2.12 Show that if the net force on a one-dimensional particle depends only on position, $F = F(x)$, then Newton's second law can be solved to find v as a function of x given by:

$$v^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x)\,dx$$

If $\dot{v} = \frac{dv}{dt}$, and $v = \frac{dx}{dt}$, then $\dot{v} = \frac{dv}{dt}\frac{dx}{dx} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}$

$$F = m\dot{v} = m\frac{1}{2}\frac{d(v^2)}{dx} = F(x)$$

$$F(x) = m\frac{1}{2}\frac{d(v^2)}{dx}$$

Separate Variables:

$$m\,d(v^2) = 2F(x)\,dx$$

Integrate Both Sides:

$$\int_{V_0}^{V} m\, d(v^2) = \int_{x_0}^{x} 2F(x)\,dx$$

$$m\int_{V_0}^{V} d(v^2) = 2\int_{x_0}^{x} F(x)\,dx$$

$$m\left[v^2 - V_0^2\right] = 2\int_{x_0}^{x} F(x)\,dx$$

$$v^2 - V_0^2 = \frac{2}{m}\int_{x_0}^{x} F(x)\,dx$$

$$\boxed{v^2 = V_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x)\,dx}$$

If $F(x)$ is a constant, the integral will become $Fx\Big|_{x_0}^{x}$

$$v^2 = V_0^2 + \frac{2}{m}\int_{x_0}^{x} F\,dx$$

$$v^2 = V_0^2 + \frac{2}{m}\left[Fx\right]_{x_0}^{x}$$

$$v^2 = V_0^2 + \frac{2}{m}\left(Fx - Fx_0\right)$$

$$v^2 = V_0^2 + \frac{2F}{m}\left(x - x_0\right)$$

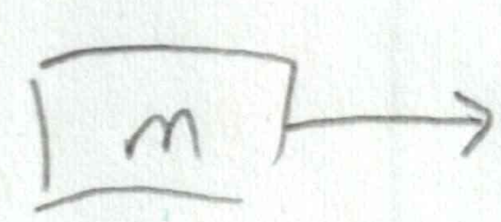
Nice

$$\boxed{F(x - x_0) = \frac{m(v^2 - V_0^2)}{2} \qquad \text{Work} = \Delta KE}$$

2.13)

$x(0) = x_0$
$v(0) = 0 = v_0$
$g(x) = |v(x)|$

FBD



✓

$$v^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} F(x')\,dx'$$

$$v(x)^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} -kx'\,dx'$$

$$v(x)^2 = v_0^2 + \frac{2}{m}\left(\frac{-k}{2}\right)(x^2 - x_0^2)$$

$$v(x)^2 = v_0^2 - \frac{k}{m}(x^2 - x_0^2)$$

$$v(x) = \sqrt{v_0^2 - \frac{k}{m}(x^2 - x_0^2)}$$

$$v(x) = \sqrt{\frac{k}{m}(x_0^2 - x^2)} \qquad \omega^2 = \frac{k}{m}$$

$$v(x) = \pm\omega\sqrt{x_0^2 - x^2}$$

$$v(x) = \frac{dx}{dt}$$

$$\int_0^t dt' = \int_{x_0}^{x} \frac{1}{v(x')}\,dx'$$

$$t = \pm\frac{1}{\omega}\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x'^2}}\,dx'$$

$$\pm\omega t = \int_{x_0}^{x} \frac{1}{x_0\sqrt{1 - \frac{x'^2}{x_0^2}}}\,dx'$$

$$\pm\omega t = \left[\arcsin\left(\frac{x}{x_0}\right)\right]_{x_0}^{x}$$

$$\pm\omega t = \left[\arcsin\left(\frac{x}{x_0}\right) - \arcsin(1)\right]$$

$$\pm\omega t = \arcsin\left(\frac{x}{x_0}\right) - \frac{\pi}{2}$$

Good!

$$\arcsin\left(\frac{x}{x_0}\right) = \frac{\pi}{2} \pm \omega t$$

$$\frac{x}{x_0} = \sin\left(\frac{\pi}{2} \pm \omega t\right)$$

Sine offset by $\frac{\pi}{2}$ = cosine

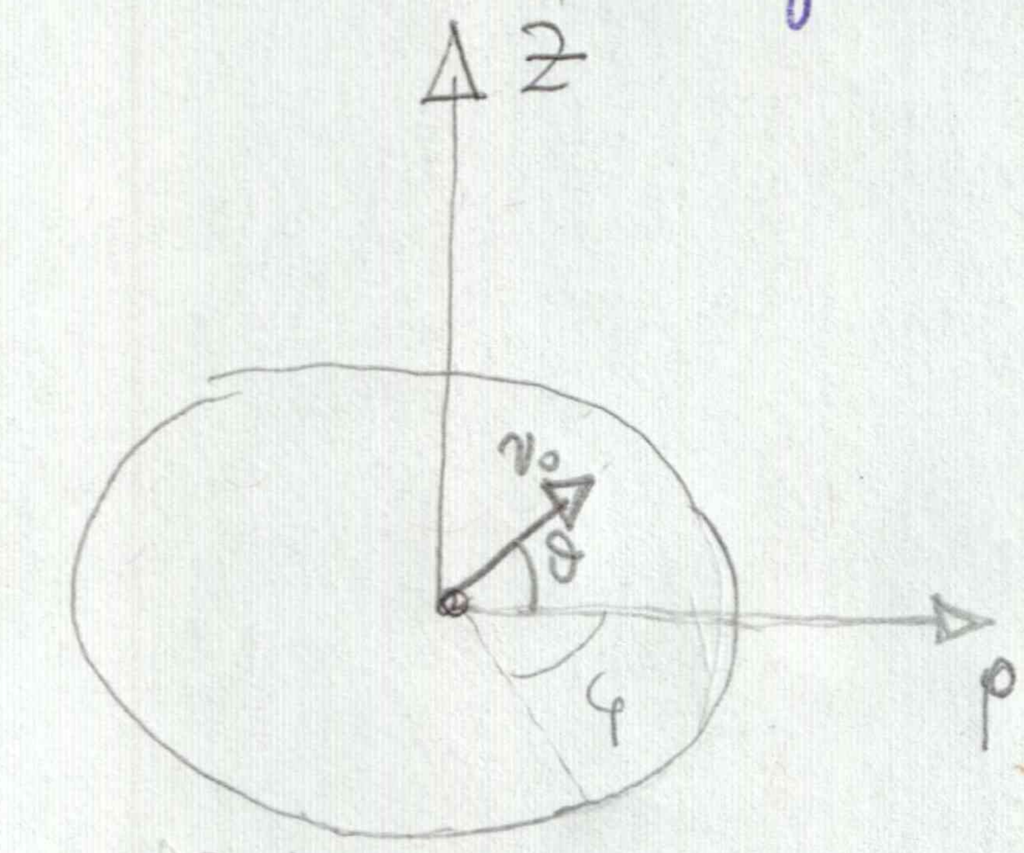
$$\frac{x}{x_0} = \cos(\pm\omega t) = \cos(\omega t)$$

$$\boxed{x = x_0\cos(\omega t)}$$

✓

Problem 3 (2.21, Taylor)

gun, shells in any direction, v_0 initial velocity
no air resistance
with cyl. polar coordinates
show that the gun can hit any object inside the surface $z = \frac{v_0^2}{2g} - \frac{g}{2v_0^2}\rho^2$.

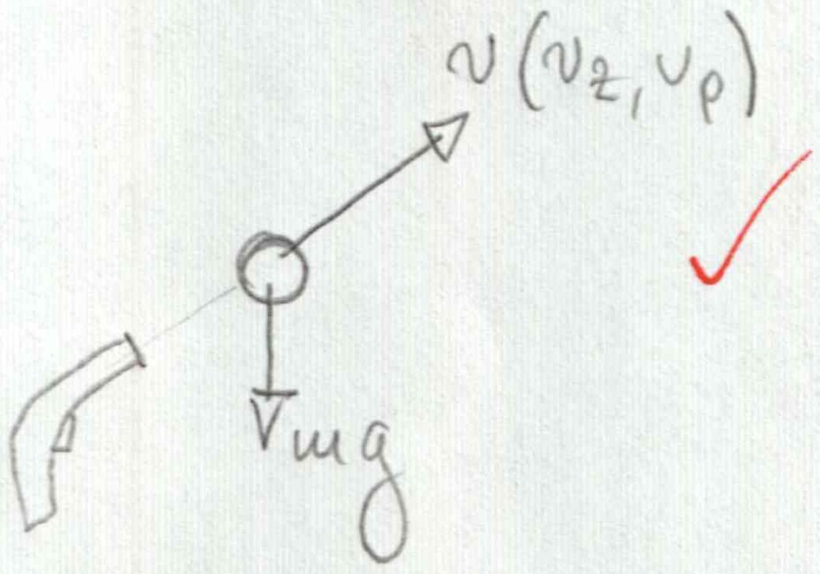


Scheme

We consider this problem as a 2D, involving only z and ρ, due to its axisymetrical nature around $\hat{z}$ axis. ✓

The final result can easily be generalized for every φ angle.

F.B.D.



✓

L.M.B.

$$\Sigma\vec{F} = m\vec{a} \rightarrow$$

$$\left.\begin{aligned} \cancel{m}\ddot{z} &= -\cancel{m}g \\ m\ddot{\rho} &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} \ddot{z} &= -g \\ \ddot{\rho} &= 0 \end{aligned}\right\} \rightarrow \left.\begin{aligned} \dot{z} &= -gt + v_{z0} \\ \dot{\rho} &= v_{\rho 0} \end{aligned}\right\} \rightarrow$$

$$\rightarrow \left.\begin{aligned} z &= -\tfrac{1}{2}gt^2 + v_{z0}t + z_0 \\ \rho &= v_{\rho 0}t + \rho_0 \end{aligned}\right\} \rightarrow ✓$$

$$\xrightarrow[\rho_0 = 0]{z_0 = 0} \left.\begin{aligned} z &= -\tfrac{1}{2}gt^2 + v_{z0}t \\ \rho &= v_{\rho 0}t \end{aligned}\right\} \rightarrow \left.\begin{aligned} z &= -\tfrac{1}{2}g\frac{\rho^2}{v_{\rho 0}^2} + v_{z0}\cdot\frac{\rho}{v_{\rho 0}} \\ t &= \frac{\rho}{v_{\rho 0}} \end{aligned}\right\} \rightarrow$$

$$\rightarrow z = -\frac{g}{2v_{\rho 0}^2}\rho^2 + \frac{v_{z0}}{v_{\rho 0}}\rho \quad (1) ✓$$

$$t = \frac{\rho}{v_{\rho 0}}$$

From the Scheme, we get: $v_{z0} = v_0 \sin\theta$ and $v_{\rho 0} = v_0\cos\theta$. Therefore:

$$\frac{v_{z0}}{v_{\rho 0}} = \frac{\cancel{v_0}\sin\theta}{\cancel{v_0}\cos\theta} = \tan\theta \quad (2)$$

From (1) and (2), we get:

$$z(t) = -\frac{g}{2v_0^2\cos^2\theta}\rho^2 + \tan\theta\cdot\rho \quad (3)$$

✓ Good!

But we know that: $\cos^2\theta + \sin^2\theta = 1 \xrightarrow{\times 1/\cos^2\theta} 1 + \tan^2\theta = \frac{1}{\cos^2\theta} \rightarrow 1 + \tan^2\theta = \sec^2\theta$ (4)

Hence: $z(t) = -\frac{g}{2v_0^2}(1+\tan^2\theta)\cdot\rho^2 + \tan\theta\cdot\rho$ or $z(t) = -\frac{g}{2v_0^2}\sec^2\theta\,\rho^2 + \tan\theta\,\rho$

(5) 3

We want to find the surface which surrounds every possible projectile flight. This is equivalent to finding the maximum z for every ρ. This corresponds to a specific θ function, which we can call $\theta_{critical}$. In order to find that, we take the partial derivative of (5) wrt θ and we set it equal to zero: ✓

$$\frac{\partial z}{\partial \theta} = 0 \Rightarrow -\frac{g\rho^2}{\cancel{2}v_0^2}\cancel{2}\sec^2\theta\tan\theta + \sec^2\theta\cdot\rho = 0 \Rightarrow$$

$$\Rightarrow -\frac{g\rho^2}{v_0^2}\sec^2\theta\tan\theta + \rho\sec^2\theta = 0 \Rightarrow$$

$$\Rightarrow -\frac{g\rho^2}{v_0^2}(1+\tan^2\theta)\tan\theta + \rho(1+\tan^2\theta) = 0 \Rightarrow$$

$$\Rightarrow -\frac{g\rho^2}{v_0^2}(\tan\theta+\tan^3\theta) + \rho\tan^2\theta + \rho = 0 \Rightarrow$$

$$\Rightarrow -\frac{g\rho^2}{v_0^2}\tan^3\theta - \frac{g\rho^2}{v_0^2}\tan\theta + \rho\tan^2\theta + \rho = 0 \Rightarrow$$

$$A = \frac{g\rho^2}{v_0^2} \qquad x = \tan\theta$$

$$\Rightarrow -A\tan^3\theta - A\tan\theta + \rho\tan^2\theta + \rho = 0 \Rightarrow$$

$$\Rightarrow -Ax^3 - Ax + \rho x^2 + \rho = 0 \Rightarrow$$

$$\Rightarrow Ax^3 - \rho x^2 + Ax - \rho = 0 \Rightarrow$$

$$\Rightarrow x^2(Ax-\rho) + (Ax-\rho) = 0 \Rightarrow$$

$$\Rightarrow (x^2+1)(Ax-\rho) = 0 \Rightarrow$$

Aborted because $\theta \in \mathbb{R}$

$$\cancel{x^2 = -1} \quad \text{or} \quad Ax = \rho \Rightarrow$$

$$x = \frac{\rho}{A} \Rightarrow \tan\theta = \frac{\cancel{\rho}}{\frac{g\rho^{\cancel{2}}}{v_0^2}} \Rightarrow$$

$$\boxed{\tan\theta = \frac{v_0^2}{\rho g}} \quad (6) \leftarrow \text{corresponds to } \theta_{critical}$$ ✓

Upon substitution of (6) to (5), we get:

$$z(t) = -\frac{g\rho^2}{2v_0^2}\left(1+\frac{v_0^4}{\rho^2 g^2}\right) + \frac{v_0^2}{\cancel{\rho} g}\cancel{\rho} \Rightarrow$$

$$\Rightarrow z(t) = -\frac{g\rho^2}{2v_0^2} - \frac{\cancel{g}\cancel{\rho^2}v_0^{\cancel{4}\,2}}{2\cancel{v_0^2}\cancel{\rho^2}g^{\cancel{2}}} + \frac{v_0^2}{g} \Rightarrow \boxed{z(t) = \frac{v_0^2}{2g} - \frac{g}{2v_0^2}\rho^2}$$ ✓

2.36) a) given density iron = $8 g/cm^3$, find terminal speed of 1 lb ball and 100 lb ball

b) if 1 cubit ≈ 2 feet, find time for heavier ball to land and position of smaller ball at that time. (use eq'n 2.58) - dropped from 100 cubits

100 lb ≈ 45.3592 kg ≈ 45.36 kg ; 1 lb = 0.4536 kg.
100 cubits = 200 ft = 60.96 m.

a) $$8 \frac{g}{cm^3} \cdot \frac{kg}{1000g} \cdot \frac{cm^3 (100^3)}{m^3} = 8000 \frac{kg}{m^3}$$

$\boxed{\text{1 lb ball:}}$ $$\frac{0.4536 kg}{8000 kg/m^3} = 5.67 \times 10^{-5} m^3 = \frac{4}{3}\pi r^3$$

$$r = 0.02383 \text{ meters.}$$
$$D = 0.04766 \text{ meters.}$$

From eq'n 2.59:
$$V_{ter} = \sqrt{\frac{mg}{\gamma D^2}} \quad \text{where } \gamma = 0.25 N \cdot s^2/m^4$$

$$V_{ter} = \sqrt{\frac{0.4536 kg \cdot 9.81 m/s^2}{0.25 N \cdot s^2/m^4 \cdot (0.04766 m)^2}}$$

$$V_{ter} = 88.5211 m/s$$

$$\boxed{V_{ter\,small} = 88.52 m/s}$$ ✓

$\boxed{\text{100 lb ball}}$ $$\frac{45.36 kg}{8000 kg/m^3} = 5.67 \times 10^{-3} m^3 = \frac{4}{3}\pi r^3$$

$$r = .1106 m$$
$$D = .2212 m.$$

$$V_{ter} = \sqrt{\frac{mg}{\gamma D^2}}$$

$$V_{ter} = \sqrt{\frac{45.36 kg \cdot 9.81 m/s^2}{0.25 N \cdot s^2/m^4 (0.2212 m)^2}} \qquad V_{ter\,big} = 190.7286 m/s$$

$$\boxed{V_{ter\,big} = 190.73 m/s}$$ ✓

2.36) continued

b) Time for heavier ball to land:

eqn 2.58

$$y = \frac{(V_{ter\,big})^2}{g} \ln\left[\cosh\left(\frac{gt}{V_{ter\,big}}\right)\right]$$

rearranging to solve for time:

$$\cosh^{-1}\left(e^{\frac{yg}{(V_{ter})^2}}\right) = \frac{gt}{V_{ter}}$$

$$t = \frac{V_{ter}}{g}\cosh^{-1}\left(e^{\frac{yg}{(V_{ter})^2}}\right)$$

$$t = \frac{(190.73\,m/s)}{(9.81\,m/s^2)}\cosh^{-1}\left(e^{\frac{60.96m \cdot 9.81m/s^2}{(190.73m/s)^2}}\right)$$

$$t = 3.53502\text{ sec}$$

$$\boxed{t \approx 3.535\text{ sec}}$$

big ball to land.

Small ball at this time:

$$y = \frac{(88.52\,m/s)^2}{9.81\,m/s^2}\ln\left(\cosh\left(\frac{9.81m/s^2 \cdot 3.535sec}{88.52\,m/s}\right)\right)$$

$$y = 59.7875\,m$$

$$\boxed{y_{small} = 59.79\,m}$$

$$\boxed{\text{They are } 60.96 - 59.79 = 1.17m \text{ apart}}$$

2.22

Based on given parameters, calculate maximum range of a projectile and compare to the projectile range in a vacuum.

Nice !

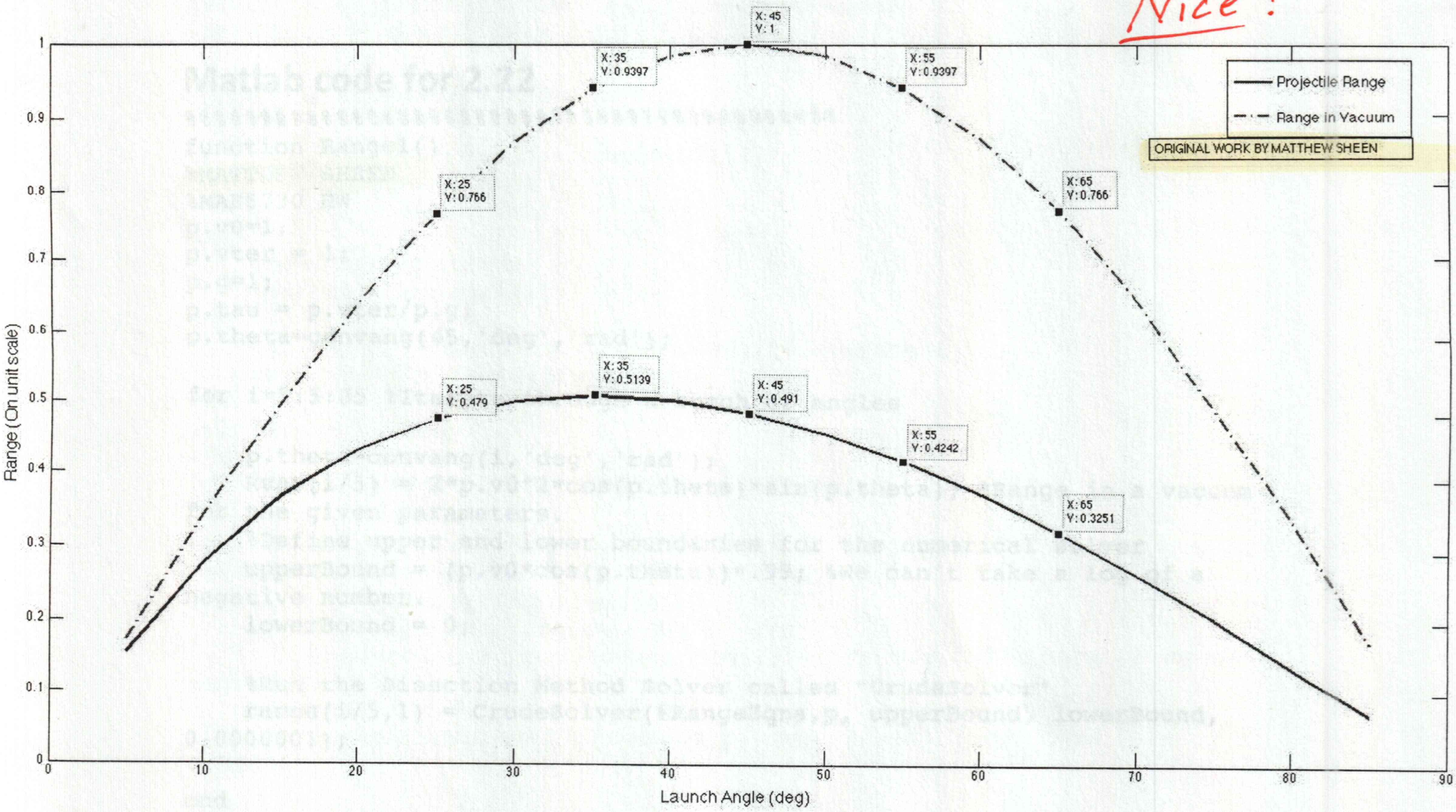


Figure 1: Maximum projectile range as a function of launch angle. The solid line is the calculation taking air resistance into account. The dotted line represents the simulation in a vacuum. Various points are marked for comparison.

A set of Matlab functions was written (attached in following pages) to represent and solve this transcendental equation for projectile range in a linear medium. The above plot shows the maximum range as a function of launch angle. The dotted line represents the projectile fired in a vacuum, and the solid line is the case in which the terminal velocity is 1. **(a)** In a vacuum, the maximum range of 1 occurs at 45 degrees.

(b) The plot shows comparisons between the range of the air resistance and vacuum cases at several angles. This reveals the trend that to achieve maximum range in real-world scenarios, a launch angle of less than the theoretical 45 degrees should be chosen.

(**c**) For the given parameters, the air-resistance case has maximum range of $\boxed{0.51}$ at 35.6 degrees. This was achieved by essentially flipping the function above and using "fminbnd" function to minimize this flipped function (which then corresponds to max range). Thus, the air-resistance case can only achieve 51% of the range possible in a vacuum.

Matlab code for 2.22

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function Range1()
%MATTHEW SHEEN
%MAE5730 HW
p.v0=1;
p.vter = 1;
p.g=1;
p.tau = p.vter/p.g;
p.theta=convang(45,'deg','rad');

for i=5:5:85 %Iterate through a bunch of angles

    p.theta=convang(i,'deg','rad');
    Rvac(i/5) = 2*p.v0^2*cos(p.theta)*sin(p.theta); %Range in a vacuum
for the given parameters.
    %Define upper and lower boundaries for the numerical solver
    upperBound = (p.v0*cos(p.theta))*.99; %We can't take a log of a
negative number.
    lowerBound = 0;

    %Run the Bisection Method Solver called "CrudeSolver"
    range(i/5,1) = CrudeSolver(@RangeEqns,p, upperBound, lowerBound,
0.0000001);

end
plot(5:5:85, range(:), 5:5:85, Rvac(:))
```

Cool!

```
[maxrngAngle, maxrng] = fminbnd(@(x)(-RangeAtAngle(x,p)),5,45); %Flip
the graph over and minimize it to get maximum range

fprintf('For the air resistance case, max range is %0.2f, which occurs
at %0.2f degrees.\n',-maxrng, maxrngAngle);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function fvals = RangeEqns( R,p )

fvals=(p.v0*sin(p.theta)+p.vter)*R/(p.v0*cos(p.theta))+(p.vter*p.tau)*l
og(1-R/(p.v0*cos(p.theta)*p.tau));


end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function  solution = CrudeSolver(rhs,p,upperBound, lowerBound,howClose)
approx1 = rhs(upperBound,p);
approx2 = rhs(lowerBound,p);

halfway = (upperBound-lowerBound)/2+lowerBound;
approxHalf = rhs(halfway,p);

while abs(approxHalf)>howClose

if (sign(approx1)~=sign(approxHalf))
    lowerBound=halfway;
    approx2 = approxHalf;
else
    upperBound=halfway;
    approx1 = approxHalf;
end
    halfway = (upperBound-lowerBound)/2+lowerBound;
    approxHalf = rhs(halfway,p);
end
solution = halfway;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Quick and dirty way to express with angle as input
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function R = RangeAtAngle(angle,p)
    p.theta = convang(angle, 'deg', 'rad');
        upperBound = (p.v0*cos(p.theta))*.99; %We can't take a log of a
negative number.
    lowerBound = 0;
        R = CrudeSolver(@RangeEqns,p, upperBound, lowerBound,
0.0000001);

end
```

⑤ Taylor 2.44

HW 2 9/11/13

<u>Iron Cannonball</u>

$D = 15\,cm$

$\rho = 7.8\,g/cm^3$

$v_0 = 300\,m/s$

$\theta = 50^\circ$

<u>Drag</u>

$f = c(y)v^2$

where $c(y) = \gamma D^2 \exp(-y/\lambda)$

$\gamma = 0.25$

$\lambda = 10000\,m$

a) <u>Write</u> eqn of motion

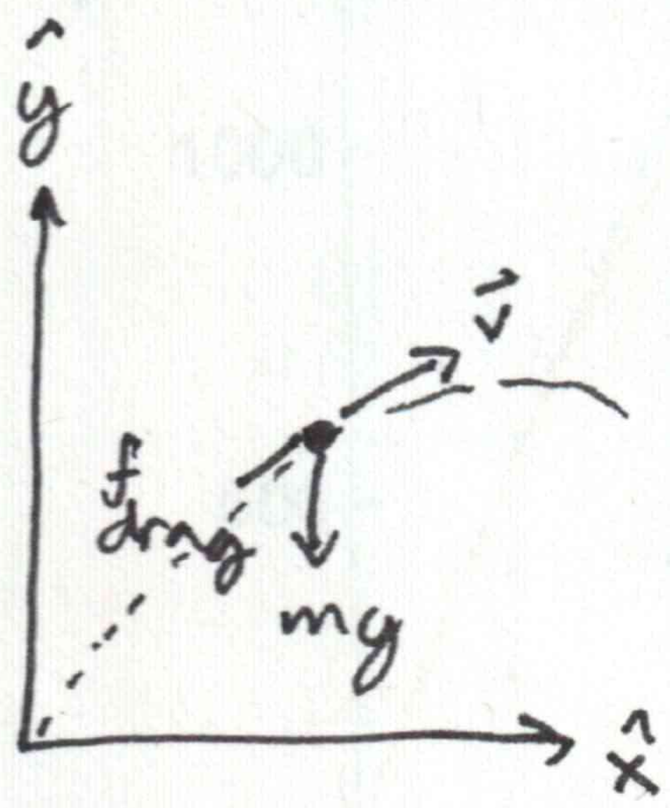
<u>Solve</u> numerically for $x(t)$ and $y(t)$

<u>plot</u> trajectory

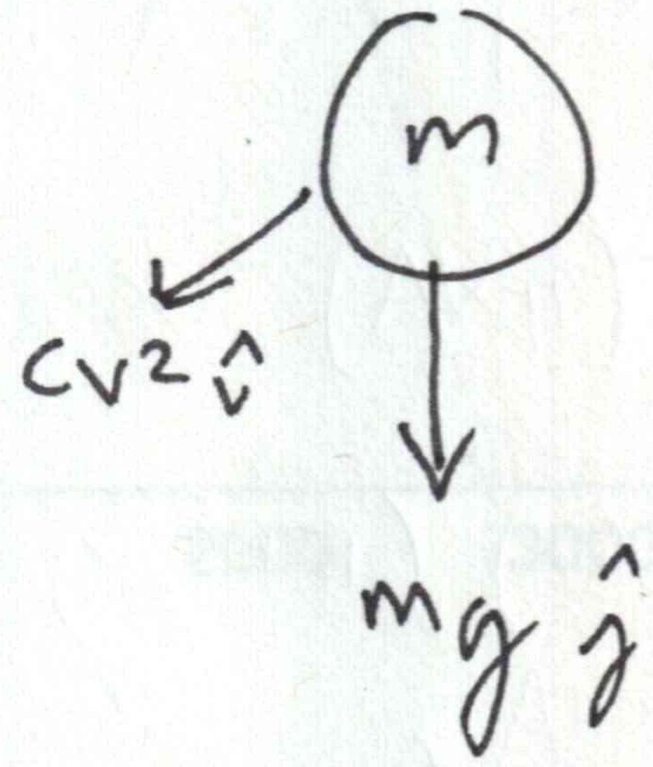
b) Same methods for $c(y) = 0$

Same thing without air resistance $c = 0$

<u>plot</u> all 3 scenarios on the same graph



FBD

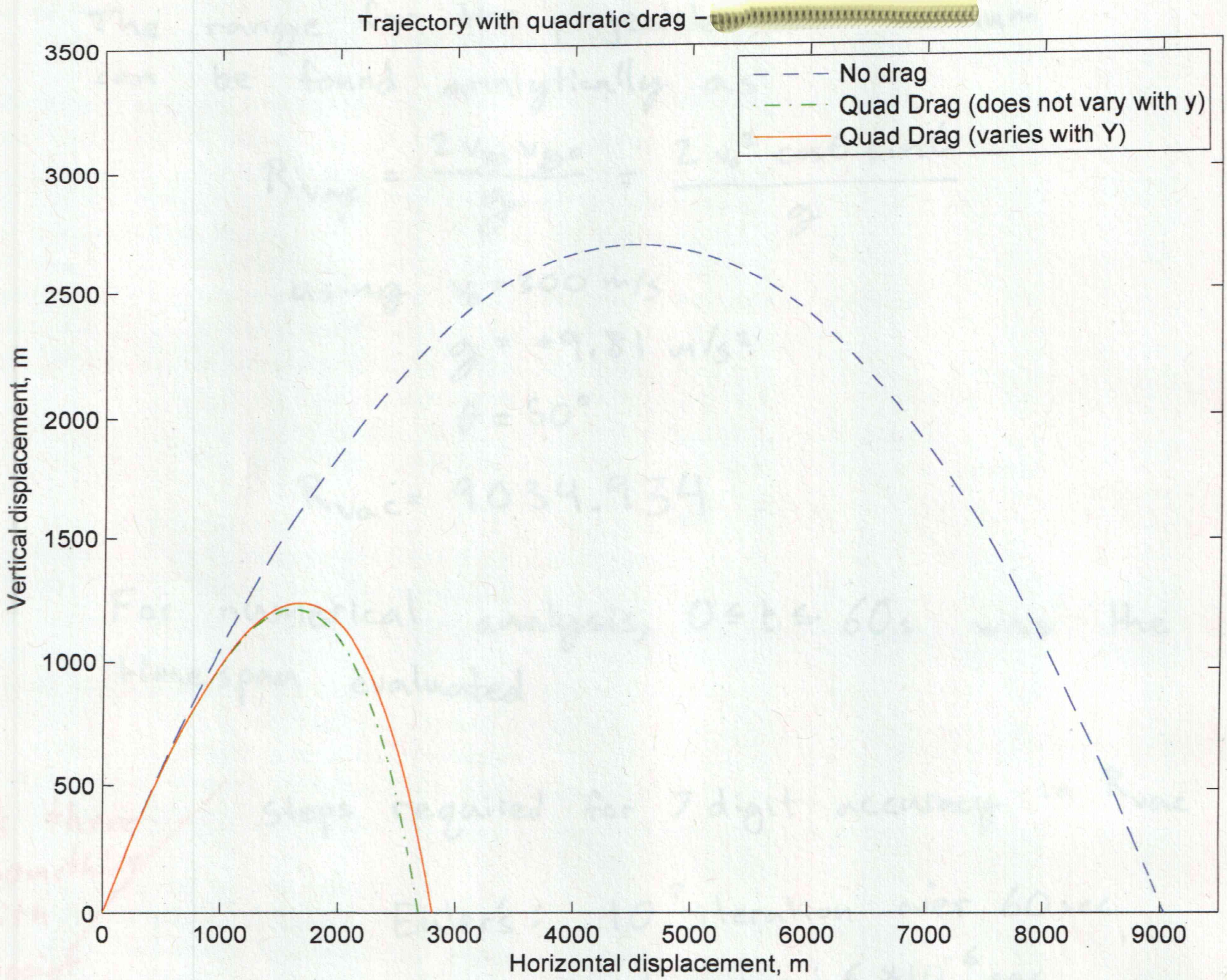


LMB

$$\sum \vec{F} = m\ddot{\vec{r}}$$

$$\boxed{m\ddot{\vec{r}} = -mg\hat{y} - \gamma D^2 \exp\left(\frac{-y}{\lambda}\right) v^2 \hat{v}}$$

NOTE: Plots of each scenario are attached, with range for each scenario shown on the plot



System	Range
No drag	9035
Quad Drag c(y=0)	2690
Quad Drag c(y)	2807

high enough.

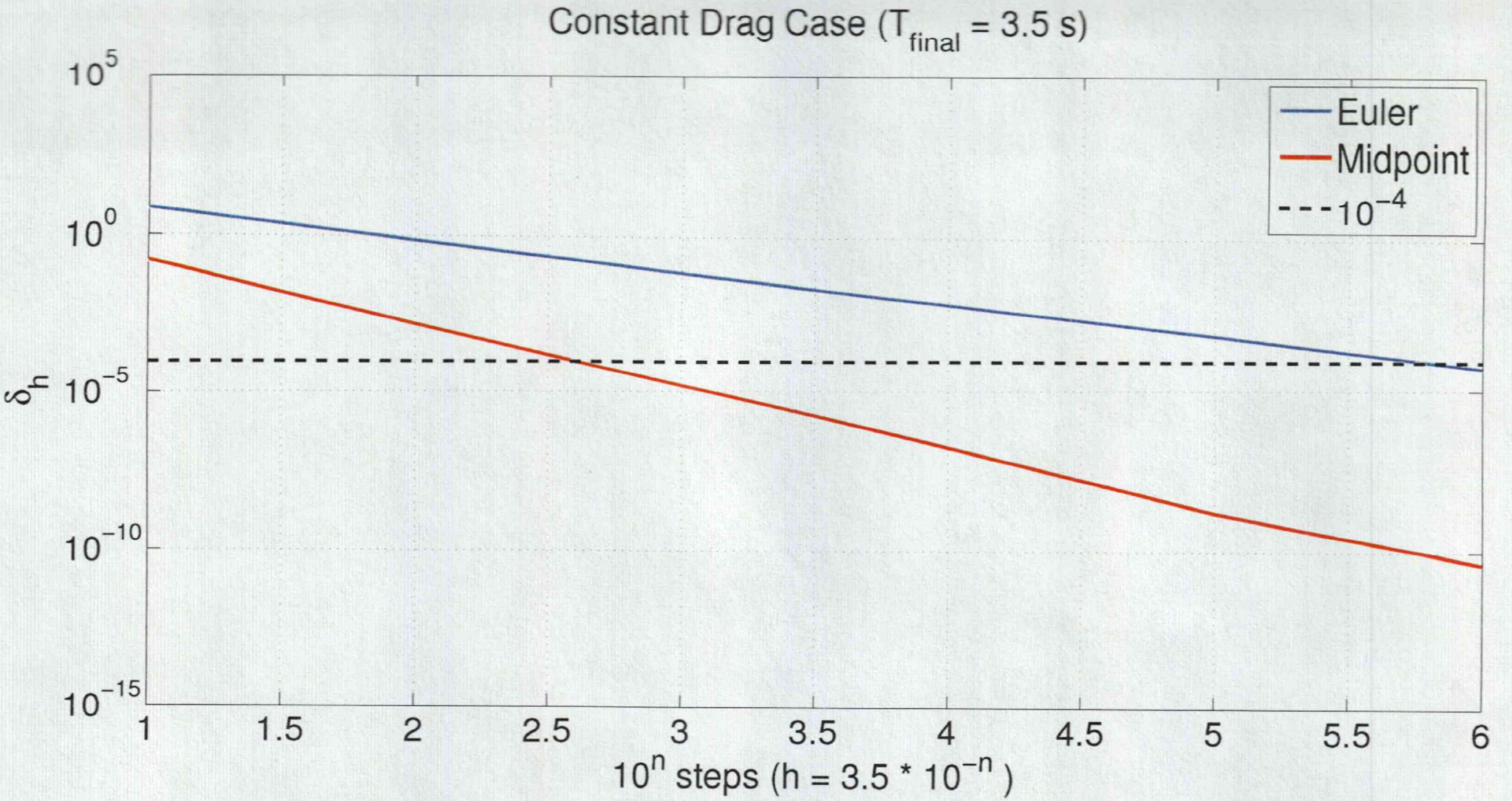
We now compare the numerical stability of Euler's method with the midpoint method in the case of constant drag (**Note**: we consider the constant drag case to show that even in complicated cases with nonanalytic functions, the midpoint method will in general converge to a solution in less steps.)

Recall that the midpoint method is defined by the following iteration scheme:

$$z_{n+1} = z_n + hf\left(t_n + \frac{h}{2}, z_n + \frac{h}{2} f(t_n, z_n)\right),$$

where f specifies the right-hand side of the set of ODEs $\dot{z} = f(t, z)$ and h is the stepsize. Numerical analysis gives us that this method has global error $\mathcal{O}(h^2)$, whereas the Euler method has global error $\mathcal{O}(h)$.

The following figure compares the two methods using the accuracy measure $\delta_h = x_{C,min}(3.5) - x_{C,h}(3.5)$, where $x_{C,min}(3.5)$ stands for the *computed* value of x using the smallest sampled step size, and $x_{C,h}(3.5)$ is the computed value of x for h bigger than the smallest sampled step size. In our experiments, we took $h = 3.5 \times 10^{-7}$ to be the smallest stepsize, so $x_{C,h} = 3.5$ has been fixed with that choice.



Great!

Figure 5: Comparison of Euler vs. midpoint methods in the constant drag case. We sampled seven different step sizes from $n = 1$ to $n = 7$. Blue denotes the Euler method and red denotes the midpoint method output. The black dashed line denotes the maximum δ_h for which we obtain seven digits of accuracy.

As we can see, we require only 10^3 steps with the midpoint method to achieve the desired seven digit accuracy with the midpoint method, whereas we require 10^6 steps to achieve the same accuracy with the Euler method. This is in line with the error analysis of the two methods stated above.