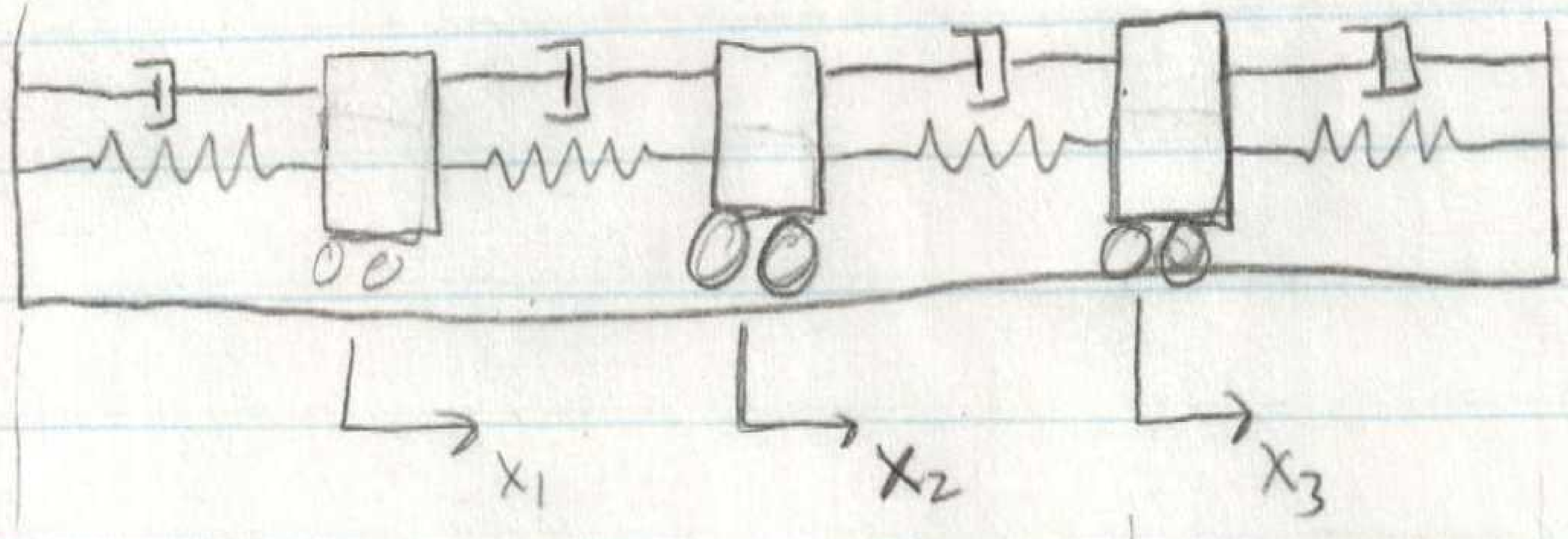
Handout 37 SOLUTION:



ⓐ

→ let $m=1$, $K=2$, $C = 0.1\sqrt{mk} = 0.14$, $F_0=4$

→ Equations of motion:

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{M} \ddot{\vec{x}} + \underbrace{\begin{bmatrix} 2C & -C & 0 \\ -C & 2C & -C \\ 0 & -C & 2C \end{bmatrix}}_{C} \dot{\vec{x}} + \underbrace{\begin{bmatrix} 2K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & 2K \end{bmatrix}}_{K} \vec{x} = \underbrace{\begin{bmatrix} F_0 \sin\sqrt{\frac{2K}{m}}\, t \\ 0 \\ 0 \end{bmatrix}}_{f}$$

$$\text{let } Z = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \rightarrow \dot{Z} = \underbrace{\begin{bmatrix} 0 \\ \hline F \end{bmatrix}}_{F} + \underbrace{\left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right]}_{A} Z$$

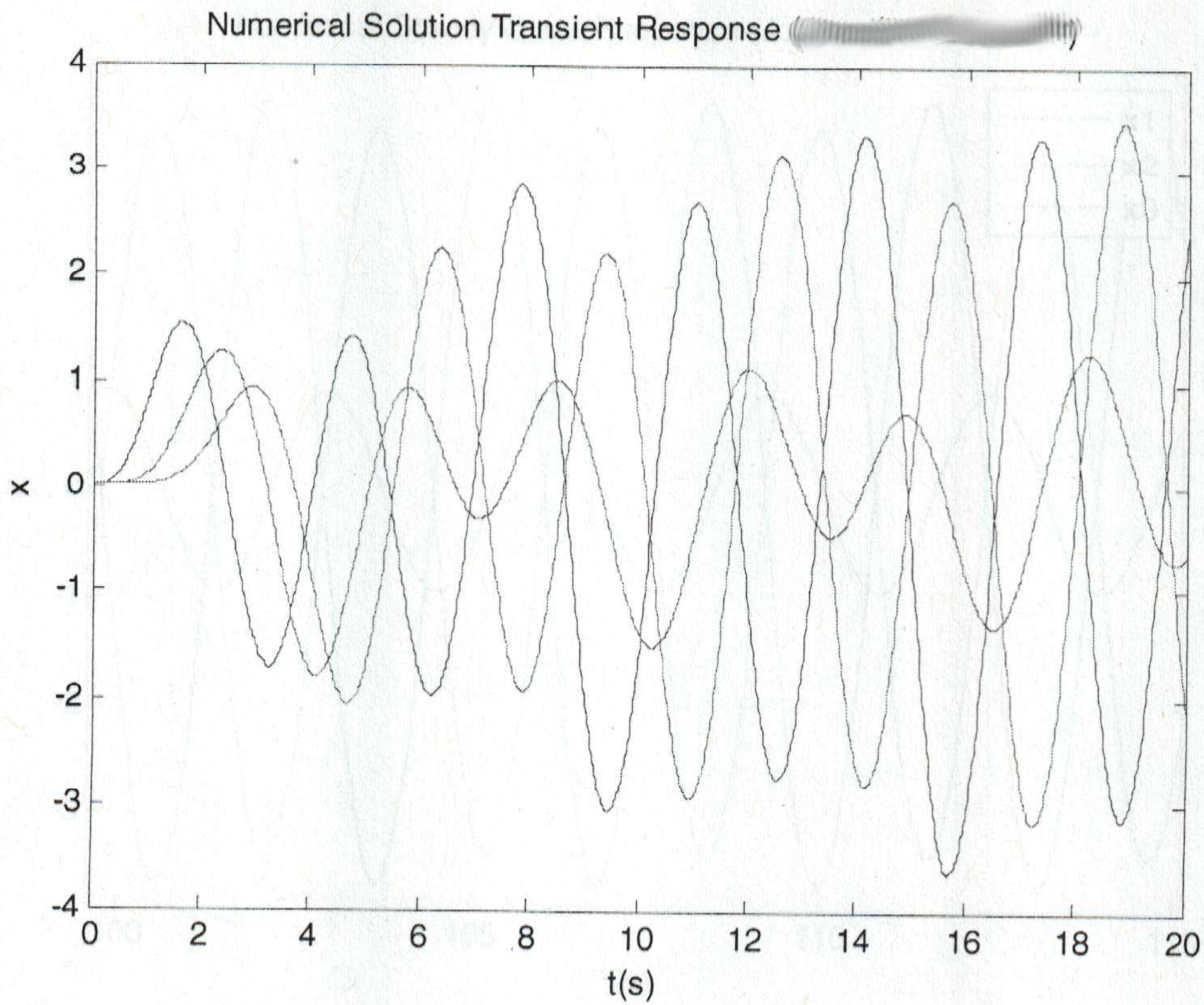
Part A:

Code:

```
function Handout37()
%[redacted]
%Handout 37

%Set up parameters
p.m = 1; p.k = 2; p.c = 0.14; p.F0 = 4;
tspan = linspace(0, 20, 1001);

%Populate M, K, C, F matrices
M = [p.m 0 0; 0 p.m 0; 0 0 p.m];
C = [2*p.c -p.c 0; -p.c 2*p.c -p.c; 0 -p.c 2*p.c];
K = [2*p.k -p.k 0; -p.k 2*p.k -p.k; 0 -p.k 2*p.k];

%Initial conditions start from rest
x0 = [0 0 0]'; v0 = [0 0 0]';

%Set up matrix A
l = length(x0);
A = zeros(2*l, 2*l);
A(1:l, l+1:end) = eye(l);
A(l+1:end, 1:l) = -M^-1*K;
A(l+1:end, l+1:end) = -M^-1*C;
p.A = A;

%Setup and run ODE45
options = odeset('reltol', 10^-6, 'abstol', 10^-6);
z0 = [x0; v0];
[tarray zarray] = ode45(@rhs, tspan, z0, options, p);
%Unpack the positions from zarray
xarray = zarray(:, 1:length(z0)/2);

plot(tarray, xarray);

end

%RHS equations simplified to use state space system
function zdot = rhs(t, z, p)
    %Set up matrix F
    l = length(z)/2;
    F = zeros(2*l, 1);
    F(l+1:end) = [p.F0*sin(sqrt(2*p.k/p.m)*t); 0; 0];

    %Calculate zdot using state space system
    zdot = F+p.A*z;
end
```

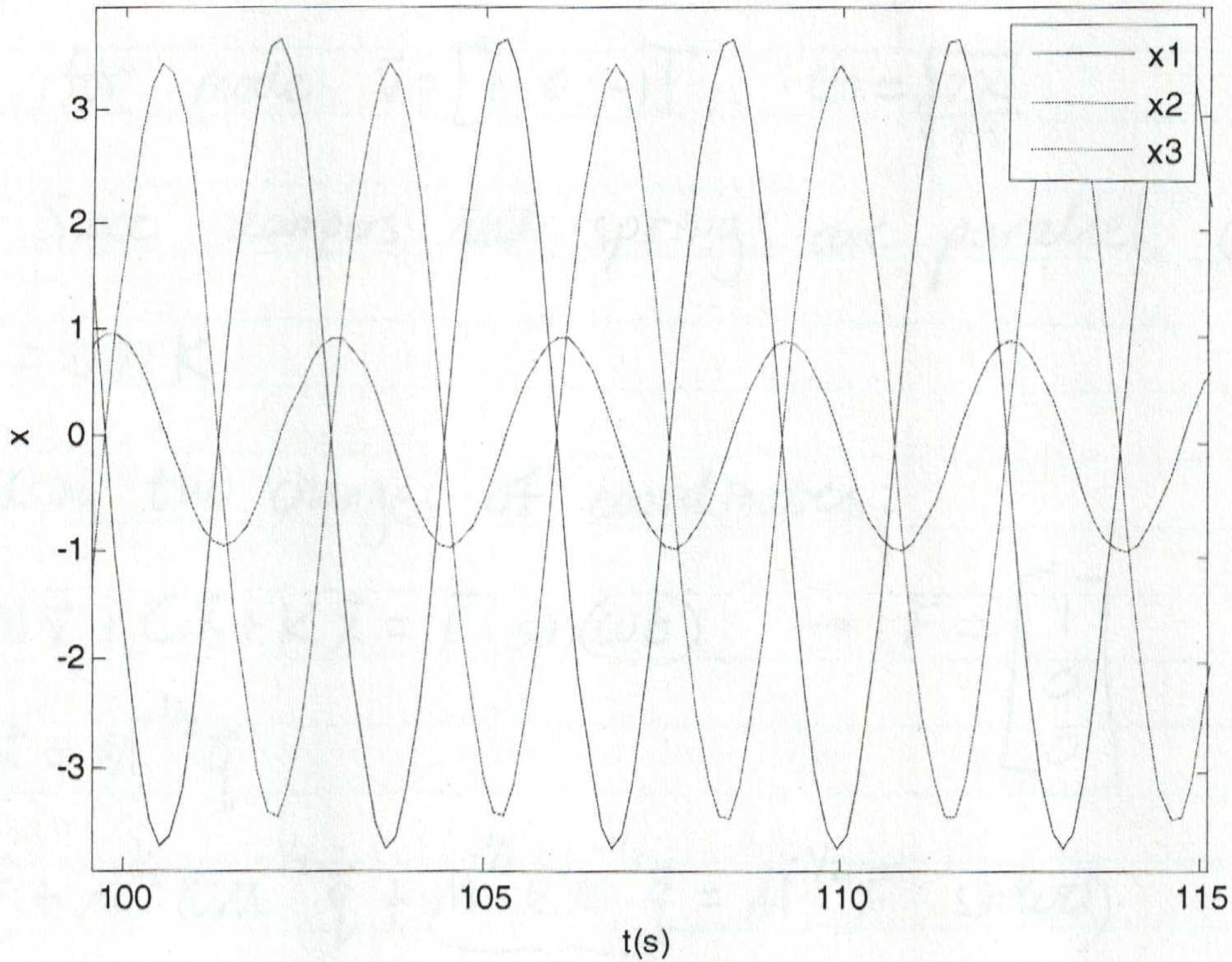
Part B:

Plotting for t < 20 seconds gives:



Plotting for longer period of time results in the following steady-state response:

Numerical Steady-State Response ([illegible])



(C) for mode $\vec{v} = [1 \;\; 0 \;\; -1]'$, $\omega = \sqrt{\frac{2K}{m}}$

Since dampers and springs are parallel, $C = \beta K = \left(\frac{\sqrt{mk}}{k}\right) K$

$= 0.71 K$

→ Using two changes of coordinates:

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + K\vec{x} = \bar{F}\sin(\omega t) \quad \rightarrow \quad \bar{F} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \vec{x} = M^{-1/2}\vec{q}$$

$$\rightarrow \ddot{\vec{q}} + M^{-1/2} C M^{-1/2}\dot{\vec{q}} + \underbrace{M^{-1/2} K M^{-1/2}}_{\tilde{K}}\vec{q} = M^{-1/2}\bar{F}\sin(\omega t)$$

→ $\vec{q} = P\vec{r}$ → P = eigenvectors of $\tilde{K}$ (normal mode shapes)

$$\rightarrow \ddot{\vec{r}} + P^{-1}M^{-1/2} C M^{-1/2} P\dot{\vec{r}} + P^{-1}\tilde{K}P\vec{r} = P^{-1}M^{-1/2}\bar{F}\sin\omega t$$

$$\rightarrow \ddot{\vec{r}} + \beta\Lambda\dot{\vec{r}} + \Lambda\vec{r} = P^{-1}M^{-1/2}\bar{F}\sin(\omega t)$$

$$\rightarrow \ddot{r}_i + \beta\lambda_i\dot{r}_i + \lambda_i r_i = [P^{-1}M^{-1/2}\bar{F}]_i \sin(\omega t) \rightarrow \quad \omega_n^2 = \lambda_i \quad 2\zeta\omega_n = \beta\lambda_i$$

→ $\omega_n = \sqrt{\lambda_i}$ $\quad \zeta = \frac{\beta\omega_n}{2} = \frac{\beta\sqrt{\lambda_i}}{2}$ $\quad$ let $P^{-1}M^{-1/2}\bar{F} = \bar{f}$

→ Particular Solution: $x = A\cos(\omega t) + B\sin(\omega t)$

$$A = \frac{-(2\zeta\omega\omega_n)\bar{f}}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \qquad B = \frac{(\omega_n^2 - \omega^2)\bar{f}}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}$$

Homogeneous Solution: $x = e^{-\zeta\omega_n t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right]$ $\rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2}$

→ $x_{gen} = x_h + x_p$ → $x(0) = 0$ → $C = -A$, $x'(0) = 0$ → $-\zeta\omega_n C + \omega_d D + \omega B = 0$

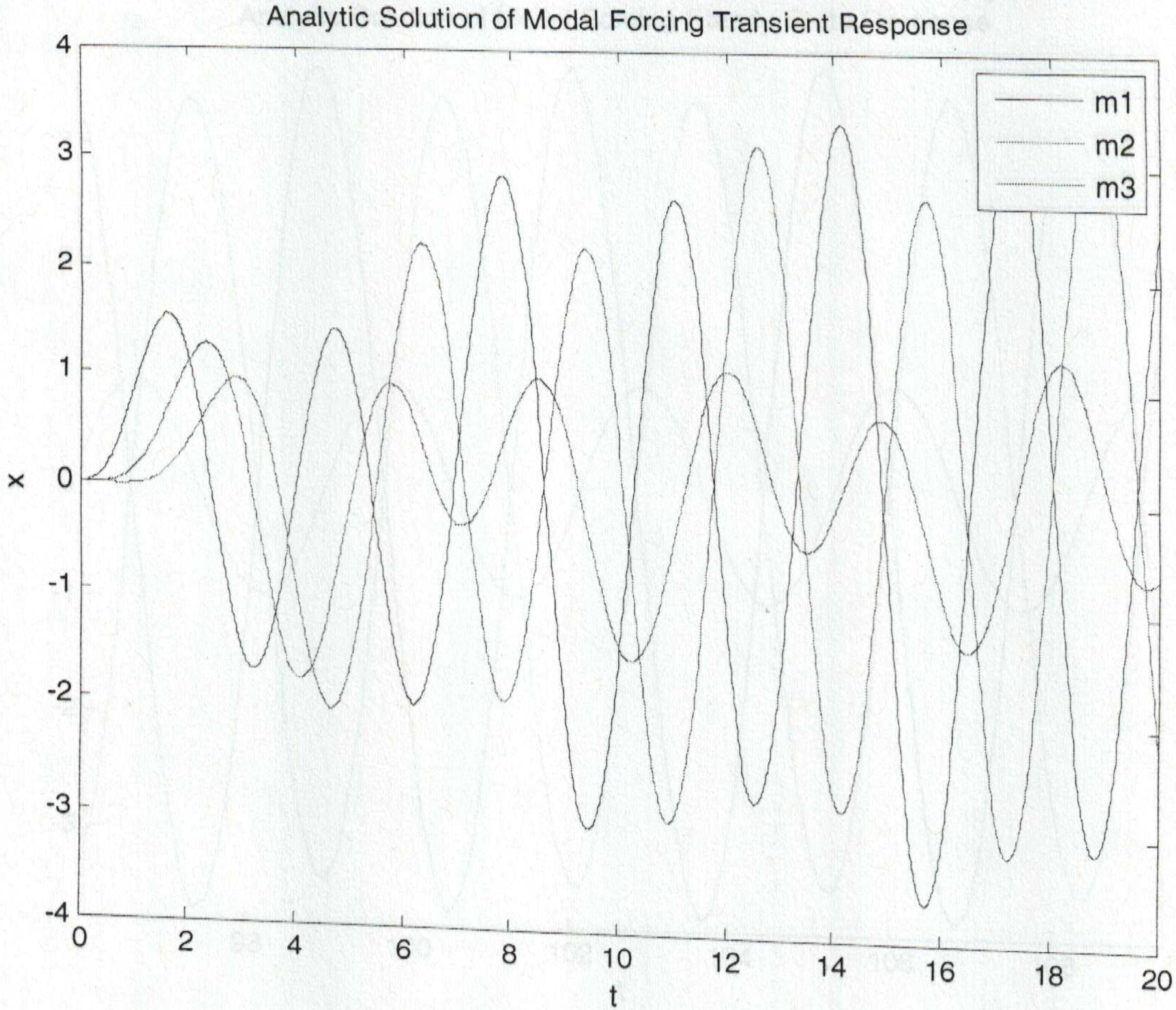
Part C:

Code:

```
function Handout37b()
% [redacted]
%Handout 37

%Set up parameters
m = 1; k = 2; c = 0.14; F0 = [4 0 0]'; w = sqrt(2*k/m); beta = c/k;
tspan = linspace(0, 20, 1001);

%Populate M, K, C, F matrices
M = [m 0 0; 0 m 0; 0 0 m];
C = [2*c -c 0; -c 2*c -c; 0 -c 2*c];
K = [2*k -k 0; -k 2*k -k; 0 -k 2*k];

%Find eigs for change of coordinates
[P L] = eig(K, M);

%Get coefficients for solving decoupled equations
wn = sqrt(diag(L));
zeta = beta*wn/2;
wd = wn.*(1-zeta.^2);
fbar = P^-1*M^-(1/2)*F0;

%Solve for constant coefficients in particular and homogeneous solution
A = -(2*zeta.*w.*wn).*fbar./((wn.^2-w.^2).^2+(2*zeta.*w.*wn).^2);
B = (wn.^2-w.^2).*fbar./((wn.^2-w.^2).^2+(2*zeta.*w.*wn).^2);
C = -A;
D = (zeta.*wn.*C-w*B)./wd;

%Allocate space for rp and rh
rp = zeros(length(wn), length(tspan));
rh = zeros(length(wn), length(tspan));

%Loop through each modal coordinate and calculate r(t)
for i = 1:length(wn)
    rp(i, :) = A(i)*cos(w*tspan)+B(i)*sin(w*tspan);
    rh(i, :) = exp(-
zeta(i)*wn(i)*tspan).*(C(i)*cos(wd(i)*tspan)+D(i)*sin(wd(i)*tspan));
end

%Combine solutions and convert back to x coordinates
r = rp + rh;
x = M^-(1/2)*P*r;

%Plot Results
plot(tspan, x);
```

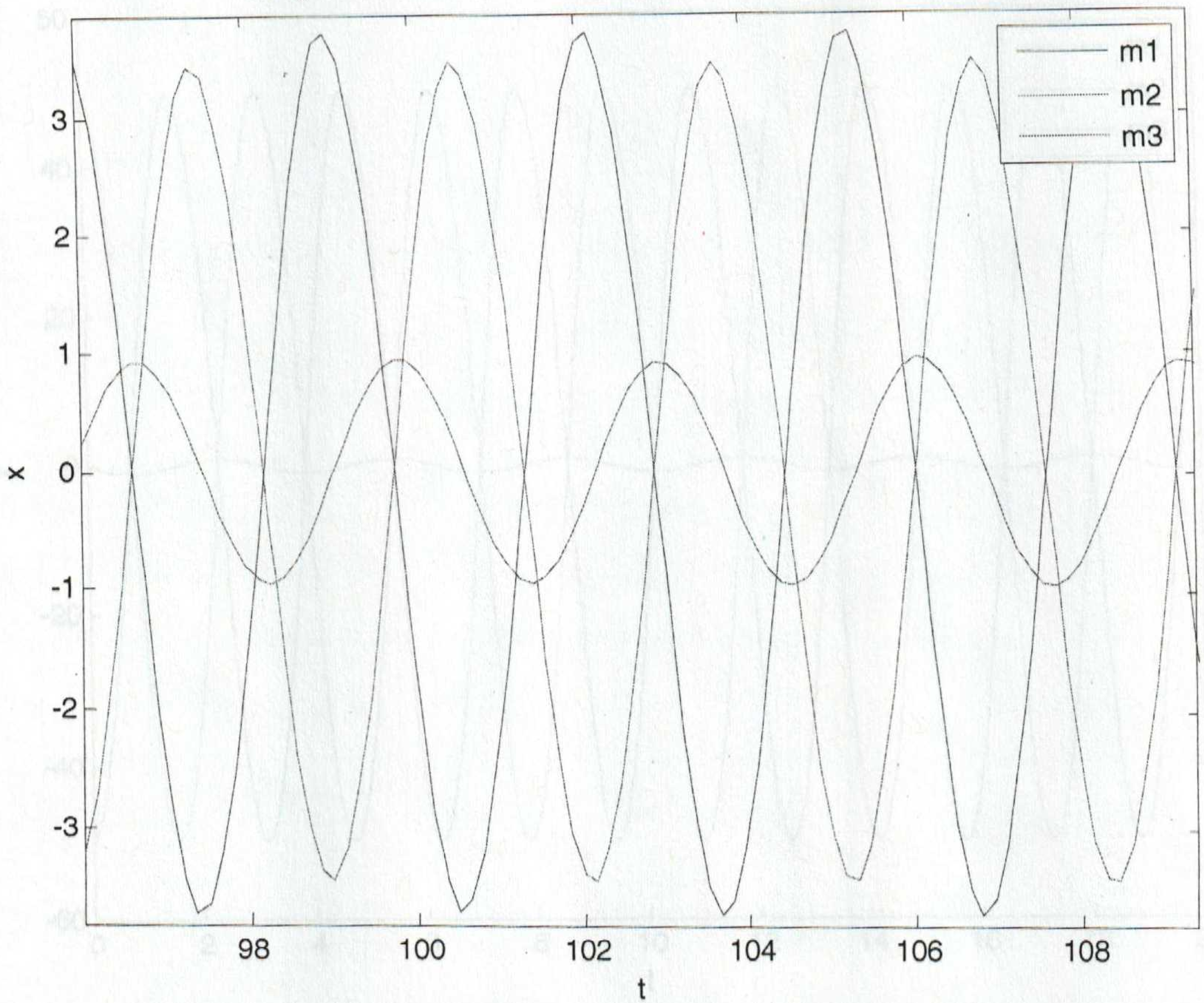
Part D:

Plotting the results for transient response gives:



This exactly agrees with the numerical solution. Plotting the steady-state response gives:

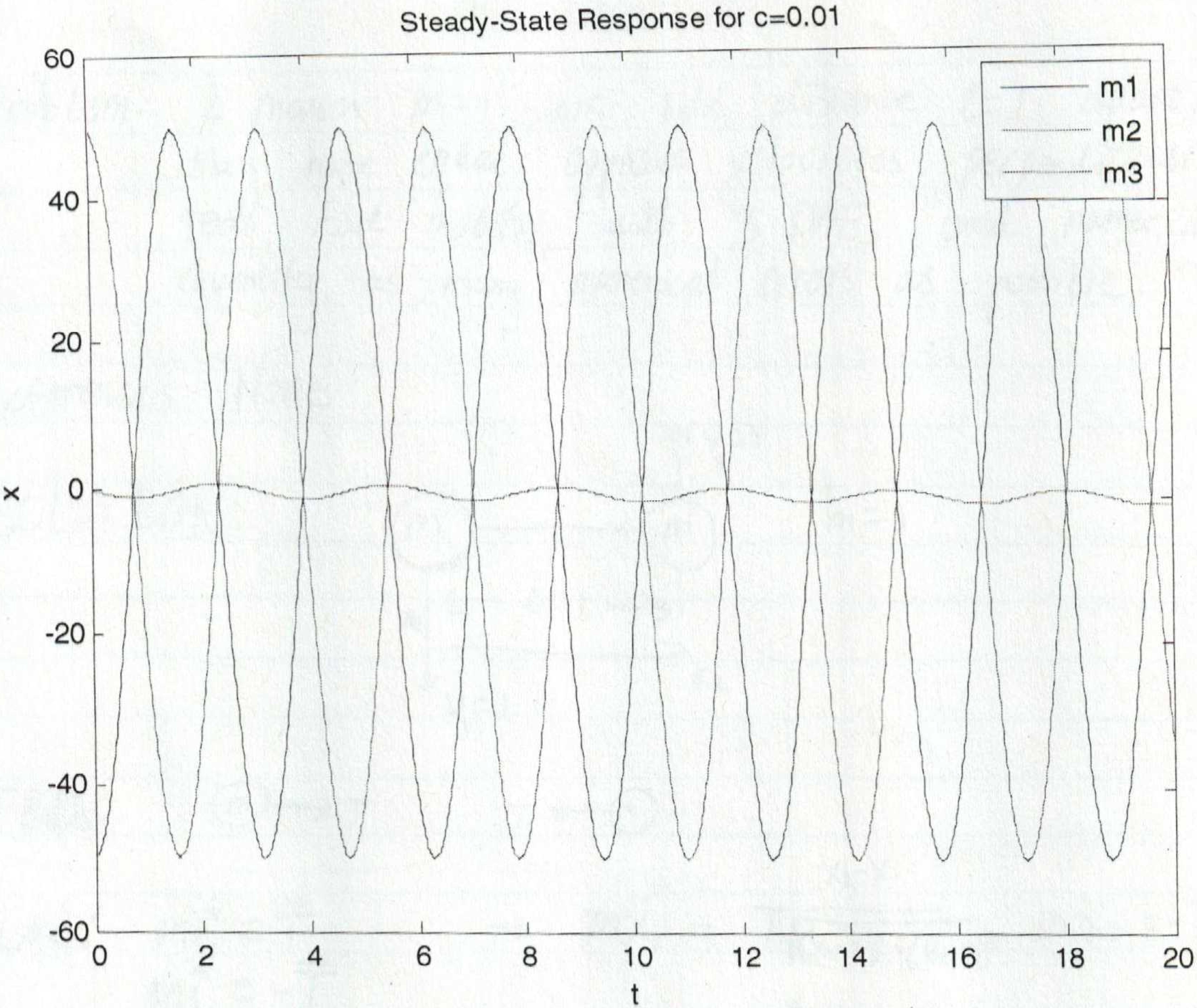
Analytic Solution of Modal Forcing Steady State Response



Note that this also agrees with the numerical solution, and the motion of the middle mass (m2) is small compared to the other two masses.

Main Question:

First, when I plot the steady state response for much lower damping, the motion of the middle mass dies away, approaching 0 as damping approaches 0. Here is a plot of the steady-state motion (only plotting particular solution and ignoring homogenous solution) for c=0.01:

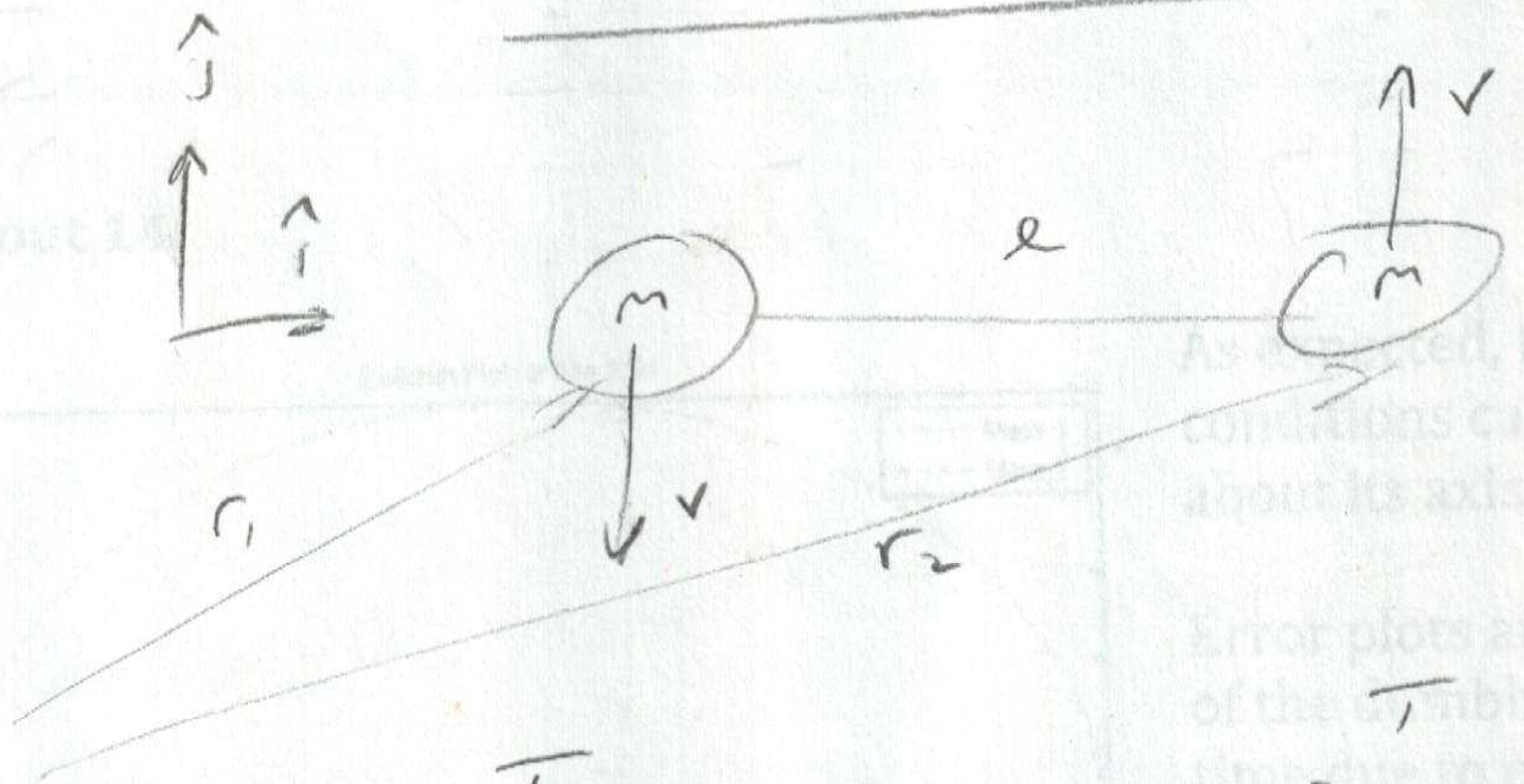


The motion of the right mass is excited at the beginning of the vibrations (from the transient motion of m2). For very low damping, this motion then continues to oscillate at its natural frequency without any disturbance from m2. The slight decay in motion due to the small damping is restored by the slight motion of the middle mass, which also is due to the small damping.

HANDOUT PROBLEM 14

$m = 1$
$l = 1$

ASSUMING NO GRAVITY



FBD

LMB

$$m\ddot{\vec{r}}_1 = T(\vec{r}_2 - \vec{r}_1)/l_0$$

$$\ddot{x}_1 = \left(T/m\right)\left[x_2 - x_1\right]/l_0$$

$$\ddot{y}_1 = \left(T/m\right)\left[y_2 - y_1\right]/l_0$$

$$m\ddot{\vec{r}}_2 = T(\vec{r}_1 - \vec{r}_2)/l_0$$

$$\ddot{x}_2 = \left(T/m\right)\left[-x_2 + x_1\right]/l_0$$

$$\ddot{y}_2 = \left(T/m\right)\left[-y_2 + y_1\right]/l_0$$

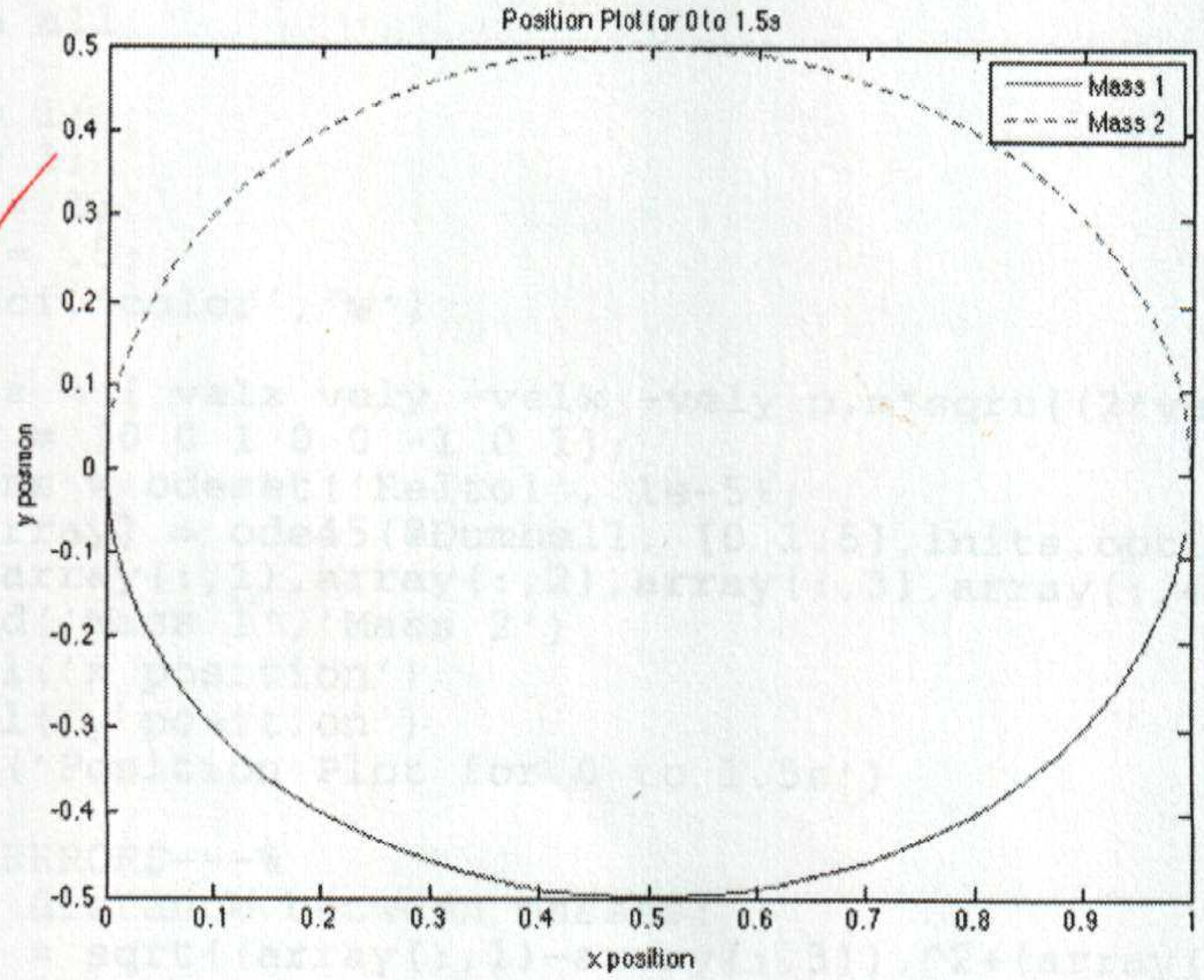
$$\begin{bmatrix} m & 0 & 0 & 0 & \frac{(x_2-x_1)}{l} \\ 0 & m & 0 & 0 & \frac{(y_2-y_1)}{l} \\ 0 & 0 & m & 0 & -\frac{(x_2-x_1)}{l} \\ 0 & 0 & 0 & m & -\frac{(y_2-y_1)}{l} \\ -2(x_2-x_1) & (y_2-y_1) & -(x_2-x_1) & -(y_2-y_1) & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -(\dot{x}_2-\dot{x}_1)^2 - (\dot{y}_2-\dot{y}_1)^2 \end{bmatrix}$$

CONSTRAINT: $(x_2-x_1)^2 + (y_2-y_1)^2 = l^2$

d/dt: $2(x_2-x_1)(\dot{x}_2-\dot{x}_1) + 2(y_2-y_1)(\dot{y}_2-\dot{y}_1) = 0$

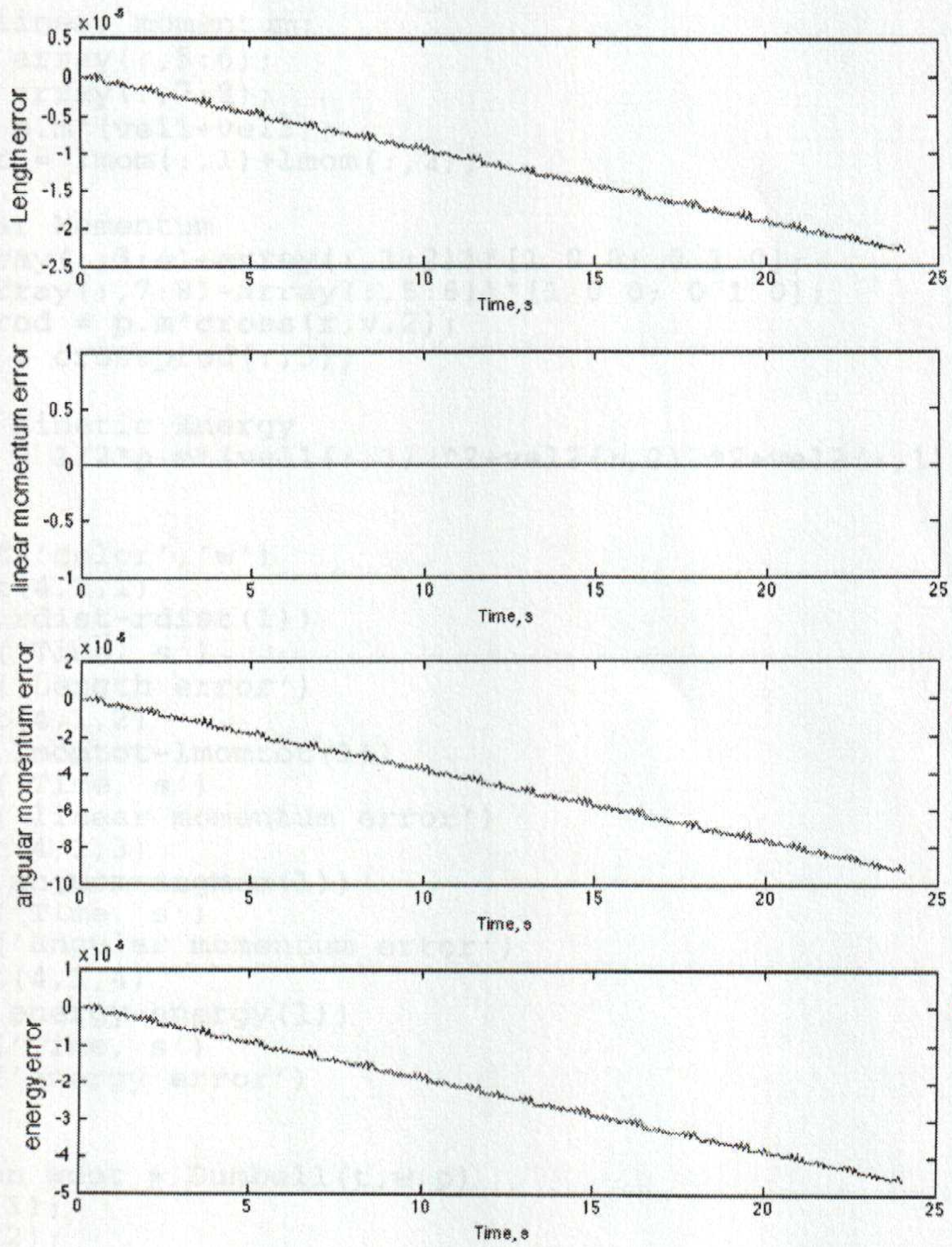
d^2/dt^2: $2(\dot{x}_2-\dot{x}_1)^2 + 2(x_2-x_1)(\ddot{x}_2-\ddot{x}_1) + 2(\dot{y}_2-\dot{y}_1)^2 + 2(y_2-y_1)(\ddot{y}_2-\ddot{y}_1) = 0$

Handout 14



As expected, the constraints and initial conditions cause the dumbbell to rotate about its axis.

Error plots are shown below. The length of the dumbbell slowly decayed over time due to error. Linear momentum remained constant since the solver seems to have kept velocities equal and opposite. Angular momentum was affected since error did slightly perturb the length. Energy is the change in momentum, so since momentum was changed by error, so was energy.



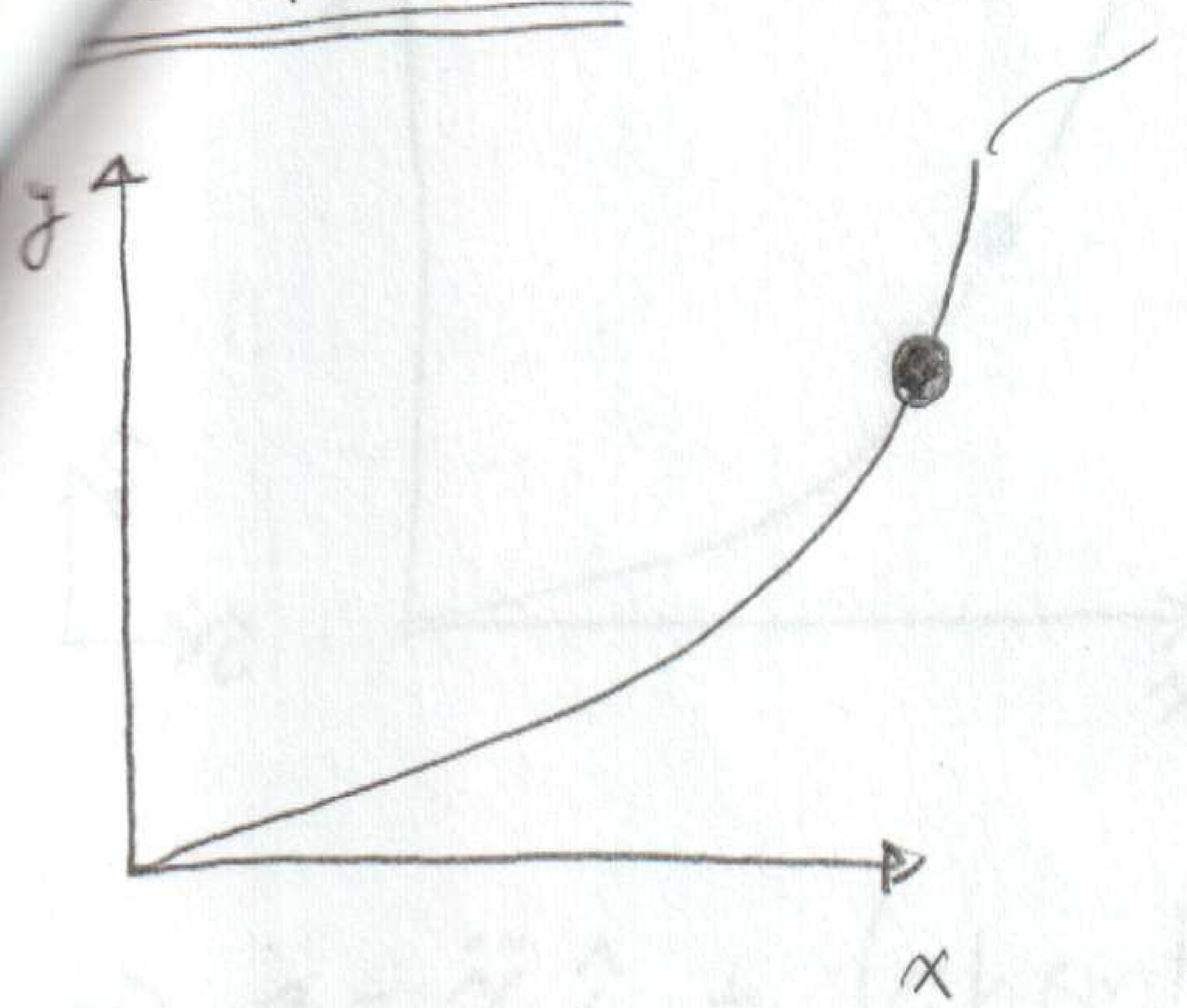
```
function Handout14
%MAE 5730 HW due nov 15
%[redacted]
clc
close all

p.m = 1;
p.l = 1;
velx = 1;
vely = .5;
set(gcf,'color','w')

%inits = [ velx vely -velx -vely p.m*sqrt((2*velx)^2+(2*vely)^2) ];
inits = [0 0 1 0 0 -1 0 1];
options = odeset('Reltol', 1e-5);
[t, array] = ode45(@Dumbell, [0 1.5],inits,options, p);
plot(array(:,1),array(:,2),array(:,3),array(:,4),'--')
legend('Mass 1','Mass 2')
xlabel('x position')
ylabel('y position')
title('Position Plot for 0 to 1.5s')

% ---ERRORS---%
%Find distance between masses:
rdist = sqrt((array(:,1)-array(:,3)).^2+(array(:,2)-array(:,4)).^2);

%Find linear momentum:
vel1 = array(:,5:6);
vel2 = array(:,7:8);
lmom = p.m*(vel1+vel2);
lmomtot = lmom(:,1)+lmom(:,2);

%Angular Momentum
r= (array(:,3:4)-array(:,1:2))*[1 0 0; 0 1 0];
v = (array(:,7:8)-array(:,5:6))*[1 0 0; 0 1 0];
crossprod = p.m*cross(r,v,2);
angmom = crossprod(:,3);

%Total Kinetic Energy
energy = 1/2*p.m*(vel1(:,1).^2+vel2(:,2).^2+vel2(:,1).^2+vel2(:,2).^2)

figure
set(gcf,'color','w')
subplot(4,1,1)
plot(t,rdist-rdist(1))
xlabel('Time, s')
ylabel('Length error')
subplot(4,1,2)
plot(t,lmomtot-lmomtot(1))
xlabel('Time, s')
ylabel('linear momentum error')
subplot(4,1,3)
plot(t,angmom-angmom(1))
xlabel('Time, s')
ylabel('angular momentum error')
subplot(4,1,4)
plot(t,energy-energy(1))
xlabel('Time, s')
ylabel('energy error')
end

function wdot = Dumbell(t,w,p)
x1 = w(1);
y1 = w(2);
x2 = w(3);
y2 = w(4);
dx1 = w(5);
dy1 = w(6);
```

```
dx2 = w(7);
dy2 = w(8);


lhs = [p.m 0 0 0 (x2 - x1)/p.l
    0 p.m 0 0 (y2-y1)/p.l
    0 0 p.m 0 -(x2-x1)/p.l
    0 0 0 p.m -(y2-y1)/p.l
    (x2-x1) (y2 - y1) -(x2-x1) -(y2 - y1) 0];

rhs = [0 0 0 0 (dx2-dx1)^2+(dy2-dy1)^2]';

dots = lhs\rhs;

wdot = [dx1 dy1 dx2 dy2 dots(1:4)']';

end
```

Prob 22 → did on my own

$y = c(1-\cos(bx))$

See back of prob 5 plots for figure



$\downarrow g$

(a) Derive E.O.M. using Lagrange Equations

$$\mathcal{L} = T - U = E_k - E_p = \tfrac{1}{2} m v^2 - mgy$$

$$\Rightarrow \underset{\sim}{v} = \dot{\underset{\sim}{r}} = \frac{d}{dt}\left(x\hat{\imath} + c(1-\cos(bx))\hat{\jmath}\right)$$

$$\Rightarrow \underset{\sim}{v} = \dot{x}\hat{\imath} + cb\sin(bx)\dot{x}\,\hat{\jmath}$$

$$v^2 = \underset{\sim}{v}\cdot\underset{\sim}{v} = \dot{x}^2 + c^2b^2\sin^2(bx)\dot{x}^2$$

$mgc\cos(bx)$

(a)

$$\Rightarrow \mathcal{L} = \tfrac{1}{2} m\left(\dot{x}^2 + c^2b^2\sin^2(bx)\dot{x}^2\right) - mg\left(c(1-\cos(bx))\right)$$

$$\mathcal{L}E: \quad \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \qquad \text{Let } q_i = x, \Rightarrow \dot{q}_i = \dot{x}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial x} = \tfrac{1}{2} mc^2b^2\dot{x}^2\, 2\sin(bx)\cos(bx)b - mgcb\sin(bx)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \tfrac{1}{2} m\,2\dot{x} + \tfrac{1}{2} mc^2b^2\sin^2(bx)\,2\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} + mc^2b^2\frac{d}{dt}\left(\sin^2(bx)\dot{x}\right)$$

$\longrightarrow \sin^2(bx)\ddot{x} + \dot{x}\,2\sin(bx)\cos(bx)\,b\dot{x}$

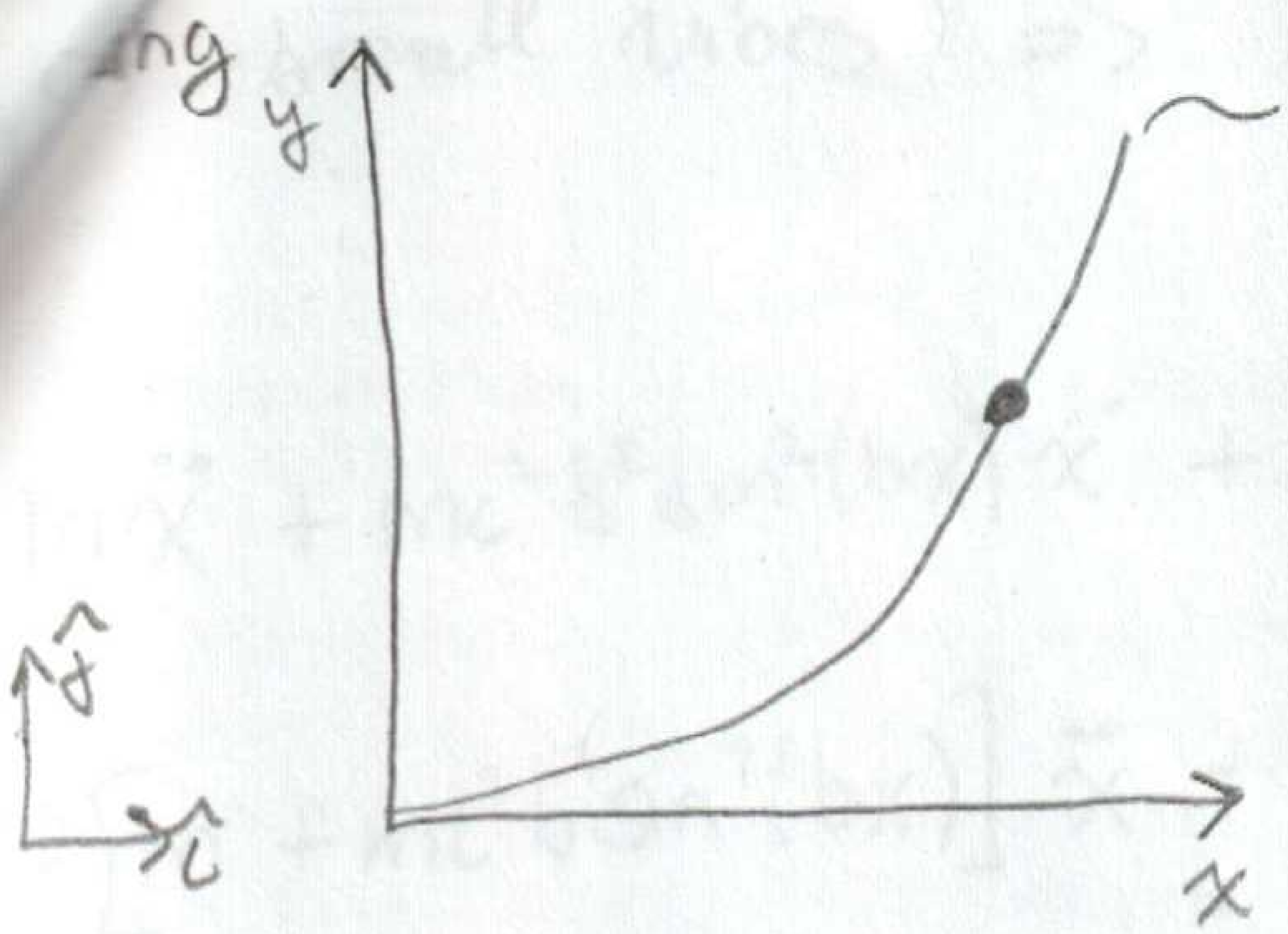
$$\Rightarrow mc^2b^3\dot{x}^2\sin(bx)\cos(bx) - mgcb\sin(bx) - m\ddot{x} - mc^2b^2\left[\sin^2(bx)\ddot{x} + 2b\dot{x}^2\sin(bx)\cos(bx)\right] = 0$$

$$\Rightarrow \cancel{m}\ddot{x} = \cancel{m}c^2b^3\dot{x}^2\sin(bx)\cos(bx) - \cancel{m}gcb\sin(bx) - \cancel{m}c^2b^2\sin^2(bx)\ddot{x} - 2\cancel{m}c^2b^3\dot{x}^2\sin(bx)\cos(bx)$$

$$\ddot{x}\left\{1 + c^2b^2\sin^2(bx)\right\} = -c^2b^3\sin(bx)\cos(bx)\dot{x}^2 - gcb\sin(bx)$$

$$\Rightarrow \boxed{\ddot{x} = \frac{-c^2b^3\sin(bx)\cos(bx)\dot{x}^2 - gcb\sin(bx)}{1 + c^2b^2\sin^2(bx)}}$$

✓



$y = c(1-\cos(bx))$ still.

derive E.O.M. using Newton's Laws.

<u>Kinematics:</u>

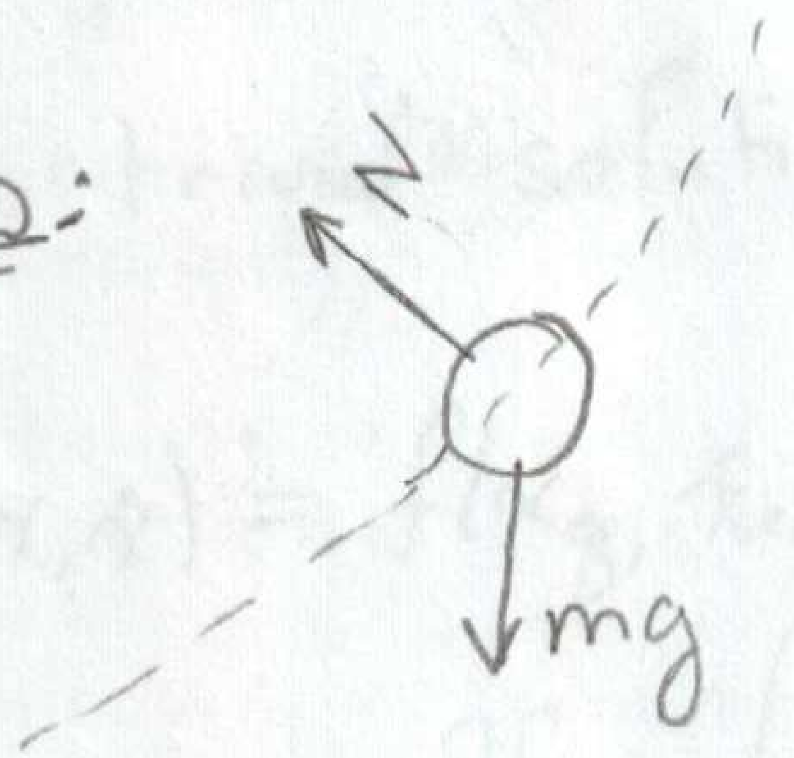
$$\underset{\sim}{r} = x\hat{\imath} + y\hat{\jmath} = x\hat{\imath} + c(1-\cos(bx))\hat{\jmath}$$

$$\Rightarrow \underset{\sim}{v} = \dot{x}\hat{\imath} + cb\sin(bx)\dot{x}\hat{\jmath}$$

$$\Rightarrow \underset{\sim}{a} = \ddot{x}\hat{\imath} + [cb\sin(bx)\ddot{x} + \dot{x}cb\cos(bx)b\dot{x}]\hat{\jmath}$$

$$\Rightarrow \underset{\sim}{a} = \ddot{x}\hat{\imath} + (cb\sin(bx)\ddot{x} + cb^2\cos(bx)\dot{x}^2)\hat{\jmath}$$

<u>FBD:</u>



<u>LMB:</u> $\sum \underset{\sim}{F} = m\underset{\sim}{a} = \underset{\sim}{N} - mg\hat{\jmath}$ b/c $\underset{\sim}{v}\cdot\underset{\sim}{N} = 0$

$$\Rightarrow \underset{\sim}{v}\cdot\{m\underset{\sim}{a} = \underset{\sim}{N} - mg\hat{\jmath}\} = \cancel{m}(\underset{\sim}{v}\cdot\underset{\sim}{a}) = \underset{\sim}{v}\cdot(-\cancel{m}g\hat{\jmath})$$

$$\Rightarrow (\dot{x}\hat{\imath} + cb\sin(bx)\dot{x}\hat{\jmath})\cdot(\ddot{x}\hat{\imath} + (cb\sin(bx)\ddot{x} + cb^2\cos(bx)\dot{x}^2)\hat{\jmath}) =$$

$$(\dot{x}\hat{\imath} + cb\sin(bx)\dot{x}\hat{\jmath})\cdot(-g\hat{\jmath})$$

$$\Rightarrow \dot{x}\ddot{x} + cb\sin(bx)\dot{x}[cb\sin(bx)\ddot{x} + cb^2\cos(bx)\dot{x}^2] = -gcb\sin(bx)\dot{x}$$

$$\Rightarrow \cancel{\dot{x}}\ddot{x} + \cancel{\dot{x}}\ddot{x}c^2b^2\sin^2(bx) + c^2b^3\sin(bx)\cos(bx)\dot{x}^{\cancel{3}\,2} = -gcb\sin(bx)\cancel{\dot{x}}$$

$$\Rightarrow \ddot{x} + \ddot{x}c^2b^2\sin^2(bx) = -c^2b^3\sin(bx)\cos(bx)\dot{x}^2 - gcb\sin(bx)$$

$$\Rightarrow \ddot{x}[1 + c^2b^2\sin^2(bx)] = -c^2b^3\sin(bx)\cos(bx)\dot{x}^2 - gcb\sin(bx)$$

$$\Rightarrow \boxed{\ddot{x} = \frac{-c^2b^3\sin(bx)\cos(bx)\dot{x}^2 - gcb\sin(bx)}{1 + c^2b^2\sin(bx)}}$$

$\rightarrow$ same as from Lagrange Equations

1. Small vibes $\Rightarrow$ linearize the Equation of motion.

$$m\ddot{x} + mc^2b^2\sin^2(bx)\ddot{x} + c^2b^3\sin(bx)\cos(bx)\dot{x}^2 + gcb\sin(bx) = 0.$$

$$\Rightarrow \left[m + mc^2b^2\sin^2(bx)\right]\ddot{x} + \left[c^2b^3\sin(bx)\cos(bx)\right]\dot{x}^2 + gcb\sin(bx) = 0.$$

$$\text{or } \ddot{x} = \frac{-c^2b^3\sin(bx)\cos(bx)\dot{x}^2 - gcb\sin(bx)}{1 + c^2b^2\sin^2(bx)} = f(x,\dot{x})$$

Know trivial solution is an equilibrum point i.e. $x_g = 0,\ \dot{x}_g = 0.$

$$\text{and } f(x,\dot{x}) = f(x_g,\dot{x}_g) + (x - x_g)\frac{\partial f}{\partial x}\Big|_g + (\dot{x} - \dot{x}_g)\frac{\partial f}{\partial \dot{x}}\Big|_g.$$

$$f(0,0) = 0,\quad \frac{\partial f}{\partial x} = \frac{\Big(1 + \underbrace{c^2b^2\sin^2(bx)}_{0}\Big)\Big\{\underbrace{\Big[-c^2b^3\dot{x}^2\big(\overbrace{-\sin(bx)\sin(bx)}^{-1} + \overbrace{\cos(bx)\cos(bx)}^{1}\big)\Big]}_{0} \overbrace{- gcb^2\cos(bx)}^{-gcb^2}\Big\} - \Big(\Big[\underbrace{-c^2b^3\sin(bx)\cos(bx)\dot{x}^2}_{0} - \underbrace{gcb\sin(bx)}_{0}\Big]\Big[2c^2b^2\underbrace{\sin(bx)}_{0}\cos(bx)b\Big]\Big)}{\Big(1 + \underbrace{c^2b^2\sin^2(bx)}_{0}\Big)^2}$$

evaluate @ $x=0$, $\dot{x}=0$

$$\Rightarrow \frac{\partial f}{\partial x}\Big|_g = -gcb.$$

$$\frac{\partial f}{\partial \dot{x}}\Big|_g = \frac{\Big(1 + \overbrace{c^2b^2\sin^2(bx)}^{0}\Big)\Big(-2c^2b^3\overbrace{\sin(bx)}^{0}\cos(bx)\overbrace{\dot{x}}^{0}\Big)}{\Big(1 + \underbrace{c^2b^2\sin^2(bx)}_{0}\Big)^2} = 0$$

$$\Rightarrow \ddot{x}_{\text{small oscillations}} = -gcb^2x \Rightarrow \ddot{x} + gcb^2x = 0$$

$\omega = 2\pi f.$

$T = 1/f \Rightarrow f = \frac{\omega}{2\pi}.$

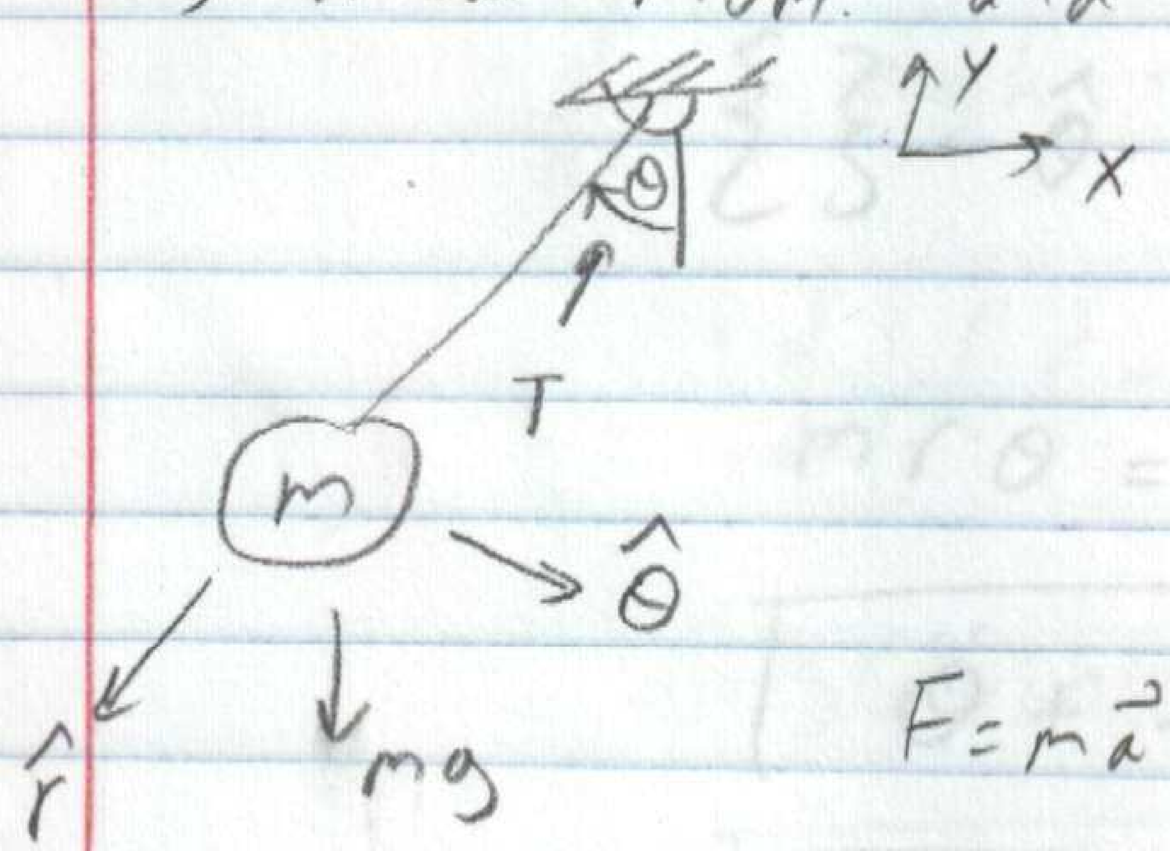
from Inspection $\boxed{\omega = \sqrt{gcb^2}}$

$\Rightarrow T = \frac{2\pi}{\omega}.$

Problem 11

Derive the simple pendulum equation $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ as many ways as you can.

a) linear mom. and manipulate eqns to eliminate constraint force



$$F = m\vec{a} = -T\hat{r} - mg\hat{j}$$

$$m\vec{a} = -T\hat{r} + mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}$$

$\vec{a}$ in polar coor. $= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$\dot{r} = \ddot{r} = 0$ pendulum rod length is constant, r,

$$\vec{a} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}$$

$$m\vec{a} = \{-mr\dot{\theta}^2\hat{r} + mr\ddot{\theta}\hat{\theta} = -T\hat{r} + mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}\}$$

$\{\}\cdot\hat{r}$ to solve for $T\hat{r}$

$$-mr\dot{\theta}^2 = -T + mg\cos\theta$$

$T = m(r\dot{\theta}^2 + g\cos\theta)$ plug into LMB

$$-\cancel{m}r\dot{\theta}^2\hat{r} + \cancel{m}r\ddot{\theta}\hat{\theta} = -\cancel{m}(r\dot{\theta}^2 + g\cos\theta)\hat{r} + \cancel{m}g\cos\theta\,\hat{r} - \cancel{m}g\sin\theta\,\hat{\theta}$$

$$r\ddot{\theta}\hat{\theta} = -g\sin\theta\,\hat{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$

Prob 11

b) linear mom dot with $\hat{\theta}$

From a) $\{-mr\dot{\theta}\hat{r} + mr\ddot{\theta}\hat{\theta} = -T\hat{r} + mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}\}$

$\{\}\cdot\hat{\theta}$ to eliminate $T\hat{r}$ term

$mr\theta = -mg\sin\theta$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$ ✓

c) linear mom cross with $\vec{r} = r\hat{r}$

From a) $\{-mr\dot{\theta}\hat{r} + mr\ddot{\theta}\hat{\theta} = -T\hat{r} + mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}\}$

$\{\}\times\vec{r}$

$mr^2\ddot{\theta}\hat{k} = -mgr\sin\theta\,\hat{k}$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$ ✓

d) angular momentum

$\dot{\vec{H}} = \Sigma\vec{M}$

$\vec{r}\times m\vec{a} = \vec{r}\times\left[(-T + mg\cos\theta)\hat{r} - mg\sin\theta\,\hat{\theta}\right]$

from a) $m\vec{a} = m(-r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta})$

$\vec{r}\times m(-r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}) = -mgr\sin\theta\,\hat{k}$

$mr^2\ddot{\theta}\hat{k} + mgr\sin\theta\hat{k} = 0$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$ ✓

prob 11

e) Conservation of energy

$$\frac{d}{dt}(PE + KE) = 0$$

$$\frac{d}{dt}\left(\cancel{m}gh + \tfrac{1}{2}\cancel{m}v^2\right) = 0$$

$$\frac{d}{dt}\left[g(r - r\cos\theta) + \tfrac{1}{2} r^2\dot{\theta}^2\right] = 0$$

$$g r \sin(\theta)\dot{\theta} + r^2 \ddot{\theta}\dot{\theta} = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$

f) power balance

$$P = \dot{E}_k$$

$$F \cdot \vec{v} = \frac{d}{dt}\tfrac{1}{2}m\vec{v}^{\,2}$$

$$\left[(-T + mg\cos\theta)\hat{r} - mg\sin\theta\,\hat{\theta}\right] \cdot r\dot{\theta}\hat{\theta} = \frac{d}{dt}\left(\tfrac{1}{2}m\,r^2\dot{\theta}^2\hat{\theta}\right)$$

$$-r\cancel{m}g\cancel{\dot{\theta}}\sin\theta\,\hat{\theta} = \cancel{m}r^2\cancel{\dot{\theta}}\ddot{\theta}$$

$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$

Problem 12

a) Lagrange Equation

$$L = \frac{1}{2} m v^2 - mgh$$

$$= \frac{1}{2} m r^2 \dot{\theta}^2 - mg(r - r\cos\theta)$$

LE: $$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$-\cancel{m} g r \sin\theta - \cancel{m} r^2 \ddot{\theta} = 0$$

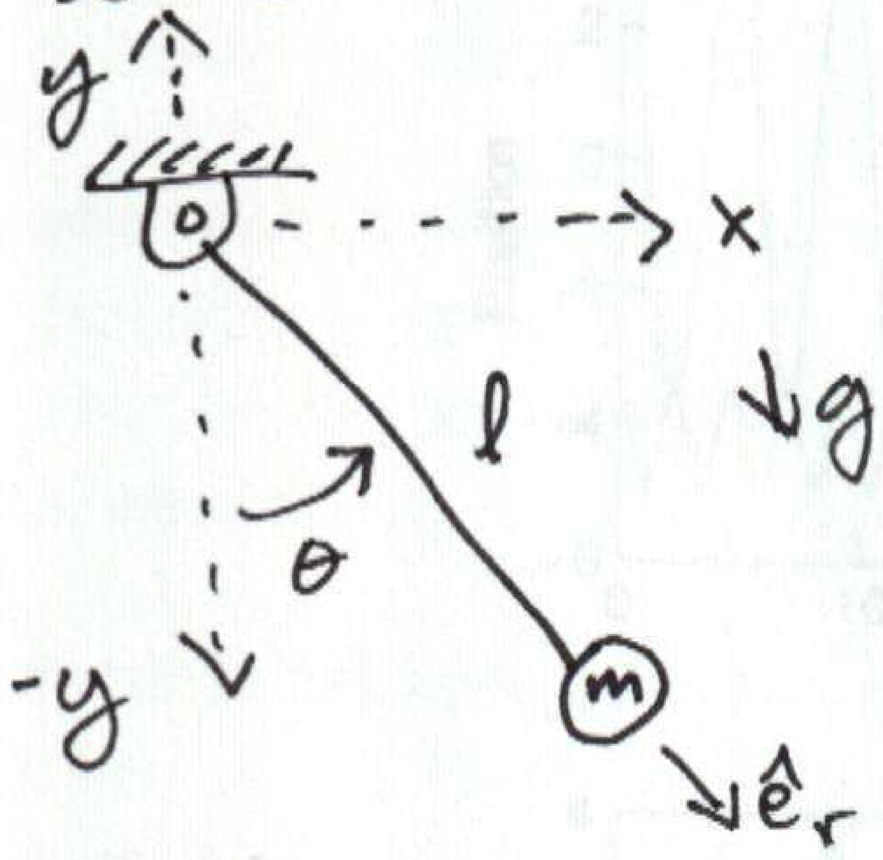
$$\boxed{\ddot{\theta} + \frac{g}{r}\sin\theta = 0}$$

✓

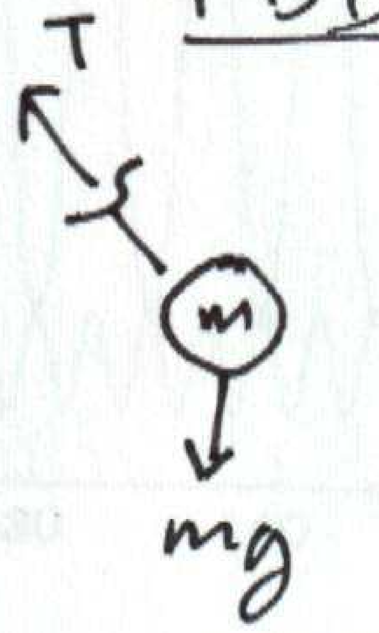
MAE 5730
HW10 11/13/13

Problem #5 Handout 12

Set up simple pendulum in cartesian coordinates and solve numerically. Compare to polar solution



FBD



LMB

$$\Sigma \vec{F} = m\vec{a}$$

$$m\ddot{\vec{x}} = -mg\hat{j} - T\hat{e}_r$$

$$\ddot{x} = -\frac{Tx}{m\sqrt{x^2+y^2}}$$

$$\ddot{y} = -g - \frac{Ty}{m\sqrt{x^2+y^2}}$$

$$\frac{d}{dt}\left\{ l^2 = x^2 + y^2 \right\}$$

$$\frac{d}{dt}\left\{ 0 = 2x\dot{x} + 2y\dot{y} = x\dot{x} + y\dot{y} \right\}$$

$$0 = x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2$$

$$0 = \frac{x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2}{\sqrt{x^2+y^2}}$$

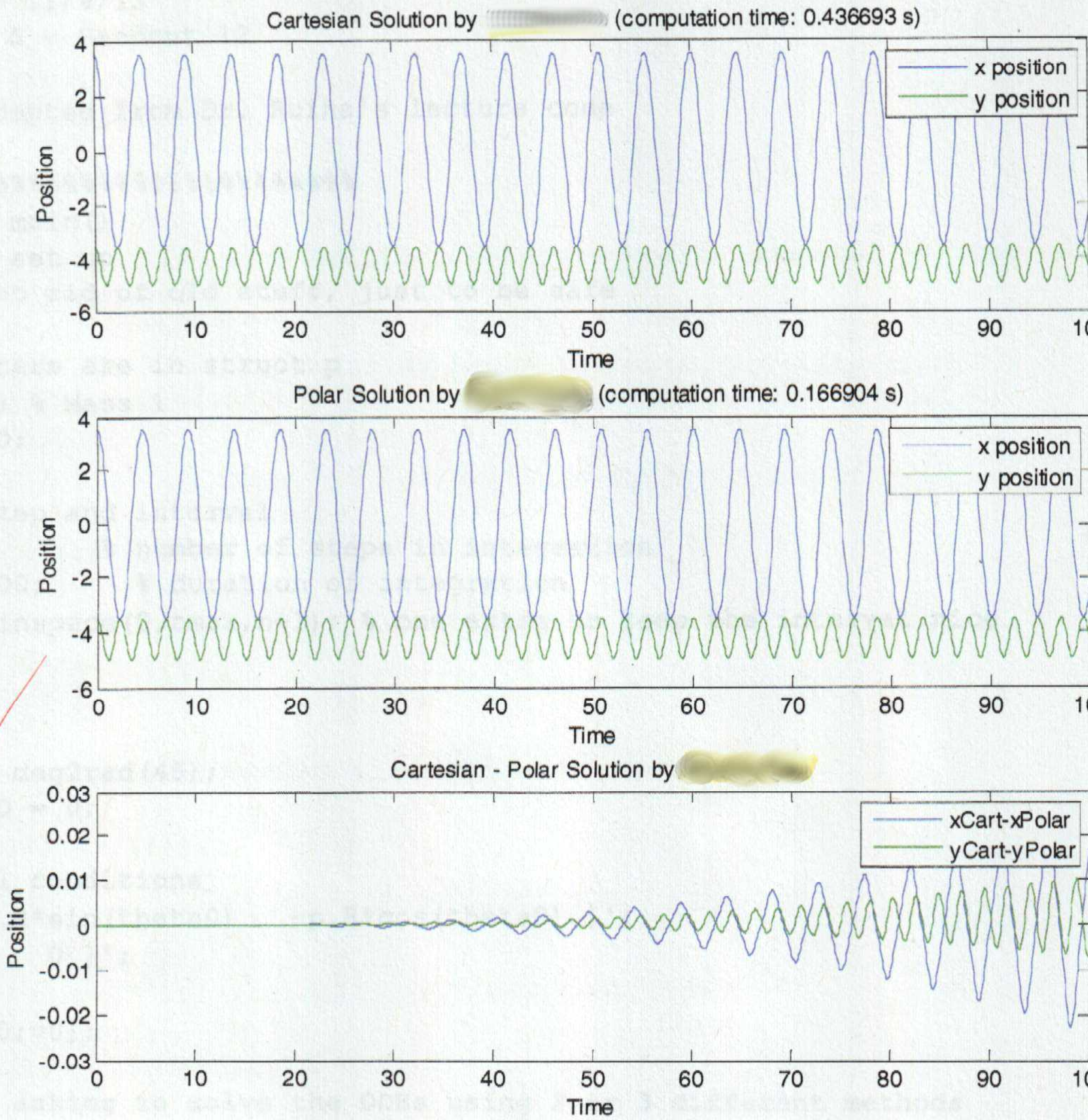
Combining DAEs in matrix form,

$$\begin{bmatrix} m & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ m & 0 & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ T \end{bmatrix} = \begin{bmatrix} mg \\ 0 \\ \frac{-\dot{x}^2-\dot{y}^2}{\sqrt{x^2+y^2}} \end{bmatrix}$$

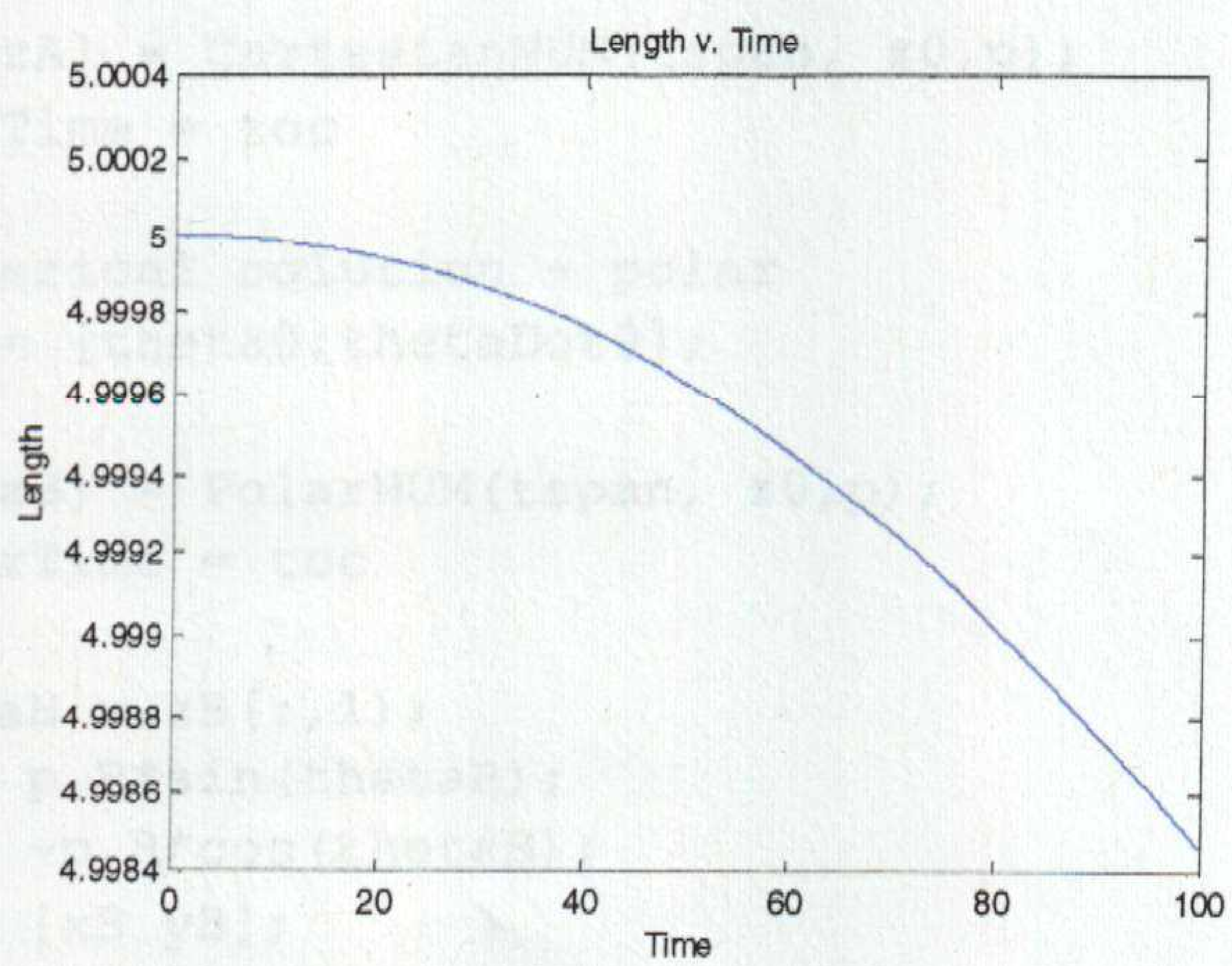
Using attached MATLAB code, system is solved with ode45()

cartesian solution is compared to polar solution

Problem 5



Overall the polar & cartesian solutions matched quite well. Over long durations, the solutions diverged as the cartesian length calculation exponentialy decreased by a small amount.



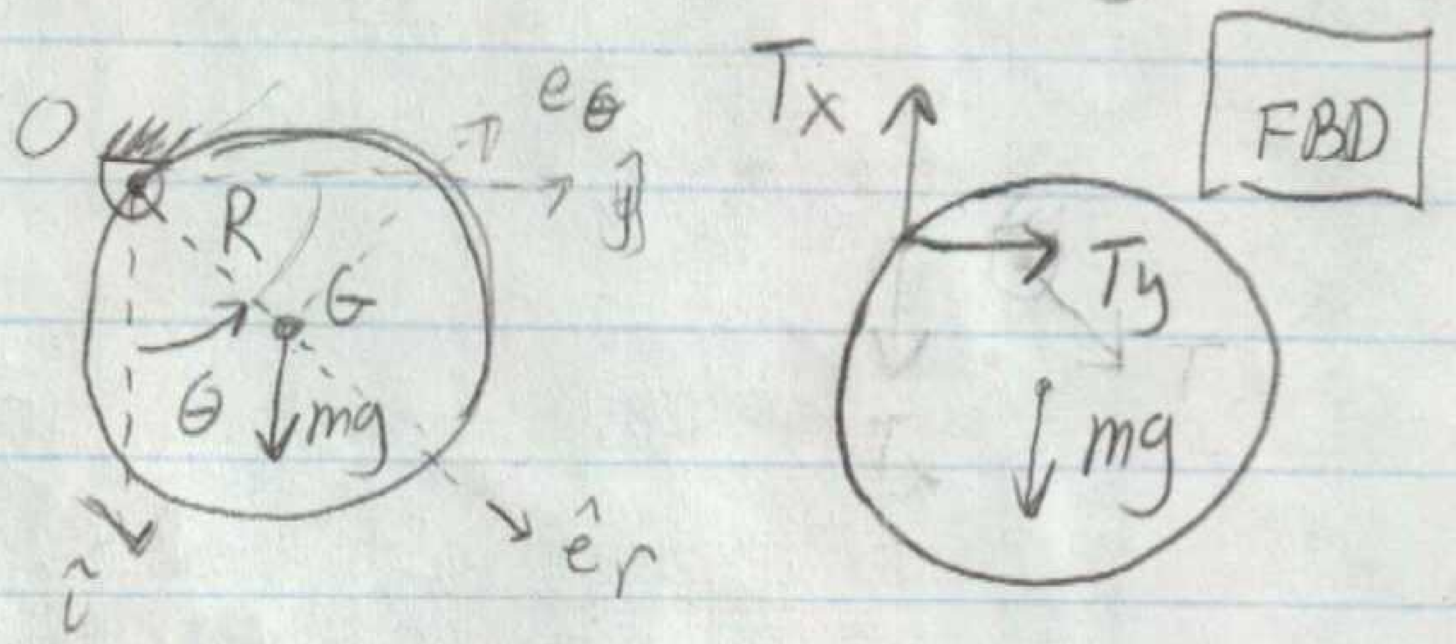
```
% [illegible]
% MAE 5730
% HW10 -- 11/9/13
%Problem 5 - Handout 12
% Code adapted from Dr. Ruina's lecture code

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function main()
%Problem set up
clear %Get rid of old stuff, just to be safe
% parameters are in struct p
p.m  = 1; % Mass 1
p.g = -10;
% time step and interval
n=1000;      % number of steps in integration
tmax = 100;      % duration of integration
tspan= linspace(0,tmax,n+1); % one extra to keep the interval nice

p.R = 5;
theta0 = deg2rad(45);
thetaDot0 = 0;
% initial conditions
r0 = [ p.R*sin(theta0)   -p.R*cos(theta0) ]';
v0 = [ 0  0 ]';

z0  = [r0;v0];
%Command asking to solve the ODEs using 2 or 3 different methods

%Numberical solution - cartesian
tic
[tA zA] = CartesianNUM(tspan, z0,p);
cartTime = toc

%Numerical solution - polar
z0  = [theta0;thetaDot0];
tic
[tB zB] = PolarNUM(tspan, z0,p);
polarTime = toc

thetaB = zB(:,1);
xB = p.R*sin(thetaB);
yB = -p.R*cos(thetaB);
zB = [xB yB];
```

```
xA = zA(:,1);
yA = zA(:,2);

xDiff = xA - xB;
yDiff = yA - yB;
zDiff = [xDiff yDiff];

figure(1)
subplot(3,1,1)
str = sprintf('Cartesian Solution by Bryan Peele (computation time: %f s)',cartTime);
makePlot(zA,tA,str);
subplot(3,1,2)
str = sprintf('Polar Solution by Bryan Peele (computation time: %f s)',polarTime);
makePlot(zB,tB,str);
subplot(3,1,3)
str = sprintf('Cartesian - Polar Solution by Bryan Peele');
plot(tA,zDiff(:,1),tA,zDiff(:,2));
legend('xCart-xPolar','yCart-yPolar');
title(str);
ylabel('Position');
xlabel('Time')
hold off;

%Animate the chosen method (or multiple methods

for i = 1:length(tA)
    figure(2)
    lineX = [0,zA(i,1)];
    lineY = [0,zA(i,2)];
    plot(0,0,'*',zA(i,1),zA(i,2),'bo',lineX,lineY,'-');
    axis square;
    axis([-10,10,-10,1]);

    title('Spring Animation');
    pause(0.01);
end

x =zA(:,1);
y =zA(:,2);
vx=zA(:,3);
vy=zA(:,4);

%Analytic Solution

L = sqrt((x).^2+(y).^2);
figure(3)
plot(tA,L);
title('Length v. Time');
xlabel('Time');
```

```
ylabel('Length');
shg


end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Plot each set of results
function makePlot(zarray,tarray,plotTitle);
   plot(tarray,zarray(:,1),tarray,zarray(:,2));
   legend('x position','y position');
   title(plotTitle);
   ylabel('Position');
   xlabel('Time')
   hold off;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [tarray zarray] = CartesianNUM(tspan, z0,p)
%Numerical solver


%Use ode45
options = odeset('RelTol',1e-6,'AbsTol',1e-6);
[tspan,zarray] = ode45(@cartesianODEs, tspan, z0,options,p);

tarray = tspan;

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zdot = cartesianODEs(t,z,p)


x  = z(1);
y  = z(2);
vx = z(3);
vy = z(4);


xdot = z(3:4);



M=[p.m    0 ;
    0  p.m ];
L = sqrt((x)^2+(y)^2);
J = [x/L y/L]';
A = [M J;J' 0];
B = [ 0 p.m*p.g (-vx^2-vy^2)/L]';
C = A\B;
```

```
vdot = C(1:2);



zdot = [xdot;vdot]; %Output new state

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [tarray zarray] = PolarNUM(tspan, z0,p)
%Numerical solver



%Use ode45
options = odeset('RelTol',1e-6,'AbsTol',1e-6);
[tspan,zarray] = ode45(@polarODEs, tspan, z0,options,p);

tarray = tspan;

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function zdot = polarODEs(t,z,p)

theta   = z(1);
thetaDot   = z(2);

thetaDotDot = p.g*sin(theta)/(p.R);

zdot = [thetaDot;thetaDotDot]; %Output new state

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

#6 Problem: A hoop (mass m, radius R) swings from frictionless pivot on hoop at O. COM G swings between 90° and -90° with vertical line from O. (a) When G is 45° from vertical down. (B) prove the direction is >, <, or = 45° from vertical (C) Find force on hoop from hinge when G is directly below hinge.



References: None

SOLUTION:

(a) At 45°, $E_p + E_K = 0 \rightarrow mgR\cos(45°) = \frac{1}{2} m V_G^2 + \frac{1}{2} I_G \omega^2$

$\rightarrow I_G = mR^2,\ V_G = R\omega \rightarrow mgR\frac{\sqrt{2}}{2} = \frac{1}{2} mR^2\omega^2 + \frac{1}{2} mR^2\omega^2 = mR^2\omega^2$

$\rightarrow \omega = \sqrt{\frac{\sqrt{2}g}{2R}}$

LMB: $\Sigma \vec{F} = m\vec{a} \rightarrow mg - T_x = m(R\ddot{\theta}\hat{e}_\theta - R\omega^2\hat{e}_r)\cdot\hat{i}$

$= mR\ddot{\theta}(-\sin\theta) - mR\omega^2\cos\theta \rightarrow \underline{T_x = mg + mR\ddot{\theta}\sin\theta + mR\omega^2\cos\theta}$

$\rightarrow T_y = mR\ddot{\theta}(\hat{e}_\theta\cdot\hat{j}) - mR\omega^2(\hat{e}_r\cdot\hat{j}) = \underline{mR\ddot{\theta}\cos\theta - mR\omega^2\sin\theta}$

AMB: $\Sigma \vec{M}_{/o} = \dot{\vec{H}}_{/o} \rightarrow \vec{r}_{G/o} \times mg\hat{i} = -mgR\sin\theta\,\hat{k} = \vec{r}_{G/o} \times m\vec{a}_G + I_G\ddot{\theta}\hat{k}$

$I_G = mR^2,\ \vec{a}_G = \ddot{\theta}\hat{k} \times \vec{r}_{G/o} - \omega^2\vec{r}_{G/o}$

$\rightarrow -mgR\sin\theta\,\hat{k} = \vec{r}_G \times m(\ddot{\theta}\hat{k} \times \vec{r}_G - \omega^2\vec{r}_G) + mR^2\ddot{\theta}\hat{k} = mR^2\ddot{\theta}\hat{k} + mR^2\ddot{\theta}\hat{k}$

$\rightarrow \ddot{\theta} + \frac{g}{2R}\sin\theta = 0$

→ I could solve this eq. of motion numerically to get T_x, T_y → direction of T at $\theta = 45°$. Instead, I will solve analytically and thereby complete proof for part (b) simultaneously.

→ at $\theta = 45° = \frac{\pi}{4}$ → $\ddot{\theta} = -\frac{g}{2R}\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}\,g}{4R}$

→ $T_x = mg + mR\left(\frac{-\sqrt{2}\,g}{4R}\right)\sin\left(\frac{\pi}{4}\right) + mR\left(\frac{\sqrt{2}\,g}{2R}\right)\cos\left(\frac{\pi}{4}\right)$

I solved ω earlier from energy balance

$= m\left[g - \frac{g}{4} + \frac{g}{2}\right] = \frac{5}{4}mg$

→ $T_y = mR\left(-\frac{\sqrt{2}\,g}{4R}\right)\cos\left(\frac{\pi}{4}\right) - mR\left(\frac{\sqrt{2}\,g}{2R}\right)\sin\left(\frac{\pi}{4}\right)$

$= -\frac{mg}{4} - \frac{mg}{2} = -\frac{3}{4}mg$

→ $\phi_T = \tan^{-1}\left(\frac{T_y}{T_x}\right) = \tan^{-1}\left(-\frac{3}{5}\right) = -31°$ → $\boxed{\phi_T = 31° \text{ to the left of vertical (straight up)}}$

(b) As proved in part (a), the direction is less than 45° to left of vertical up direction. This makes sense, because The pivot must exert a radial force on the hoop for centripetal acceleration (at 45°) as well as a vertical force to cancel gravity, thus bringing the total direction less than 45°.

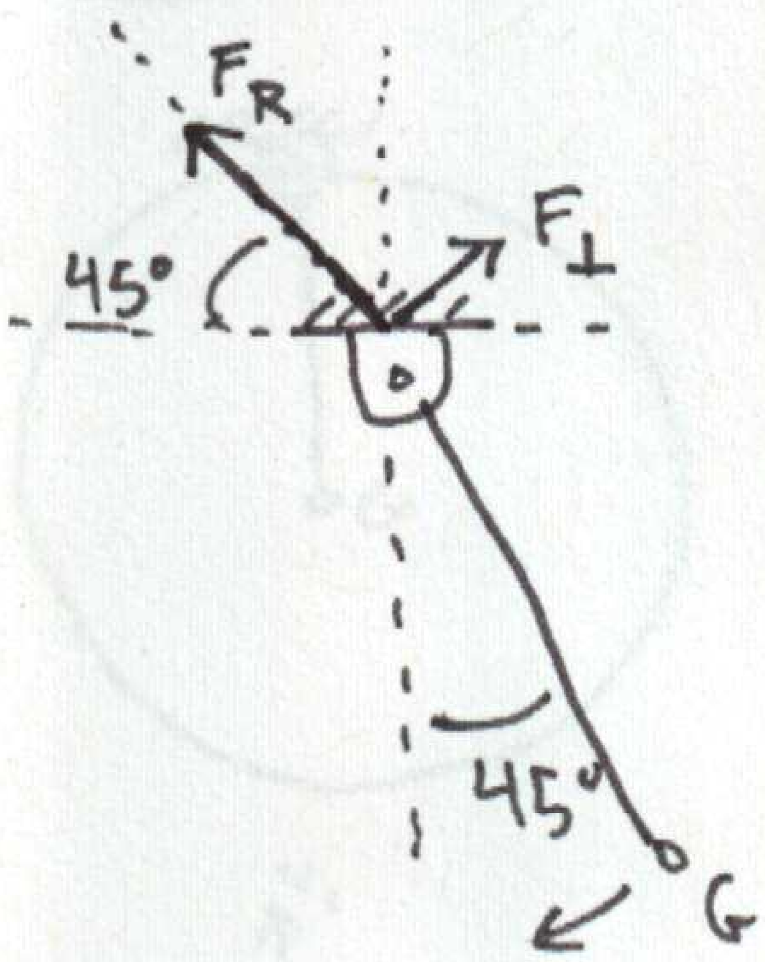
(c) At bottom, $\ddot{\theta} = 0$ → $E_K = -E_P$ → $mR^2\omega^2 = mgR$ → $\omega^2 = \frac{g}{R}$

→ $\sum \vec{F} = m\vec{a}$ → $(mg - T_x)\hat{i} + T_y\hat{j} = m\left(-\omega^2 \vec{r}_G\right) = -m\omega^2 R\,\hat{i}$

→ $T_x = mg + mR\omega^2 = mg + mR\left(\frac{g}{R}\right) = 2mg$ → $T_y = 0$

→ So force on hoop is $\boxed{T = 2mg \text{ vertically up}}$

Problem #6



We can look at the $\overbrace{\text{hinge force}}^{F_h}$ as a radial (F_R) and perpendicular ($F_\perp$) force such that $F_h = F_R + F_\perp$

Intuitively, F_R will be positive in the direction shown. To determine whether F_h will be within 45° of vertical, we need to know the sign of $F_\perp$

To do so, we can perform an <u>AMB</u> about G

$$\Sigma \vec{M}_{/G} = \dot{\vec{H}}_{/G}$$

F_R provides no moment since it is alligned radial to G. Moreover, because the pendulum is accelerating ~~swinging~~ downwards, we know that $\dot{\vec{H}}_{/G}$ must be directed in the $-\hat{k}$ direction. Therefore $\Sigma \vec{M}_{/G}$ must also be in the $-\hat{k}$ direction. Using the right hand rule, this means that $F_\perp$ must be positive in the direction picked above. Thus as long as F_R and $F_\perp$ are >0, F_h must be within (less than) 45° of vertical (regardless of the relative magnitudes of F_R and $F_\perp$)

This is true even when swinging up.