \; dm$$



ANGULAR MOMENTUM BALANCE of RIGID OBJECT $\mathcal{B}$

$$\sum_{ext} \vec{M}_{/G} = \frac{d}{dt}\left(\vec{H}_{/G}\right)$$

1st FIND AN EXPRESSION FOR $\vec{H}_{/G}$ <u>THEN</u> TAKE $\frac{d}{dt}$ RATHER THAN DOING OUT <u>ALL</u> THE CROSS PRODUCTS

<u>ASIDE</u> WHAT IS $\vec{H}_{/G}$?

* $\vec{r}_D = \vec{r}_B + \vec{r}_{/D}$

$\vec{r} = \vec{r}_G + \vec{r}_{/G}$

$$\vec{H}_{/G} = \int_{\mathcal{B}} \vec{r}_{/G} \times \vec{V}_{/G} \; dm$$

"$\int \vec{r} \times \vec{v} \; dm$"

* $\sum \vec{M}_{/G} =$

$\sum \vec{M}_{/G}$

$$\vec{H}_{/G} = \int \vec{r}_{/G} \times \underbrace{\left(\vec{\omega} \times \vec{r}_{/G}\right)}_{} \; dm$$

$$\hookrightarrow \vec{V}_{/G} = \vec{\omega} \times \vec{r}_{/G} + \cancelto{\vec{0}}{{}^{\mathcal{B}}\dot{\vec{r}}_{/G}}$$

IF $\vec{r}$ IS CONSTANT = FIXED IN BODY

$$= \vec{H}_G + \vec{H}_{/G}$$

$$= \vec{H}_{/G}$$

BECAUSE
Γ ABOUT ITSELF $= \underset{\rightarrow}{0}$!

ASIDE: WHAT IS $\vec{a}\times(\vec{b}\times\vec{c})$?

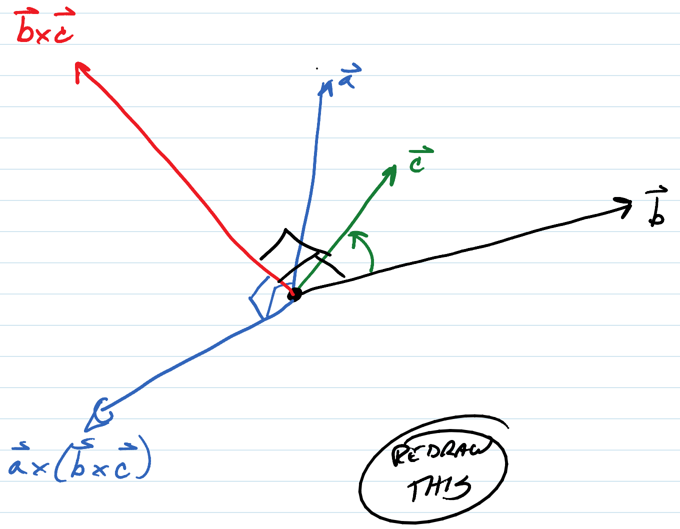
→ $\vec{a}\times\vec{b}\times\vec{c}$ IS AMBIGUOUS!

BECAUSE $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

SO LET'S TURN IT INTO DOT PRODUCTS!

$$\vec{a}\times(\vec{b}\times\vec{c}) =$$

✱ "DON LEWIS/ANDY RUINA METHOD"



REDRAW THIS

✱ a IS NOT ⊥ TO bc PLANE

NOR IS IN BC PLANE

b×c IS ⊥ to b AND c

a×(b×c) IS ⊥ TO a AND (b×c)

$\vec{a}\times(\vec{b}\times\vec{c})$ IS LINEAR IN $\vec{a}$, $\vec{b}$ & $\vec{c}$ IF OTHERS ARE FIXED

[DISTRIBUTIVE LAW OF × PRODUCT]

$$\vec{a}\times(\vec{d}\times\vec{c}) = (\vec{a}\times\vec{d}) + (\vec{a}\times\vec{e})$$ FIGURE THIS OUT!

→ SEE RUINA & PRATAP

$$\vec{a}\times(\vec{b}\times\vec{c}) = (d_1)\,\vec{b}\,(\vec{a}\cdot\vec{c}\ ) + (d_2)\,\vec{c}\,(\,\vec{b}\cdot\vec{a}\ )$$

IN THE PLANE ⊥ TO THE ⊥ OF $\vec{b}$ & $\vec{c}$

"IN THE b c PLANE"

✱ NOW FIND d_1 + d_2

ex! $\hat{\imath}\times(\hat{\imath}\times\hat{k}) = \hat{\imath}\times\hat{k} = -\hat{\jmath}$

a b c

ex 1 $\underset{a}{\hat{i}} \times (\underset{b}{\hat{i}} \times \underset{c}{\hat{k}}) = \hat{i} \times \hat{k} = -\hat{j}$

$$= d_1 \hat{i} (\hat{i} \cdot \hat{j}) + d_2 \hat{j} (\hat{i} \cdot \hat{i}) \Rightarrow \boxed{d_2 = -1}$$

$$\hat{j} \times (\hat{i} \times \hat{j}) \Longrightarrow \boxed{d_1 = 1}$$

d_1, d_2 ARE CONSTANTS!!

so $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ "bac minus cab"

//END ASIDE

$$\vec{H}_{/G} = \int \vec{r}_{/G} \times (\vec{\omega} \times \vec{r}_{/G}) \; dm$$

$$= \int \left[\vec{\omega} (\vec{r}_{/G} \cdot \vec{r}_{/G}) - \vec{r}_{/G} (\vec{r}_{/G} \cdot \vec{\omega}) \right] \; dm$$

$$=$$

Dynamics of a Rigid Object in 3D rel to C.O.M.

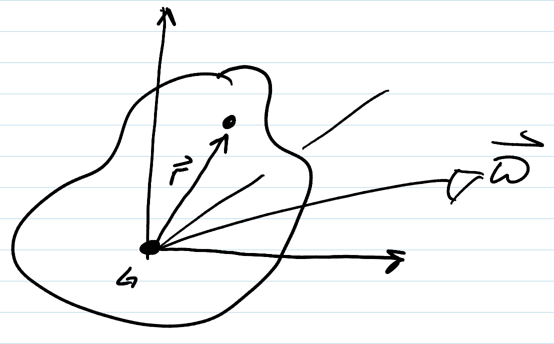
$$\vec{H} = \vec{H}_{/G}$$

↳ For today

$$= \int \vec{r} \times \vec{v} \, dm$$

$v = \vec{v}_{/G}$ $\boxed{\text{for today}}$

↳ $= \vec{r}_{/G}$ For $\boxed{\text{today}}$



$$\vec{V} = \vec{\omega} \times \vec{r}$$

↳ for today $\vec{\omega} = \vec{\omega}_{B/F}$

OK

$$\vec{H} \quad \int \vec{r} \times (\vec{\omega} \times \vec{r}) \, dm$$

"bac - cab" ⇒ $$= \int \vec{\omega} \underbrace{(\vec{r} \cdot \vec{r})}_{\text{scalar}} - \underbrace{\vec{r}(\vec{r} \cdot \vec{\omega})}_{(\vec{r}\vec{r}) \cdot \vec{\omega}} \quad dm$$

factor out $\vec{\omega}$

$$\left[\int \vec{r} \cdot \vec{r}) \cdot \underset{=}{1} \quad dm - \int \vec{r}\vec{r} \, dm \right] \cdot \vec{\omega}$$

eye(3)

$\delta_{ij} \hat{e}_i \hat{e}_j$

$\forall$ $\vec{r}\vec{r}$

"TEN

$[\vec{r}\vec{r}]_{\mathcal{F}}$

$= \vec{r} \otimes \vec{r}$

SOR PRODUCT"

$$= \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix}$$

$d_{ij} \ddot{e}_i \ddot{e}_j$

col

$$\boxed{\vec{H} = \underline{\underline{I}} \cdot \vec{\omega}}$$

$$\hookrightarrow \underline{\underline{I}} = \int \vec{r}\cdot\vec{r}\, dm \cdot \underline{\underline{1}} - \int \vec{r}\,\vec{r}\, dm$$

$\underline{\underline{I}}$ = INERTIA

$\underline{\underline{1}}$ = IDENTITY

$$[\underline{\underline{I}}]_{ij_F} = \int \underset{\substack{\uparrow \\ r_k r_k}}{r^2} \cdot \delta_{ij}\, dm - \int r_i r_j\, dm$$

$$[\underline{\underline{I}}] = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yz & x^2+z^2 & -yz \\ -xz & -yz & x^2+z^2 \end{bmatrix} dm$$

"INERTIA MATRIX"

$\boxed{\rightarrow \text{WHICH IS SYMMETRIC}}$

DIAGONALIZE

e VALUES EXIST

e VECTORS ARE REAL → ORTHO NORMAL!

→ WE CAN FIND A COORDINATE SYSTEM $\mathcal{B}$

SO THAT

$$[\underline{\underline{I}}]_{\mathcal{B}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

AND

$$\underline{\underline{I}}_{i'j'} = I_1 \hat{e}_1' \hat{e}_2' + I_2 \hat{e}_2' \hat{e}_2' + I_3 \hat{e}_3' \hat{e}_3'$$

"PRIME SYSTEM" WHERE ALL OFF-DIAGONALS ARE 0

$$\underbrace{\qquad\qquad\qquad\qquad}_{[\vec{r}]\cdot[\vec{r}]'}$$

diAD

× row

EACH BIT of MASS HAS A FIXED IN TIME COORDINATE $x'\ y'\ z'$

"FIXED ON BODY"

$\rightarrow \hat{e}_1'\ \hat{e}_2'\ \hat{e}_3'$ ARE FIXED ON BODY

$$I_1 = \int (y'^2 + z'^2)\, dm$$

$$I_2 = \int (x'^2 + z'^2)\, dm$$

$$I_3 = \int (x'^2 + y'^2)\, dm$$

TO RECAP

$$\vec{H} = \underline{\underline{I}} \cdot \vec{\omega}$$

$$[\vec{H}]_F = [\underline{\underline{I}}]_F \cdot [\vec{\omega}]_F$$

TENSORS ARE ABSTRACT MATRICES

MATRICES ARE EXPLICIT TENSORS

THEN,

ROTATIONAL MATRICES ARE TENSORS SINCE THEY ARE NOT TIED TO A SPECIFIC COORDINATE SYSTEM

ABOUT $[\underline{\underline{I}}]$

- $I_1 > 0 \qquad I_2 > 0 \qquad I_3 > 0$

  * POSITIVE DEFINATE

- SUM of ANY 2 MUST BE BIGGER THAN THE 3^{RD}

$$I_1 + I_2 = \int x'^2 + y'^2 + 2z'^2\ dm$$

$$\geq I_3 = \int x'^2 + y'^2\ dm$$

$= I_3 - J \; x \; + y \; m$

SAME HOLDS TRUE for OTHER 2

$$I_2 + I_3 \geq I_1$$

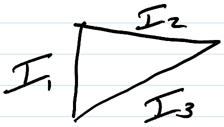
$$I_1 + I_3 \geq I_2$$

$$I_1 + I_2 \geq I_3$$

"TRIANGLE INEQUALITY"

ex) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ok!

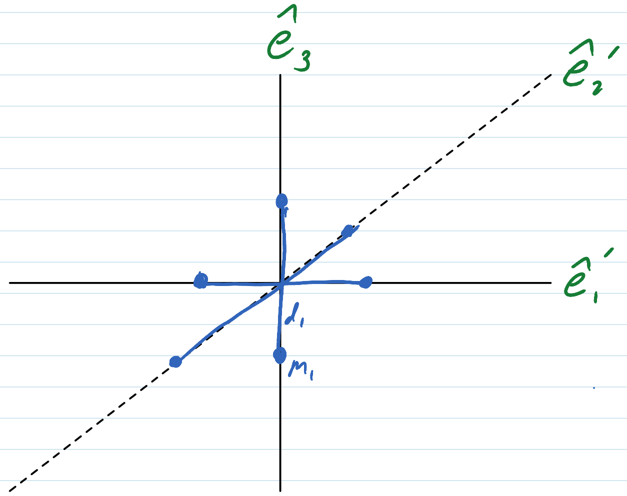
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ NOT ok!



CLASSIC EXAMPLES:

- CANONICAL OBJECT

1) JACK A: $d_1 = d_2 = d_3 \quad m_1 \neq m_2 \neq m_3$



- JACK 2: $m_1 = m_2 = m_3 = \frac{m}{6} \qquad d_1 \neq d_2$ ----

A RIGID OBJECT HAS 7 FREE PARAMETERS

COMPONENTS OF $[\underline{\underline{I}}]_F$ (6)

MASS m (1)

GIVEN PRINCIPAL DIRECTIONS, (3)

THERE ARE 4 FREE PARAMETERS

$I_1 \quad I_2 \quad I_3 \, , m$

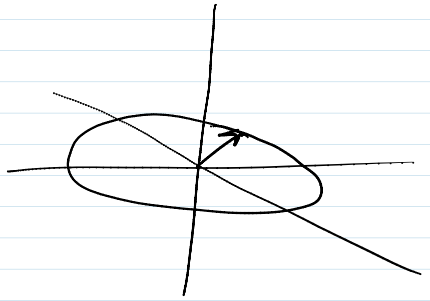
OR

$m_1 \quad m_2 \quad m_3 \quad d$

OR

$d_1 \quad d_2 \quad d_3 \quad m$

EX) hoop RADIUS r MASS m



LOOK AT Z AXIS

$$[I] = \begin{bmatrix} mr^2/2 & & 0 \\ & mr^2/2 & \\ 0 & & r^2 m \end{bmatrix}$$

$$\uparrow$$

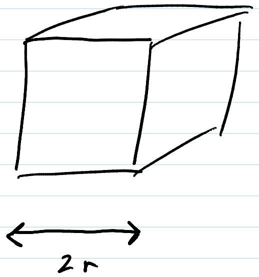
$$\begin{bmatrix} x^2 & & \\ & y^2 & \\ & & x^2+y^2 \end{bmatrix}$$

FOR PLANAR OBJECTS:

$$\int z^2 \, dm = 0 \quad , \quad I_1 + I_2 = I_3$$

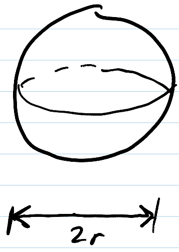
"PERPENDICULAR AXIS THEOREM"

PUZZLE PROBLEM:



m

CUBE



m

SPHERE

WHICH I IS BIGGER? SHOW

$I = \frac{2}{3} m r^2$ $I = \frac{2}{5} m r^2$

FIND I_1 I_2 I_3

Rigid Object Rotation,2/19/2013

Tuesday, Feb ua y 19, 2013

2 56 PM

A PUZZLE PROBLEM,.......

P 157 FEYNMAN'S BOOK

$\boxed{\text{* HE WAS WRONG}}$

PLATE IN AIR

- WOBBLING
- SPINNING

SPINNING TWICE AS FAST AS IT WOBBLES

CAN WE PROVE IT?

TODAY: RIGID OBJECT ROTATION

BASIC AMB

$$\Sigma M = \int \vec{r}_{/c} \times m\vec{a}_{/F} \, dm$$

$$\Rightarrow \Sigma \vec{M}_{/G} = \frac{d}{dt}\left(\vec{H}_{/G}\right)$$

$$\hookrightarrow \vec{H}_{/G} = \int \vec{r}_{/G} \times \vec{V}_{/G} \, dm$$

FOR A RIGID OBJECT:

$$\vec{H}_{/G} = \underline{\underline{I}} \cdot \vec{\omega}$$

$$\vec{\omega} \hookrightarrow \vec{\omega}_{B/F}$$

$$\Rightarrow \vec{M}_{/G} = \frac{{}^{F}d}{dt}\left(\underline{\underline{I}} \cdot \vec{\omega}\right) = \frac{{}^{B}d}{dt}\left(\underline{\underline{I}} \cdot \vec{\omega}\right) + \vec{\omega} \times \left(\underline{\underline{I}} \cdot \vec{\omega}\right)$$

$\uparrow$ $\ddot{Q}$ FORMULA

$$\boxed{M_{/G} = \underline{\underline{I}} \cdot \dot{\vec{\omega}} + \underbrace{\vec{\omega} \times \left(\underline{\underline{I}} \cdot \vec{\omega}\right)}_{}}$$

EULER EQUATION

IN 2D

$\vec{\omega}$ is in $\hat{k}$ $\quad \underbrace{\left(\underline{\underline{I}}\cdot\vec{\omega}\right)}_{\hat{k}}$

$$\hat{k}\times\hat{k} = 0$$

What about the dot?

$$\dot{\vec{\omega}}$$

$${}^{F}\dot{\vec{\omega}} = {}^{B}\dot{\vec{\omega}} + \cancel{\vec{\omega}\times\vec{\omega}} \to 0$$

$$\vec{\omega} = \vec{\omega}$$

$$\omega_i\,\hat{e}_i = \omega_{i'}\,\hat{e}_i'$$

$${}^{F}\dot{\vec{\omega}} = {}^{B}\dot{\vec{\omega}}$$

$$\dot{\omega}_i\,\hat{e}_i = \dot{\omega}_{i'}\,\hat{e}_i'$$

$\boxed{\text{SOLVE}}$ $\quad \vec{M}_{/G} = \underline{\underline{I}}\,\dot{\vec{\omega}} + \vec{\omega}\times\left(\underline{\underline{I}}\cdot\vec{\omega}\right)$ $\quad$ in Body fixed coordinates

Assume

$\hat{e}_i'$ are aligned

with the e-vectors of $\underline{\underline{I}}$

$$\underline{\underline{I}} = I_1\,\hat{e}_1'\hat{e}_1' + I_2\,\hat{e}_2'\hat{e}_2' + I_3\,\hat{e}_3'\hat{e}_3'$$

or $$\underline{\underline{I}}_{B} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

$$\left[\vec{\omega}\right]_{B} = \begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix}$$

☆ $$\dot{\vec{\omega}} = \underline{\underline{I}}^{-1}\cdot\left[\vec{M}_{/G} - \vec{\omega}\times\left(\underline{\underline{I}}\cdot\vec{\omega}\right)\right]$$

FOR EXAMPLE, TO DO THIS EXPLICITLY → MUST PUT IN VECTOR FORM

$$[\dot{\vec{\omega}}]_{\mathcal{B}} = [\underline{\underline{I}}]_{\mathcal{B}}^{-1}\left[[\vec{M}_{/G}]_{\mathcal{B}} - [\vec{\omega}]_{\mathcal{B}}\times\left([\underline{\underline{I}}]_{\mathcal{B}}\cdot[\vec{\omega}]_{\mathcal{B}}\right)\right]$$

← IN BODY FRAME

→ FOR SIMPLICITY: ALL IN BODY FRAME

$$[\dot{\omega}] = [I]^{-1}\left[[M] - [\omega]\times\left[[I]\cdot[\omega]\right]\right]$$

$$\hookrightarrow = \begin{bmatrix} 1/I_1 & & 0 \\ 0 & 1/I_2 & \\ & & 1/I_3 \end{bmatrix}$$

$$\text{SET } [M] = 0$$

YOU GET FEYNMAN PLATE PROBLEM

NO TORQUE APPLIED TO A SPINNING OBJECT

THIS IS

3 NONLINEAR ODE'S

→ CAN SOLVE IN MATLAB

☆ ω IS MOVING VECTOR EXPRESSED IN MOVING FRAME W/R/T FIXED FRAME

NOT W/R/T BODY!

FAMOUS PROBLEM #1

FIND ANY NON-TRIVIAL SOLUTION!

TRY: $\vec{\omega}$ = CONSTANT

↑

MEAN $[\vec{\omega}]_{\mathcal{B}}$

THUS $[\dot{\omega}] = 0$

PLUG IT IN

$$0 = I^{-1} \underbrace{\left[0 - \omega \times (I \cdot \vec{\omega})\right]}$$

IS IN NULL SPACE of I^{-1}

→ THIS MUST BE 0

→ IMPLIES $\vec{\omega} \times (I \times \vec{\omega}) = \vec{0}$

MEANS: ω IS PARALLEL TO $I \cdot \omega$

$\omega =$ (SOME CONSTANT) $I \cdot \omega$

$c\vec{\omega} = I \cdot \vec{\omega}$

ω IS AN e-VECTOR of I

3 SOLUTIONS: $\hat{e}_1' \quad \hat{e}_2' \quad \hat{e}_3'$

$\vec{\omega} = \omega_1 \hat{e}_1'$

OR

$\vec{\omega} = \omega_2 \hat{e}_2'$

OR

$\vec{\omega} = \omega_3 \hat{e}_3'$

SO// $\vec{\omega}$ = CONSTANT MEANS IT SPINS ABOUT A PRINCIPAL AXIS OR THERE ARE TORQUES

FAMOUS PROBLEM #2 ARE THESE ROTATIONS STABLE?

[POSITIVE OR NEGATIVE EVALUES IN LINEARIZED VERSION of SOLUTION]

* CONSTANT $\vec{\omega}$ STABLE? [w/o LOSS of GENERALITY]

$$[\vec{\omega}] = \begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix}$$

$\hat{\omega}_1 << \omega$

$\hat{\omega}_2 << \omega$

$\hat{\omega}_3 << \omega$

$\hat{\omega}$ IS A PERTURBATION

(LOOK BACK TO LYAPANOV ANALYSIS)

$\dot{\omega}=0$

$$\begin{bmatrix} \dot{\hat{\omega}}_1' \\ \dot{\hat{\omega}}_2' \\ \dot{\hat{\omega}}_3' \end{bmatrix} = \begin{bmatrix} 1/I_1 & & 0 \\ 0 & 1/I_2 & \\ & & 1/I_3 \end{bmatrix} \left[0 - \begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix} \times \underbrace{\begin{bmatrix} I_1 & & 0 \\ 0 & I_2 & \\ & & I_3 \end{bmatrix} \begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix}}_{\begin{bmatrix} I_1(\omega + \hat{\omega}_1) \\ I_2(\hat{\omega}_2) \\ I_3(\hat{\omega}_3) \end{bmatrix}} \right]$$

$$\dot{\hat{\omega}}_1 = \frac{1}{I_1}\left[-(\hat{\omega}_2 \hat{\omega}_3 I_3 - \hat{\omega}_3 \hat{\omega}_2 I_2)\right] \Rightarrow 0$$

BECAUSE $\hat{\omega}_2, \hat{\omega}_3$ ARE SMALL

$$\dot{\hat{\omega}}_2 = \frac{1}{I_2}\left[\omega \hat{\omega}_3 (I_3 - I_1)\right] = \omega \hat{\omega}_3 \frac{(I_1 - I_3)}{I_2}$$

$$\dot{\hat{\omega}}_3 = \frac{1}{I_3}\left[\omega \hat{\omega}_2 (I_1 - I_2)\right] = \omega \hat{\omega}_2 \frac{(I_1 - I_2)}{I_3}$$

$$\vec{\omega} = \vec{\omega}$$

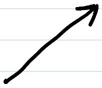
$$\omega_i \hat{e}_i = \omega'_i \hat{e}'_i$$

$$[a]_F = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$[a]_B = \begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix}$$

Notation

Vector



$\mathbf{a}$ (Boldface)

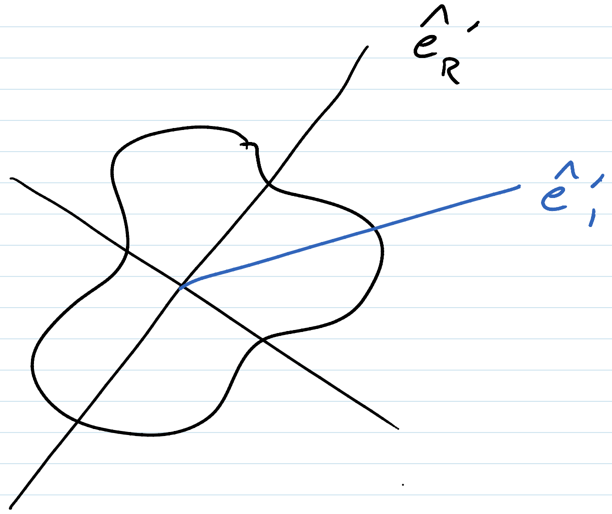
$\underline{a}$ Typeset Boldface

$\vec{a}$ Notation used

$\underline{\underline{A}}$ Tensor

$\underline{e}$ or $\hat{e}$ unit vectors

Rigid Object Continued



$$\vec{M}_{/A} = \underline{\underline{I}}\,\dot{\vec{\omega}} + \vec{\omega} \times (\underline{\underline{I}} \times \vec{\omega})$$

Body Coordinates Only

$$\dot{\omega}_1 = \frac{\omega_2 \omega_3 (I_2 - I_3)}{I_1}$$

SOLVE THESE IN
FIXED COORDINATES
SO THAT THEY ARE
NOT DECOUPLED
$[I]$ WILL NOT BE
DIAGONAL!!

$$\dot{\omega}_1 = \frac{\omega_2 \omega_3 (I_2 - I_3)}{I_1}$$

$$\dot{\omega}_2 = \frac{\omega_1 \omega_3 (I_3 - I_1)}{I_2}$$

$$\dot{\omega}_3 = \frac{\omega_2 \omega_3 (I_1 - I_2)}{I_3}$$

$\vec{M}_{/A} = \vec{0}$

* <u>DO IN MATLAB!</u>

IF $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

ONLY 1 ω CAN BE NONZERO

SO you have ROTATION ABOUT A PRINCIPAL AXIS

→ You CAN ONLY ROTATE IN EIGEN DIRECTIONS

<u>Now</u> LOOK AT SMALL PERTURBATION (FROM YESTERDAY)

$$[\vec{\omega}] = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix} \qquad [\vec{\omega}] = \begin{bmatrix} \omega + \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\omega}_3 \end{bmatrix}$$

$$\dot{\hat{\omega}}_1 = \frac{1}{I_1}\left[-\left(\hat{\omega}_2 \hat{\omega}_3 I_3 - \hat{\omega}_3 \hat{\omega}_2 I_2\right)\right]$$

FROM YESTERDAY

→ = 0 TO 2ND ORDER

$$\dot{\hat{\omega}}_2 = \frac{1}{I_2}\left[\omega \hat{\omega}_3 (I_3 - I_1)\right] = \omega \hat{\omega}_3 \frac{(I_3 - I_1)}{I_2}$$

$$\dot{\hat{\omega}}_3 = \frac{1}{I_3}\left[\omega \hat{\omega}_2 (I_1 - I_2)\right] = \omega \hat{\omega}_2 \frac{(I_1 - I_2)}{I_3}$$

DIAGONAL !!

$$\dot{\hat{\omega}}_2 = \underbrace{\omega \left[\frac{I_3 - I_1}{I_2}\right]}_{\text{CONSTANT}} \hat{\omega}_3$$

$$\dot{\hat{\omega}}_3 = \underbrace{\left[\omega \frac{I_1 - I_2}{I_3}\right]}_{\text{CONSTANT}} \hat{\omega}_2$$

PAIR OF CONSTANT COEFFICIENT 1ST ORDER ODE

→ INCREASE ORDER

$$\ddot{\hat{\omega}}_2 = \left[\omega \frac{I_3 - I_1}{I_2}\right] \underbrace{\left[\omega \frac{I_1 - I_2}{I_3}\right] \hat{\omega}_2}_{\dot{\hat{\omega}}_3}$$

$$\ddot{\hat{\omega}}_2 = \left[\omega^2 \frac{(I_3 - I_1)}{I_2} \frac{(I_1 - I_2)}{I_3}\right] \hat{\omega}_2$$

If $[\,] > 0$

$$\omega = e^{\sqrt{[\,]}t}, e^{-\sqrt{[\,]}t}$$

OR equivalently

$$\sinh\sqrt{[\,]}t, \cosh\sqrt{[\,]}t$$

If $[\,] < 0$, harmonic Oscillator

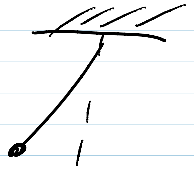
$$\hat{\omega}_2 = \sin(\sqrt{[\,]}t) \text{ or } \cos(\sqrt{[\,]}t)$$

✱ REALIZE:

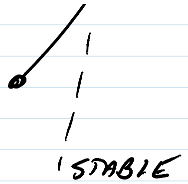
for stability, $[\,]$ must < 0

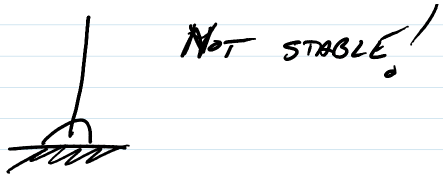
$e^{-\sqrt{\;}t}$ is NOT stable because there is only 1 condition that can exist

THINK: A PENDULUM



NOT STABLE!





"STABLE"

$$\left[(I_3-I_2)(I_1-I_2)\right] < 0$$

1 MUST BE POSITIVE AND 1 NEGATIVE

IE $I_1 < I_3$ AND $I_1 < I_2$

OR

$I_1 > I_3$ AND $I_1 > I_2$

THEN STABLE,

OTHERWISE, NOT STABLE

UNSTABLE $I_2 < I_1 < I_3$ OR $I_3 < I_1 < I_2$

CONVENTIONALLY $\left[\begin{array}{l} I_2 < I_1 < I_3 \\ I_2 > I_1 > I_3 \end{array}\right.$

SPIN ABOUT BIGGEST = STABLE

SPIN ABOUT SMALLEST = STABLE

SPIN ABOUT INTERMEDIATE = UNSTABLE

EX

AXI-SYMMETRIC OBJECT

SYMMETRY AXIS $= I_1$

I_1 $I_2 = I_3$ IE A PLATE OR DISK

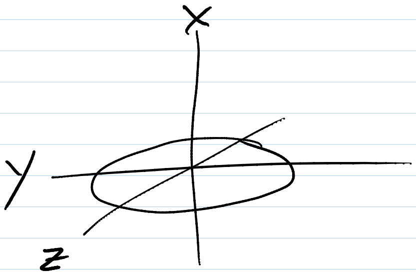
PERP. AXIS THEOREM

$$I_1 = 2I_2 = 2I_3$$

$$I_1 = \int y^2 + z^2 \, dm$$

$$I_2 = \int \underset{\searrow 0}{x^2} + z^2 \, dm$$

$$I_3 = \int \underset{\downarrow 0}{x^2} + y^2 \, dm$$



MASS IS IN $x = 0$ PLANE

PERTURBATION EQ'S

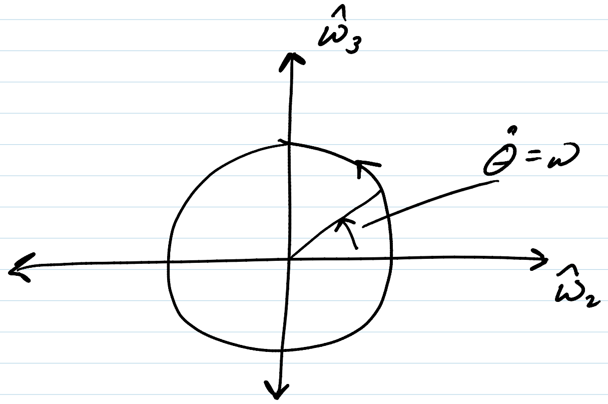
SAY $I_2 = 1$

$$\frac{I_3 - I_1}{I_2} = -1$$

$$\frac{I_1 - I_2}{I_3} = +1$$

$$\left[\begin{array}{l} \dot{\hat{\omega}}_2 = \omega\,[-1]\,\hat{\omega}_3 \\ \dot{\hat{\omega}}_3 = \omega\,[+1]\,\hat{\omega}_2 \end{array}\right.$$

$$\left[\begin{array}{l} \dot{\hat{\omega}}_2 = -\omega\,\hat{\omega}_3 \\ \dot{\hat{\omega}}_3 = \omega\,\hat{\omega}_2 \end{array}\right.$$



AXIS OF ROTATION
($\vec{\omega}$ DIRECTION)

PRECESSES ABOUT $\hat{e}_1'$ DIRECTION AT RATE ω

NOW, HOW DOES IT LOOK FROM A FIXED FRAME?

WHAT ABOUT BODY "ORIENTATION"?

$\hat{e}_1'(t) = ?$

GIVEN: $[\vec{\omega}(t)]_B$

↳ SOLUTION TO EULER EQs

$\hat{e}_2'(t) = ?$

$\hat{e}_3'(t) = ?$

GIVEN: $\hat{e}_i'(0)$

$$\boxed{{}^{F}\dot{\hat{e}}_1' = \vec{\omega} \times \hat{e}_1'}$$

* what is $\dot{\hat{e}}_1'$?

NOT BODY DERIVATIVE!

$${}^{B}\dot{\hat{e}}_1' = 0$$

COORDINATES ON THE BODY DON'T CHANGE w/r/t ITSELF!

$$[\dot{\hat{e}}_1']_F = [\vec{\omega}]_F \times [\hat{e}_1']_F$$

↑

3 DIFF EQ's for EACH UNIT VECTOR!

$\hat{e}_1'$ IS 1st COLUMN OF $\underline{\underline{R}}$

$$[\vec{\omega}]_F = [R][\vec{\omega}]_B$$

$$\boxed{[R] \text{ NOT } [R]^{-1}}$$

$$[R] = \left[[\hat{e}_1']_F \,\middle|\, [\hat{e}_2']_F \,\middle|\, [\hat{e}_3']_F \right]$$

$$R_{ij} = \hat{e}_i \cdot \hat{e}_j' = \hat{e}_j' \cdot \hat{e}_i$$

DO AS DYAD!

"SECOND SUBSCRIPT GOES WITH THE PRIME — DOES NOT MATTER RIGHT OR LEFT"

$$\vec{\omega} = \omega_i' \hat{e}_i'$$

PRIME IS IN BODY FRAME

$$\omega_j = \vec{\omega} \cdot \hat{e}_j$$

$$\omega_j = \boxed{\vec{\omega}_i' \hat{e}_i' \cdot \hat{e}_j}$$

TURN i's INTO j's

$$\omega_i = \underbrace{\hat{e}_1 \cdot \hat{e}_j'}_{R_{ij}} \cdot \omega_j'$$

DO LIKEWISE w/ $\hat{e}_2'$ AND $\hat{e}_3'$

$$[\dot{R}] = \underbrace{\Big[[R][\omega]_{\mathcal{B}}\Big]}_{\text{VECTOR}} \times \underset{\text{MATRIX}}{\text{"}R\text{"}}$$

$$\boxed{[\dot{R}] = \mathcal{S}\left(R[\omega]_{\mathcal{B}}\right) \quad R}$$

9 ODE's
SOLVED w/ EULER EQ's

$\mathcal{S}$ IS the FUNCTION THAT GENERATES THE ANTI SYMMETRIC MATRIX

IF YOU INTEGRATE FORWARD IN TIME, [R] WILL EVENTUALLY FALL APART, SO YOU HAVE TO "RESET" THE R MATRIX

IE $\boxed{R^T R \neq [1]}$

[GRAMM-SCHMDTT ORTHOGONALIZATION]

BUT THIS weights 1st VECTOR

[R-U DECOMPOSITION

TAKE ROTATION PART of POLAR PROJECTION]

→ CAN DRAW WOBBLING AXES
IN PLOT3

<u>PLOT</u> $\hat{e}_1'$ IN F

$\hat{e}_2'$ IN F

<u>WOBBLE</u> IS
ANGULAR VELOCITY
of $\hat{e}_1'$ AXIS
w/r/t A FIXED AXIS

$$\underline{\underline{R}} = R^{F}_{ij}\, \hat{e}_i\, \hat{e}_j = R_{ij}\, \hat{e}_i'\, \hat{e}_j'$$

$$R_{ij}^{F} = R_{ij}^{B}$$

OR TO MIX BASES

$$R_{BB} = R_{FF}$$ ← SAME ROTATION MATRIX IN 2 BASES !

$$[\underline{\underline{R}}]_{FF} = [\underline{\underline{R}}]_{BB}$$

* TENSOR OR MATRIX ?

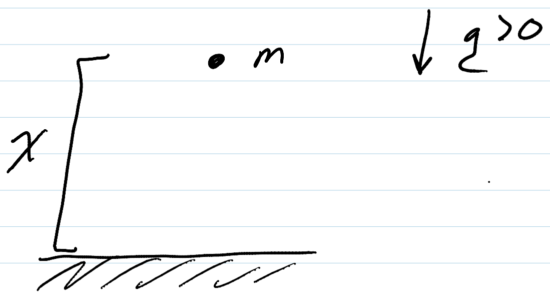
GUEST LECTURE TODD MURPHY

VARIATIONAL METHODS

"A PARTICLE IN GRAVITY"

* ANSWER

$$\ddot{x} = -g$$



Where do Euler/Lagrange EQ's come from

→ Some magical minimization of some sort of path integral

equivalent to taking a derivative of "said path integral thing"

AND SETTING EQUAL TO ZERO

→ BUT IT'S NOT QUITE SO

What IS A DERIVATIVE? [FRECHET VS GATEAUX]

WE WANT TO TAKE A DIRECTIONAL DERIVATIVE

[TO MINIMIZE AT A LOCALITY]

$f(x^*)$ IS AN EXTREME IF

$$\frac{\partial f}{\partial x} \cdot v = 0, \ \forall v$$

* TRUE FOR ALL V

SO $\frac{\partial f}{\partial x}$ MUST ALWAYS $= 0$

WE differentiate the "ACTION"

$$A = \int \underbrace{L(x, \dot{x})}_{\text{Le GRANGIAN}} dt$$

↳ $A(x(t))$

What do we MEAN By "VARIATIONS"?

So NOW LOOK AT A SPECIFIC PATH

$$A = \int_0^T L(x, \dot{x})\, dt$$

IF f IS A FUNCTION OF 1 REAL VARIABLE

ex) $f(x) = x^2$

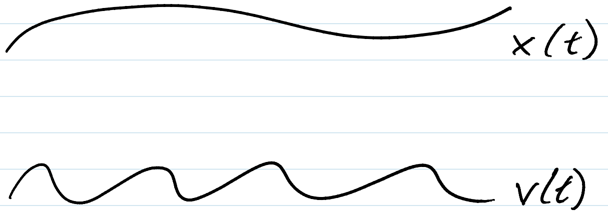
A PERTURBATION IS A SCALAR

BUT WHAT IF ITS A VECTOR SPACE

$$f(x) = x^T x \; , \quad x \in \mathbb{R}^n$$

the DIRECTIONAL DERIVATIVE IS IN THE SPACE $\mathbb{R}^n$

SO, IT MUST BE A CURVE



CURVES, BUT
CAN'T ADD "+1"
TO IT →
DATA TYPE MISMATCH

TO TAKE A DIRECTIONAL DERIVATIVE
of $A(x(t))$ IN DIRECTION of V,
V must BE A CURVE $v(t)$ * And $v(t)$ be diff w/r/t time

ASIDE

$$D f(x) \cdot v = \underbrace{\frac{d}{d\varepsilon} f(x + \varepsilon v)}\Big|_{\varepsilon = 0}$$

ε IS A SCALAR

→ DEFINES GATEAUX DERIVATIVE

D SHALL BE diff w/r/t The ARGUMENT

$$D f(x) = \frac{d}{dx} f(x)$$

for our simple system

$$\mathcal{L} = K_e - V = \frac{1}{2}m\dot{x}^2 - mgx$$

Replace x w/ $(x+\epsilon v)$

$$\frac{d}{d\varepsilon}\int\left[\frac{1}{2}m\left[\underbrace{\frac{d}{dt}(x+\epsilon v)}_{\substack{\| \\ (\dot{x}+\varepsilon\dot{v})}}\right]^2 - mg(x+\varepsilon v)\right]dt\Bigg|_{\varepsilon=0}$$

SQUARE IT

$$\frac{d}{d\varepsilon}\int\left[\frac{1}{2}m\left(\overset{\downarrow}{\dot{x}^2+2\varepsilon\dot{v}\dot{x}+\varepsilon^2\dot{v}^2}\right)-mg(x+\varepsilon v)\right]dt\Bigg|_{\varepsilon=0}$$

$$= \underbrace{\int\frac{d}{d\varepsilon}\left[\frac{1}{2}m(\dot{x}^2+2\varepsilon\dot{x}\dot{v}+\varepsilon^2\dot{v}^2)-mg(x+\varepsilon v)\right]dt}_{\text{TAKE this}}\Bigg|_{\varepsilon=0}$$

$$= \int \frac{1}{2}m(2\dot{v}\dot{x})+2\varepsilon\dot{v}^2 - mg(v)\quad dt\Bigg|_{\varepsilon=0}$$

when we set $\varepsilon = 0$

$$\int \underbrace{m\,\dot{v}\,\dot{x}}_{} - mgv\quad dt$$

↳ Integration By Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

(to get rid of $\dot{v}$)

$$\int m\underset{dv}{\dot{v}}\,\underset{u}{\dot{x}}\;dt \;\longrightarrow\; = \left[\dot{x}v - \int\ddot{x}\,v\,dt\right]m$$

dv $\quad \hookrightarrow = \left[\dot{x}v - \int \ddot{x}\, v\, dt\right] m$

IF MADE EXPLICIT: 0 to T

$$= m\left[\dot{x}\, v\Big|_0^T - \int_0^T \ddot{x}\, v\, dt\right]$$

WE OBTAIN

$$m\dot{x}v\Big|_0^T + \int -m\ddot{x}v - mgv \; dt = 0, \ \forall\, v \ (v \text{ is differentiable})$$

$$\underbrace{\cancel{m\dot{x}v\Big|_0^T}}_{0} - \int m(\ddot{x}+g)v\, dt \overset{SET}{=} 0$$

How do we get this to = 0?

REQUIRE $v(0) = v(T) = 0$

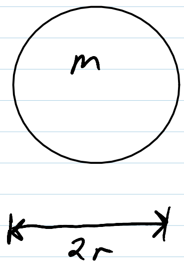
$$\Rightarrow \ddot{x} = -g$$

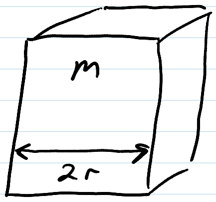
* WE ARE RESTRICTING THE DIRECTION
YOU CAN DIFFERENTIATE

① MOMENT of INERTIA

② SIMPLE 3D SOLUTIONS

Q1:





$I = ?$ $\qquad\qquad$ $I = ?$

$$I_{CUBE} > I_{SPHERE}$$

BECAUSE MORE MASS IS FURTHER FROM CENTER

FOR SPHERE $I_1 = I_2 = I_3 = \frac{2}{3}\int r^2 \, \underbrace{dm}_{\rho \, dV} \longrightarrow \iiint \rho \, dV$

$$\longrightarrow (x^2+y^2+z^2) = r^2$$

$$I_{zz} = \int (x^2+y^2) \, dm$$

$$= I_{xx} = \int (y^2+z^2) \, dm \qquad r^2 = \vec{r} \cdot \vec{r}$$

$$= I_{yy} = \int (x^2+z^2) \, dm$$

Q2) SPHERICAL SHELL $\quad I_1 = I_2 = I_3 = \frac{2}{3}\int r^2 \, dm$

$$I = \frac{2r^2}{3} m$$

$$[I]_{BB} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \frac{2r^2}{3} m$$

Q3)
SPHERE

$$I = \frac{2}{3}\int r^2 \, dm$$

$$I = \frac{2}{3}\int r^2 dm$$

$$= \frac{2}{3}\int_0^r \underbrace{r'^2 \rho \underbrace{4\pi r'^2 dr'}_{dV}}_{dm} = \frac{8}{3}\rho\pi\int_0^r r'^4 dr'$$

$4\pi r^2$ = Area of shell

$dV = A\,dr$

$I = \frac{8}{3}\rho\pi\frac{r^5}{5}$ Aside: $m = \rho\frac{4}{3}\pi r^3$

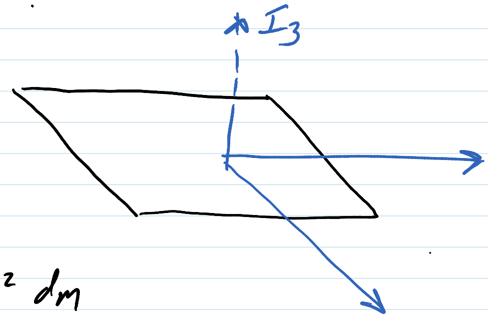
$$\boxed{I = \frac{2}{5}mr^2}$$

---

ex) Cube

$$I_3 = \int_{-r}^{r}\int_{-r}^{r}\int_{-r}^{r}(x^2+y^2)\rho\; dx\; dy\; dz$$

or



$= I_3$ for a cube

$I_3 = \int x^2+y^2\, dm$

→same for square or cube

Now we could look at I_{xx} or I_{yy}

But $I_{zz} = 2I_{xx}$

since $\int(x^2+y^2)$ ....

$$= 2\int_{-r} y^2 \, dm$$

↳ for a line

↳ ⊥ axis theorem

so $I_3 = 2 I_{\text{line segment}}$ w/ length r mass m

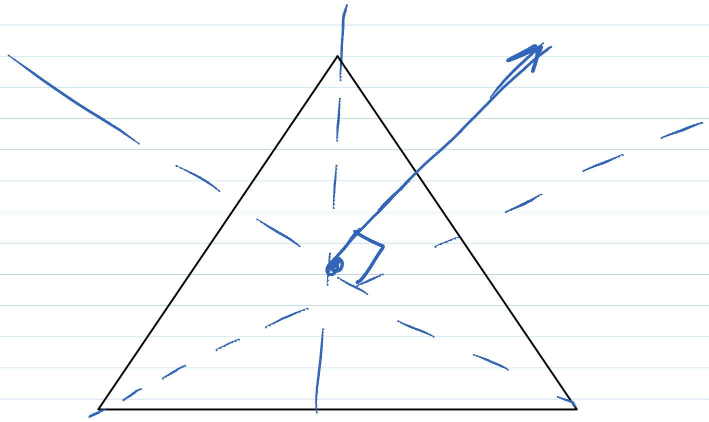
$$= 2\int_{-r}^{r} x^2 \rho \, dy$$

↳ $m/2r$

$$= \frac{2m}{2r} \frac{y^3}{3}\Big|_{-r}^{r}$$

$$I_3 = \frac{2}{3} m r^2$$

EQUALATERAL Δ



$$\underline{\underline{I}} \, \vec{v}_1 = \lambda_1 \vec{v}_1$$

$$\underline{\underline{I}} \, \vec{v}_2 = \lambda_1 \vec{v}_2$$

$$\underline{\underline{I}} (a\vec{v}_1 + b\vec{v}_2) = \lambda_1 (a\vec{v}_1 + b\vec{v}_2)$$

implies that an EQUALATERAL triangle is a circle

IN 3D

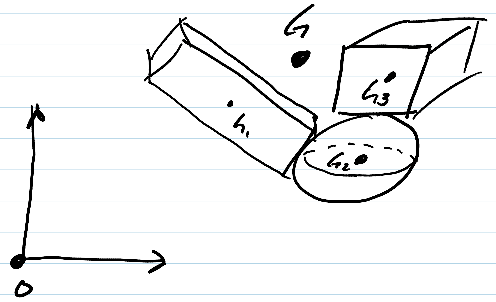
* If you have $\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3$ have same λ_1 then OBJECT = SPHERE

→ OBJECT = REGULAR Polyhedron

Ex)

Given a bunch of objects welded together, Find I_{TOTAL}

For each you know I, m, location of COM w/r/t some reference point.



* you must know $\underline{\underline{I}}$ in "crooked" orientation!!

→ Have to do change of coordinates for each object!!

Step 1 Find G:

$$\vec{r}_G = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

Step 2

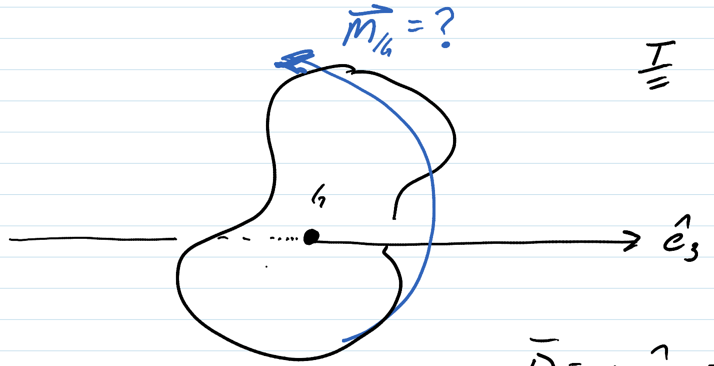
$$\left[\underline{\underline{I}}\right]_G = \sum_{\substack{\text{All}\\ \text{objects}}} \left[\left[\underline{\underline{I}}_{G_i}\right] + \underbrace{-\left[\vec{r}_{G_i/G}\right]\left[\vec{r}_{G_i/G}\right]^T + \left[\vec{r}_{G_i/G}\right]^T\left[\vec{r}_{G_i/G}\right]\left[\underline{\underline{1}}\right]}_{\substack{\text{moment of inertia}\\ \text{of point mass at } G_i\\ \text{relative to } G}} \right] m_i$$

* This could be an iterative process

where this composite object is one of several composite objects that make up the whole

---

Spin about a fixed Axis



* Torque is fixed in body frame

But

varies in fixed frame

$$\bar{\omega} = \omega \hat{e}_3 = \text{constant}$$

What is torque?

Use Euler EQ's and we know torques now!

$$\vec{M} = \dot{\vec{H}} = \vec{\omega} \times \vec{H} + \frac{d}{dt}^B \vec{H}$$

$$\hookrightarrow \frac{d}{dt}^F\left(\vec{H}\right)$$

↑

$$\hookrightarrow \frac{d^+}{dt}(\vec{H}) \qquad dt \;\ldots$$

$$\uparrow$$

IN BODY FRAME $\vec{H}$ IS CONSTANT

$$\frac{d^{B}}{dt}\left[\underline{\underline{I}} \cdot \vec{\omega}\right] = 0$$

CONSTANT IN BODY FRAME

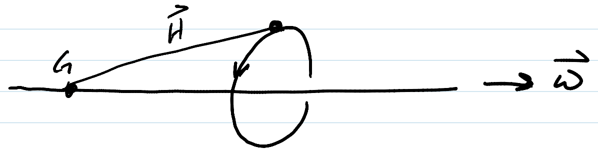
CONSTANT IN BODY FRAME

$$= \vec{\omega} \times \underbrace{\left[I \cdot \vec{\omega}\right]}$$

↳ NOT ∥ TO ω, GENERALLY

$$= \vec{\omega} \times \vec{H}$$

$\vec{H}$ SPINS AROUND



$$\left[\vec{M}_{/G}\right]_F = \omega \begin{bmatrix}0\\0\\1\end{bmatrix} \times \left[\begin{bmatrix}\sim & \sim & I_{xz}\\ \sim & \sim & I_{yz}\\ \sim & \sim & I_{zz}\end{bmatrix}\begin{bmatrix}0\\0\\\omega\end{bmatrix}\right]$$

DON'T CARE ABOUT THESE BECAUSE WE MULTIPLY BY

WARNING:

$$I_{xy} = \begin{cases}\int xy\,dm\\ -\int xy\,dm\end{cases}$$

SO

$$[\underline{\underline{I}}] = \begin{bmatrix} & -I_{xy}\\ & \end{bmatrix}$$

$$= \begin{bmatrix} & +I_{xy}\\ & \end{bmatrix}$$

Do CROSS-PRODUCT

$$[\vec{M}] = \omega^2 \begin{bmatrix}\int yz\,dm\\ \int xz\,dm\\ 0\end{bmatrix}$$

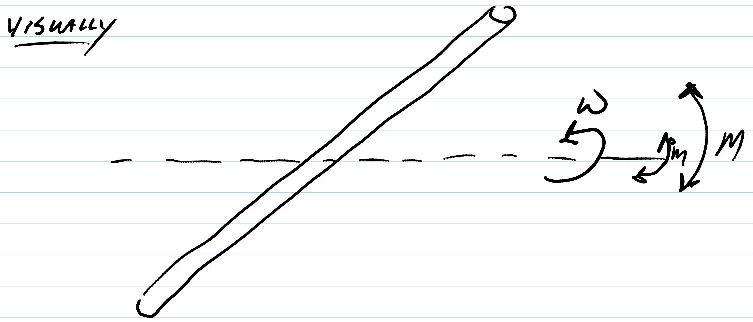
SO YOU DON'T PROVIDE TORQUE ABOUT THE SPINNING AXIS

BUT ABOUT THE OTHER 2!

SOME
BOOKS

OTHER
BOOKS

VISUALLY

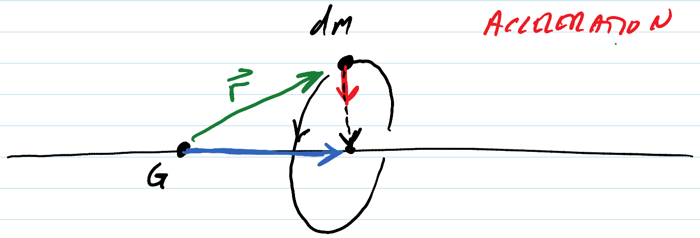


Now

$$M_{/G} = \int \vec{r}_{/G} \times \vec{a} \quad dm$$

$$\hookrightarrow \quad a = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \int \vec{r} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r})) \quad dm$$



ACCELERATION IS IN TO CENTER

$$= \int \omega^2 d \, d^\perp \quad dm$$

Wobbling of unbalanced spinning is imbalance of centripital torques

$$\Longrightarrow \left[\underline{\underline{I}}\right]_F = \begin{bmatrix} \circ & \square & \square \\ \square & \circ & \square \\ \square & \square & \circ \end{bmatrix}$$

"CENTRIFUGAL TERMS IN I"

Tensor arithmatic Examples

Tuesday, March 05, 2013
3:51 PM

$$\underline{\underline{R}} = R_{ij}\, \hat{e}_i\, \hat{e}_j \Big| \cdot \hat{e}_k$$

$$\underset{\approx}{R} \cdot \hat{e}_k = R_{ij}\, \hat{e}_i\, \delta_{jk}$$

$$\underset{\sim}{R} \cdot \hat{e}_k = R_{ik}\, \hat{e}_i \Big| \cdot \hat{e}_l$$

$$\left(\underline{\underline{R}} \cdot \hat{e}_k\right) \cdot \hat{e}_l = R_{il}$$

$$\hat{e}_l \cdot \left(\underset{\approx}{R} \cdot \hat{e}_k\right) = R_{lK}$$

$$\hat{e}_l \cdot \left(\underset{\approx}{R} \cdot \hat{e}_k\right) = R_{lK}$$

$$\hat{e}_i \cdot \left(\underline{\underline{R}}\ \hat{e}_j\right) = R_{ij}$$

$\hat{e}_i \hat{e}_j$ AND R

$$\underset{\sim}{R} \cdot \hat{e}_l = \hat{e}_l'$$

$$R_{ij} \hat{e}_i \hat{e}_j \cdot \hat{e}_l = \hat{e}_l'$$

$$\underset{\sim}{R}_{il} \hat{e}_i = \hat{e}_l' \;\Big|\; \cdot \hat{e}_k$$

$$\underset{\approx}{R}_{KL} = \hat{e}_{L}' \cdot \hat{e}_{k}$$

$$R_{ij} = \hat{e}_{i}\,\hat{e}_{j}'$$

$$\hat{e}_{i} \cdot \hat{e}_{k} = \delta_{ik}$$

NET ROTATION

ROTATE
ψ ABOUT Z
ϕ ABOUT X
θ ABOUT ORIG Z

$$\underline{\underline{R}} = \underset{\uparrow \text{ 3RD}}{\underline{\underline{R}}(\hat{e}_3, \theta)}\ \underset{\uparrow \text{ 2ND}}{\underline{\underline{R}}(\hat{e}_1, \phi)}\ \underset{\uparrow \text{ 1ST}}{\underline{\underline{R}}(\hat{e}_3, \psi)}$$

$$\left\{ \underset{3}{\underline{\underline{R}}(\hat{e}_3'', \psi)}\ \underset{2}{\underline{\underline{R}}(\hat{e}_1', \phi)}\ \underset{1}{R(\hat{e}_3, \theta)} \right\}$$

RECALL $$\underline{\underline{R}}(\hat{n}, \beta) = \cos\beta\, \underline{\underline{I}} + (1 - \cos\beta)\, \hat{n}\hat{n} + \sin\beta\, S(\hat{n})$$

NOW RECALL DYNAMICS → ω's AND $\dot{\omega}$'s

$$\alpha = \dot{\omega}$$

$$\vec{M}_{/G} = \underbrace{\underline{\underline{I}} \cdot \vec{\alpha} + \omega \times (I \cdot \omega)}_{\to\ \dot{\vec{H}}_{/G}}$$

$$\alpha = \underline{\underline{I}}^{-1} \left(\vec{M}_{/G} - \vec{\omega} \times \underline{\underline{I}} \cdot \vec{\omega} \right)$$

"CLOSED SET of ODES" for EVOLUTION of POSE

12 ODE's

$$\left[\begin{array}{l} \dot{\omega} = \vec{\alpha} \\ \dot{\underline{\underline{R}}} = S(\underbrace{\vec{\omega}}_{\to\ \vec{\omega}_{B/G}})\ \underline{\underline{R}} \end{array} \right.$$

$\uparrow$

12 ODE's

So what's the issue with using Body coordinates?

When we solve, we need to have specific sets of #'s

↳ $\vec{\omega}_{B/F}$

$$\hookrightarrow \dot{\underline{R}} = \dot{R}_{F_{ij}} \hat{e}_i \hat{e}_j$$

$$= \dot{R}_{B_{ij}} \hat{e}_i' \hat{e}_j' + R_{B_{ij}} \dot{\hat{e}}_i' \hat{e}_j' + R_{B_{ij}} \hat{e}_i' \dot{\hat{e}}_j'$$

Y'all have to keep track of everything

$\Rightarrow$

MIXED BASIS
IN VS. OUT

$$\underline{R} = \delta_{ij} \hat{e}_i \hat{e}_j = \hat{n}_1 \hat{N}_1 + \hat{n}_2 \hat{N}_2 + \hat{n}_3 \hat{N}_3$$

$\hat{N}$ = IN
$\hat{n}$ = OUT

$$\underline{R} = \begin{cases} R_{ij}\, \hat{e}_i\, \hat{e}_j \\ = R_{B_{ij}}\, \hat{e}_i'\, \hat{e}_j' \end{cases}$$

WE WANT TO REPLACE

$\dot{\underline{R}} = f(\vec{\omega})\ \underline{R}$ $\xrightarrow{\text{WITH}}$ $\dot{\underline{\Phi}} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix} =$ SOMETHING WHICH DEPENDS ON $\vec{\omega}$

ROTATION MATRIX $\xrightarrow{\text{TO}}$ EULER ANGLES

So we know

$\underline{\underline{R}}$ BECAUSE WE know EULER Angles

WE NOW WANT ω VECTORS

$$\vec{\omega}_{B/F} = \dot{\theta}\,\hat{e}_3 + \dot{\phi}\,\hat{e}_1' + \dot{\psi}\,\hat{e}_3''$$

infintessimally small Angles ADD

ROTATE ABOUT TOP BEARING

$$\hat{e}_3 = \hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{e}_1' = \underline{R}(\hat{e}_3, \theta)\,\hat{e}_1$$

1st COLUMN of ROTATION MATRIX

$$\underline{R}(\hat{e}_1', \phi)\;\hat{e}_3$$

$$\hat{e}_1' = \underline{R}(\hat{e}_3, \theta)\,\hat{e}_1$$

$$\omega_F = \underbrace{\left[\; [\hat{e}_3]_F \;\middle|\; [\hat{e}_1']_F \;\middle|\; [\hat{e}_3'']_F \;\right]}_{A} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix}$$

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = A\,\dot{\underline{\Phi}}$$

IF WE know θ, ϕ, ψ
WE CAN CALCULATE A

$$\dot{\underline{\Phi}} = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = A^{-1}\,\omega_F$$

This says we have 6 eq's

THIS SAYS WE HAVE 6 EQS

CLOSED SET OF EQ'S

$$\left[\begin{array}{l} \dot{\vec{\omega}} = \underline{\underline{I}}^{-1}\left(M_G - \vec{\omega}\times(\underline{\underline{I}}\,\vec{\omega})\right) \\ \\ \dot{\underline{\Phi}} = A^{-1}\,\omega \end{array}\right.$$

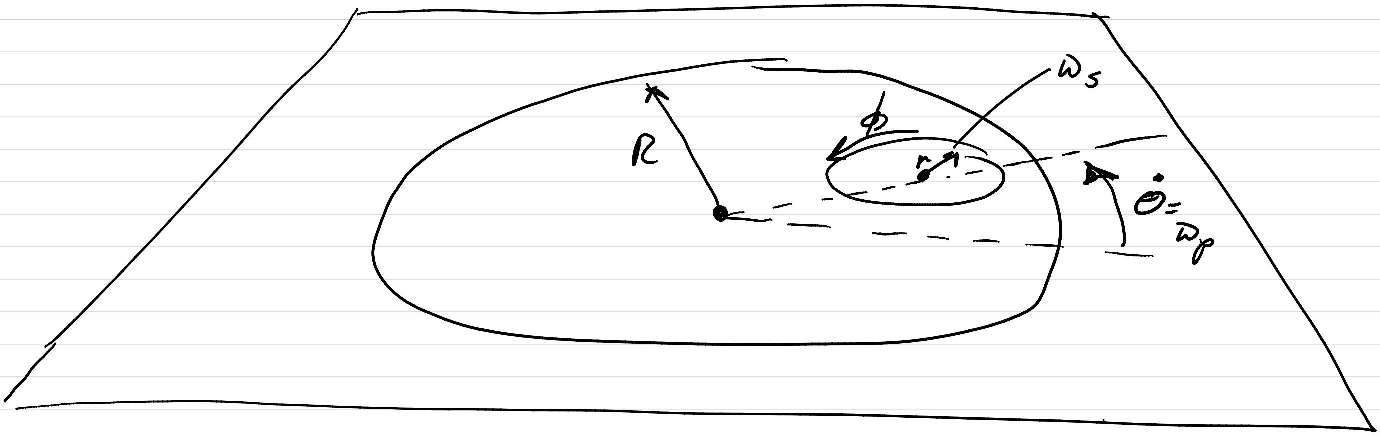
$\longrightarrow$ A CALCULATED FROM θ ϕ ψ

* When $\phi = 0$ A IS NOT INVERTIBLE

"GIMBAL LOCK"

DO THIS

DISK ON A PLANE USING SPIN + PRECESSION



r = radius of Disc

$$I = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 2I \end{bmatrix}$$

MASS = m

g = gravity

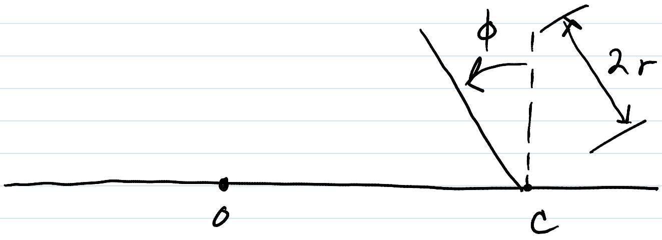
WHAT ARE RESTRICTIONS ON R, ω_s, ω_p $\quad \phi$ = lean angle

a) NO SLIP

b) NO FRICTION

c) BOTH

REDRAW



USE AMB/C

COULD GO

- STRAIGHT
- BIG CIRCLES (WHAT RADII?) } NO SLIP

ON AN ICE RINK } NO FRICTION

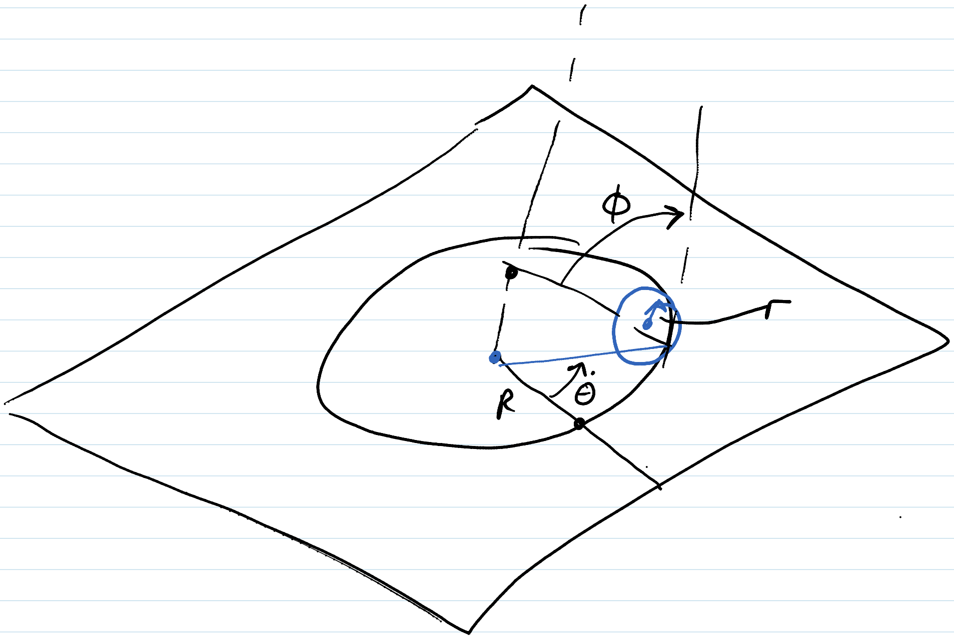
SPIN IN PLACE

GO IN STRAIGHT LINE

SPIN LIKE A COIN

NO ROTATION MATRICES

RESTATING HW PROBLEMS

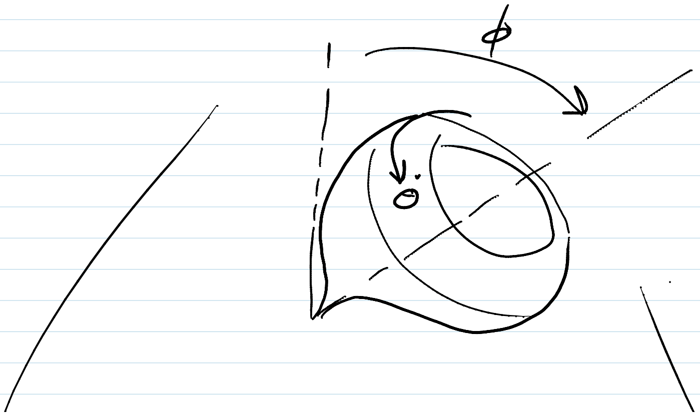


$\overset{\checkmark}{r}, \phi, R, \overset{\checkmark}{m}, \overset{\checkmark}{g}, \overset{\checkmark}{I}, \omega_p, \omega_s$ ✓ = FIXED

- * THE SPIRAL SOLUTION IS NOT THIS HERE!
  "CIRCLE ON CIRCLE" SOLUTIONS ONLY → ALGEBRAIC
  ie) R AND ϕ ARE CONSTANT
  → THERE ARE 2 FREE PARAMETERS SO YOU GET A 2-D SET OF SOLUTIONS
  → LOOK FOR INTERSECTIONS
  → THESE ARE SPECIAL CASES → "GUESSES" IN CLOSED FORM [ALGEBRAIC

NEW PROBLEM

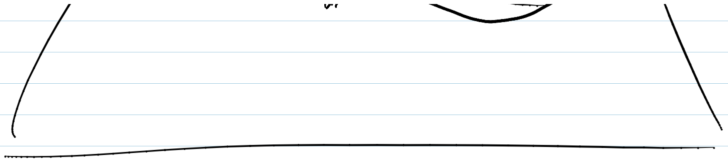
HOW FAST DO YOU HAVE TO SPIN A TOP FOR IT TO STAY UP?



- * WE HAVE A FIXED POINT AND SHOULD BE ABLE TO WRITE FULL EQ'S

- * LINEARIZE ABOUT UPRIGHT SOL'N ARE THEY STABLE?
  ie) EXPONENTIAL >0?

ALLY!]



(e) EXPONENTIAL SO ?

BUT CAN YOU DO IT W/O WRITING OUT DIFF EQ'S?

CONSIDER A SPHERICAL PENDULUM......

[CAN YOU FIND SOLUTIONS IF THE PENDULUM IS UP?]

NEW CHAPTER

"ANALYTICAL DYNAMICS"

DISCOVER VARIATIONAL PRINCIPLES FROM F=ma

FROM THERE FIND LE GRANGE EQ'S

SORT OF....

THE VARIATIONS OF THE PATH BETWEEN THE 2 POINTS

(VARIATIONS OF THE LENGTH OF THE CURVE)

ARE ZERO TO 1ST ORDER

START $\vec{F} = m\vec{a}$ FOR PARTICLES

ASSUME ALL THINGS ARE MADE OF PARTICLES

NOTATION:

$$n \text{ eq's} \left\{ \begin{array}{l} \vec{F}_1 = m_1 \vec{a}_1 \\ \vec{F}_2 = m_2 \vec{a}_2 \\ \vdots \\ \vec{F}_n = m_n \vec{a}_n \end{array} \right.$$

$$\left\{ \begin{array}{l} F_1 = m_1 \ddot{x}_1 \\ F_2 = m_2 \ddot{x}_2 \\ F_3 = m_3 \ddot{x}_3 \end{array} \right\} \quad m_1 = m_2 = m_3$$

$$\vdots$$

3D SPACE

↓

3n eq's

SCALAR

DIVIDE FORCES INTO 2 types

- CONSTRAINT (often internal)
- EXTERNAL (often conservatives)

START

$$\vec{F}_i = m_i \vec{a}_i$$

← FOR EACH MASS

$$\vec{F}_i - m_i \vec{a}_i = \vec{0}$$

$$\underbrace{\left(\vec{F}_i - m_i \vec{a}_i\right)}_{\text{this}} \cdot \underbrace{\delta \vec{V}_i}_{\text{Regardless of this}} = 0$$

DOT PRODUCT

this = 0

Regardless of this

$\delta \vec{V}_i$ = VIRTUAL VARIATION IN VELOCITY

= ANY VECTOR

could use $\delta \vec{r}_i$

so $\left(\vec{F}_i - m_i \vec{a}_i\right)$ EQUALS 0 ALWAYS!

$$\Rightarrow \sum \left(\vec{F}_i - m_i \vec{a}_i\right) \cdot \delta \vec{V}_i = 0$$

$$\vec{F}_i = \vec{F}_i^{\text{CONSTRAINTS}} + \vec{F}_i^{\text{EXTERNAL}}$$

POSTULATE A

$$\boxed{\sum \vec{F}_i^{\text{CONST}} \cdot \delta \vec{V}_i = 0}$$

For ALL $\delta \vec{V}_i$ that satisfy the constraints to first order

— OR —

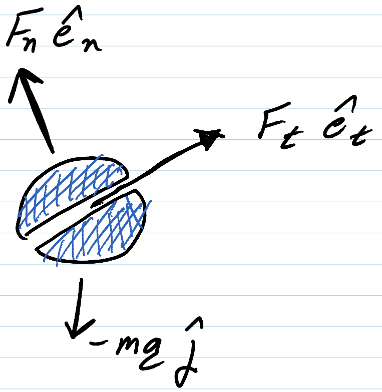
Virtual Work of Constraint forces is ZERO For Virtual displacement/Velocity that satisfy the constraints

EXAMPLE

Bead on Rigid Wire

Constraint Bead stays on wire

FBD



$$\vec{F}^{CONST} = F_n \hat{e}_n + F_t \hat{e}_t$$

$$0 = \delta W_{ORK} \qquad \vec{F}^{CONST} \cdot \delta \vec{V} = 0$$

SO $$(F_t \hat{e}_t + F_n \hat{e}_n) \cdot \underbrace{V\hat{e}_t}_{\text{ALONG WIRE}} = 0$$

$$\boxed{F_t = 0}$$

BUT F_t IS NOT NECESSARILY TRUE!

SO WE CALL IT A MODEL OR ASSUMPTION

WE KNOW IT CAN BE WRONG

→ WE "ASSUME NO FRICTION"!

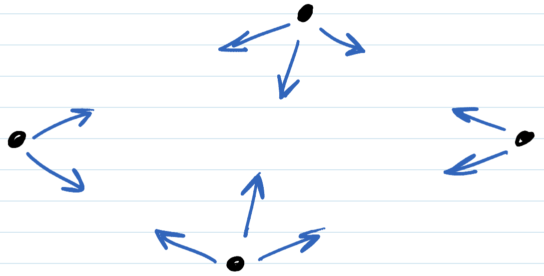
FOR EXAMPLE

EXAMPLE #2

RIGID OBJECT [MADE of PARTICLES]

CONSTRAINT FORCES

Keep All d_{ij} = CONSTANT



$$0 = \delta W = \sum \vec{F}_i^{\,CONST} \cdot \underbrace{\delta V_i}_{} = 0$$

FOR VIRTUAL VELOCITY THAT SATISFIES CONSTRAINT

$$= \sum \vec{F}_i^{\,CONST} \cdot \left[\underbrace{\vec{V}_G}_{\text{ANY VECTOR}} + \underbrace{\vec{\omega}}_{\text{ANY VECTOR}} \times \vec{r}_{i/G}\right]$$

CONSIDER $\vec{\omega} = \vec{0}$ $\quad \vec{V}_G = \hat{e}_1$

CONSIDER $\vec{\omega} = \vec{0}$ $\quad \vec{V}_{/a} = \hat{e}_1$

IMPLIES

$$\sum \vec{F}_{const} \cdot \hat{e}_1 = 0 \Rightarrow \sum \vec{F}_1 = 0$$

LIKEWISE IF $\vec{V}_{/a} = \hat{e}_2$ OR $\hat{e}_3$

$$\Rightarrow \boxed{\sum \vec{F}_i^{\,const} = \vec{0}}$$

NOW CONSIDER $\vec{V}_{/a} = \vec{0}$

WE HAVE

$$\left.\begin{aligned} \vec{0} = \delta W &= \sum \vec{F}_i^{\,const} \cdot \vec{\omega} \times \vec{r}_{i/a} \\ &= \sum \left(\vec{\omega} \times \vec{r}_{i/a}\right) \cdot \vec{F}_i^{\,const} \\ &= \sum \vec{\omega} \cdot \left(\vec{r} \times \vec{F}_i^{\,const}\right) \end{aligned}\right]$$

$(A \times B) \cdot C =$

CONSIDER $\vec{\omega} = \hat{e}_1$

then

$\vec{\omega} = \hat{e}_2$

AND

$\vec{\omega} = \hat{e}_3$

$$\delta W = 0 \Rightarrow \quad 0 = \sum \vec{r}_{i/a} \times \vec{F}^{\,const}$$

$$\Updownarrow$$

NET INTERNAL MOMENT

$= \vec{0}$

$$A \cdot (B \times C)$$

* Look @ Craig Book

Analytic Dynamics Cont'd

$\vec{F} = m\vec{a}$

from last class

$$\sum \left[F_i^{EXT} - m a_i\right] \underbrace{\delta x_i}_{\text{or } \delta v_i} = 0$$

"The Fundamental EQ of Analytical Dynamics"

$$\underbrace{\sum F_i^{EXT} \delta x_i} - \underbrace{\sum m_i \ddot{x}_i \delta x_i} = 0$$

Let's Assume All Forces are conservative

Conservative means:

- All forces depend on x's
  $F_i = F_i(x_1, x_2, x_3 \dots\dots)$
- $\int_{\vec{x}_1}^{\vec{x}_2} \sum F_i \, dx_i$ is path independent
- $\oint \sum F_i \, dx_i = 0$
- $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ "Everywhere"
- V exists such that
  $V(\vec{x}) = -\int_{\vec{x}_1}^{\vec{x}_2} \sum F_i \, dx_i$
- $\boxed{F_i = \dfrac{\partial V_i}{\partial x_i}}$ we use this

etc .....

$$\sum F_i \delta x_i = \sum \frac{-\partial V}{\partial x_i} \delta x_i = -\delta V$$

$\alpha \, x_i$

$$\underbrace{\quad}_{\varepsilon \, \underbrace{\eta_i(t)}_{\text{"some function"}}}$$

$$\delta \bar{V} = \frac{\partial}{\partial \varepsilon} \bar{V}(x_i + \varepsilon \eta_i)$$

↑ TRUE for ALL η's

$\delta \bar{V}$ is how much the potential energy changes with a small variation in x

$$\sum m_i \ddot{x}_i \delta x_i = \sum \frac{d}{dt} (\dot{x}_i \delta x_i) m_i - \underbrace{\sum (\dot{x}_i \delta \dot{x}_i) m_i}$$

ERROR GENERATED FROM PRODUCT RULE of DIFFERENTIATION

$$\left[\frac{d}{dt}(fg) = \dot{f}g + f\dot{g} \right]$$

Look at error form

$$m_i \dot{x}_i \underbrace{\delta \dot{x}_i} = \frac{m_i \delta [\dot{x}_i^{\,2}]}{2}$$

So $$\sum m_i \ddot{x}_i \delta x_i = \frac{d}{dt} \sum [m_i \dot{x}_i \delta x_i] - \underbrace{\sum \delta \left(m_i \frac{\dot{x}_i^{\,2}}{2} \right)}_{\delta T \;\to\; E_K}$$

BACK TO: *

$$\int_{t_0}^{t_1} * \; dt = \int_{t_0}^{t_1} \left[\sum F_i \delta x_i - \sum m_i \ddot{x}_i \delta x_i \right] \frac{d}{dt} = 0$$

$$= \int_{t_0}^{t_1} -\delta \bar{V} - \left[\underbrace{\frac{d}{dt} \sum m_i \dot{x}_i \delta x_i}_{**} - \delta T \right] dt$$

$$= \int_{t_0} -\delta V - \Big[\underbrace{\tfrac{d}{dt} \sum m_i \dot{x}_i \delta x_i}_{**} - \delta T \Big] dt$$

$$** = \int_{t_0}^{t_1} \frac{d}{dt} \sum m_i \dot{x}_i \delta x_i \quad dt$$

$$= \sum m_i \dot{x}_i \delta x_i \Big|_{t_0}^{t_1}$$

$\hookrightarrow$ ADD AN ASSUMPTION ABOUT VARIATION IN x_i

$\rightarrow$ ASSUME $\delta x_i = 0$ @ $t = 0$ AND $t = t_1$

$** = 0$

THEREFORE

$$\underbrace{0 = \underbrace{\delta}_{\text{for } \delta x_i = 0 \text{ @ } t=0 \text{ AND } t_1} \underbrace{\int_{t_0}^{t_1} (T - V)\, dt}_{\text{"ACTION"}}}_{\text{PRINCIPLE of LEAST ACTION}}$$

for $\delta x_i = 0$
@ $t=0$ AND t_1

PRINCIPLE of LEAST ACTION → "HAMILTON'S PRINCIPLE"

"STATIONARY"

$(T - V) = \mathcal{L}$ = "THE LAGRANGIAN"

ASSUMPTIONS

- CONSERVATIVE FORCES
- CONSTRAINTS DO NO WORK
- VARIATION of x_i SATISFIES CONSTRAINTS

SO WHAT DOES THE LAGRANGE EQ MEAN?

GIVEN A SYSTEM of PARTICLES AND RULES TO CALCULATE

$$E_p = V(x_1 \; x_2 \; x_3 \; \ldots\ldots)$$

$$E_K = T(x_1, x_2 \; \ldots\ldots)$$

KINEMATIC CONSTRAINTS

STATEMENT

For A soln of $\vec{F} = m\vec{a}$ for ALL PARTS

"IF soln Goes through $\vec{X}_0(t_0)$ AND $\vec{X}_1(t_1)$,

THEN the soln has PROPERTY that:

$$\frac{d}{d\epsilon} A = 0$$

$$\hookrightarrow A = \int_{t_0}^{t_1} \mathcal{L}\,(x_1+\epsilon\eta_1,\ x_2+\epsilon\eta_2,\ x_3+\epsilon\eta_3 \,......\, \dot{x}_1+\epsilon\dot{\eta}_1,\ \dot{x}_2+\epsilon\dot{\eta}_2,\ \dot{x}_3+\epsilon\dot{\eta}_3 \,.....)\ dt$$

For ALL η that satisfy constraints and have $\eta(t_0) = \eta(t_1) = 0$

---

2 ASIDES:

① GIVEN $\vec{V} \cdot \vec{b} = 0$ for ALL $\vec{b}$

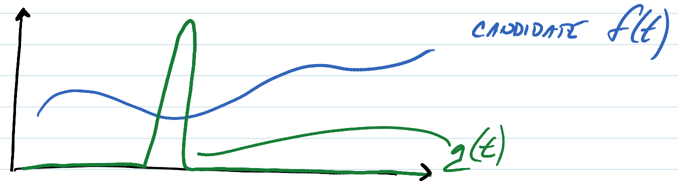
$\therefore\ \vec{V} = \vec{0}$

Let $\vec{b}$ = 1 basis vector
$\vec{V}$ is in
so that element of $V = 0$
repeat n times

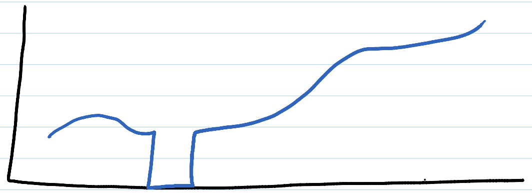
② Given $\int_0^1 f(t)\, g(t)\, dt = 0$

for all $g(t)$

Look at $g(t)$ like this:



$\Downarrow$ implies



Can put these "blips" anywhere (and everywhere!)
so $\boxed{f(t) = 0}$

Now we can derive Lagrange Equations
From Hamilton's Principle

Particles at positions $x_i(t)$

Constraints on x_i

We want to parameterize
the intersections of constraint

→ Assume we have minimal/General coordinates

such that

$$x_1 = x_1(q_1, q_2, q_3 \cdots q_n)$$

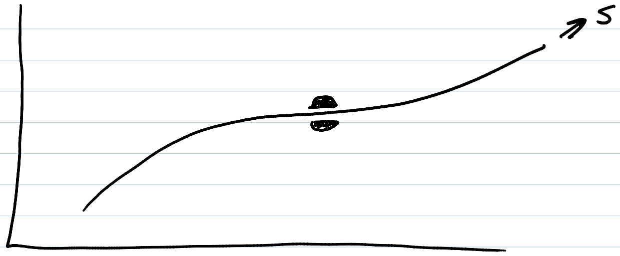
$$x_2 = x_2(q_1, q_2, q_3 \cdots q_n)$$

→ All positions + velocities can be found
from q_i + $\dot{q}_i$

q_i are independent (No constraints on q_i)

q_i parameterize the kinematically allowed configurations

ex



q could $= s$

q could $= x$ or y

or

$q = s^2$

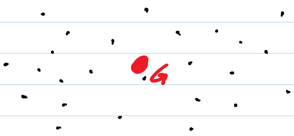
Bead on wire

$$\left.\begin{aligned} x &= f(s) \\ y &= g(s) \end{aligned}\right\}$$ Allows you to calculate position of Bead [particle]

ex

100 Particles

d_{ij} = constant



300 degrees of freedom

$\approx$ 5000 constraints
[many redundant]

But we have 6 Dimensions [3 positions, 3 angles of orientation]

$q_1 = x_G$

BUT WE HAVE 6 DIMENSIONS [3 POSITIONS, 3 ANGLES OF ORIENTATION]

$q_1 = x_G$
$q_2 = y_G$
$q_3 = z_G$
$q_4 = \phi$
$q_5 = \theta$
$q_6 = \psi$

$\left.\begin{matrix} q_4 = \phi \\ q_5 = \theta \\ q_6 = \psi \end{matrix}\right\}$ EULER ANGLES $\left\{\begin{matrix} \phi \\ \psi \\ \theta \end{matrix}\right.$

START W/

$$\delta A = 0$$

$$A = \int_{t_1}^{t_2} \mathcal{L} \, dt$$

$$\mathcal{L} = T - V$$

$$\hookrightarrow V(q_1, q_2 \cdots q_n)$$

$$\hookrightarrow (q_1, q_2, \cdots q_n, \dot{q}_1, \dot{q}_2 \cdots\cdots \dot{q}_n)$$

IF WE CALCULATE
THIS INTEGRAL

WE CAN ADD AN ARBITRARY FUNCTION THAT SATISFIES CONDITION
THAT IT = 0 AT $t = 0, t_1$

$$\mathcal{L}(q_i, \dot{q}_i)$$

$$0 = \delta \int_{t_0}^{t_1} \mathcal{L}(q_i, \dot{q}_i) \, dt$$

$$= \int_{t_0}^{t_1} \delta \mathcal{L}(q_i, \dot{q}_i) \, dt$$

$$= \int_{t_0}^{t_1} \sum \frac{\partial \mathcal{L}}{\partial q_i} \delta q_i + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta \dot{q}_i}_{} \quad dt$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i\right] - \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] \delta q_i$$

$$\Rightarrow \quad 0 = \int_{t=0}^{t=t_1} \sum \left[\left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right] \delta q_i + \underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i\right)}_{}\right] dt$$

$$\left[\partial \mathcal{L} \quad \right]^{t_1}$$

$t=0$

$$\Big[\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\,\delta q_i}_{}\Big]_{t_0}^{t_1} \longrightarrow 0$$

$= 0$ AT ENDPOINTS

$$0 = \int_{t_0}^{t_1} \sum \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] \delta q_i \, dt$$

1 INDEX AT A TRIAL

1 BLIP AT A TRIAL

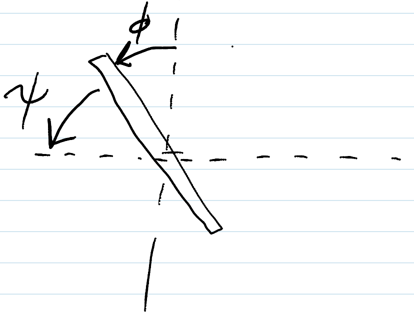
$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] = 0}$$

The Ultimate LaGrange Derivation 4/2/2013
Tuesday, April 02, 2013
3:00 PM

✱ BE MORE FORMAL IN HW!

MEASURE TIP ANGLE FROM HORIZONTAL

AS TIP ANGLE → 0   PRECESSION → ∞



MEASURE ψ, NOT ϕ

✱ DIFFERENTIATE BETWEEN
ω_θ of AXIS vs ω of OBJECT

"THE LAGRANGE DERIVATION" [w/o HAMILTON PRINCIPLE]

✱ "LAGRANGE EQ IS A LINEAR COMBINATION of ALL of THE F=ma's OF ALL of the PARTICLES of the system"

✱ "PARTIAL DERIVATIVE MAGIC"

N PARTICLES   [WE WILL USE N VECTOR EQ's RATHER THAN 3N SCALAR EQ's]

n MINIMAL/GENERALIZED COORDINATES $[q_i]$
$(1, 2, 3 \ldots\ldots n)$

$$\vec{r}_j = \vec{r}_j(q_1, q_2, \ldots\ldots q_n, t)$$

$$\dot{\vec{r}}_j = \sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_j}{\partial t} \cong \dot{\vec{r}}_j\underbrace{(q, \dot{q}, t)}$$

FOR PURPOSE of PARTIAL DERIVATIVES, think of these $(q, \dot{q}$ & $t)$ AS INDEPENDENT FUNCTIONS

If confuse

✱ think of $(q_1 \ldots q$

as q_1 & q_2

where $q_1 = \theta$ q_2

AS IN POLAR

D :

$q_n)$

$= r$

$$\boxed{\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_k} = \frac{\partial \vec{r}_j}{\partial q_k}}$$

Look AT

$$\frac{\partial \dot{\vec{r}}_j}{\partial q_k} = \sum \frac{\partial^2 \vec{r}_j}{\partial q_k q_i}\dot{q}_i + \frac{\partial^2 \vec{r}_j}{\partial q_k \partial t} \qquad Ⓐ$$

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) = \sum_{i=1}^{n} \frac{\partial\left(\frac{\partial \vec{r}_j}{\partial q_k}\right)}{\partial q_i}\dot{q}_i + \frac{\partial^2}{\partial t}\frac{\vec{r}_j}{\partial q_k} \qquad Ⓑ$$

$$\left[\frac{d}{dt} f(x,t) = \frac{\partial f}{\partial x}\dot{x}\right.$$

$$\hookrightarrow \frac{\partial \vec{r}_j}{\partial q_k}$$

ASSERT $A = B$

$$A = B \Longrightarrow \boxed{\frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) = \frac{\partial \dot{\vec{r}}_j}{\partial q_k}} \qquad ②$$

As you RECALL

$$\vec{F}_i = m\vec{a}_i \Longrightarrow \vec{F}_i - m\vec{a}_i = 0$$

AND IT FOLLOWED:

forces other than constrained forces

$$\sum_{j=1}^{n}\left(\vec{F}_j^{\,*} - m\vec{a}_j\right)\cdot \underbrace{\delta \vec{r}_j}_{\text{VARIATIONS THAT SATISFY CONSTRAINT}} = \vec{0}$$

$$\delta \vec{r}_j = \sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i}\delta q_i$$

LOOK AT:

$$\sum_{j=1}^{n} \vec{F}_j^{\,*}\cdot\delta\vec{r}_j = \sum_{j=1}^{N}\sum_{i=1}^{n} \vec{F}_j^{\,*}\cdot\frac{\partial \vec{r}_j}{\partial q_i}\delta q_i$$

$$+ \frac{\partial f}{\partial E} \Big]$$

$$= \sum_{i=1}^{n}\left[\sum_{j=1}^{N} \vec{F}_j^{*} \frac{\partial \vec{r}_j}{\partial q_i}\right]\delta q_i$$

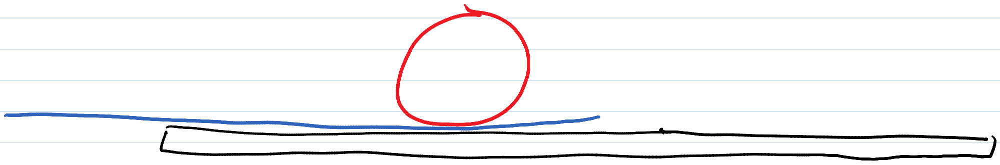
$$= \sum_{i=1}^{n} Q_i \, \delta q_i$$

$$Q_i = \sum_{j=1}^{N} \vec{F}_j^{*} \cdot \frac{\partial \vec{r}_j}{\partial q_i} \equiv \text{“}i^{th}\text{ GENERALIZED FORCE”}$$

* ANOTHER Q - PROBLEM:

→ THE NAPKIN RING PROBLEM ←

→ WHAT IS FINAL ROLLING VELOCITY



* OR

Pull TABLECLOTH out however you LIKE!

N particles 1, 2, 3 . . . . j . . . . N

1, 2, 3 SPATIAL DIMENSIONS

n - MINIMAL/ACTUAL COORDINATES 1, 2, 3 . . . . . i . . .

$$\vec{r}_j = (q_1, q_2, q_3 \ldots q_n, t)$$

$$\dot{\vec{r}}_j = (\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, t)$$

$$\sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i} \dot{q}_i + \frac{\partial \vec{r}_j}{\partial t}$$

LONG CALCULATION
REASONING

$\sum_{i=1} \quad \partial q_i \quad q_i \quad + \quad \partial t$

KEY QUANTITY

$$\boxed{\frac{\partial \vec{r}_j}{\partial q_k} \quad \text{"THE JACOBIAN"}}$$

PLUS ["IDENTITIES"]

$$\boxed{\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_i} = \frac{\partial \vec{r}_j}{\partial q_k}}$$

① "CANCELLATION of the DOTS"

$$\boxed{\frac{\partial \dot{\vec{r}}_j}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right)}$$

② "DISTRIBUTION of the DOTS"

PRINCIPLE of VIRTUAL WORK / D'ALEMBERT'S PRINCIPLE

"THE FUNDAMENTAL EQ of ANALYTICAL DYNAMICS"

$$\sum \left(\underbrace{\vec{F}_i^{\text{NON CONS}}}_{\sum \textcircled{1}} - \underbrace{m\vec{a}_i}_{\sum \textcircled{2}}\right)\cdot \delta \vec{r}_i = \vec{0}$$

$$\textcircled{1}\ \sum \vec{F}_i^{\text{NONCONSTR}} \cdot \delta \vec{r}_i = \underbrace{\sum Q_i \underbrace{\delta q_i}_{\text{GENERALIZED DISPLACEMENT}}}_{\delta W = \text{VIRTUAL WORK}} \qquad \underbrace{Q_i \equiv \sum_{j=1}^{n} \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i}}_{\text{DEF. of GENERALIZED FORCE}}$$

FORCE · DISPLACEMENT = WORK!

$$\delta \vec{r}_j = \sum_{i}^{n} \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i$$

$$\left[\delta \vec{r}_j = \sum_{i=1} \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i \right]$$

FORCE · DISPLACEMENT = WORK !

LOOK AT VARIOUS TERMS IN FINAL ANSWER [WE HOPE TO GET]

$$T = E_k$$

$$\frac{\partial T}{\partial q_k}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}$$

T: $$T = \sum_{j=1}^{N} \frac{1}{2} m_j \vec{V}_j \cdot \vec{V}_j$$

↳ $\vec{V}_j = \dot{\vec{r}}_j$

$$T = \sum_{j=1}^{N} \frac{1}{2} m_j \dot{\vec{r}}_j \cdot \dot{\vec{r}}_j$$

* PRODUCT RULE of DIFF

$\frac{\partial T}{\partial q_k}$: $$\frac{\partial T}{\partial q_k} = \sum_{j=1}^{N} m_j \underbrace{\frac{\partial \dot{\vec{r}}_j}{\partial q_k}}_{\text{PLUG ②}} \cdot \dot{\vec{r}}_j$$

$$\frac{\partial T}{\partial q_k} = \sum m_j \overbrace{\frac{d}{dt}\frac{\partial \vec{r}_j}{\partial q_k}} \cdot \dot{\vec{r}}_j \qquad Ⓒ$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} = \frac{d}{dt}\left[\sum_{j=1}^{N} m_j \left(\frac{\partial \dot{\vec{r}}_j}{\partial \dot{q}_k}\right) \cdot \dot{\vec{r}}_j\right]$$

$$= \frac{d}{dt}\left(\sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k} \cdot \dot{\vec{r}}_j\right)$$

$$= \sum_{j=1}^{N} m_j \frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right) \cdot \dot{\vec{r}}_j + \sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k} \cdot \ddot{\vec{r}}_j \qquad Ⓓ$$

$$= \sum_{j=1}^{N} m_j \frac{d}{dt}\left(\frac{\partial \vec{r}_j}{\partial q_k}\right)\cdot \dot{\vec{r}}_j + \sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k}\cdot \ddot{\vec{r}}_j \qquad Ⓓ$$

✱ Ⓒ AND Ⓓ SHARE COMMON TERM → SUBTRACT AND IT DROPS OUT

$$\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_k}\cdot \ddot{\vec{r}}_j\right\}$$

SWITCH INDEX $k \Rightarrow i$

MULTIPLY THIS [BOTH SIDES] BY ANY FUNCTION WE LIKE! IN THIS CASE $\delta q_i \Rightarrow \sum_{i}^{n} \{\} q_i$

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right]\delta q_i = \sum_{i=1}^{n}\sum_{j=1}^{N}\left[m_j \frac{\partial \vec{r}_j}{\partial q_i}\cdot \ddot{\vec{r}}\right]\delta q_i$$

NOW RECALL THAT WE STARTED WITH:

$$\vec{F}_j = m_j \ddot{\vec{r}}_j \quad \xRightarrow{\text{BECOMES}} \quad \vec{F} \text{ NON CONSTRAINT}$$

ONLY IN THE SUM!

✱

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}\right)\delta q_i = \sum_{i=1}^{i} Q_i\, \delta q_i$$

ex) $\delta q_1 \neq 0$, $\delta q_2 = \delta q_3 = \dots\dots = 0$

likewise, for $i = 1 \dots\dots n$

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i}$$

$$\boxed{Q_i \equiv \sum_{j=1}^{n} \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i}}$$

"LAGRANGE FOR NON-CONSERVATIVE FORCE"

INSERT AT Ⓐ

$$\sum_{i=1}^{n}\sum_{j=1}^{N} m_j \frac{\partial \vec{r}_j}{\partial q_i}\ddot{\vec{r}}_j \cdot \delta q_i \longrightarrow \delta\vec{r}_j$$

ONCE YOU SUM OVER i

$$= \sum_{j=1}^{N}\underbrace{\left[\sum_{i=1}^{n} \frac{\partial \vec{r}_j}{\partial q_i}\delta q_i\right]}\cdot m_j \ddot{\vec{r}}_j$$

$$= \sum_{j=1}^{N} m_j \ddot{\vec{r}}_j \cdot \delta\vec{r}_j$$

$$= \sum_{j=1}^{N} \vec{F}_j \cdot \delta\vec{r}$$

↳ $\vec{F}_j^{\text{NON CONSTRAINT}} + \vec{F}_j^{\text{CONSTRAINT}}$

$$= \sum_{j=1}^{N} \vec{F}_j^{\text{NONCONSTRAINT}} \cdot \delta\vec{r}_j = \sum_{i=1}^{n} Q_i\,\delta q_i$$

$$Q_i = \sum_{j=1}^{3N} \frac{\partial r_j}{\partial q_i} F_j^{NC}$$

3N WILL

REMOVE DOT PRODUCT

$r_j = r_j(q_1, q_2 \ldots\ldots 3N)$

INSTEAD ~~of~~ N

ASSUME:

CONSERVATIVE FORCES $\left[F_j = \frac{-\partial V}{\partial r_j}\right] \leftarrow E_p = V$

* IF NON-CONS FORCES
ADD BACK IN Q

$$V = V(r_1, r_2 \dots r_N)$$
$$= V(q_1, q_2, \dots\dots q_n)$$

$$Q_i = \sum_{j=1}^{3N} \frac{\partial r_j}{\partial q_i}\left(-\frac{\partial V}{\partial r_j}\right) = -\frac{\partial V}{\partial q_i}$$

LAGR. EQ:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}$$

V might depend on t,
But NOT $\dot{q}$!

$$V(q, \cancel{\dot{q}}, t)$$

$$\frac{d}{dt}\frac{\partial (T-\underbrace{V}_{\text{No Effect}})}{\partial \dot{q}_i} - \frac{\partial (T-V)}{\partial q_i} = 0$$

Axioms of Mechanics 4/9/2013

Tuesday, April 09, 2013
2:57 PM

"Axioms" = Assumptions

Basics ("Facts")

By "facts" we mean accurate [true] to 1 part in $\approx 10^{-8}$ for terrestrial mechanics

$$10^{-9}\,m < d < 10^{9}\,m$$ [characteristic lengths]

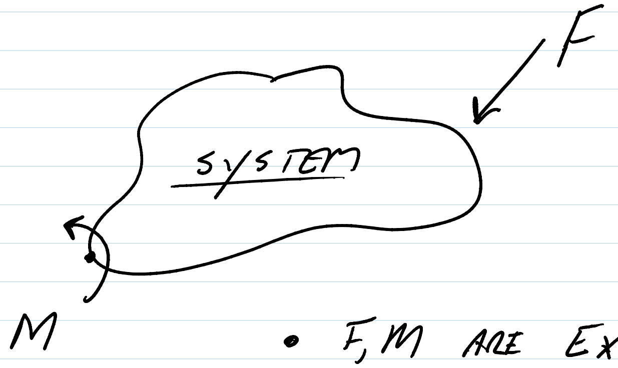
$$v < 10^{5}\ m/s$$

Assume

- Space and time are "flat" and obey all the laws you know + like (Euclid, continuity of time)
- Mass is immutable
  - Doesn't come or go
  - It's local: you can label it
- A Newtonian frame exists where laws hold!

"Andy Recommends" the following postulates:

① Force (and moment) are "THE" means of system interaction



- F, M are External
- No internal forces
- System of Mass Particles

LAWS of MECHANICS

0) ACTION - REACTION

IF SYSTEM A CAUSES F & M ON B,

B CAUSES −F & −M ON A

1) FOR ANY SYSTEM

a) <u>LMB</u>:

$$\sum \vec{F}^{ext} = \begin{cases} \sum m_i \vec{a}_i \\ \int \vec{a}\, dm \end{cases} \qquad \boxed{\vec{a} \equiv \vec{a}_{/F}}$$

b) <u>AMB</u>:

$$\sum \vec{M}_{/c}^{\,ext} = \begin{cases} \sum \vec{r}_{i/c} \times m_i \vec{a}_i \\ \int \vec{r}_{i/c} \times \vec{a}\, dm \end{cases}$$

* ALREADY, THESE ARE <u>NOT</u> INDEPENDENT!

<u>ex</u>) $AMB_{/c_1} + AMB_{/c_2} + AMB_{/c_3} \Rightarrow LMB$

* $AMB_{/c_1} \Longrightarrow \sum \vec{r}_{i/c_1} \times \left(\vec{F}_i^{\,ext} - m_i \vec{a}_i\right) = \vec{0}$

↳ NET EXT FORCE ON PART i

* * $AMB_{/c_2} \longrightarrow \sum \vec{r}_{i/c_2} \times \left(\vec{F}_i^{\,ext} - m_i \vec{a}_i\right) = 0$

* * * $AMB_{/c_3}$ 〃

$\vec{r}_{c2} - \vec{r}_{c1} = d_1$ $\vec{r}_{c3} - \vec{r}_{c1} = d_2$

SUBTRACT EQ's

$$\vec{d}_1 \times \underbrace{\sum\left(\vec{F}_i^{\,ext} - m_i\vec{a}_i\right)}_{\vec{A}} = \vec{0}$$

$$\vec{d}_2 \times \underbrace{\sum\left(\vec{F}_i^{\,ext} - m_i\vec{a}_i\right)}_{\vec{A}} = \vec{0}$$

$\Rightarrow \vec{d}_1 \times \vec{A} = 0$ IMPLIES $\vec{A}$ IS $\parallel \vec{d}_1$

$\vec{d}_2 \times \vec{A} = 0$ → $\vec{A}$ IS $\parallel \vec{d}_2$

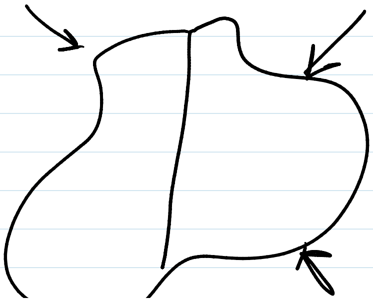
IF C_1, C_2, C_3 ARE NOT CO-LINEAR

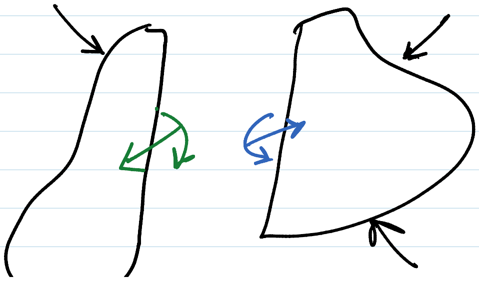
$[\vec{d}_1$ NOT $\parallel$ to $\vec{d}_2] \Rightarrow A = \vec{0} \Rightarrow$ LMB !

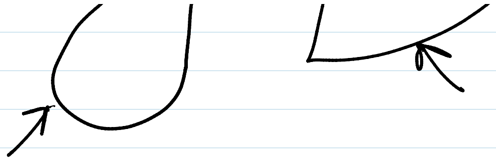
[F=ma]!

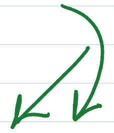
Applying AMB, LMB to Arbitrary Sub systems

implies Action & Reaction







SO  MUST CANCEL 

BECAUSE rhs's ADD

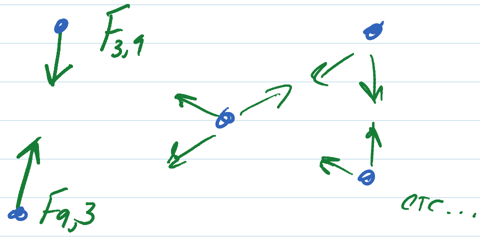
* "THE CULTURE OF FORCE"

FRANK WILCZEK (SP?)

USUAL MODEL

ACCEPT $\vec{F} = m\vec{a}$ for PARTICLES

USE: ALL INTERNAL FORCES ARE PAIRWISE & OPPOSITE FORCES



ASSUME

$F_{3,9} = F_{9,3}$

IMPLIES:

$$\sum\left(\vec{F}_i^{\,EXT} + \underbrace{\vec{F}_i^{\,INT}}\right) = \sum m_i \vec{a}_i \qquad \text{LMB}$$

$$\hookrightarrow \sum \vec{F}_i^{\,INT} = \vec{0}\,!$$

$$\sum \vec{r}_{i/c} \times \left(\vec{F}_i^{\,EXT} + \vec{F}_i^{\,INT}\right) = \sum \vec{r}_{i/c} \times m_i \vec{a}_i \qquad \text{AMB}$$

WHY ARE THESE "BAD" DERIVATIONS?

② DOESN'T AGREE WITH ACTUAL MICROSCOPIC PARTICLES
(MORE TO PHYSICS than ELECTROSTATICS & $\frac{mMG}{r^2}$)

① NOT CONSISTENT TO MAKE MECHANICS REST ON PHYSICS which you don't KNOW ABOUT !

③ IMPLIES RESTRICTIONS ON moduli;
CALLED the ___?___ (forgot) relationships

for isotropic MATERIALS

$\nu = 1/4$ $(1/3)$?

ν → POISSON RATIO

BAD IDEA TO REST MECHANICS ON AN ASSUMPTION THAT GIVES A BAD MACROSCOPIC PREDICTION

ALTERNATIVE APPROACH

START w/: $\vec{F} = m\vec{a}$ for particles

AND ADD

a) INTERNAL forces ADD to ZERO AND have NO moment

(NOT NECESSARILY PAIRWISE)

b) INTERNAL FORCES DO NO WORK IN VIRTUAL TRANSLATIONS AND ROTATION

(FROM CALCULUS OF VARIATIONS)

WE NEED a) OR b)

"STYROFOAM"

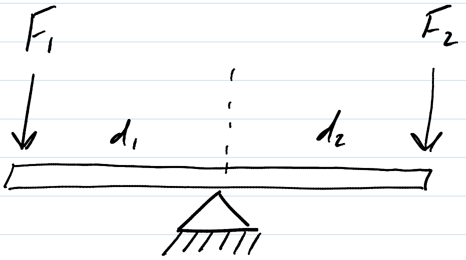
KONIG'S THEOREMS

C.O.M. SIMPLIFICATIONS of LMB, AMB, E_k

$\dot{L}$ $\dot{\vec{H}}$ $\dot{\vec{E}}_k$

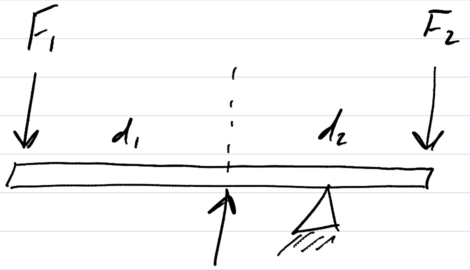
STATICS AS AN APPROACH

START w/ PRINCIPLE of LEVER

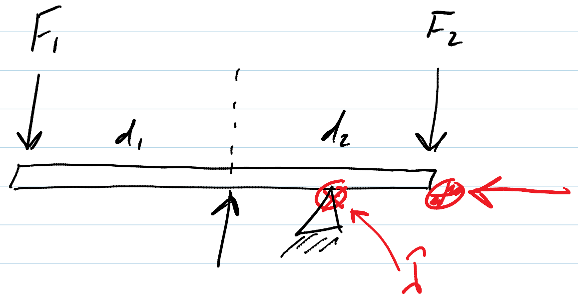


$$d_1 F_1 = d_2 F_2$$

IMPLIES WE CAN REPLACE FULCRUM w/ A HINGE



AND CAN MOVE THE hinge Anywhere



IF WE ADD FORCES PARALLEL + ⊥ TO hinge

FORCES PARALLEL TO hinge have NO EFFECT
⊥ TO hinge have NO effect

IMPLIES ⟹

$$\underbrace{\hat{\lambda}\cdot\sum \vec{M}_{/c} = \vec{0}}_{\text{MOMENTS ABOUT ALL AXES} = 0}$$

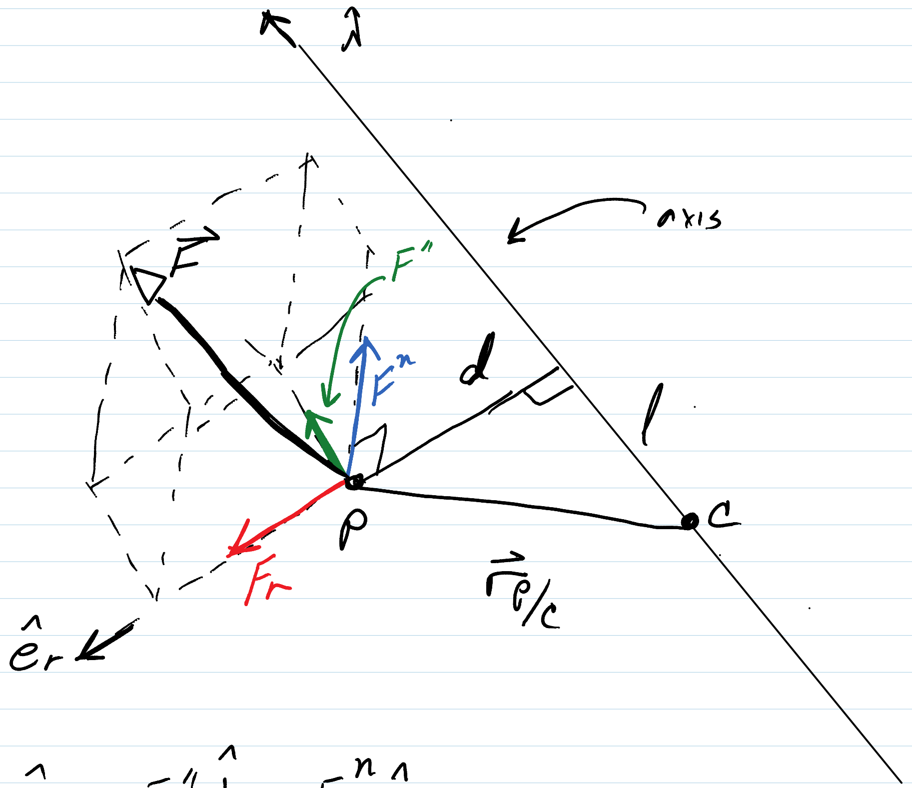
C IS A POINT ON the AXIS
$\hat{\lambda}$ IS ALONG the AXIS] for ALL C & $\hat{\lambda}$

$$\longrightarrow \sum \vec{M}_{/c} = 0$$

* Look up × PRODUCTS IN CH 2 IN UNDERGRAD Book

AXIOMS OF MECHANICS, CONT'D

BACK TO MOMENTS



$$\vec{F} = \hat{e}_r + F''\hat{\lambda} + F^n\hat{n}$$

$$\vec{r} = l\,\hat{\lambda} + d\,\hat{e}_r$$

STATIC EQUILIBRIUM

PRINCIPLE OF LEVER : FOR ANY AXIS

$$\sum F_i^n d_i = 0$$

Claim $F^n d = \hat{\lambda} \cdot (\vec{r}_{P/l} \times F_i)$

$= F^n d$ ✓

$\hookrightarrow l\hat{\lambda} + d\hat{e}_r$

$\hookrightarrow \vec{F} = F^r\hat{e}_r + F''\hat{\lambda} + F^n\hat{n}$

LEVER PRINCIPLE

$$\sum M_{\text{axis}} = 0$$

$$\hat{\lambda} \cdot \sum \vec{M}_{/c} = 0$$

$\llcorner$ ANY $\hat{\lambda}$

$$\Rightarrow \sum \vec{M}_{/c} = 0 \quad \Rightarrow \quad \sum \vec{F} = \vec{0}$$

BY PICKING
3 NON COLINEAR
$c_1 \quad c_2 \quad c_3$

ALL OBJECTS ARE IN STATIC EQUILIBRIUM
BUT
HAVE TO PUSH AROUND BITS of MASS
WHICH OBEY:

$\vec{F} = m\vec{a}$ AND ACTION/REACTION

AND THESE PUSH BACK WITH $-m\vec{a}$

$$\sum \vec{M}_{/c} = \vec{0} \Rightarrow \sum\left(\vec{r}_{i/c} \times \vec{F}_i + -\vec{r}_{j/c} \times m_j \vec{a}_j\right) = \vec{0}$$

DYNAMICS

$\hookrightarrow i \rightarrow$ ALL EXTERNAL FORCES

$\hookrightarrow j \rightarrow$ ALL POINTS of MASS

"MATTER IS MADE of STRUCTURE which IS MASSLESS
AND OBEYS STATICS AND HAS EXTERNAL LOADS
AND INERTIAL REACTIONS"

IDEA

IDEA

Styrofoam in equalibrium
Loaded by forces
BB's pushing back

⟺ internal forces have NO net moment

Done w/Axioms

3D w/ Constraints

"Harder Problems"

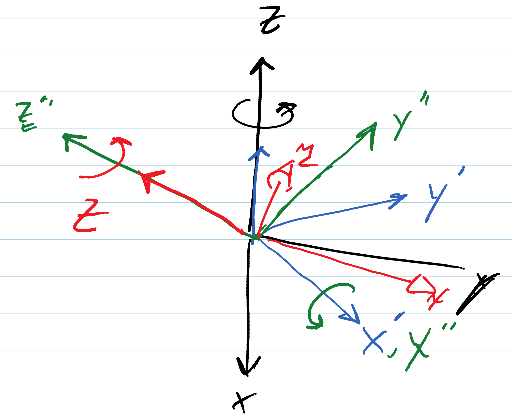
"Rolling Disc with Euler Angles"

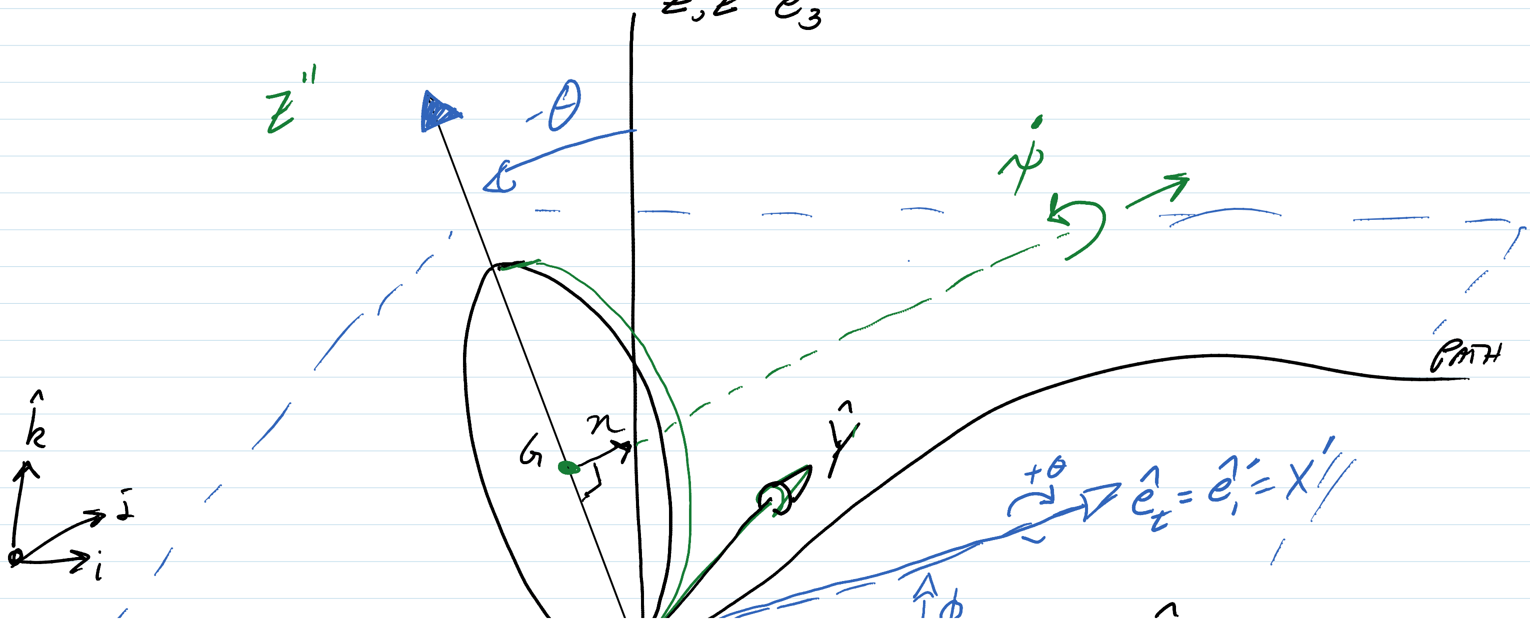
Parameterization of Axes

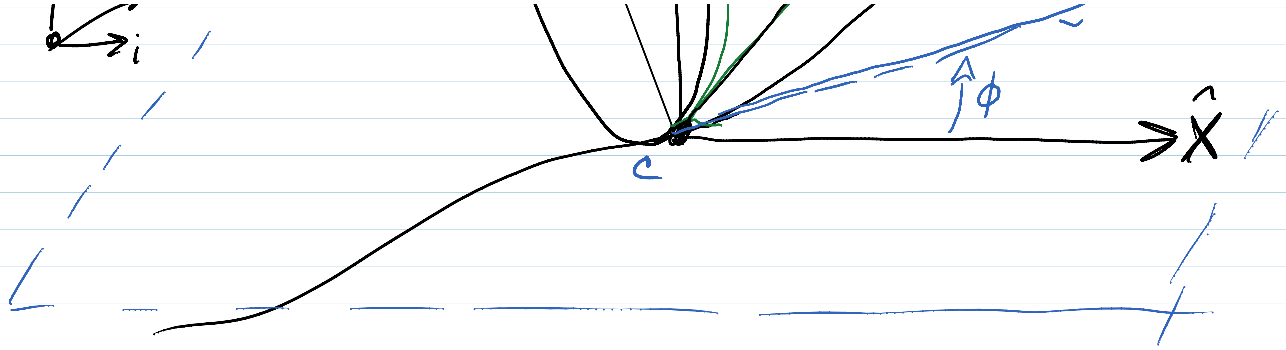
①

Z Y Z'

3-2-1- Euler Angles







n is $\perp$ to
plane of Disc

STARTING $\hat{n} = \hat{e}_2$ = normal to plane of Disc

$\hat{\lambda} = \hat{k} = \hat{e}_3$

ϕ = steer, yaw : Rotation about Z Axis

θ = lean, "roll" : Rotation about $\underbrace{\text{"New X"}}_{Y'}$ Axis

ψ = spin, pitch, : Rotation about $\underbrace{\text{"Newer X"}}_{\hat{n}}$ Axis

∗ Configuration Space is 5 Dimension

Velocity space is 3

Non holonomic system is where
Dimension of accessible Configuration Space

EXCEEDS

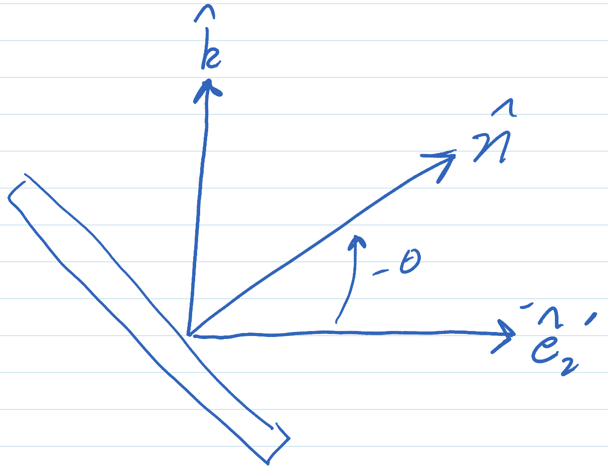
DIMENSION OF ACCESSIBLE VELOCITY SPACE

* We will use AMB/C [3 EQ's]

$$\vec{\omega} = \vec{\omega}_{B/F} = \underset{\hookrightarrow y'}{\dot{\theta}\,\hat{e}_t} + \dot{\phi}\,\hat{k} + \dot{\psi}\,\hat{n}$$

$$\dot{\vec{\omega}} = \underbrace{\ddot{\theta}\,\hat{e}_t}_{\text{GOOD,}} + \dot{\theta}\,\dot{\hat{e}}_t + \ddot{\phi}\,\hat{k} + \cancelto{0}{\dot{\phi}\,\dot{\hat{k}}} + \ddot{\psi}\,\hat{n} + \dot{\psi}\,\dot{\hat{n}}$$

Define others from k, $\hat{e}_t$

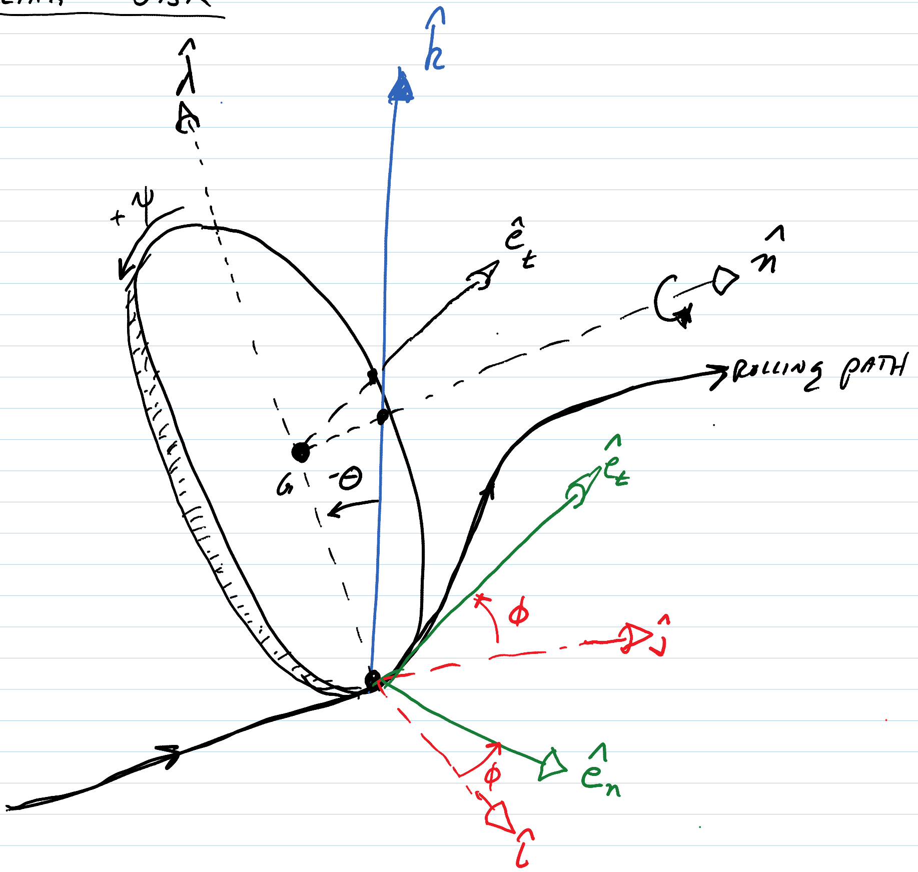


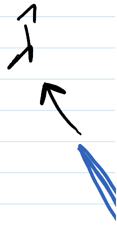
ROLLING DISK

STEER = YAW = ϕ

LEAN = ~~ROLL~~ = $-\theta$

PITCH = SPIN = ψ





REST

$\phi = \theta = \psi = 0$

$\hat{n} = \hat{i} = \hat{e}_n$

$\hat{e}_t = \hat{j}$

$\hat{\lambda} = k$

$\hat{n}$ IS NORMAL TO DISK

$\hat{e}_t$ is tangent to path, IN DISK PLANE AND x-y PLANE

$\hat{e}_n = -\hat{k} \times \hat{e}_t$ projection of $\hat{n}$ onto XY PLANE

$\hat{\lambda} = \vec{r}_{A/C} / |\vec{r}_{A/C}|$

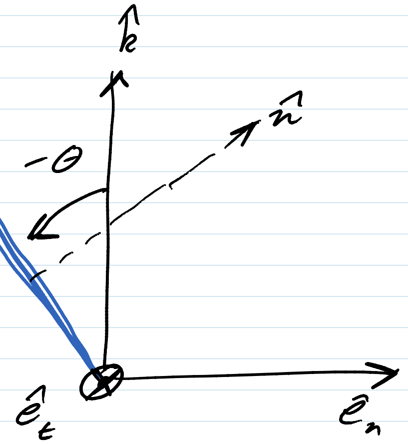
4 REFERENCE FRAMES

	• FIXED, F	$\hat{i}, \hat{j}, \hat{k}$
P	• PRECESSING	$\hat{e}_n, \hat{e}_t, \hat{k}$
T	• TIPPED, PRECESSING	$\hat{n}, \hat{e}_t, \hat{\lambda}$
B	• BODY	$\hat{n}$, ?, ?

SAME FRAME BUT 2 DIFFERENT COORDINATE SYSTEMS IF θ IS CONSTANT

(IE ROLLING DISK FROM PROBLEM 15)

* WE DO ALL CALCS IN PRECESSING FRAMES


$\hat{k}$
$\hat{n}$
$-\theta$
$\hat{e}_t$
$\hat{e}_n$

THEN TRANSLATE TO $\mathcal{F}$

~~GEOMETRY~~ [AS NEEDED]

FRAME 2 IS LINED UP w/ RATES of change of angles

FRAME 3 IS LINED UP w/ INERIA

$$\hat{k} = \cos\theta \ \hat{\lambda} + (-\sin\theta)\ \hat{n}$$

$$\dot{\hat{e}}_t = \underbrace{-\dot{\phi}\hat{e}_n} \rightarrow \dot{\phi}\hat{k} \times \hat{e}_t = \vec{\omega}_{P/F} \times \hat{e}_t$$

$$\hat{e}_n = \cos\theta \ \hat{n} + \sin\theta \hat{\lambda}$$

$$\vec{r}_{G/C} = r\hat{\lambda}$$

$$A_{MB/C} = (r \times ma) + I\alpha + \omega \times I\alpha$$

$$\vec{\omega} = \dot{\phi}\ \hat{k} + \dot{\theta}\hat{e}_t + \dot{\psi}\ \hat{n}$$

STEERING ABOUT $\hat{k}$

WRITE $\hat{k}$ IN terms of $\hat{\lambda}, n$

$$\llcorner \hat{k} = \cos\theta\ \hat{\lambda} - \sin\theta\ \hat{n}$$

$$\vec{\omega}(\underbrace{\dot{\phi}, \dot{\theta}, \dot{\psi}}, \underbrace{\hat{n}, \hat{\lambda}, \hat{e}_t})$$

EULER Angles $\qquad \underline{\underline{I}}$ IS NICE

[THIS WILL BLOW UP INTO A MESS $\rightarrow \vec{\alpha} = \dot{\vec{\omega}} = $ MESS]

$$\dot{\hat{k}} = -\sin\theta\dot{\theta}\hat{\lambda} + \cos\theta\ \dot{\hat{\lambda}} + \cos\theta\ \dot{\theta}\ \hat{n} - \sin\theta\ \dot{\hat{n}} = \vec{0}$$

How?

$$\vec{\omega} = \dot{\phi}\ \hat{k} + \dot{\theta}\ \hat{e}_t + \dot{\psi}\ \hat{n}$$ AND IS True for ALL TIME

SO you CAN take DERIVATIVE → BUT $\hat{k}$ IS CONSTANT

SO $\dot{\hat{k}} = \vec{0}$ $\qquad$ or Take d/dt

or $\omega \times \omega k$

$\hookrightarrow = \vec{\omega}_{BODY}$ w/r/t $\mathcal{F}$

(take d^F)

ROAD

$A_{MB/C}$

$$\Sigma M_{/C} = \dot{H}_{/C}$$

$$= \vec{r}_{G/C} \times ma$$

GOAL: $\vec{\omega}, \dot{\vec{\omega}}$

IN TERM

☆ 6 $\vec{\omega}$'s

$\vec{\omega}_{B/F}$, $\vec{\omega}_{\psi/F}$, $\vec{\omega}_{B/F}$ etc

MAP, STRATEGY

$a + \underline{\underline{I}}\,\dot{\vec{\omega}} + \vec{\omega} \times \underline{\underline{I}}\,\omega$

, $\vec{a}_G$, $\vec{r}_{a/c}$

ns of Euler Angles

↳ = ω_{BODY} w/r/t F or $\omega \times \omega k$

take $\frac{d^F}{dt}\left(\vec{\omega}_{B/F}\right)$

$$\boxed{\vec{\alpha} = \ddot{\phi}\hat{h} + \ddot{\theta}\hat{e}_t + \dot{\theta}\underbrace{\dot{\hat{e}}_t} + \ddot{\psi}\hat{n} + \dot{\psi}\underbrace{\dot{\hat{n}}}}$$

$\hat{h} = \cos\theta\,\hat{\lambda} - \sin\theta\,\hat{n}$

$$\dot{\hat{e}}_t = -\dot{\phi}\left[\cos\theta\,\hat{n} + \sin\theta\,\hat{\lambda}\right]$$

$\hat{n}$ changes about $\hat{h}$ and $\hat{e}_t$

$$\dot{\hat{n}} = \underbrace{-\dot{\theta}\,\hat{\lambda} + \dot{\phi}\cos\theta\,\hat{e}_t}$$

$\vec{\omega}_{T/F} \times \hat{n}$

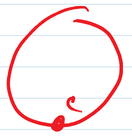
or

$\vec{\omega}_{B/F} \times \hat{n}$

Both work because $\hat{n}$ is in both frames

$$\boxed{\vec{r}_{G/C} = r\hat{\lambda}}$$

Rolling Constraint



$V_c = V_c$

$$\vec{0} = \vec{V}_{/C}$$

$$0 = \vec{V}_G + \left[\vec{\omega}_{B/F} \times \underbrace{-r\hat{\lambda}}_{\vec{r}_{C/G}}\right]$$

① $\vec{V}_G = \vec{V}_G$(Euler angles and rates J base vectors

$\frac{d}{dt}(①) = \vec{a}_G = \vec{a}_G$(Euler angles & rates, J base vectors)

$$\sum \vec{M}_{/C} = \dot{\vec{H}}_{/C}$$

$$\vec{r}_{G/C} \times -mg\hat{k} = \vec{r}_{G/C} \times m\vec{a}_{G} + \underline{\underline{I}}\,\dot{\vec{\omega}} + \vec{\omega} \times \underline{\underline{I}}\,\vec{\omega}$$

$$\underline{\underline{I}} \;\llcorner\; \begin{bmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$\underbrace{\begin{array}{l} {}^{3}\,\text{EQs to solve for } \ddot{\phi},\ \ddot{\theta},\ \ddot{\psi} \\ \text{in terms of } \theta,\ \dot{\theta},\ \dot{\phi},\ \dot{\psi} \end{array}}$$

<u>THEN</u> TRANSLATE TO f to MAKE AN ANIMATION of A ROLLING COIN

WANTED TOPICS

- CONSTRAINTS
- COLLISION + FRICTIONS
- VIBRATIONS
- CAPSTONE PROJECT → 3 LINK PEND IN 3D

SPECIAL CASES ("EASIEST")

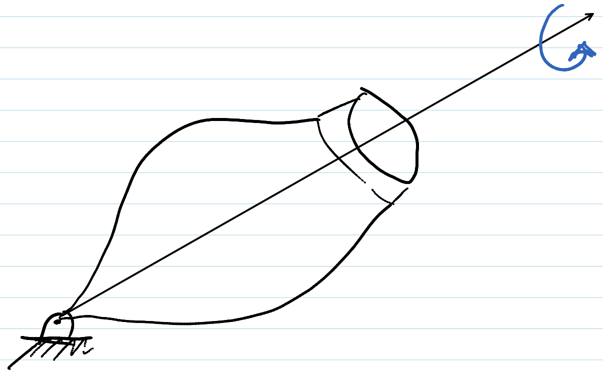
* FIXED AXIS ROTATION, CONSTRAINED

* ALMOST ALL AXI-SYMMETRIC OBJECTS
  - SPINNING ABOUT SYMMETRY AXIS AT A CONSTANT RATE

HARDER:

- MOTION OF RIGID BODY IN SPACE
- COMPLICATION OF MOTION

SPINNING TOP



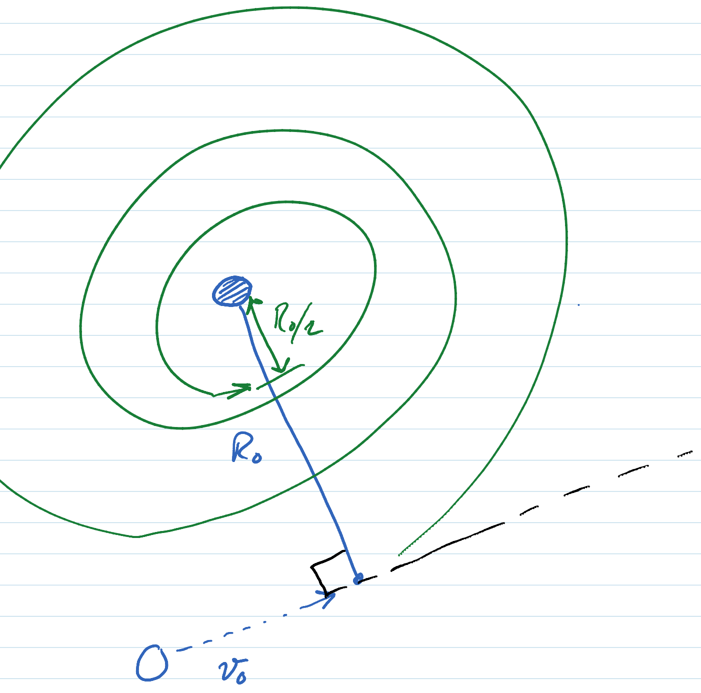
PUZZ

LOOKING

- FRICTIONLESS
  ICE RINK, SKATER

LE PROBLEM

DOWN ON AN ICE RINK



QUESTION

HOW FAST ARE YOU GOING?

r of POLE = 'SMALL'

$$V_1 = ?$$

$$V_1 = V_1 \hat{e}_t$$

$\frac{1}{2}$

$\frac{1}{2}$

$R\dot{\theta} =$

$R/2\,\dot{\theta}$

$$\vec{r}_1 \times m\vec{a}_{/a\ 1} = \vec{r}_2 \times m\vec{a}_2$$

$$\vec{R}\hat{e}_r \times m$$

$$m V^2 = \frac{1}{2} m V^2$$

$$m \left(V e_n + V e_t \right) = \frac{1}{2} m V$$

$$= V$$

$$= V_1$$

$$\vec{V} = V_T + V_N$$

$$V_T = \frac{V_T}{R} \rightarrow \quad 2V_0$$

① $\boxed{\text{"AMB"} \quad V_1 = 2V_0}$

$$\boxed{?} = \vec{F} = m\vec{a}$$

$$\left[F_t = m \left[\dot{v}\, \hat{e}_t + \frac{v^2}{2} \hat{e}_n \right] \right]$$

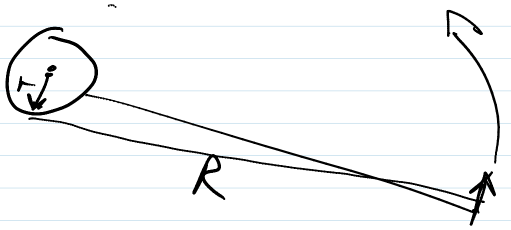
DOT w/ $\hat{e}_t$

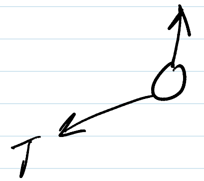
$$0 = \dot{v} \Rightarrow \quad V_1 = V_0$$

3) CONS ENERGY $\Rightarrow V_1 = V_0$

LA GRANGE MULTIPLIERS

SKATER





$T \approx \frac{mv^2}{r}$

$$M = \vec{H}$$

$$r\frac{mv^2}{R} = \frac{d}{dt}(mvR)$$

$$\frac{rmv^2}{R} = mvR + mv\dot{R}$$

↳ $\dot{R} = \frac{-vr}{R}$

$$\overset{2}{v} = 0$$

---

* TENSION ⊥ VELOCITY → NO WORK IS DONE

ADDING KINEMATIC CONSTRAINTS TO LAGRANGE EQ

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q} = Q_i$$

$$\mathcal{L} = E_K - E_T$$
$$= T - V$$

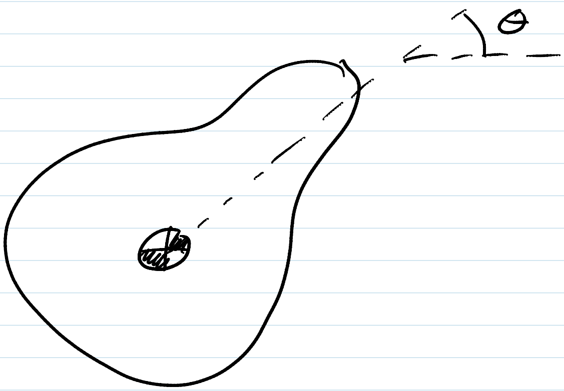
$$Q_i = \sum_{j=1}^{3N} \overbrace{\frac{\partial r_j}{\partial q_i} F_j}^{\text{WORK DONE PER UNIT CHANGE IN } q_i \text{ BY FORCE SYSTEM}} {}^{*}$$

COULD BE
TORQUE × MOMENT

FORCES OTHER THAN THOSE
OF CONSTRAINT
BY THE HOLONOMIC
CONSTRAINTS
AND NOT YET TAKEN
ACCOUNT OF IN
$\overline{V} = E_R$

EXAMPLE SYSTEM

CHAPLYGIN SLEIGH



#of DIMENSIONS

3=2

N=1

$$\sum_{j=1}^{3N}$$

$q_1 = x_G$
$q_2 = y_G$
$q_3 = \theta$

$E_p = 0$

NO SKATE => LAGRANGE EQ

$V = E_p = 0$

$T = E_K = m\frac{1}{2}(\underset{\dot{x}}{V_x}^2 + \underset{\dot{y}}{V_y}^2) + \frac{1}{2} I \dot{\theta}^2$

WRITE LAGRANGE EQ

$\mathcal{L} = T$

1) $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_G} - \frac{\partial \mathcal{L}}{\partial x_G} = Q_1 = \sum F_x^*$

$Q_1 = \sum_{j}^{2} \frac{\partial r_i}{\partial q_i} F_j = \sum F \hat{x}$

r IS HOW MUCH IT MOVES IN X

q IS DIRECTION

i=1 = 1

j=1=X

j=2=Y

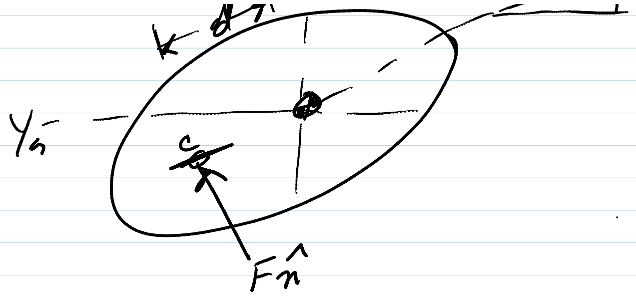
j=3 ∠ G

$m\ddot{x}_G = \sum F_x^*$

2) $m\hat{y}_G = \sum F_y^*$

3) $I\ddot{\theta} = \sum_i M_G^*$

NOW ADD A SKATE



$F\hat{n}$ IS A FORCE WITH A FUNCTION IN TIME

→ W/O LOOKING AT CONSTRAINT, WE CAN WRITE LAG EQ'S w/ this force

$$m\ddot{x} = \underbrace{-F\sin\theta}_{Q_x}$$

FORCE X COMPONENT

$$my = \underbrace{-F\cos\theta}_{Q_y}$$

$$I\ddot{\theta} = \underbrace{-Fd}_{Q_\theta}$$

IN LIMIT

SPINS AROUND

3 EQ FOR $\ddot{x}_a$, $\ddot{y}_a$, $\ddot{\theta}_a$, $F(t)$

BUT WE DON'T KNOW $F(t)$

WE KNOW KINEMATICS!

$$\vec{V}_c \cdot \hat{n} = 0$$

$$\left[\dot{x}_a \hat{i} + \dot{y}_a \hat{j} + \left[\dot{\theta}\hat{k} \times (-d\hat{\lambda})\right]\right] \cdot \hat{n} = 0$$

4^{th} EQ BUT ITS IN VELOCITIES

$$\frac{d}{dt}\left[EQ\ 4\right] = ④$$

LaGrange Multipliers Cont'd, 4/25/2013
Thursday, April 25, 2013
2:57 PM

LAGRANGE MULTIPLIERS

CONTINUED

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

$\mathcal{L} = T - V$

or 2 in 2D

$$Q_i = \sum_{j=1}^{3N} \frac{\partial r_j}{\partial q_i} F_j^*$$

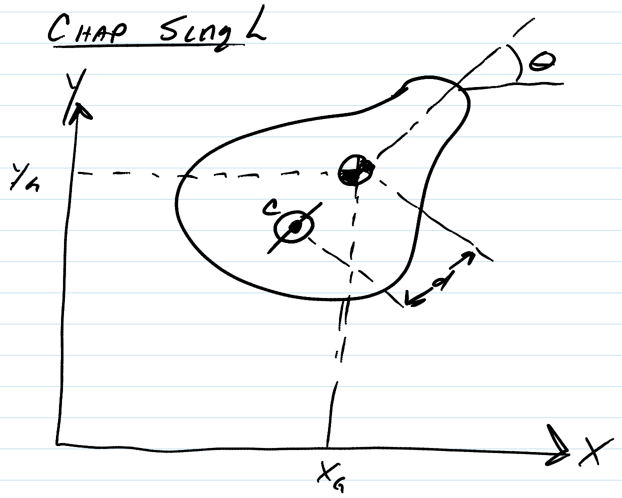
OTHER THAN THE CONSTRAINT FORCES USED TO FORM L EQ'S

$Q_i \, \delta q_i$ = VIRTUAL WORK (δW) of F_j^*

"Q_i IS WORK PER UNIT DISPLACEMENT"

RECALL EXAMPLE

CHAP STICK



$q_1 = x_G$

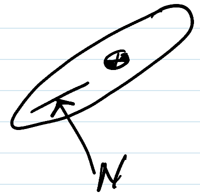
$q_2 = y_G$

$q_3 = \theta$

METHOD 1

WE 'KNOW' DIRECTION OF CONSTRAINT FORCES BY REASON

ex)



$\Rightarrow Q_i$

$Q_1 = -N \sin\theta$

$Q_2 = N \cos\theta$

$Q_3 = -Nd$

These Are

LMB/x

LMB/y

AMB/G

'N

$Q_3 = -Nd$

↳ WITH THESE GENERALIZE EQ'S

Q_3 IS ALWAYS THE TORQE

&

AND SOLVE CONSTRAINT EQUATIONS

$\Longrightarrow$ DAE

METHOD 2

"METHOD of LAGRANGE MULTIPLIERS"

- ASSUME CONSTRAINTS ARE of FORM:

$$\sum a_i \dot{q}_i + a_t(t) = 0$$

↳ $a_i(q_1, q_2, \ldots, t)$

for clarity we use this

"We have m constraints of form" ← IN texts

$$\sum a_{ji} \dot{q}_i + a_{jt}(t) = 0$$

↳ $j = 1 \ldots m$

* WE CONSIDER 1 CONSTRAINT FORCE AND REPEAT PROCESS TO generate EQ'S

KINEMATICALLY ALLOWED VARIATIONS

$$\sum a_i \delta q_i = 0$$

w/ sleigh

Recall:

$\vec{V}_c \cdot \hat{n} = 0$ (CAN'T MOVE ⊥ TO SKATE)

→ WRITE IN terms of $\dot{q}$ etc

* WE COULD WRITE $\hat{n}$ EXPLICITLY

AmB/g

→ write in terms of $\dot{q}$ etc

$$\hat{n} = \hat{k} \times \underbrace{\hat{\lambda}} \rightarrow \hat{\lambda} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

* We could write $\hat{n}$ explicitly but writing as × prod, you validate right-hand-systems

$$\vec{V}_c = \vec{V}_G + \vec{V}_{c/G}$$

↳ $\vec{v}_{c/G} = (\dot{\theta}\hat{k} \times -d\hat{\lambda})$

Result $-\dot{x}_G \sin\theta + \dot{y}_G \cos\theta - d\dot{\theta} = 0$ (Do this)!!

$$\Rightarrow \underbrace{\begin{bmatrix}-\sin\theta & \cos\theta & -d\end{bmatrix}}_{a} \underbrace{\begin{bmatrix}\dot{x}_G \\ \dot{y}_G \\ \dot{\theta}\end{bmatrix}}_{\dot{q}} = \vec{0}$$

$a_1 = \sin\theta$
$a_2 = \cos\theta$
$a_3 = -d$

↳ $\sum a_i q_i + 0 = \vec{0}$

Put together 2 ideas

1) Set of allowed variations q_i are all those which satisfy

$$\boxed{\sum a_i \delta q_i = 0}$$

2) Constraint forces Q_i do no work in allowed motions

$$\boxed{\sum Q_i \delta q_i = 0}$$ "for allowed motions"

("Author" cannot see how to derive this from above)

⟹ Before we had constraints, we thought of

↓—# of generalized coordinates

q_i were a vector in $\mathbb{R}^N$

→ set of allowed variations ("δq_i") is ⊥ to $\vec{a}$ in $\mathbb{R}^N$

[IN this case a is $[\sin\theta \;\; \cos\theta \;\; -d]$]

AND is $\perp$ TO $[\dot{x}_a \;\; \dot{y}_a \;\; \dot{\theta}]$

$\boxed{*}$ $Q_i \perp \delta q_i$ AND $a_i \perp \delta q_i$

SO $Q_i \parallel a_i$

AND thus $\underline{\underline{Q_i = \lambda a_i}}$

SINCE δq_i ARE ORTHOGONAL COMPLIMENT of $\vec{a}_i$

Q_i ARE orthogonal compliment of δq_i

$\Rightarrow Q_i = \lambda q_i$

Back to example [sleigh]

$$\begin{aligned} m_a \ddot{x}_a &= \overbrace{-\lambda \underbrace{\sin\theta}_{a_1}}^{Q_i} && Q_1 \\ m_a \ddot{y}_a &= \lambda \underbrace{\cos\theta}_{a_2} && Q_2 \\ I_a \ddot{\theta} &= -\lambda \underbrace{d}_{a_3} && Q_3 \end{aligned}$$

AND CONSTRAINT EQ

4 EQ's in $\ddot{x}_a \;\; \ddot{y}_a \;\; \ddot{\theta}_a \;\; \lambda$

AND $\boxed{\lambda = N}$ (*)

HW

DO Rolling Wheel/Disc/Coin

Using Lagrange EQ's → CONSTRAINT is VERTICAL FORCE

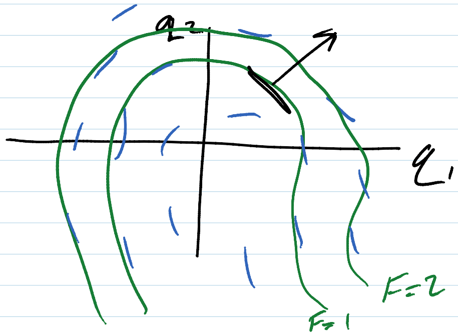
So WE HAVE 3 VERSIONS

- Steady state
- Euler EQ
- Lagrange EQ

Why can nonholonic condition ~~not~~ exist in 2D?

What if $n=2$?

$$a_1 \dot{q}_1 + a_2 \dot{q}_2 = 0 \quad ①$$



This could be a direction field where $\vec{a} \perp \dot{\vec{q}}$ everywhere

Curves cannot cross!

These are contour lines of an integration

① ⟹ F = Constant

Constraint is "integrated"

* In 2D, a velocity constraint is also a position constraint

[Recall: All non-linear [and linear] 1st order EQ's are integrable]

$$\text{"}a_1 dq_1 + a_2 dq_2 = 0\text{"}$$

BUT in 3D things are different

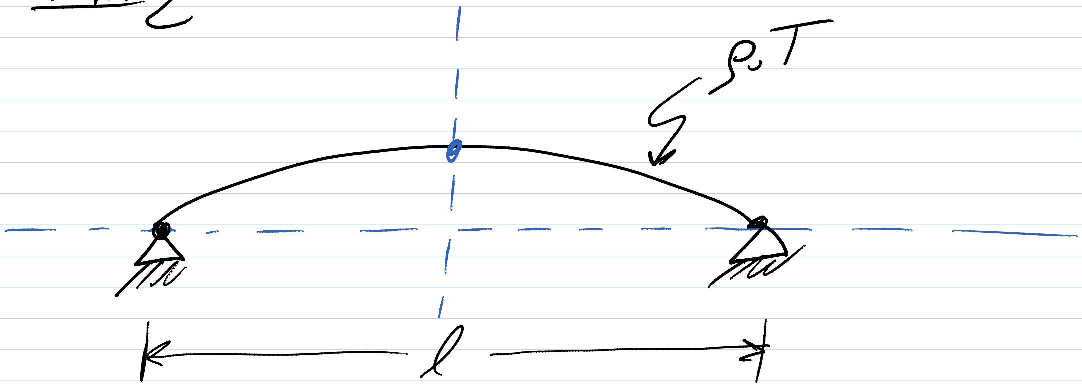
Vibes of cont systems

In principle, we have ∞ D.O.F.

→ But pretend we only have, say 1 & 3 that matter

Find Normal Modes would be good,
But if that's too hard → GUESS! mode shapes

ex) STRING



say we guessed that instead of a sine wave (correct) the shape was parabolic (incorrect)

$$\left(\frac{l}{2}-x\right)\left(\frac{l}{2}+x\right) = \phi(x)$$

$$u(x,t) = q\,\phi(x)$$

↳ Lagrange Eq $q(t)$

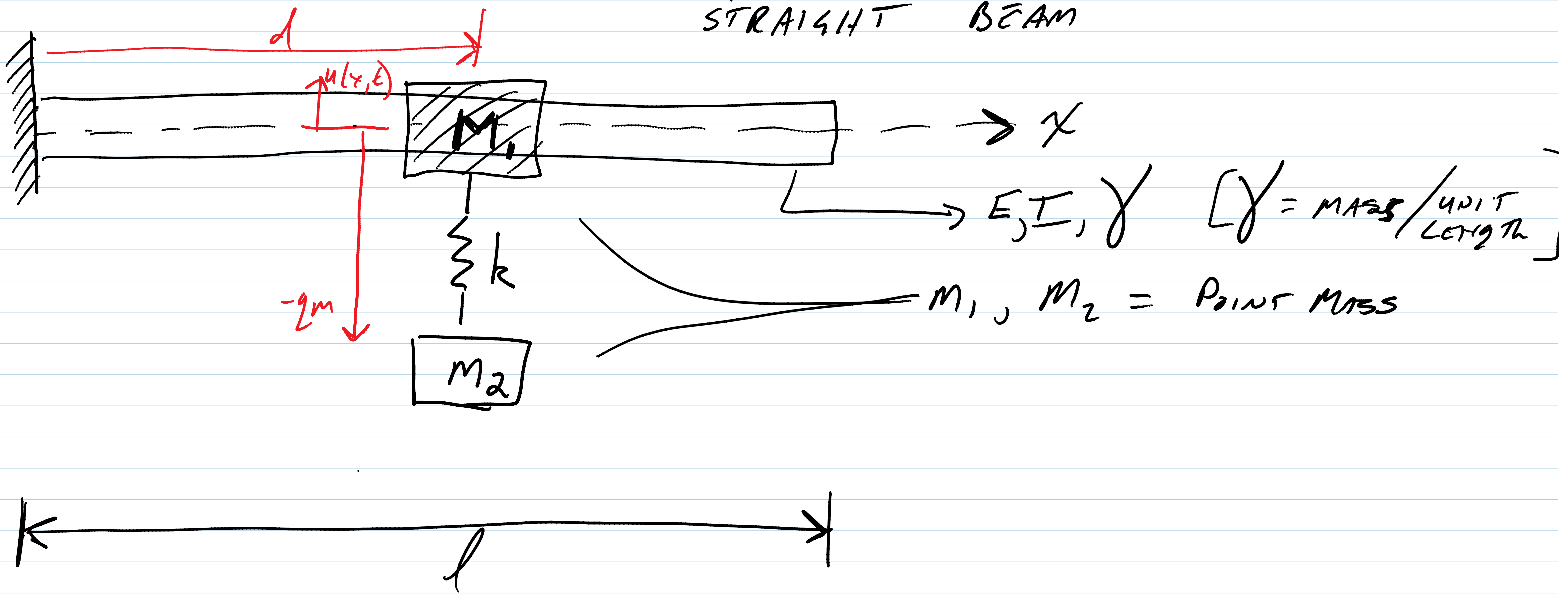
Given shape ϕ, we can calculate T, V, from q, $\dot{q}$

→ write Lagr Eq's
→ EOM
→ approximate calc of freq
→ ≈ 1/2 % error in frequency

Beam Vibration, 4/30/2013
Tuesday, April 30, 2013
3:02 PM

HW

APPROXIMATE SOLUTION
STRAIGHT BEAM



$q_i = 0$ = POTENTIAL ENERGY MINIMUM

How DOES IT VIBRATE?

guess ↓

$$q(x,t) = \sum_i^{\#\text{ of shapes}} q_i \, \phi_i(x) \, f_i(t) + \boxed{\sum_{n+1}^{n+?} q_i(t)}$$

DISCRETE MODES

q_i └→ GENERALIZED COORDINATES

ϕ_i └→ ASSUMED "MODE" SHAPES

WRITE LAGRANGE EQs TO GET DEQ's

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \boxed{\text{YIELDS}} \Rightarrow [M][\ddot{q}] + [K][q] = 0$$

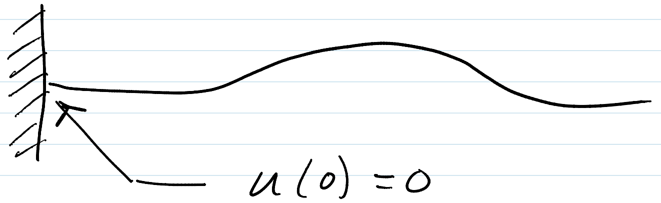
FIND $\mathcal{L}$

$$\mathcal{L} = T - V$$

$T \to ?$ $V \to ?$

V = STRAIN ENERGY IN BEAM

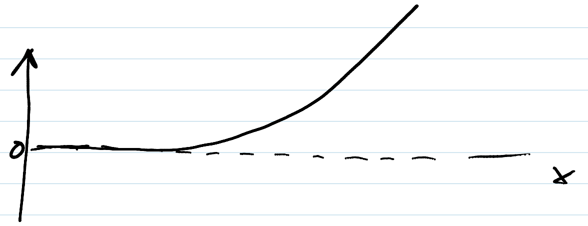
BEST IS TO RESPECT BOUNDARY CONDITIONS



$u(0) = 0$

$u'(0) = 0$

ex) $\phi_1(x) =$



x^2

$e^{cx} - 1 - cx$

$\cosh(cx) - 1$

$1 - \cos(cx) \Longrightarrow x = l;\ cx = \frac{\pi}{2} \rightarrow c = \pi/2l$

BEAM DEFLECTION w/ POINT LOAD ON END
3RD ORDER CUBIC

BEAM DEFLECTION w/ UNIFORM LOAD
4TH ORDER CUBIC

$\phi_2(x)$

$\cos c_2 x = =$ $\quad c_2 = \pi/l$

$\phi_3(x) \quad \cos(c_3 x) \quad c_3 = \frac{3}{2}\pi$

q_4 = DEFLECTION at MASS 2

GIVEN these 4 PARAMETERS
GENERATE E_p , E_k

$$T = E_k = \sum_{\substack{\text{ALL}\\ \text{PARTICLES}}} \frac{1}{2} m_i \dot{u}_i^2$$

$$= \underbrace{\int \frac{1}{2} \dot{u}^2 \, dm}_{\substack{\text{CONTINUOUS}\\ \text{PARTS}\\ \text{of}\\ \text{SYSTEM}}} + \underbrace{\sum_{\substack{\text{BIG}\\ \text{POINT}\\ \text{MASSES}}} \frac{\dot{u}_i^2}{2} m_i}_{\substack{\text{DISCRETE}\\ \text{PARTS}\\ \text{of}\\ \text{SYSTEM}}}$$

For Point mass m_1 :

$$E_k = \frac{1}{2} m_2 \dot{q}_n^2$$

BEAM $\quad \frac{1}{2} \int_0^\ell \dot{u}^2 \rho \, dx + \frac{1}{2} \dot{u}(\ell)^2 m_1$

$$\dot{u} = \sum \dot{q}_i \, \phi_i(x)$$

$$\frac{1}{2} \int_0^\ell \dot{u}^2 \rho \, dx = \underbrace{\int_0^\ell \left(\sum \dot{q}_i \, \phi_i(x) \right)^2 \rho \, dx}$$

$$= \frac{1}{2} [\dot{q}_1 , \dot{q}_2 , \dot{q}_3 \cdots] \Big[M \Big] \begin{matrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \end{matrix}$$

CONTIN.

L

$$\boxed{[M]^{\text{CONTIN.}}_{ij} = \int_0^l \phi_i(x)\,\phi_j(x)\,\gamma\,dx} \longleftarrow$$

ugly PART

$$M^{\text{M ON BEAM}}_{ij} = \phi_i(d)\,\phi_j(d)$$

$$\boxed{M^{\text{BEAM}} = M^{\text{CONT}}_{ij} + M^{\text{MASS ON BEAM}}_{ij}}$$

$$E_k \text{ of } M_2 = \frac{1}{2} m_2 \dot{q}_2^{\,2}$$

$$E_K^{\text{TOT}} = \underbrace{\begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \cdots & \dot{q}_m \end{bmatrix} \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ \hline 0 & 0 & 0 & 0 & M_B \end{array}\right] \dot{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_m \end{bmatrix}}}_{\text{"T"}}$$

$$V_{\text{SPRING}} = \frac{1}{2} k \left(\sum q_i\,\phi_i(d) - q_m \right)^2$$

$$V_{\text{BEAM}} = \int_0^l \frac{1}{2} EI\,(\underbrace{u''}_{\to \frac{\partial^2 u}{\partial x^2}})^2 \, dx$$

M, I, L

BENDING MOMENT

M

SANG

"BENDIN

, $\underset{\text{CURVATURE}}{\kappa} = 1/\rho$ ← RADIUS of CURVATURE

$$\boxed{= EI/\rho}$$

to old MAC DONALD

g MOMENT'S GIVEN BY

$\hookrightarrow \frac{\partial^2 u}{\partial x^2}$

"BENDIN

EI

$$= \frac{1}{2} EI \int_0^l \left[\sum_i \left(q_i \, \phi_i''(x)\right)\right]^2 dx$$

$$M = EI\kappa = E$$

$$V = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix} \begin{bmatrix} K^{BEAM} + K^{SPRING} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}$$

$$K_{ij_{BEAM}} = EI \int_0^l \phi_i'' \, \phi_j'' \, dx$$

Pick a good function

WE TALKED ABOUT $1 - \cos(cx)$

$$\Rightarrow \quad M\ddot{\vec{q}} + K\vec{q} = 0$$

FIND NORMAL MODES, freq, etc

AT END, SUB IN

$l = 1 \qquad m_1 = 1 \qquad m_2 = 1$

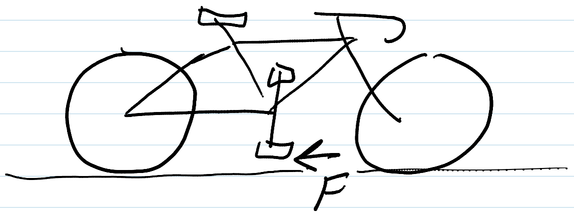
$E = 1 \qquad I = 1 \qquad d = 1/2$

$\gamma = 1 \qquad k = 1$

g MOMENT'S GIVEN BY
E OVER RHO....''

$$I \frac{\partial^2 y}{\partial x^2}$$

Q - QUESTION



Which way does bike go?

PENDULA IN 3D

Logic Assume we know all θ's $\dot{\theta}$'s, θ's

and all mass properties

→ Can we write enough EQ's to extract $\ddot{\theta}_1$ $\ddot{\theta}_2$?

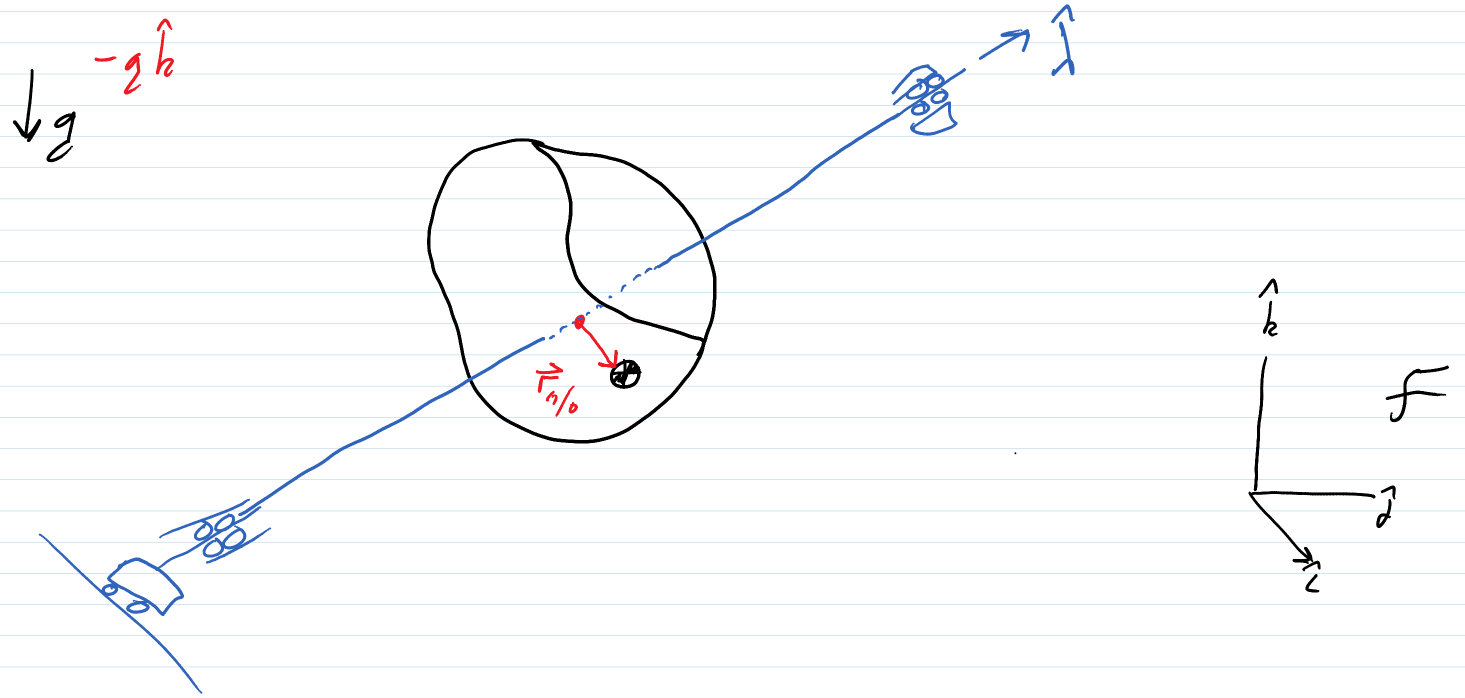
ex1 Simple pendulum in "3D"

Arbitrary rigid object with a reference configuration

$$\vec{r}^{\,REF}_{G/O} \qquad \underline{I}^{REF}$$

$$\underbrace{\underline{\underline{R}} = \mathbb{1}}_{\text{no rotation}}, \quad m$$

→ Rotates about a fixed axis through O

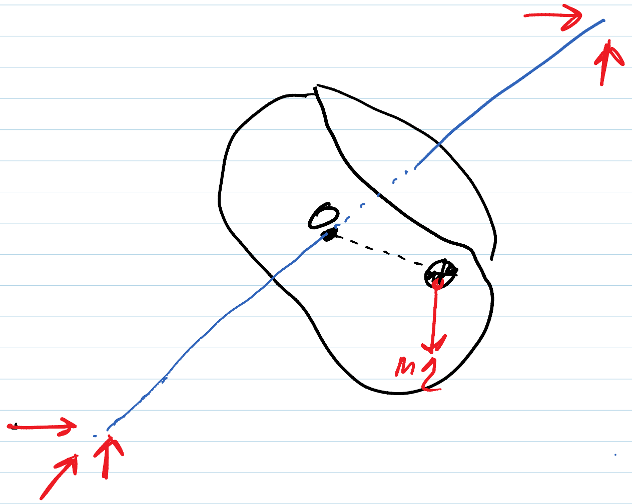


1 Degree of freedom [could be done as a separate exercise]

$$\ddot{\theta} + c\sin\theta = 0$$

C IS A FUNCTION OF
$\hat{\lambda}\cdot\hat{b}$, $\hat{\lambda}\cdot\underline{I}\hat{k}$, d^{4}

FBD



1 KNOWN FORCE
5 U.K REACTION FORCES

IN PRINCIPLE
LMB & AMB $\Longrightarrow$
6 EQ's for 5 REACTIONS
+ $\ddot{\theta}$

$$\left\{\underline{AMB_{/O}}\right\}\cdot\hat{\lambda} \Longrightarrow \ddot{\theta}$$

$$\left\{\sum \vec{M}_{/O} = \dot{\vec{H}}_{/O}\right\}\cdot\hat{\lambda}$$

$$\left[\vec{r}_{G/O} \times -mg\hat{k}\right]\cdot\hat{\lambda} = \left[\vec{r}_{G/O} \times m\vec{a}_{G} + \underline{\underline{I}}\dot{\vec{\omega}} + \vec{\omega}\times\left(\underline{\underline{I}}\vec{\omega}\right)\right]\cdot\hat{\lambda}$$

CAN WE WRITE ALL IN TERMS OF "GIVENS" + $\theta, \dot{\theta}, \ddot{\theta}$

WHAT IS $\underline{\underline{R}}$?

$$\underline{\underline{R}} = \underline{\underline{R}}(\theta) = 1 - \cos(\theta)\hat{\lambda}\hat{\lambda} + \cos(\theta)\underline{\mathbb{1}} + \sin(\theta)\underline{\underline{S}}(\hat{\lambda})$$

$$\vec{r}_{G/O} = \underline{\underline{R}}\cdot\vec{r}_{G/O}^{\,REF}$$

$$\vec{\omega} = \dot{\theta}\hat{\lambda} \qquad \dot{\vec{\omega}} = \ddot{\theta}\hat{\lambda}$$

$$\underline{\underline{I}} = \underline{\underline{R}}\,\underline{\underline{I}}^{REF}\,\underline{\underline{R}}^{T}$$

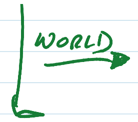
ANDY'S WORLD $\rightarrow$

$$\underline{\underline{R}} = \hat{e}_i'\hat{e}_i$$

$$T^{REF} = T_{ij}^{REF}\hat{e}_i\hat{e}_j \qquad T = T$$

Def $\hat{e}' \quad \hat{e}'$ $\quad \underline{I} = R\,\overset{R}{\underline{I}}\,R^T$

$\underline{\underline{I}} = \underline{\underline{R}}\ \underline{\underline{I}}\ \ \underline{\underline{R}}'$



$\underline{\underline{R}} -$ $\quad e_i\ e_i$

$\underline{\underline{I}}^{REF} = I_{ij}^{REF}\,\hat{e}_i\,\hat{e}_j$ $\qquad \underline{\underline{I}} = I_{,}$

$\Rightarrow$ ① $\quad$ 1 EQ w/ 1 U.K., $\ddot{\theta}$

QED : WE HAVE ANSWER !

EXTRACTION : [IN PRACTICE]

① Do symbolic derivation and do symbolic "solve" [Brute Force]

② Organize EQ's such that $\ddot{\theta}$ is isolated

③ Some "on the fly" method

→ Use that EQ's are LINEAR IN $\ddot{\theta}$

Solve for $\ddot{\theta} = 0$, get answer

" $\quad \ddot{\theta} = 1$, get answer

Take difference, that is the coefficient for $\ddot{\theta}$!

EXPLICITLY (sits in "RHS" file)

1) Put in #'s for all constants like

$m, \hat{\lambda}, I^{ref}, \vec{r}_{G/O}$

AND for θ and $\dot{\theta}$

2) Set $\ddot{\theta} = 1 \Rightarrow$ EVALUATE

$$\vec{O} = -\left(\sum \vec{M}_{/O} + \dot{\vec{H}}_{/O}\right) \cdot \hat{\lambda} \Rightarrow -f(\text{parameters}, \theta, \dot{\theta}) + M_{\theta\theta}\,\ddot{\theta}$$

$\hookrightarrow f(\text{Parameters}, \theta$

$$A_1 = -(\sum M + \dot{H})$$

Set $\ddot{\theta} = 0$ $\quad$ Gives A_0

$$A_1 - A_0 = M_{\theta\theta}$$

$$A_0 = -f(\,)$$

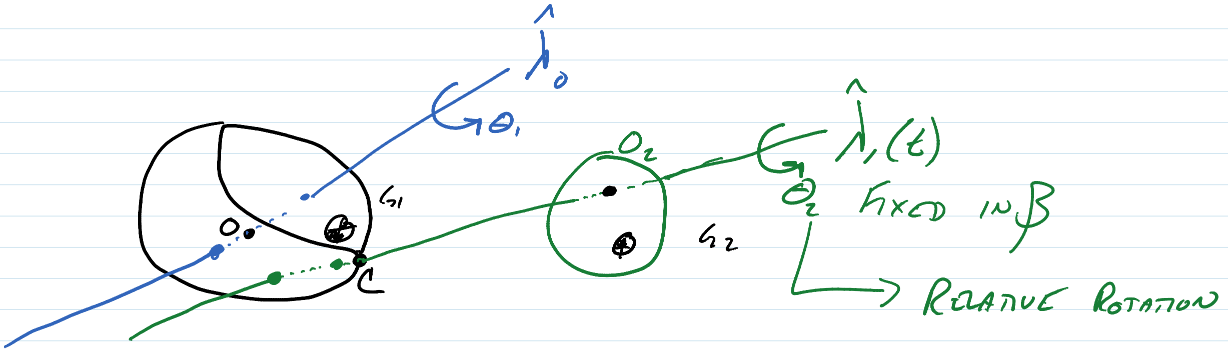
$$M_{\theta\theta}\,\ddot{\theta} = f$$

Def

${}_{ij}\ \hat{e}_i'\ \hat{e}_j'$ $\qquad$ $\underline{\underline{I}} = \underline{\underline{R}}\ \underline{\underline{I}}^R\ \underline{\underline{R}}^T$

↑ KNOWN ↑ KNOWN

DOUBLE PENDULUM

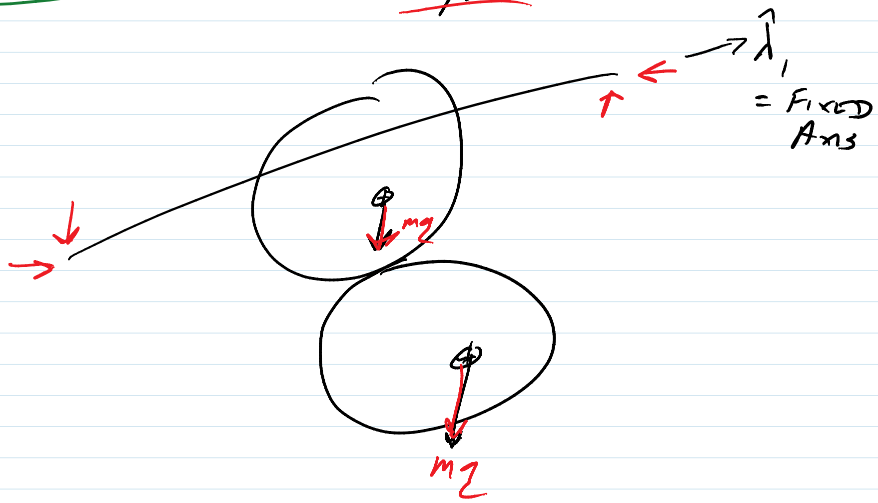


C = FIXED POINT ON BODY 1
LOCATED AT HINGE POINT

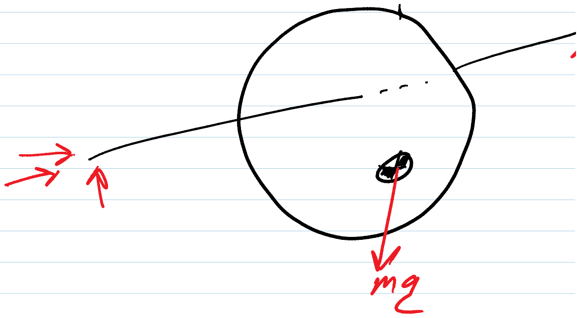
2 DOF, "2 LINK ROBOTIC ARM"

FBD'S

SYSTEM

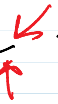


B_2 FBD



PARAMS:

$$P\left\{\begin{array}{lllll} m_1 & m_2 & g & \hat{\lambda}_0 & \hat{\lambda}_1^{Ref} \\ \underline{\underline{I}}_1^{REF} & \underline{\underline{I}}_2^{REF} & & & \\ \vec{r}_{G1/O_1}^{Ref} & \vec{r}_{C/O}^{Ref} & \vec{r}_{G2/C}^{Ref} & & \end{array}\right.$$



$\longrightarrow \lambda_2$

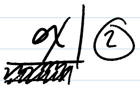
GIVEN: $P,\ \theta_1,\ \dot{\theta}_2,\ \theta_2,\ \dot{\theta}_2$

FIND $\ddot{\theta}_1 \quad \ddot{\theta}_2$

$$\left\{ AMB_{/C} \text{ for } \beta_2 \right\} \cdot \hat{\lambda}_1 \qquad \text{(I)}$$

$$\left\{ AMB_{/O} \text{ for SYSTEM} \right\} \cdot \hat{\lambda}_0 \qquad \text{(II)}$$

EVALUATE IN TERMS of $\ddot{\theta}_1 \ \ddot{\theta}_2$



$$\left\{ \Sigma \vec{M}_{/O} = \dot{\vec{H}}_{/O} \right\} \cdot \hat{\lambda}_0$$

$$\left\{ \Sigma \vec{M}_{/O} \right\} \cdot \hat{\lambda}_0 = \left[\underbrace{\dot{\vec{H}}_{1/O}}_{\text{BODY 1}} + \underbrace{\dot{\vec{H}}_{2/O}}_{\text{BODY 2}} \right] \cdot \hat{\lambda}_0$$

↑HARD TERM

EVALUATE $\left[\dot{\vec{H}}^{2}_{/O} \right] \cdot \hat{\lambda}_0$ for AN EXAMPLE

$$\dot{\vec{H}}^{2}_{/O} \cdot \hat{\lambda}_0 = \left[\vec{r}_{G2/O} \times m_2 \vec{a}_{G2} \right] + \left[\vec{\omega}_2 \times \underline{\underline{I}}_2 \vec{\omega}_2 \right] + \underline{\underline{I}}_2 \cdot \dot{\vec{\omega}}_2$$

$$\underline{\underline{R}}_{\beta_1/F} = \underline{\underline{R}}_{\beta_1/F}(\theta_1) = \hat{\lambda}_0 \hat{\lambda}_1 \ldots -$$

$$\hat{\lambda}_1 = \underline{\underline{R}}_{\beta_1/F} \cdot \hat{\lambda}_1^{ref}$$

$$\vec{r}_{G1/O} = \underline{\underline{R}}_{B_1/F} \cdot \vec{r}_{G/O}^{\,ref} \qquad \vec{r}_{C/O} = \underline{\underline{R}}_{B_1/F} \cdot \vec{r}_{C/O}^{\,ref}$$

$$\underline{\underline{R}}_{B_2/} = \underline{\underline{R}}_{B_2/}(\theta, \hat{\lambda}_1)$$

$$\underline{\underline{R}}_{B2/F} = \underline{\underline{R}}_{B1/F} \cdot \underline{\underline{R}}_{B2/B1}$$

$$\vec{r}_{G2/C} = \underline{\underline{R}}_{B2/F} \cdot \vec{r}^{\,ref}_{G2/C}$$

$$\vec{r}_{G2/O} = \vec{r}_{G2/C} + \vec{r}_{C/O}$$

$$\vec{\omega}_{B2/B1} = \dot{\theta}_2 \hat{\lambda}_1$$

$$\vec{\omega}_{B1/F} = \dot{\theta}_1 \hat{\lambda}_0$$

$$\vec{\omega}_{B2/F} = \vec{\omega}_{B1/F} + \underbrace{\vec{\omega}_{B2/B1}}$$

$$\dot{\vec{\omega}}_{B2/F} = \dot{\vec{\omega}}_{B1/F} + \underbrace{\left[\vec{\omega}_{B1/F} \times \vec{\omega}_{B2/B1} + \ddot{\theta}_2 \hat{\lambda}_1\right]}_{\vec{Q}\text{ DOT FORMULA}}$$

$$\underline{\underline{\underline{I}}}_2 = \underline{\underline{R}}_{B2/F} \; \underline{\underline{I}}_2^{ref} \; \underline{\underline{R}}^T_{B2/F}$$

$$\vec{a}_{/C} = \left[\dot{\vec{\omega}}_{B1/F} \times \vec{r}_{C/O}\right] + \left[\vec{\omega}_{B1/F} \times \left(\vec{\omega}_{B/F} \times \vec{r}_{C/O}\right)\right)$$

$$\underbrace{\vec{a}_{G2/C}}_{\text{FIXED FRAME}} = \left[\dot{\vec{\omega}}_{B1/F} \times \vec{r}_{G2/C}\right] + \left(\vec{\omega}_{B2/F} \times \vec{\omega}_{B2/F} \times \vec{r}_{G2/C}\right)$$

<u>DONE</u> for $\dot{\vec{H}}_{2/O}$

(OTHERS ARE EASIER....)

$\xrightarrow{\text{IMPLIES}}$ 2 EQ's IN 2 UK, $\ddot{\theta}_1, \ddot{\theta}_2$

$$\boxed{\star\ \dot{\underline{R}} = \vec{\omega} \times \underline{R}}$$ RECALL