Taylor 1.17b) prove the product rule $\frac{d}{dt}(\vec{r}\times\vec{s}) = \vec{r}\times\frac{d\vec{s}}{dt} + \frac{d\vec{r}}{dt}\times\vec{s}$

definition of derivative ; $f'(x) = \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$

Similarly:

$$\frac{d}{dt}(\vec{r}\times\vec{s}) = \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)\times\vec{s}(t+\Delta t) - \vec{r}(t)\times\vec{s}(t)}{\Delta t}$$

$$\lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)\times\vec{s}(t+\Delta t) + \overbrace{\vec{r}(t+\Delta t)\times\vec{s}(t) - \vec{r}(t+\Delta t)\vec{s}(t)}^{\text{add and subtract same term}} - \vec{r}(t)\times\vec{s}(t)}{\Delta t}$$

because a limit of a sum is the sum of the limits:

$$\lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)\times\vec{s}(t+\Delta t) - \vec{r}(t+\Delta t)\times\vec{s}(t)}{\Delta t} + \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)\times\vec{s}(t) - \vec{r}(t)\times\vec{s}(t)}{\Delta t}$$

by dist. property:

$$\lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)\times[\vec{s}(t+\Delta t) - \vec{s}(t)]}{\Delta t} + \lim_{\Delta t\to 0} \frac{[\vec{r}(t+\Delta t) - \vec{r}(t)]\times\vec{s}(t)}{\Delta t}$$

because limit of products is product of limits:

$$\left[\lim_{\Delta t\to 0} \vec{r}(t+\Delta t)\right]\times\left(\lim_{\Delta t\to 0} \underbrace{\frac{\vec{s}(t+\Delta t) - \vec{s}(t)}{\Delta t}}_{\text{definition of deriv.}}\right) + \left[\lim_{\Delta t\to 0} \underbrace{\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}}_{\text{definition of deriv.}}\right]\times\left(\lim_{\Delta t\to 0} \vec{s}(t)\right)$$

yields: $\vec{r}(t)\times\frac{d\vec{s}}{dt} + \frac{d\vec{r}}{dt}\times\vec{s}(t)$ ✓

Great Explanations!

Problem 2. (Taylor 1.23)

$\vec{b} \cdot \vec{v} = a$ / If a, $\vec{b}$ and $\vec{c}$ are fixed and known, find $\vec{v}$ in terms
$\vec{b} \times \vec{v} = \vec{c}$ / of them.

I discussed this problem with Bryan Peele; Nazomi Hitomi and Jonathan Daudeline.
I also found some helpful properties at: www-math.mit.edu/~djk/18_022/chapter02/section02/html. ✓

Given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, it can be found that:

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$ [double cross product]. ✓

Therefore, in this problem, we can have:

$$\vec{b} \times \vec{c} = \vec{b} \times (\vec{b} \times \vec{v}) = \overbrace{(\vec{b} \cdot \vec{v})}^{a} \vec{b} - (\vec{b} \cdot \vec{b}) \vec{v} = a\vec{b} - |\vec{b}|^2 \vec{v} \Rightarrow$$

$$\Rightarrow \vec{b} \times \vec{c} = a\vec{b} - |\vec{b}|^2 \vec{v} \xRightarrow{\vec{b} \neq \vec{0}} \boxed{\vec{v} = \frac{a\vec{b} - \vec{b} \times \vec{c}}{|\vec{b}|^2}}$$ ✓

1.23 $\vec{b}\cdot\vec{v}=\lambda$ and $\vec{b}\times\vec{v}=\vec{c}$

Find $\vec{v}$ in terms of λ, $\vec{b}$, and $\vec{c}$

Let $\vec{a}=\vec{c}\times\vec{b}$

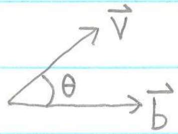
Thus, $\vec{a}$ and $\vec{b}$ are orthogonal and $\vec{v}$ lies in the plane that contains $\vec{a}$ and $\vec{b}$.

Draw a BIG Picture!

Therefore $\vec{v}$ can be written as:

$$\vec{v}=\beta\frac{\vec{b}}{|\vec{b}|}+\alpha\frac{\vec{a}}{|\vec{a}|}$$

where β is the magnitude of the component of $\vec{v}$ that is parallel to $\vec{b}$ and α is the magnitude of the component of $\vec{v}$ that is perpendicular $\vec{b}$ (parallel to $\vec{a}$)



$\beta=|\vec{v}|\cos\theta \qquad \alpha=|\vec{v}|\sin\theta$

$$\vec{b}\cdot\vec{v}=|\vec{b}||\vec{v}|\cos\theta$$

$$\lambda=|\vec{b}||\vec{v}|\cos\theta$$

$$|\vec{v}|\cos\theta=\frac{\lambda}{|\vec{b}|}$$

$$\beta=\frac{\lambda}{|\vec{b}|}$$

$$\vec{b} \times \vec{v} = |\vec{b}|\,|\vec{v}| \sin\theta \, \hat{n}$$

$$\vec{c} = |\vec{b}|\,|\vec{v}| \sin\theta \, \frac{\vec{c}}{|\vec{c}|}$$

$$1 = |\vec{b}|\,|\vec{v}| \sin\theta \, \frac{1}{|\vec{c}|}$$

$$|\vec{v}| \sin\theta = \frac{|\vec{c}|}{|\vec{b}|}$$

$$\alpha = \frac{|\vec{c}|}{|\vec{b}|}$$

Recall: $\vec{v} = \beta \frac{\vec{b}}{|\vec{b}|} + \alpha \frac{\vec{a}}{|\vec{a}|}$

$$\Rightarrow \quad \vec{v} = \frac{\lambda}{|\vec{b}|} \frac{\vec{b}}{|\vec{b}|} + \frac{|\vec{c}|}{|\vec{b}|} \frac{\vec{a}}{|\vec{a}|}$$

Recall: $\vec{a} = \vec{c} \times \vec{b}$

$$\Rightarrow \quad \vec{v} = \frac{\lambda}{\vec{b} \cdot \vec{b}} \vec{b} + \frac{|\vec{c}|}{|\vec{b}|\,|\vec{c} \times \vec{b}|} (\vec{c} \times \vec{b})$$

Recall: $\vec{b} \times \vec{v} = \vec{c} \Rightarrow \vec{b} \perp \vec{c}$

$$\Rightarrow |\vec{c} \times \vec{b}| = |\vec{c}|\,|\vec{b}|$$

$$\Rightarrow \quad \vec{v} = \frac{\lambda}{\vec{b} \cdot \vec{b}} \vec{b} + \frac{\cancel{|\vec{c}|}}{|\vec{b}|\,\cancel{|\vec{c}|}\,|\vec{b}|} (\vec{c} \times \vec{b})$$

$$\Rightarrow \boxed{\vec{v} = \frac{\lambda \vec{b}}{|\vec{b}|^2} - \frac{\vec{b} \times \vec{c}}{|\vec{b}|^2}}$$

Problem 1.45

a) Given that for an arbitrary vector $\underset{\sim}{v}(t)$, $\|\underset{\sim}{v}(t)\| = c$, $c \in \mathbb{R}^+$, for all t, prove that $\underset{\sim}{v}(t) \perp \dot{\underset{\sim}{v}}(t)$.

SOLUTION

Suppressing the t, in order to prove that $\underset{\sim}{v} \perp \dot{\underset{\sim}{v}}$, we must show that $\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}} = 0$.

We know that $\|\underset{\sim}{v}\|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v} = d$, where $d = c^2 \in \mathbb{R}^+$, for all t. Then by the properties of the dot product

$$\frac{d}{dt}(\underset{\sim}{v} \cdot \underset{\sim}{v}) = \frac{d}{dt}\underset{\sim}{v} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \frac{d}{dt}\underset{\sim}{v} = \frac{d}{dt}(d) = 0$$

and by the property $a \cdot b = b \cdot a$

$$2\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}} = 0$$

$$\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}} = 0$$

✓

Q.E.D.

b) Given that $\underset{\sim}{v}(t) \perp \dot{v}(t)$, prove that $\|\underset{\sim}{v}(t)\| = c$, $c \in \mathbb{R}^+$, for all t.

SOLUTION

The same as above but working backwards

I know that $\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}} = 0 \implies 2\underset{\sim}{v} \cdot \dot{\underset{\sim}{v}} = 0$

$$\parallel$$

$$\underset{\sim}{v} \cdot \dot{v} + \dot{\underset{\sim}{v}} \cdot \underset{\sim}{v} = 0$$

$$\parallel$$

$$\frac{d}{dt}(\underset{\sim}{v} \cdot \underset{\sim}{v}) = 0$$

$$\parallel$$

$$\frac{d}{dt}(\underset{\sim}{v} \cdot \underset{\sim}{v}) = \frac{d}{dt}(d) \implies \underset{\sim}{v} \cdot \underset{\sim}{v} = \|\underset{\sim}{v}(t)\|^2 = d$$

$d \in \mathbb{R}^+$

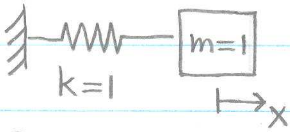
$$\parallel$$

✓

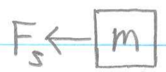
$$\|v(t)\| = c, \ c \in \mathbb{R}^+$$

Q.E.D.

4) a)



FBD:



$$\Sigma F = m\ddot{x}$$

$$-F_s = m\ddot{x} \qquad F_s = kx$$

$$-kx = m\ddot{x}$$

$$\ddot{x} = -\frac{k}{m}x$$

$$x = A\cos(wt) + B\sin(wt) \qquad w = \sqrt{\frac{k}{m}}$$

$$x(0) = 1 \Rightarrow 1 = A + 0$$

$$A = 1$$

$$\dot{x} = -wA\sin(wt) + wB\cos(wt)$$

$$\dot{x}(0) = 0 \Rightarrow 0 = 0 + wB$$

$$B = 0$$

$$x = \cos\left(\sqrt{\frac{k}{m}}t\right) \qquad k=1,\ m=1$$

$$x = \cos t$$

$$x(2\pi) = \cos(2\pi)$$

$$\boxed{x(2\pi) = 1}$$

b) $\ddot{x} = -\frac{k}{m} x$

$\Rightarrow \dot{x} = v$

$\dot{v} = -\frac{k}{m} x$

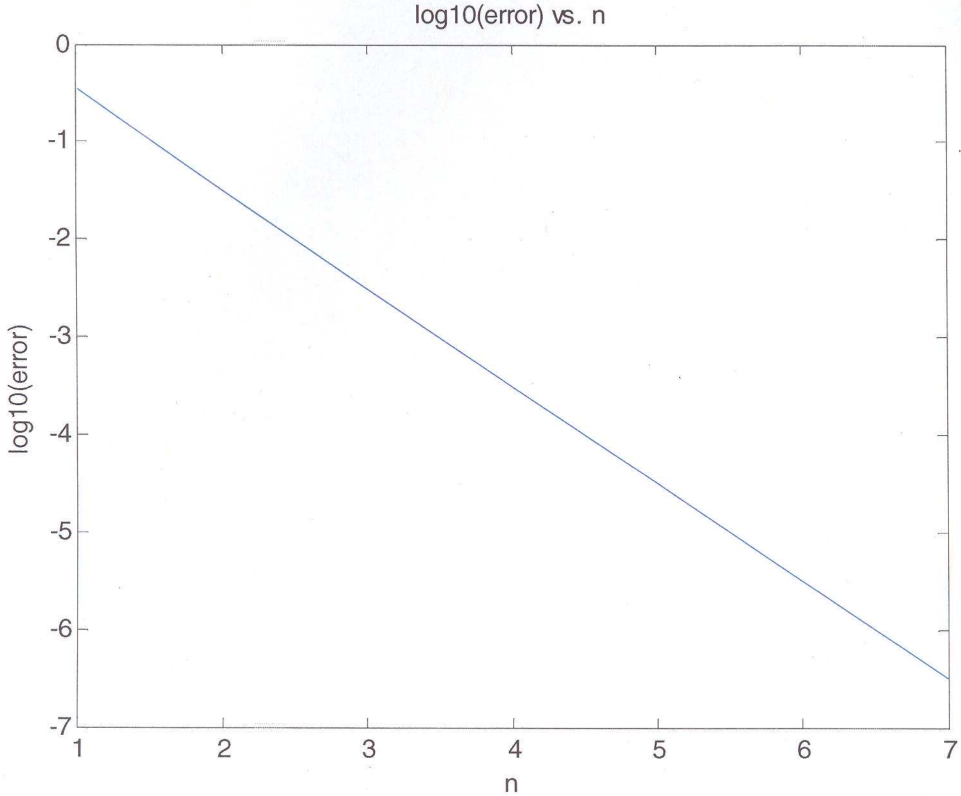
(see attached code, plot, and explaination)

Problem 4b

MATLAB code:

```
% HW 1 Problem 4b

% Define constants
m = 1;
k = 1;
t = 2*pi;

% Choose max n value
nmax = 7;
nspan = 1:nmax;

% Define analytic solution
ana_sol = 1;
error = zeros(1,nmax);

% Loop through each n value
for i = 1:nmax

    % Define step size and the number of steps
    n = i;
    h = 10^(-n);
    steps = t/h;

    % Initial conditions for x and v
    x = 1;
    v = 0;

    % Euler loop
    for j = 1:steps
        vdot = -k/m*x;
        xdot = v;
        v = v + vdot*h;
        x = x + xdot*h;
    end

    % Error at each n value
    error(i) = abs(x - ana_sol);
end

% Plot log10 of error vs. n
plot(nspan, log10(error))
xlabel('n')
ylabel('log10(error)')
title('log10(error) vs. n')
```

Plot (for max n = 7):



The accuracy stops getting better when n increases to a certain point. This happens due to the fact that the total amount of roundoff error is increasing as more steps are taken. Even though the amount of truncation error decreases with increasing n, the amount of roundoff error increases with increasing n and overcomes the reduction in truncation error. Roundoff error occurs due to the fact that computers use float point notation to estimate the result of each calculation, so this error occurs after each step. Thus, the greater the number of steps, the greater the total roundoff error.