Problem #1 (Addition to the periodic motion problem)

In the previous HW I showed that for a central force of the form $F = |\bar{r}\cdot\dot{\bar{r}}|\cdot k\cdot\hat{r}$ the particle's motion appeard to be almost periodic (Fig. 1).

In order to find the initial conditions that would result in a pure periodic motion I defined a cost function as follows:

$$f = |\text{azimuth}_{t=0} - \text{azimuth}_{t=k}|\cdot|\text{Velocity}_{t=0} - \text{Velocity}_{t=k}|$$

where $t=0$ is the initial time and $t=k$ is the time when the particle returnes to its initial position (subject to numerical errors)

For each value of k and V_{0y} the function would find its minimal value.

Fig. 2 shows the values obtaind for f for all combinations of k and V_{0y}. Note the the f axis scale is logarithmic.

The "valleys" indicate those combinations of k and V_{0y} where the product of the azimuth error and the velocity error are minimal.

The "red zones" are acctually a cut of of greater values, indicating that no periodic motion exists there.

I picked $K=1$ and $V_{y_0}=1$ for my final trajectory (Fig. 3). The results are quite convincing.

Fig 1.

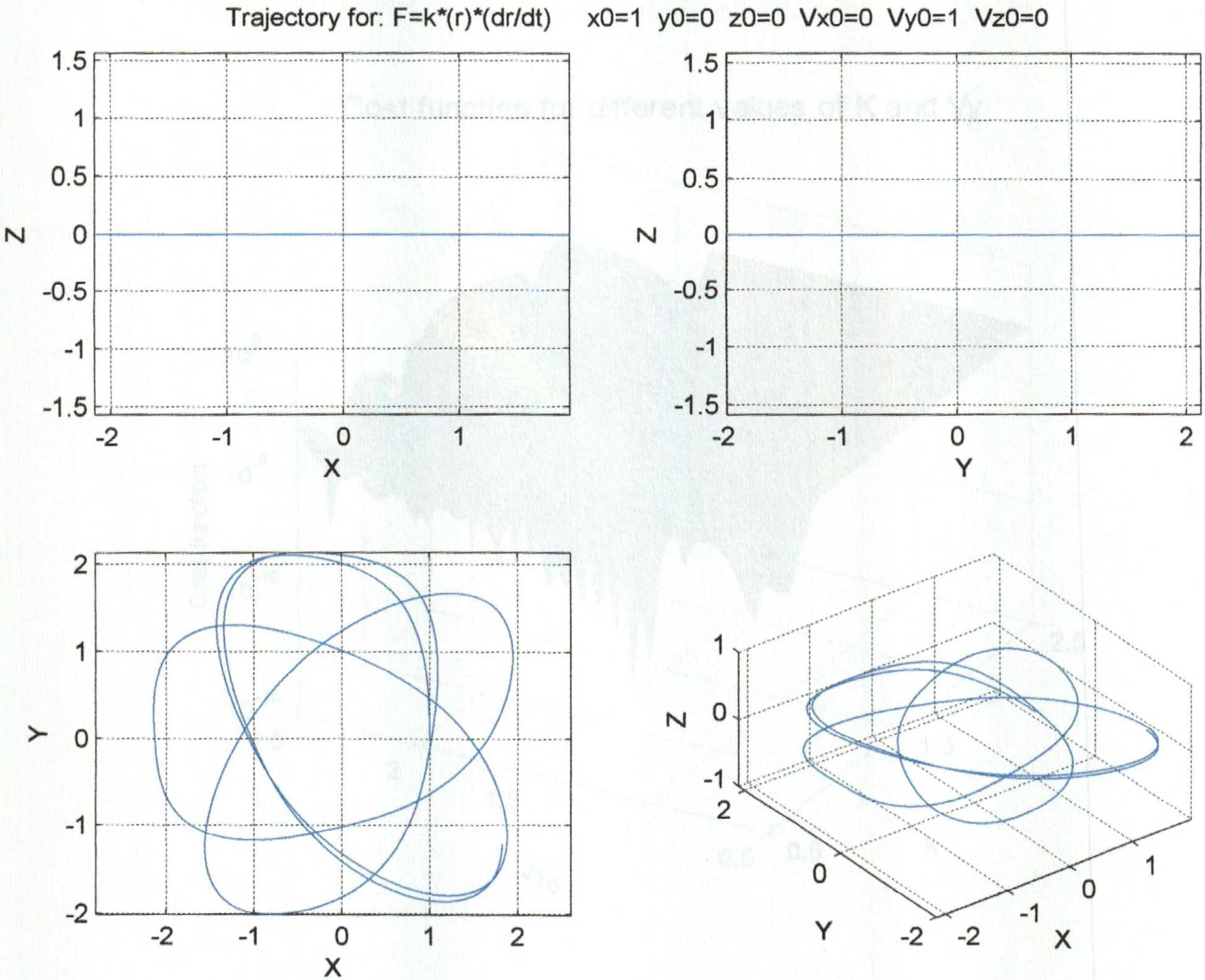
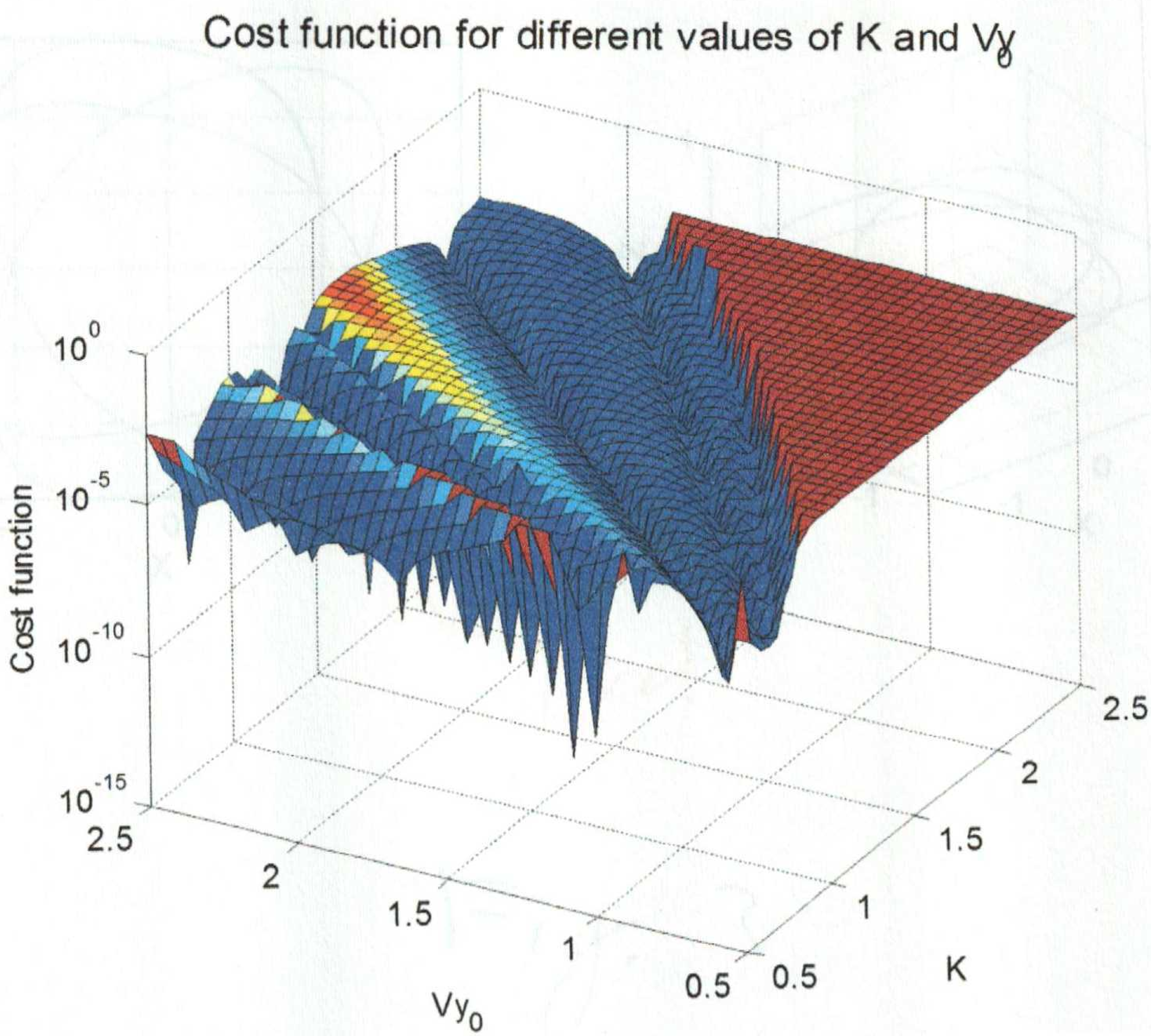
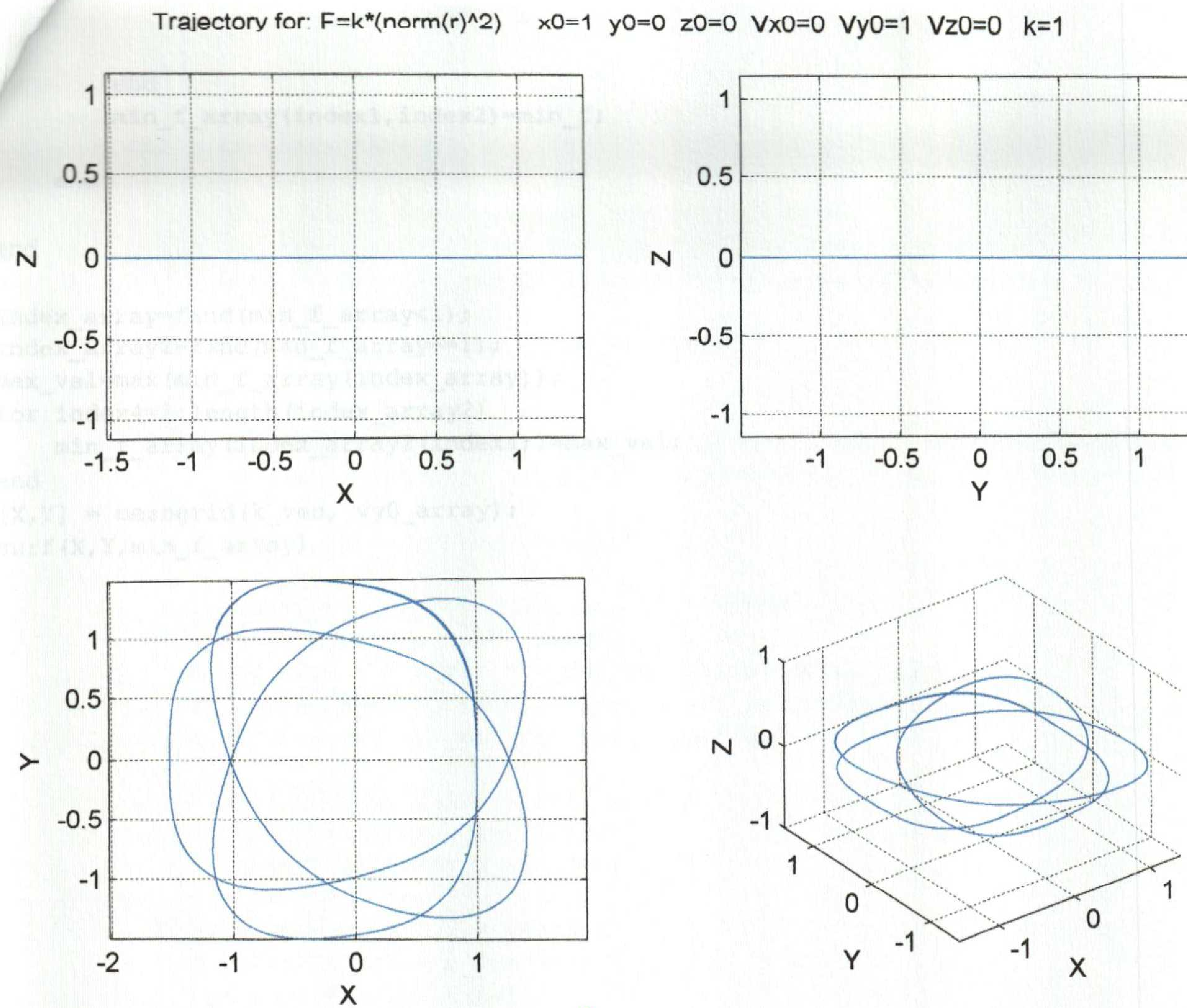


Fig. 2


Cost function for different values of K and Vy0
Cost function
10^0
10^-5
10^-10
10^-15
2.5
2
1.5
1
0.5
Vy0
0.5
1
1.5
2
2.5
K

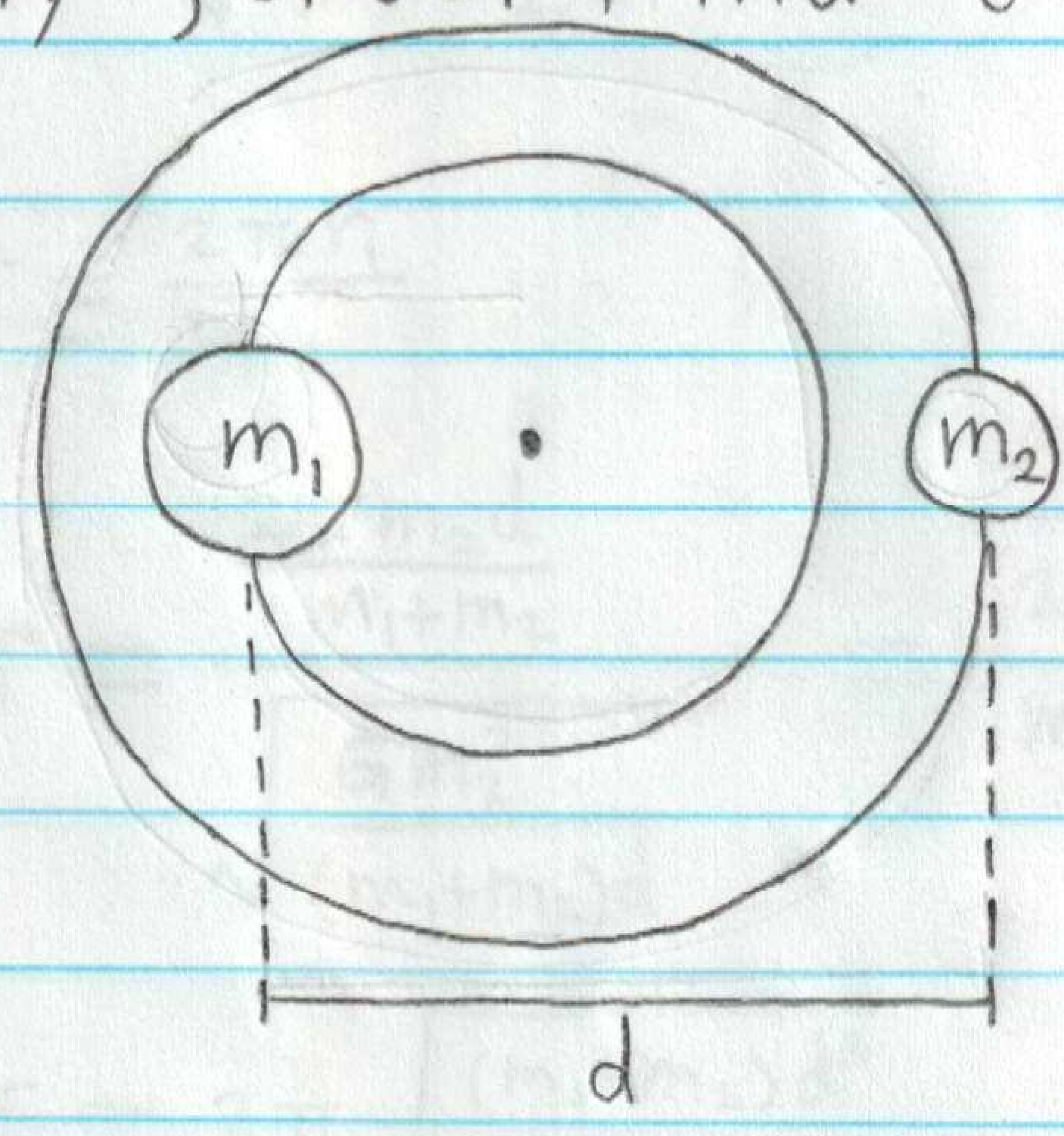


Very Nice!

Fig. 3

2. Handout #6

(a) Two particles (m_1 and m_2) are in circular motion with the gravity between them being the only force. Find the period.



FBD: $m_1 \longrightarrow F = \frac{Gm_1m_2}{d^2}$ $\quad F = \frac{Gm_1m_2}{d^2} \longleftarrow m_2$ ✓

$$r_{cm} = \frac{\Sigma r_i m_i}{\Sigma m_i}$$

$m_1 \longrightarrow r$

$$r_{cm/m_1} = r_1 = \frac{m_2 d}{m_1+m_2}$$

$$\frac{m_1 v_1^2}{r_1} = \frac{Gm_1m_2}{d^2}$$

$$\frac{v_1^2}{\frac{m_2 d}{m_1+m_2}} = \frac{Gm_2}{d^2}$$

$$\frac{(m_1+m_2)v_1^2}{m_2 d} = \frac{Gm_2}{d^2}$$

$$(m_1+m_2)d\,v_1^2 = Gm_2^2$$

$$v_1^2 = \frac{G m_2^2}{(m_1 + m_2) d}$$

$$v_1 = \sqrt{\frac{G m_2^2}{(m_1+m_2) d}}$$

$$T = \frac{2\pi r_1}{v_1}$$

$$T = \frac{\frac{2\pi m_2 d}{m_1+m_2}}{\sqrt{\frac{G m_2^2}{(m_1+m_2)d}}} = \frac{2\pi m_2 d \sqrt{(m_1+m_2)d}}{m_2 (m_1+m_2)\sqrt{G}}$$

$$T = 2\pi \sqrt{\frac{(m_1+m_2) d^3}{(m_1+m_2)^2 G}}$$

$$\boxed{T = 2\pi \sqrt{\frac{d^3}{G(m_1+m_2)}}}$$

(b) Pick $\boxed{G = 1,\ m_1 = 2,\ m_2 = 1,\ d = 1}$

Initial conditions:

$$\boxed{\vec{r}_{10} = \vec{0},\quad \vec{r}_{20} = 1\hat{\imath}}$$

$$v_1 = v_{10} = \sqrt{\frac{G m_2^2}{(m_1+m_2)d}} = \sqrt{\frac{1(1)^2}{(2+1)(1)}} = \frac{1}{\sqrt{3}}$$

$$\boxed{\vec{v}_{10} = -\frac{1}{\sqrt{3}}\hat{\jmath}}$$

$$v_2 = v_{20} = \sqrt{\frac{G m_1^2}{(m_1+m_2)d}} = \sqrt{\frac{1(2)^2}{(2+1)(1)}} = \frac{2}{\sqrt{3}}$$

$$\boxed{\vec{v}_{20} = \frac{2}{\sqrt{3}}\hat{\jmath}}$$

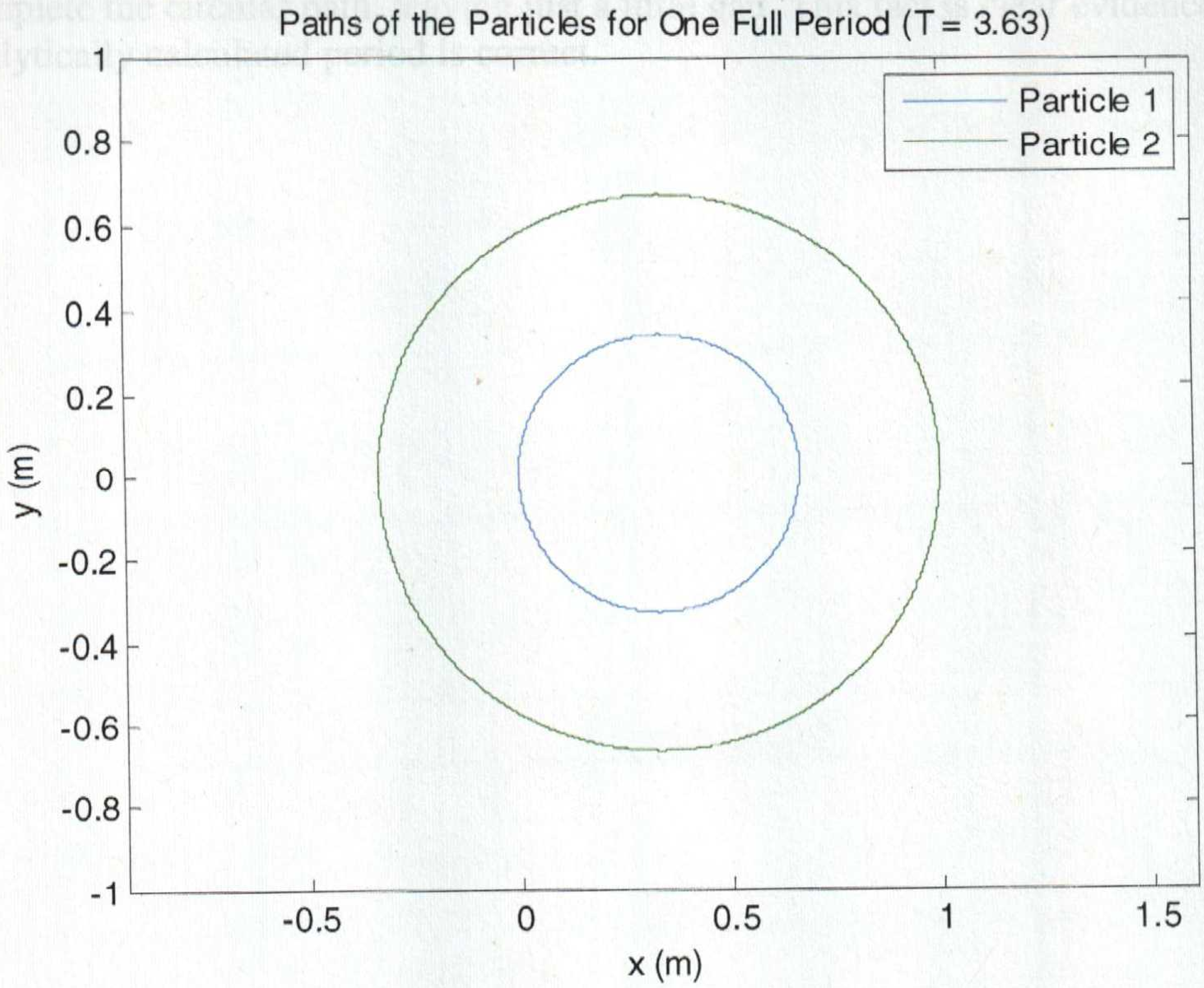
Problem 2 (Handout #6) part (b)

MATLAB code:

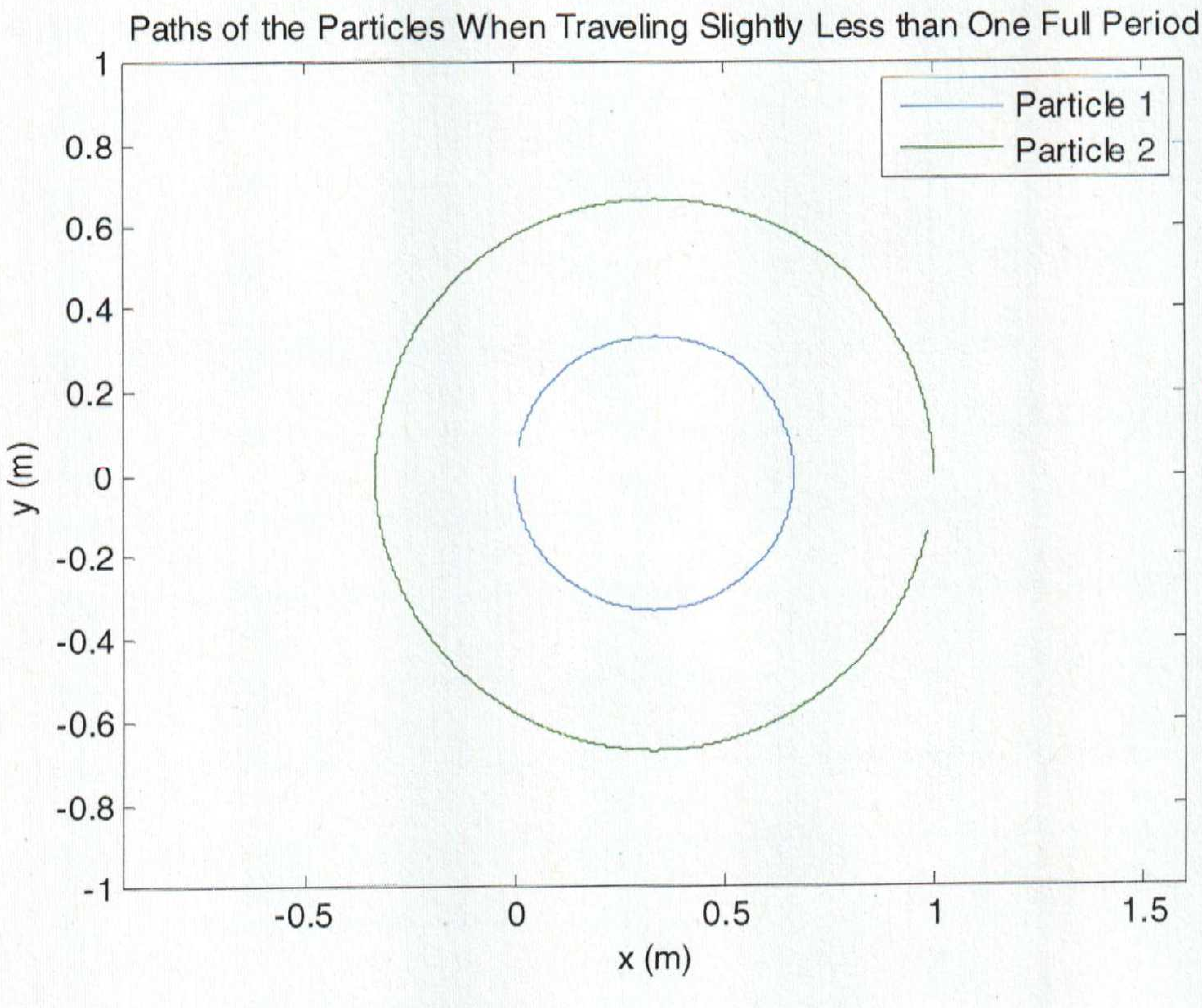
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function TwoParticles()
close all
clear
clc

% Put parameters in struct p
p.G = 1;
p.m1 = 2;
p.m2 = 1;
p.d = 1;

n = 100000; % number of steps in integration
tarray = linspace(0,3.63,n+1); % time span

% Initial conditions
r10 = [0,0];
r20 = [1,0];
v10 = [0,-1/sqrt(3)];
v20 = [0,2/sqrt(3)];
z0 = [r10,r20,v10,v20];

% Command to solve the ODEs
zarray = euler(@ODEs,tarray,z0,p);

x1 = zarray(:,1);
y1 = zarray(:,2);
x2 = zarray(:,3);
y2 = zarray(:,4);

% Plot trajectory
plot(x1,y1,x2,y2)
axis('equal');
title('Paths of the Particles')
xlabel('x (m)')
ylabel('y (m)')
legend('Particle 1', 'Particle 2')

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zdot = ODEs(t,z,p)
% The ODEs for quadratic drag

% Unpack the list of numbers into position and vel
r1 = z(1:2);
r2 = z(3:4);
v1 = z(5:6);
v2 = z(7:8);

% Unpack constants in struct p
G = p.G;
m1 = p.m1;
```

```
m2 = p.m2;
d = p.d;

% Differential equations
r1dot = v1;
r2dot = v2;
v1dot = G*m2/d^2*(r2-r1);
v2dot = G*m1/d^2*(r1-r2);

% pack the ODEs back into the list of numbers z
zdot = [r1dot,r2dot,v1dot,v2dot];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zarray = euler(rhs,tarray,z0,p)
% General ODE solver using Euler's method
% Solution will be in zarray. One row for each instant in time.

nrows = length(tarray);
ncols = length(z0);
zarray = zeros(nrows,ncols); % initialize the solution matrix
zarray(1,:) = z0; % first row is the initial condition

for i = 1:nrows-1
    h = tarray(i+1) - tarray(i); % time step
    zdot = feval(rhs,tarray(i),zarray(i,:),p); % ODEs(zarray(i))
    zarray(i+1,:) = zarray(i,:) + zdot*h; %Euler's method
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Paths of the particles traveling one full period (tmax = T = 3.63):



Paths of the particles traveling slightly less than one full period (tmax = 3.5):



The analytically calculated period is 3.63. When plotting with 3.63 as the end time, the complete circular path is observed. If a slightly smaller end time is used, say 3.5, then the plot will not complete the circular path, leaving just a little gap. This fact is clear evidence that the analytically calculated period is correct.

FBD:

$$m_1 \longrightarrow F = \frac{Gm_1m_2}{d^2} \qquad F = \frac{Gm_1m_2}{d^2} \longleftarrow m_2$$

Equations of motion:

$$m_1\ddot{\vec{r}}_1 = \frac{Gm_1m_2}{d^2}(\vec{r}_2 - \vec{r}_1) \qquad m_2\ddot{\vec{r}}_2 = \frac{Gm_1m_2}{d^2}(\vec{r}_1 - \vec{r}_2)$$

$$\ddot{\vec{r}}_1 = \frac{Gm_2}{d^2}(\vec{r}_2 - \vec{r}_1) \qquad \ddot{\vec{r}}_2 = \frac{Gm_1}{d^2}(\vec{r}_1 - \vec{r}_2)$$

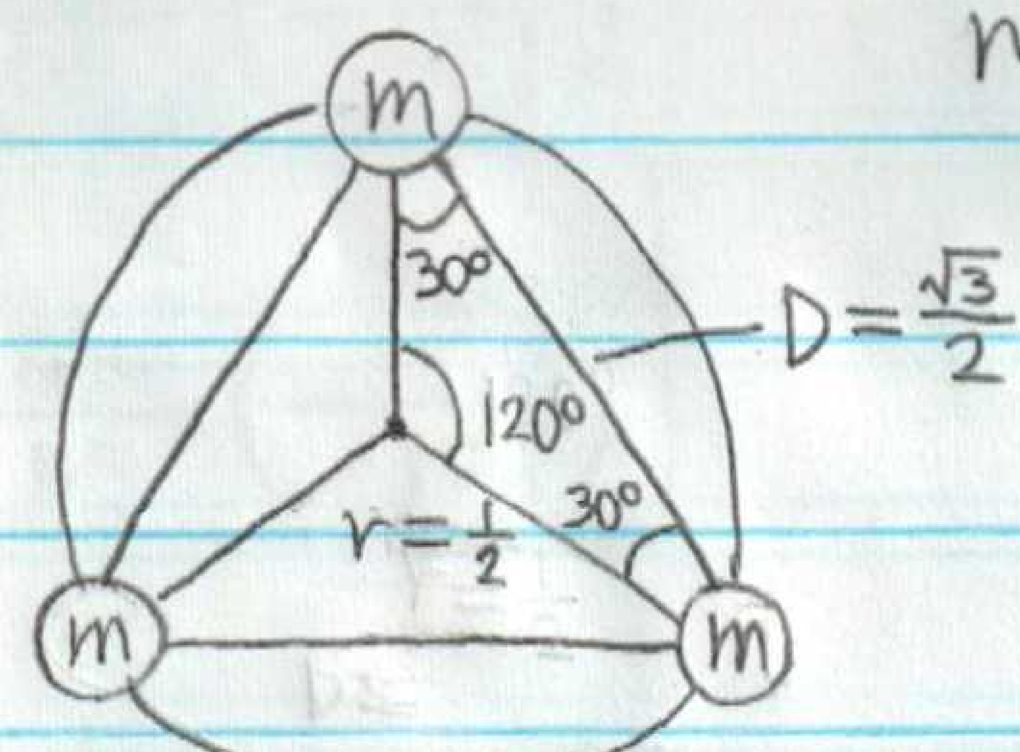
$$\boxed{\begin{array}{ll} \dot{\vec{r}}_1 = \vec{v}_1 & \dot{\vec{r}}_2 = \vec{v}_2 \\ \dot{\vec{v}}_1 = \frac{Gm_2}{d^2}(\vec{r}_2 - \vec{r}_1) & \dot{\vec{v}}_2 = \frac{Gm_1}{d^2}(\vec{r}_1 - \vec{r}_2) \end{array}}$$

$$T = 2\pi\sqrt{\frac{d^3}{G(m_1+m_2)}} = 2\pi\sqrt{\frac{1}{2+1}}$$

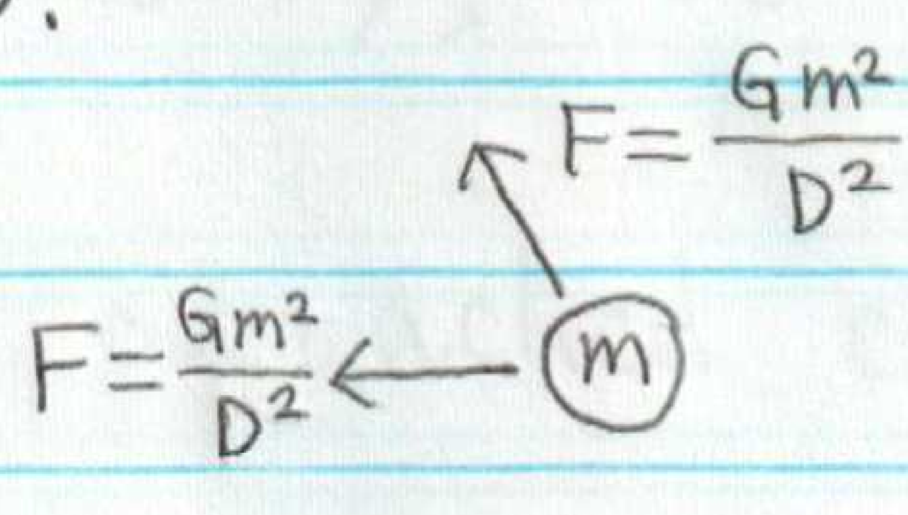
$$\boxed{T = \frac{2\pi}{\sqrt{3}} = 3.63\,s}$$

See attached code, plots, and explanation.

(c) $m_1 = m_2 = m_3 = m = 1$



FBD:



$$F_{central} = 2F\cos 30°$$

$$F_{central} = 2\left(\frac{\sqrt{3}}{2}\right)\frac{Gm^2}{D^2}$$

$$F_{central} = \frac{\sqrt{3}\,Gm^2}{D^2}$$

$$\Sigma F = ma$$

$$F_{central} = mrw^2$$

$$\frac{\sqrt{3}\,Gm^2}{D^2} = mrw^2$$

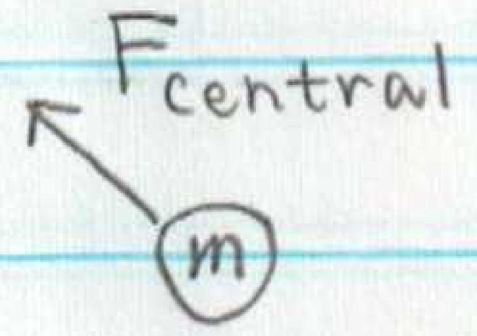
$$w^2 = \frac{\sqrt{3}\,Gm}{D^2 r}$$

$$w = \sqrt{\frac{\sqrt{3}\,Gm}{D^2 r}}$$

$$w = \sqrt{\frac{\sqrt{3}(1)(1)}{\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)}}$$

$$\boxed{w = 2.15 \text{ rad/s}}$$

(d) FBD:



$$m\ddot{\vec{r}}_i = \frac{\sqrt{3}\,Gm^2}{D^2}\left(-\frac{\vec{r}_i}{|\vec{r}_i|}\right)$$

$$\ddot{\vec{r}}_i = \frac{\sqrt{3}\,Gm}{D^2}\left(-\frac{\vec{r}_i}{|\vec{r}_i|}\right)$$

$$\boxed{\begin{aligned} \dot{\vec{r}}_i &= \vec{V}_i \\ \dot{\vec{V}}_i &= \frac{\sqrt{3}\,Gm}{D^2}\left(-\frac{\vec{r}_i}{|\vec{r}_i|}\right) \end{aligned}}$$

Initial conditions are determined based on geometry, see attached MATLAB code.

See attached code and plot for the results

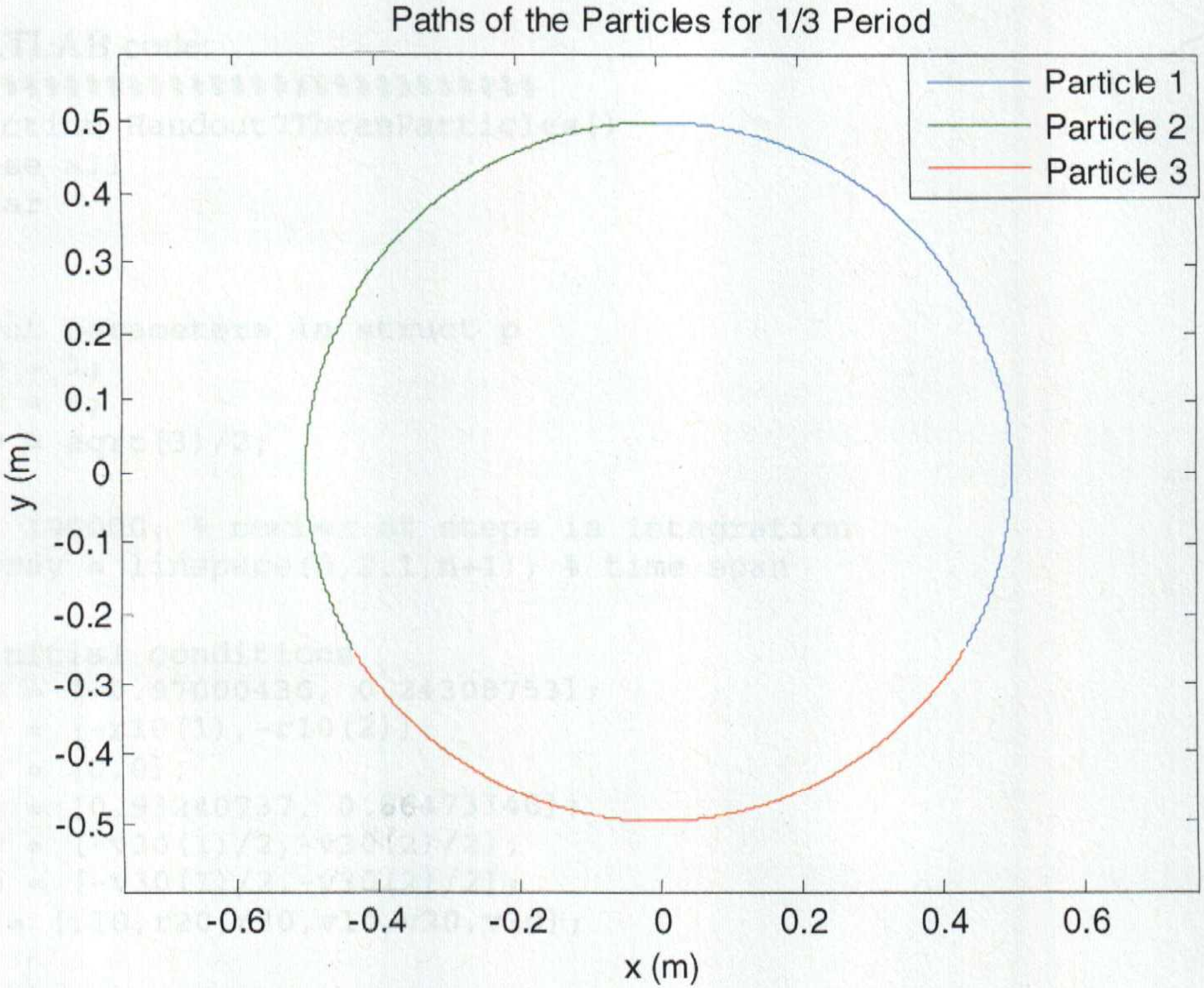
Problem 2 (Handout #6) part (d)

MATLAB code:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function ThreeParticles()
close all
clear
clc

% Put parameters in struct p
p.G = 1;
p.m = 1;
p.D = sqrt(3)/2;

n = 100000; % number of steps in integration
tarray = linspace(0,0.975,n+1); % time span

% Initial conditions
r10 = [sqrt(3)/4,-1/4];
r20 = [0,1/2];
r30 = [-sqrt(3)/4,-1/4];
omega = 2.149;
r = 1/2;
v = r*omega;
v10 = [v*cosd(60),v*sind(60)];
v20 = [-v,0];
v30 = [v*cosd(300),v*sind(300)];
z0 = [r10,r20,r30,v10,v20,v30];

% Command to solve the ODEs
zarray = euler(@ODEs,tarray,z0,p);

x1 = zarray(:,1);
y1 = zarray(:,2);
x2 = zarray(:,3);
y2 = zarray(:,4);
x3 = zarray(:,5);
y3 = zarray(:,6);

% Plot trajectory
plot(x1,y1,x2,y2,x3,y3)
axis([-0.6,0.6,-0.6,0.6])
axis('equal')
title('Paths of the Particles for 1/3 Period')
xlabel('x (m)')
ylabel('y (m)')
legend('Particle 1', 'Particle 2', 'Particle 3')

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zdot = ODEs(t,z,p)
% The ODEs for quadratic drag

% Unpack the list of numbers into position and vel
r1 = z(1:2);
```

```
r2 = z(3:4);
r3 = z(5:6);
v1 = z(7:8);
v2 = z(9:10);
v3 = z(11:12);

% Unpack constants in struct p
G = p.G;
m = p.m;
D = p.D;

% Differential equations
r1dot = v1;
r2dot = v2;
r3dot = v3;
v1dot = sqrt(3)*G*m/D^2*(-r1/norm(r1));
v2dot = sqrt(3)*G*m/D^2*(-r2/norm(r2));
v3dot = sqrt(3)*G*m/D^2*(-r3/norm(r3));

% pack the ODEs back into the list of numbers z
zdot = [r1dot,r2dot,r3dot,v1dot,v2dot,v3dot];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zarray = euler(rhs,tarray,z0,p)
% General ODE solver using Euler's method
% Solution will be in zarray. One row for each instant in time.

nrows = length(tarray);
ncols = length(z0);
zarray = zeros(nrows,ncols); % initialize the solution matrix
zarray(1,:) = z0; % first row is the initial condition

for i = 1:nrows-1
    h = tarray(i+1) - tarray(i); % time step
    zdot = feval(rhs,tarray(i),zarray(i,:),p); % ODEs(zarray(i))
    zarray(i+1,:) = zarray(i,:) + zdot*h; %Euler's method
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Paths of the particles traveling for 1/3 period:



The paths of the particles are a perfect circle. This indicates that the angular speed calculated in part (c) is correct.

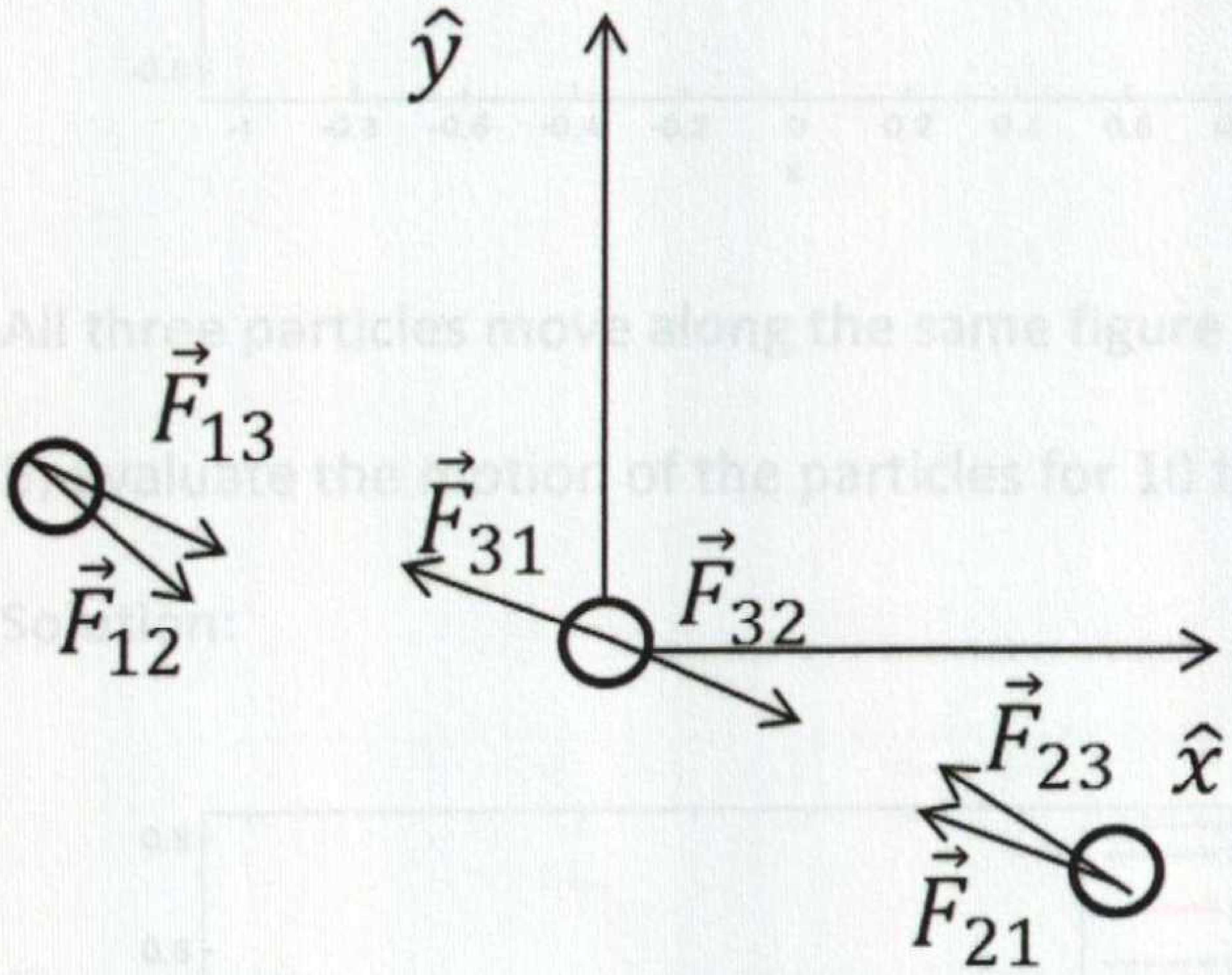
Problem 3 (#7 from Fall 2012PDF)

Evaluate the particle trajectories for the three particles experiencing gravitational force from the other two particles

A) Evaluate the motion of the particles for 2.1 time units

Solution:

FBD



$m1 = 1, m2 = 2$, $m3 = 2$, G=1,

$$r_0 1 = \begin{bmatrix} -0.97000436 \\ 0.24308753 \end{bmatrix}, r_0 2 = \begin{bmatrix} -r_0 1_x \\ -r_0 1_y \end{bmatrix}, r_0 3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_0 1 = \frac{\begin{bmatrix} -v_0 3_x \\ -v_0 3_y \end{bmatrix}}{2}, v_0 2 = \frac{\begin{bmatrix} -v_0 3_x \\ -v_0 3_y \end{bmatrix}}{2}, v_0 3 = \begin{bmatrix} 0.93240737 \\ 0.86473146 \end{bmatrix}$$

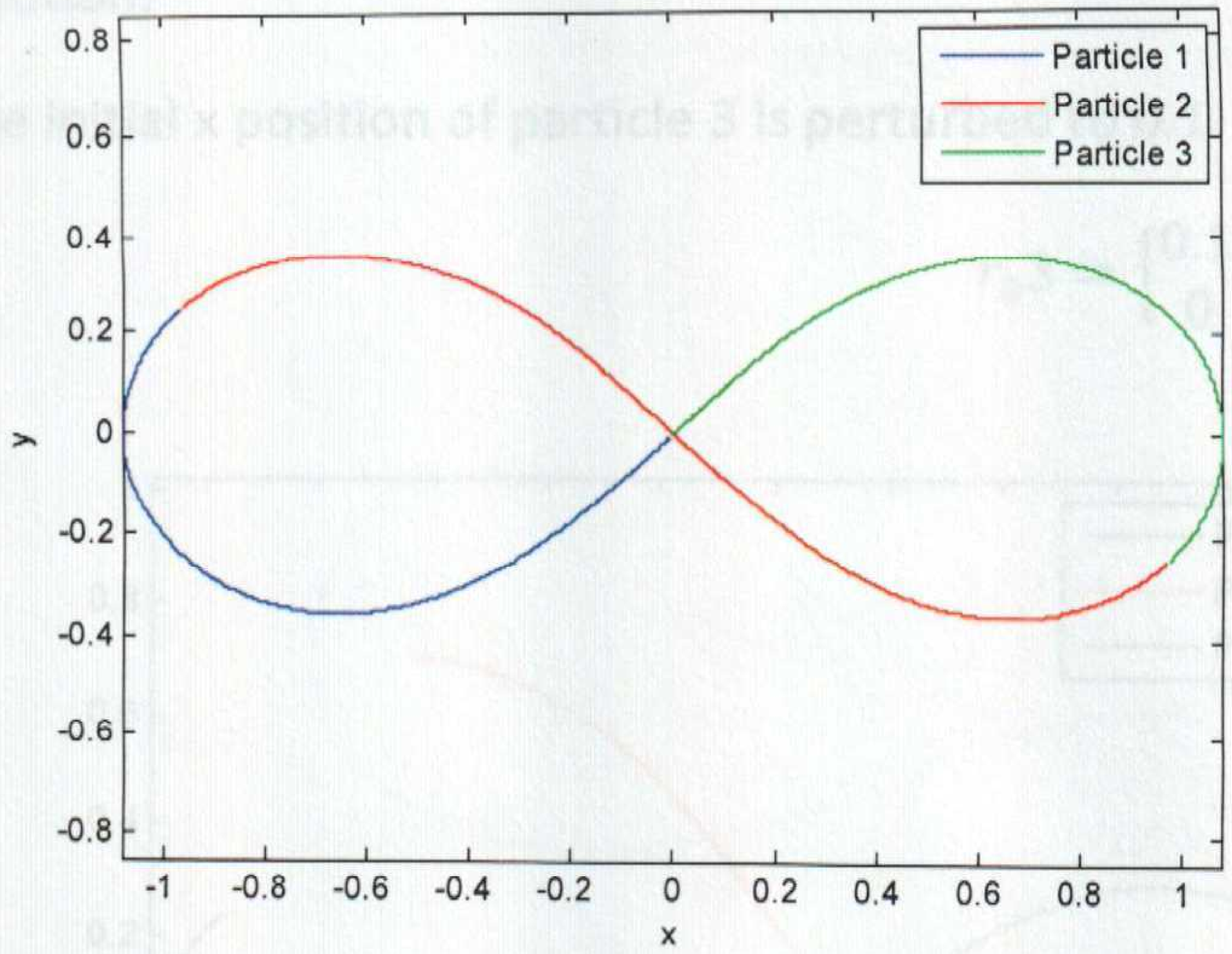
LMB

$$\vec{F_1} = \frac{1}{|\vec{r_1} - \vec{r_2}|^3}\vec{r_2} - \vec{r_1} + \frac{1}{|\vec{r_1} - \vec{r_3}|^3}\vec{r_3} - \vec{r_1} = \ddot{\vec{r_1}}$$

$$\vec{F_2} = \frac{1}{|\vec{r_2} - \vec{r_1}|^3}\vec{r_1} - \vec{r_2} + \frac{1}{|\vec{r_2} - \vec{r_3}|^3}\vec{r_3} - \vec{r_2} = \ddot{\vec{r_2}}$$

$$\overrightarrow{F_3} = \frac{1}{|\overrightarrow{r_3} - \overrightarrow{r_2}|^3}\overrightarrow{r_2} - \overrightarrow{r_3} + \frac{1}{|\overrightarrow{r_1} - \overrightarrow{r_3}|^3}\overrightarrow{r_1} - \overrightarrow{r_3} = \ddot{\vec{r}}_3$$

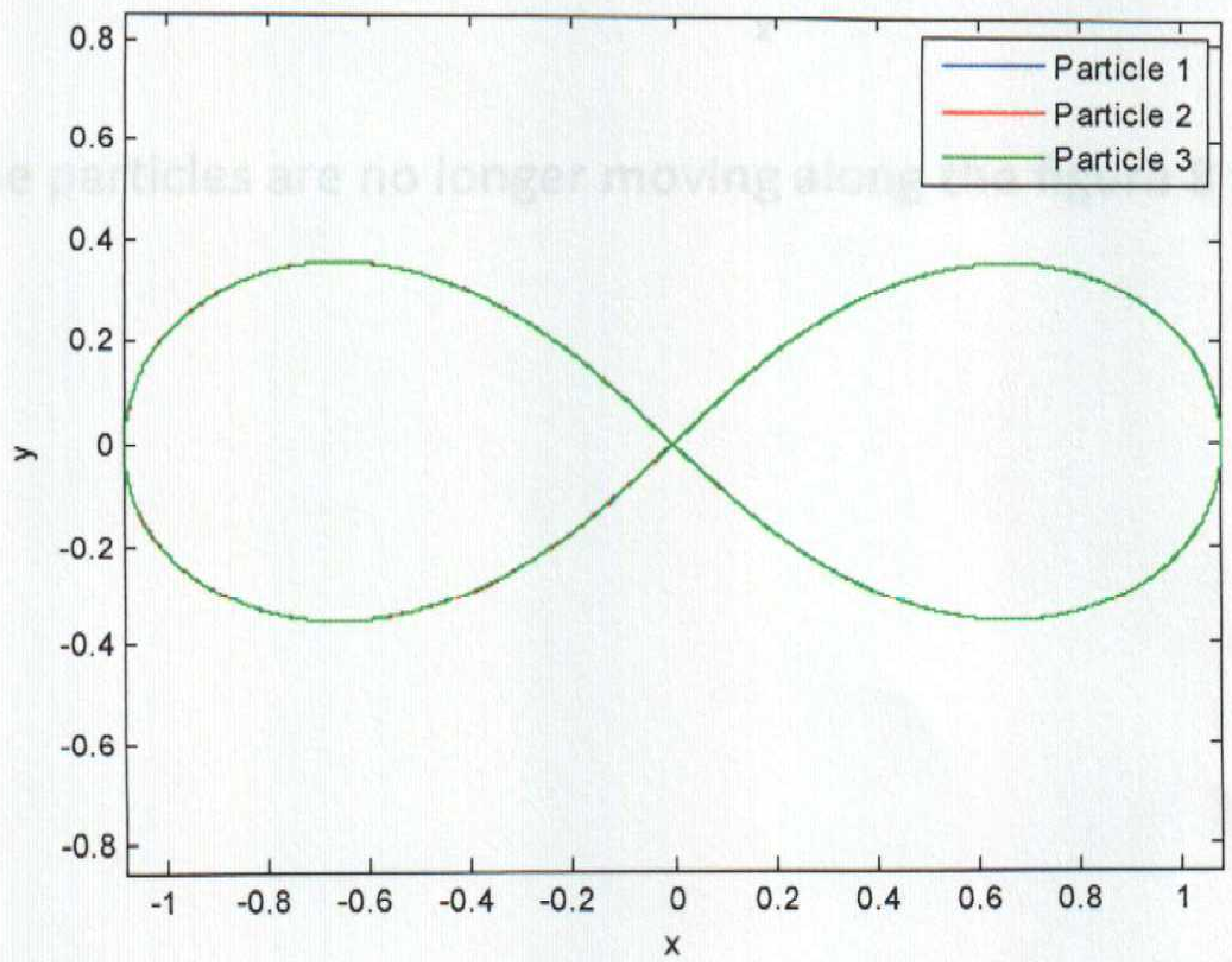
The



All three particles move along the same figure 8 orbit

B) Evaluate the motion of the particles for 10 time units

Solution:



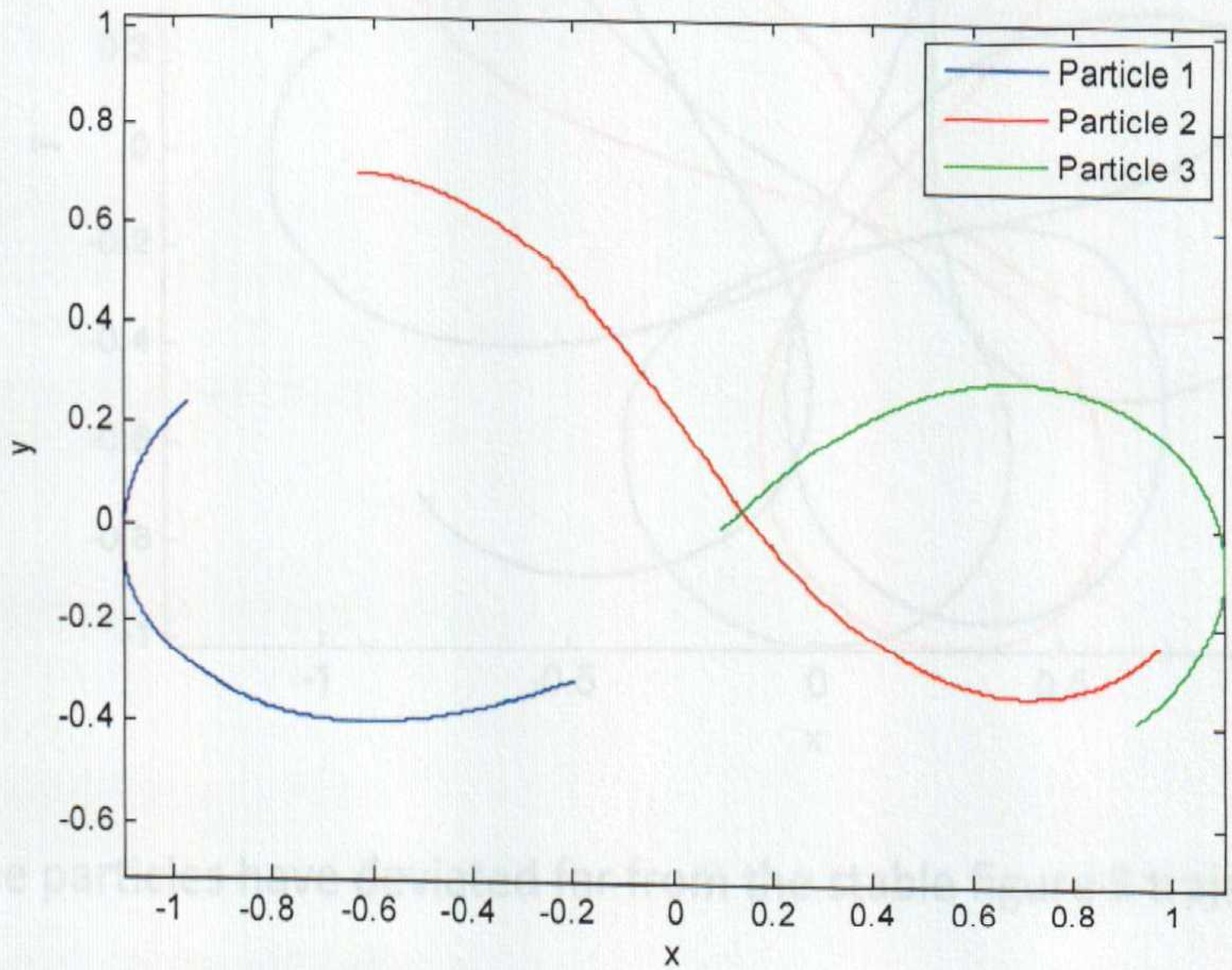
The three trajectories have overlapped.

C) Make small changes to the initial conditions and evaluate the motion of the particles and evaluate for 2.1 time units.

Solution:

The initial x position of particle 3 is perturbed to 0.1

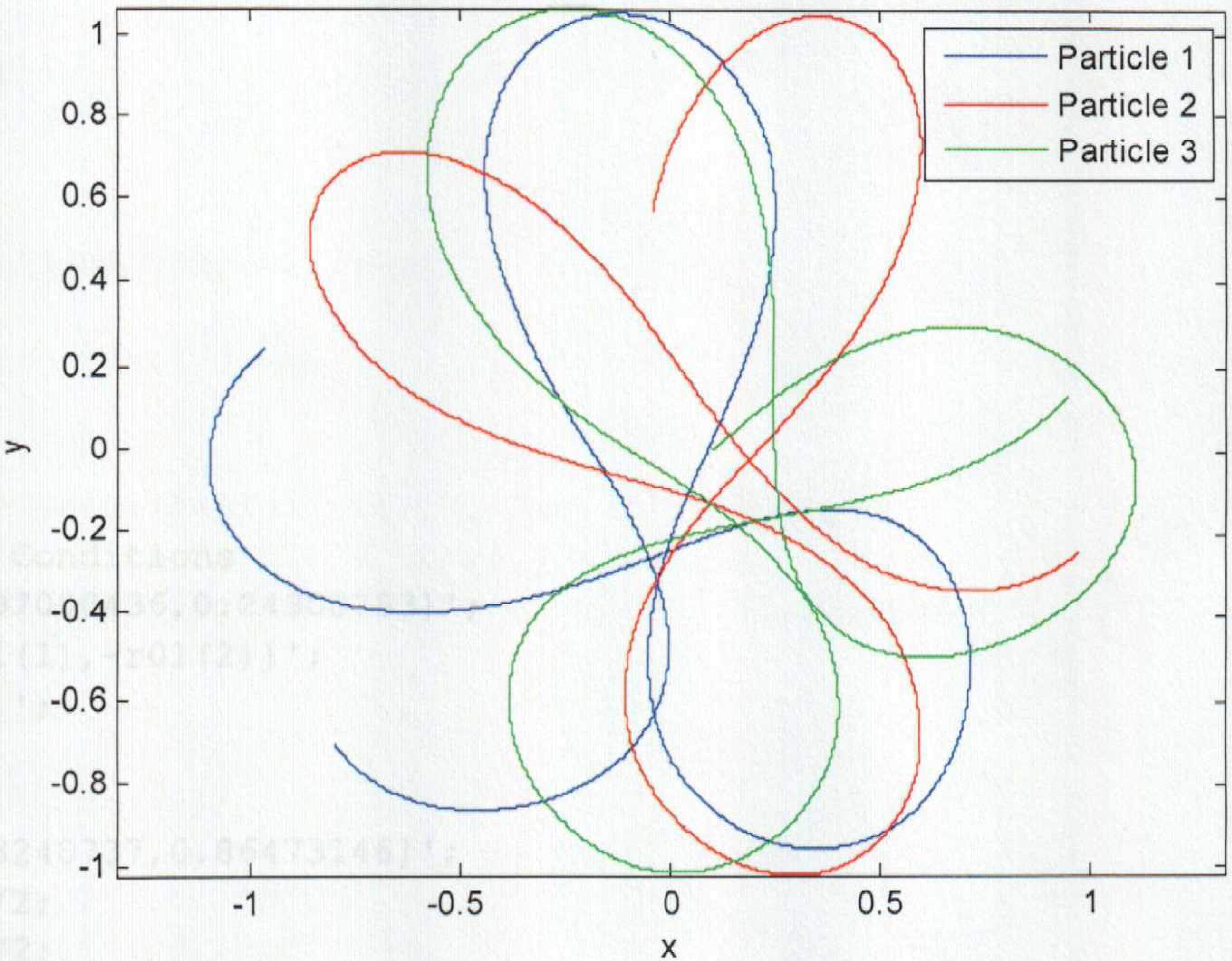
$$r_0 3 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$



The particles are no longer moving along the figure 8 trajectories.

D) Make small changes to the initial conditions and evaluate the motion of the particles and evaluate for 10 time units.

Solution:



The particles have deviated far from the stable figure 8 trajectories in parts A and B.

```
%%
%
%MSE 5730
%HW 4 - p3 (#7 From PDF)

clear
clc
close all

p.m1=1;
p.m2=1;
p.m3=1;
p.G=1;

%% A
%Initial Conditions
r01=[-0.97000436,0.24308753]';
r02=[-r01(1),-r01(2)]';
r03=[0,0]';

v03=[0.93240737,0.86473146]';
v01=-v03/2;
v02=-v03/2;

z0=[r01;r02;r03;v01;v02;v03];

%Inital and Final Times
t0=0;
tf=2.1;
options=odeset('RelTol',1e-10,'AbsTol',1e-10);
%Solve The Three Particle Gravity Equations
[t z] = ode45(@threeParticleGravity,[t0 tf],z0,options,p);

%Plot Trajectory
figure(1)
plot(z(:,1),z(:,2))
hold on
plot(z(:,3),z(:,4),'r')
plot(z(:,5),z(:,6),'g')
xlabel('x')
ylabel('y')
axis equal
legend('Particle 1','Particle 2','Particle 3')

%% B
```

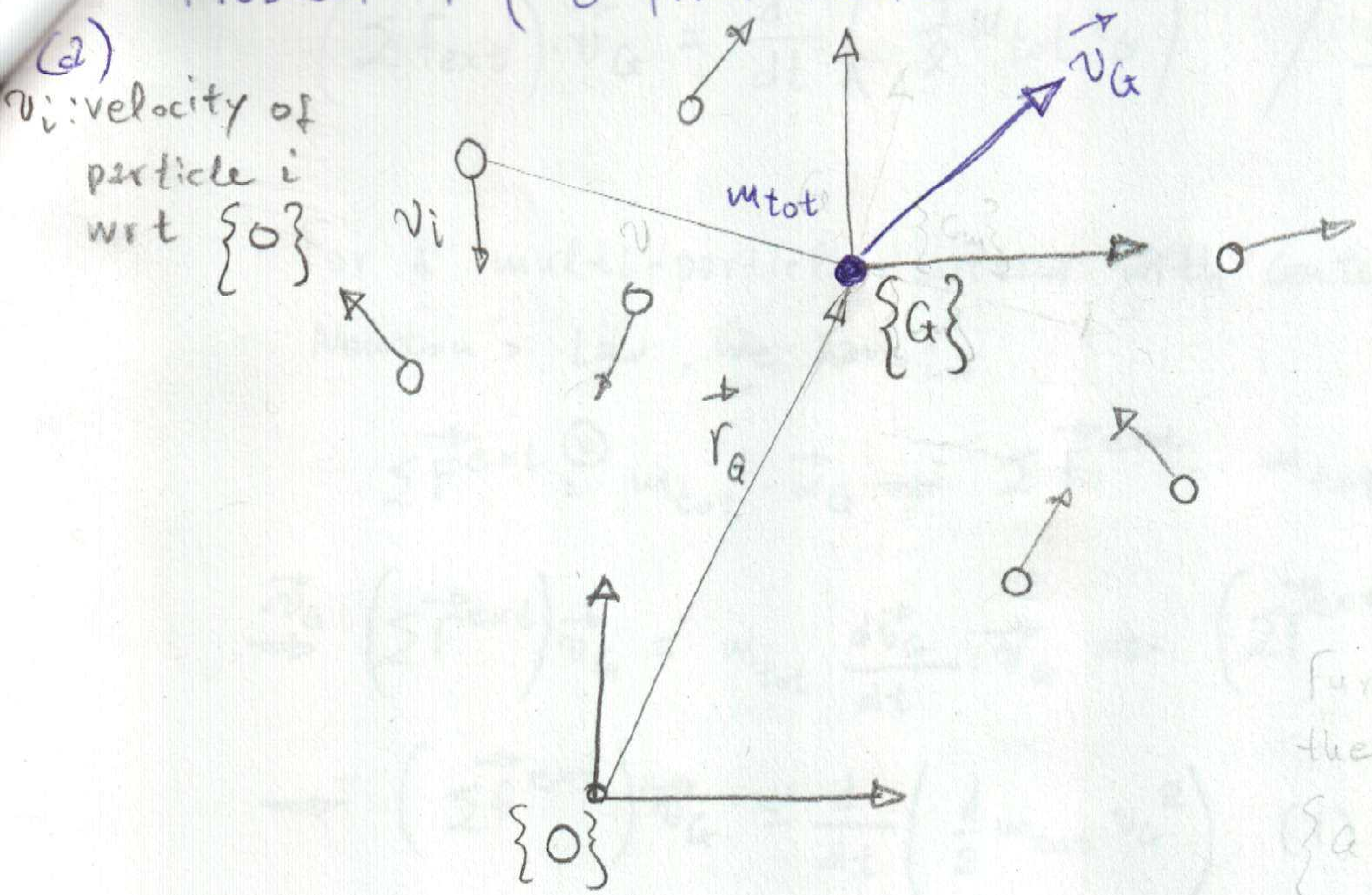
```
%Initial Conditions
r01=[-0.97000436,0.24308753]';
r02=[-r01(1),-r01(2)]';
r03=[0,0]';

v03=[0.93240737,0.86473146]';
v01=-v03/2;
v02=-v03/2;

z0=[r01;r02;r03;v01;v02;v03];

%Inital and Final Times
t0=0;
tf=10;
options=odeset('RelTol',1e-10,'AbsTol',1e-10);
%Solve The Three Particle Gravity Equations
[t z] = ode45(@threeParticleGravity,[t0 tf],z0,options,p);
%Plot Trajectory
figure(2)
plot(z(:,1),z(:,2))
hold on
plot(z(:,3),z(:,4),'r')
plot(z(:,5),z(:,6),'g')
xlabel('x')
ylabel('y')
axis equal
legend('Particle 1','Particle 2','Particle 3')
%% C
%Initial Conditions
r01=[-0.97000436,0.24308753]';
r02=[-r01(1),-r01(2)]';
r03=[.1,0]';

v03=[0.93240737,0.86473146]';
v01=-v03/2;
v02=-v03/2;


z0=[r01;r02;r03;v01;v02;v03];

%Inital and Final Times
t0=0;
tf=2.1;
options=odeset('RelTol',1e-10,'AbsTol',1e-10);
```

```
%Solve The Three Particle Gravity Equations
[t z] = ode45(@threeParticleGravity,[t0 tf],z0,options,p);

%Plot Trajectory
figure(3)
plot(z(:,1),z(:,2))
hold on
plot(z(:,3),z(:,4),'r')
plot(z(:,5),z(:,6),'g')
xlabel('x')
ylabel('y')
axis equal
legend('Particle 1','Particle 2','Particle 3')

%% D
%Initial Conditions
r01=[-0.97000436,0.24308753]';
r02=[-r01(1),-r01(2)]';
r03=[.1,0]';

v03=[0.93240737,0.86473146]';
v01=-v03/2;
v02=-v03/2;

z0=[r01;r02;r03;v01;v02;v03];

%Inital and Final Times
t0=0;
tf=10;
options=odeset('RelTol',1e-10,'AbsTol',1e-10);
%Solve The Three Particle Gravity Equations
[t z] = ode45(@threeParticleGravity,[t0 tf],z0,options,p);

%Plot Trajectory
figure(4)
plot(z(:,1),z(:,2))
hold on
plot(z(:,3),z(:,4),'r')
plot(z(:,5),z(:,6),'g')
xlabel('x')
ylabel('y')
axis equal
legend('Particle 1','Particle 2','Particle 3')
```

```
function zdot = threeParticleGravity(t,z,p)

    r1=z(1:2);
    r2=z(3:4);
    r3=z(5:6);
    v1=z(7:8);
    v2=z(9:10);
    v3=z(11:12);

    rdot1 = v1;
    rdot2 = v2;
    rdot3 = v3;

    F1 = +p.G*p.m1*p.m2/norm(r2-r1)^3*(r2-r1)+p.G*p.m1*p.m3/norm(r3-r1)^3*(r3-r1);
    a1 = F1/p.m1;
    vdot1 = a1;

    F2 = +p.G*p.m1*p.m2/norm(r1-r2)^3*(r1-r2)+p.G*p.m3*p.m2/norm(r2-r3)^3*(r3-r2);
    a2 = F2/p.m2;
    vdot2 = a2;

    F3 = +p.G*p.m1*p.m3/norm(r1-r3)^3*(r1-r3)+p.G*p.m3*p.m2/norm(r2-r3)^3*(r2-r3);
    a3 = F3/p.m3;
    vdot3 = a3;

    zdot = [rdot1;rdot2;rdot3;vdot1;vdot2;vdot3];
end
```

Problem 4 (8 from Handout of 09/18/2013)

(a)



We define a Newtonian frame $\{G\}$ to have its origin at the center of mass of the system.

Then it should be:

$m_{tot}\vec{r}_G' = \Sigma m_i \vec{r}_i'$, where the symbol (') denotes relation to frame $\{G\}$.

Therefore, it should be:

$0 = \Sigma m_i \vec{r}_i'$ (1)

We can also write that:

$\vec{v}_i' = \vec{v}_i - \vec{v}_G$ (2)

An expression for the kinetic energy of the system is:

$E_k = \frac{1}{2}\Sigma m_i v_i^2$ (velocities wrt $\{O\}$) $\rightsquigarrow$

$$E_k = \frac{1}{2}\Sigma m_i v_i^2 \overset{(2)}{=} \frac{1}{2}\Sigma m_i(\vec{v}_i' + \vec{v}_G)^2 = \frac{1}{2}\Sigma m_i(\vec{v}_i' + \vec{v}_G)(\vec{v}_i' + \vec{v}_G) =$$

$$= \frac{1}{2}\Sigma m_i(v_i'^2 + \vec{v}_i'\vec{v}_G + \vec{v}_G v_i' + v_G^2) = \frac{1}{2}\Sigma m_i v_i'^2 + \frac{1}{\cancel{2}}\cancel{2}\Sigma m_i \vec{v}_i'\vec{v}_G + \frac{1}{2}\Sigma m_i v_G^2 =$$

$$\overset{(2)}{=} \frac{1}{2}\Sigma m_i v_i'^2 + \frac{1}{2}\underbrace{\Sigma m_i}_{m_{tot}} v_G^2 \overset{(2)}{=} \frac{1}{2}m_{tot}v_G^2 + \frac{1}{2}\Sigma m_i(\vec{v}_i - \vec{v}_G)^2 =$$

$$\boxed{= \frac{1}{2}m_{tot}v_G^2 + \frac{1}{2}\Sigma m_i |\vec{v}_i - \vec{v}_G|^2} \ \square \ ✓$$

(b)

$$\left(\sum \vec{F}_{ext}\right) \cdot \vec{v}_G = \frac{d}{dt}\left(\frac{1}{2} m_{tot} v_G^2\right)$$ / Always true?

For a multi-particle system with center of mass G, from the 2nd Newton's Law, we have;

$$\sum \vec{F}^{ext} \overset{(*)}{=} m_{tot} \cdot \vec{\alpha}_G \Rightarrow \sum \vec{F}^{ext} = m_{tot} \cdot \frac{d\vec{v}_G}{dt} \Rightarrow$$

$$\overset{\cdot \vec{v}_G}{\Rightarrow} \left(\sum \vec{F}^{ext}\right) \vec{v}_G = m_{tot} \frac{d\vec{v}_G}{dt} \cdot \vec{v}_G \Rightarrow \left(\sum \vec{F}^{ext}\right) \vec{v}_G = \frac{d}{dt}\left(m_{tot} \frac{1}{2} \vec{v}_G^{\,2}\right) \Rightarrow$$

$$\Rightarrow \left(\sum \vec{F}^{ext}\right) \vec{v}_G = \frac{d}{dt}\left(\frac{1}{2} m_{tot} v_G^2\right) \quad (1)$$

We can also see that $\frac{d}{dt}\left(\frac{1}{2} m_{tot} \vec{v}_G^{\,2}\right) = \frac{1}{\cancel{2}} m_{tot} \cancel{2} v_G \cdot \vec{\alpha}_G = \left(m_{tot} \vec{\alpha}_G\right) \cdot \vec{v}_G =$

$= \left(\vec{F}^{ext}\right) \vec{v}_G \quad (2)$

Therefore the expression is always true.

✓

④ Handout #8

c) Is it <u>always</u> true that $p^{int} = \frac{d}{dt}\left[\frac{1}{2}\sum m_i v_{i/G}^2\right]$

From part ⓐ

$$E_K = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2} m_{tot} v_G^2 + \frac{1}{2}\sum m_i (v_{i/G})^2$$

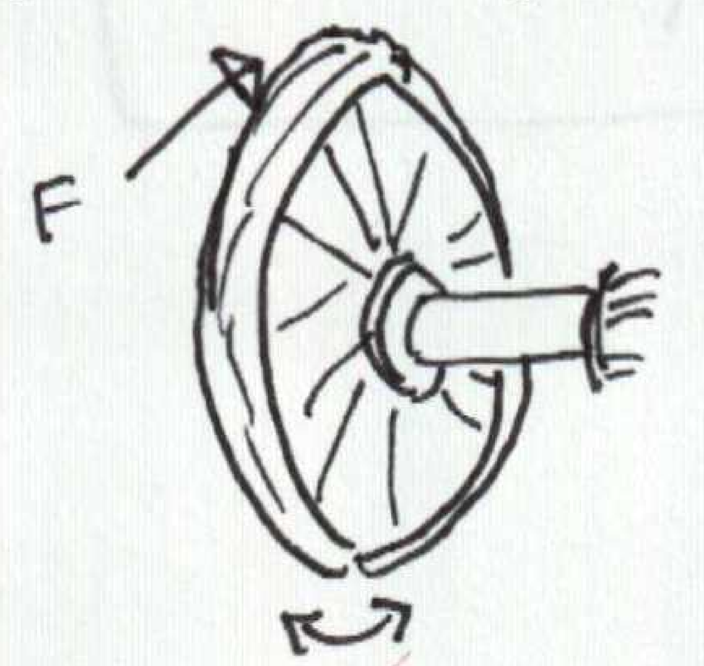
$$\dot{E}_K = \underbrace{m_{tot}\vec{a}_G \vec{v}_G}_{\text{from part ⓑ}} + \sum m_i \vec{a}_{i/G} \vec{v}_{i/G}$$

$$\dot{E}_K = \sum \vec{F}^{ext} \vec{v}_G + \sum m_i \vec{a}_{i/G} \vec{v}_{i/G}$$

$$\dot{E}_K = \sum \vec{F}^{ext} \vec{v}_G + \frac{d}{dt}\left[\frac{1}{2}\sum m_i v_{i/G}^2\right] = p^{tot} = p^{int} + p^{ext}$$

If $p^{int} = \frac{d}{dt}\left[\frac{1}{2}\sum m_i v_{i/G}^2\right]$, then $\underbrace{p^{ext} = \sum \vec{F}^{ext} \vec{v}_G}_{\text{Let's examine this claim}}$

Imagine a initially motionless bicycle wheel that has boundary forces fixing it in position at its axle.



The wheel is free to rotate but has no initial rotation ($\omega = 0$)

A force (external) is then applied tangentially to the tire, causing it to spin. Clearly the kinetic energy increases according to $E_{k,rot} = \frac{1}{2}I\omega^2$

Looking at $p^{ext} = \sum \vec{F}^{ext} \vec{v}_G$. p^{ext} is non-zero due to the added energy.

However, because the axle is constrained $\vec{v}_G = 0$

Therefore, for this case $p^{ext} \neq \sum \vec{F}^{ext} \vec{v}_G$

This means that $p^{int} \neq \frac{d}{dt}\left[\frac{1}{2}\sum m_i v_{i/G}^2\right]$ for all cases.

<u>NOTE:</u> For this proof, I discussed my thoughts with Adam Trofa and Nozomi Hitomi before coming up with a good counter example.

Problem 3.20

Consider a system comprising two extended bodies, which have masses M_1 and M_2 and centers of mass at $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$. Prove that the center of mass of the whole system is

$$\underset{\sim}{R} = \frac{M_1 \underset{\sim}{R}_1 + M_2 \underset{\sim}{R}_2}{M_{tot}}$$

SOLUTION

From the definition of CM

$$\underset{\sim}{R}_1 = \frac{1}{M_1} \sum_i m_i \underset{\sim}{r}_i$$, where m_i is a particle of extended body 1 and $\underset{\sim}{r}_i$ is its corresponding position vector

$$\underset{\sim}{R}_2 = \frac{1}{M_2} \sum_j m_j \underset{\sim}{r}_j ,$$ " ... body 2 ... "

For a system comprised of these two extended bodies then

$M_{tot} = M_1 + M_2$ and by def'n

$$\underset{\sim}{R} = \frac{1}{M_{tot}} \sum_k m_k \underset{\sim}{r}_k$$, where m_k is a particle of the system and r_k is its corresponding position vector.

$$= \frac{1}{M_{tot}} \left(\sum_i m_i \underset{\sim}{r}_i + \sum_j m_j \underset{\sim}{r}_j \right)$$, where by the prop. of summation it has been broken into 2 sums with the particles assoc. with body 1 are grouped & those with body 2 are grouped.

$$= \frac{1}{M_{tot}} \left(M_1 \underset{\sim}{R}_1 + M_2 \underset{\sim}{R}_2 \right)$$

$$\underset{\sim}{R} = \frac{M_1 \underset{\sim}{R}_1 + M_2 \underset{\sim}{R}_2}{M_1 + M_2}$$