MAE 325 Prelim 2 Sol.

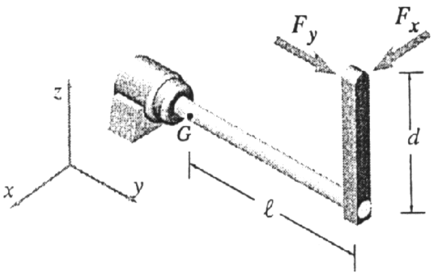
Note on (c): solution below is max tension stress, not max magnitude of tension stress, which would depend on the variables in a more complex way; θ may be negative

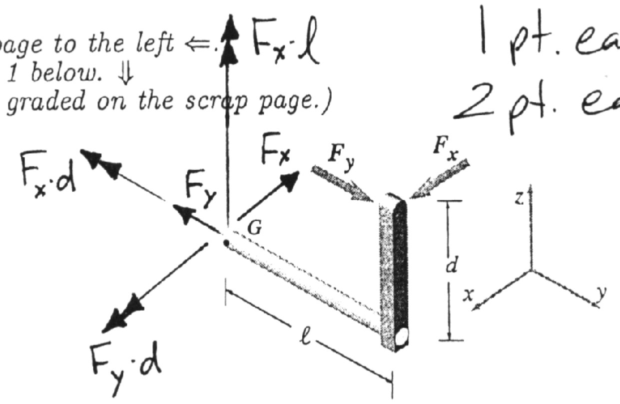
1/4

1) (35 pt) A round shaft with length ℓ and radius r has a lever arm (length d) attached to it with two loads applied. The shaft has area moment of inertia $I = \pi r^4/4$, polar moment of inertia $J = \pi r^4/2$, and cross sectional area $A = \pi r^2$. Neglect gravity. Assume linear elastic behaviour and the usual strength-of-solids assumptions about loading of long narrow structures. Neglect the effect of stress concentrations. Point G is at the outer edge of the bar in the front (a distance r in the x direction from the center line).

⇐ Please put scrap work for problem 1 on the page to the left ⇐.
⇓ Put neat, clear work to be graded for problem 1 below. ⇓
(If you need the space, clearly mark work to be graded on the scrap page.)

1 pt. each force
2 pt. each moment





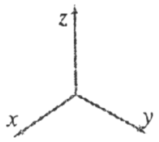
Free Body Diagram

a) (8 points) Complete the free body diagram of the structure with a cut at the section G. Use statics to find the forces and moments at this cut in terms of ℓ, d, F_x and F_y. (Parts (b) and (c) below depend on this being correct and will not be graded if this part is incorrect.)

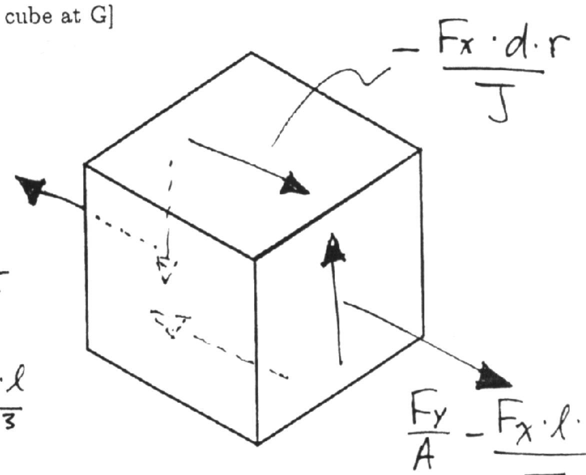
3 pt. each in lower triangle, 3 each for symmetric blocks

b) (27 points) Fill in the components of the stress matrix at point G, assign values to all of the tractions shown. The rows and columns have the standard x, y, z interpretations. Signs are important. [hint: you may want to draw the stress components on a small cube at G]

$$[\sigma] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{F_y}{A} - \frac{F_x \cdot \ell \cdot r}{I} & -\frac{F_x \cdot d \cdot r}{J} \\ 0 & -\frac{F_x \cdot d \cdot r}{J} & 0 \end{bmatrix}$$

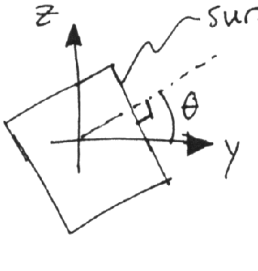


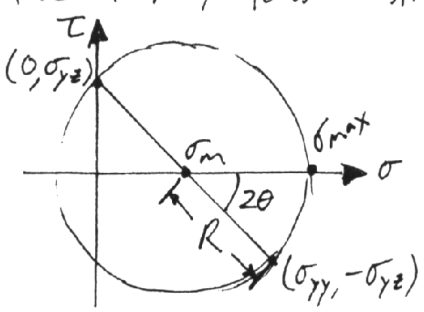
$$\sigma_{yy} = \frac{F_y}{A} - \frac{F_x \cdot \ell \cdot r}{I} = \frac{F_y}{\pi r^2} - \frac{4 F_x \cdot \ell}{\pi r^3}$$



$$\sigma_{yz} = \sigma_{zy} = -\frac{F_x \cdot d \cdot r}{J} = \frac{-2 F_x \cdot d}{\pi r^3}$$

c) (Extra credit, 5 points) Assuming F_x and F_y are positive and that ℓ and d are much bigger than r. What is the magnitude of the maximum tension stress at G? On what surface does it act (make a sketch and write a formula for any angle(s) that you mark). Give your answers in terms of ℓ, d, r, I, J, A, F_x and F_y [You may use Mohr's circle(s) or any other method. The answer depends on care with signs. The answer is not tidy.]





$$\sigma_m = \frac{\sigma_{yy}}{2} \qquad R = \frac{1}{2}\left[(2\sigma_{yz})^2 + \sigma_{yy}^2\right]^{1/2}$$

$$\sigma_{max} = \sigma_m + R = \frac{1}{2}\left(\frac{F_y}{\pi r^2} - \frac{4F_x \ell}{\pi r^3}\right) + \frac{1}{2}\left[\left(\frac{4F_x d}{\pi r^3}\right)^2 + \left(\frac{F_y}{\pi r^2} - \frac{4F_x \ell}{\pi r^3}\right)^2\right]^{1/2}$$

$$\theta = \frac{1}{2}\arctan\left(\frac{\sigma_{yz}}{\sigma_m}\right) = \frac{1}{2}\arctan\left(\frac{\frac{-2F_x d}{\pi r^3}}{\frac{1}{2}\left(\frac{F_y}{\pi r^2} - \frac{4F_x \ell}{\pi r^3}\right)}\right)$$

2) (35 pt) For student projects structures are often built of balsa wood instead of with stronger engineering materials like steel. A student team makes a solid square cross section cantilever beam (clamped at one end, loaded transversely at the other) and finds that its failure load is 100 lbf. They then plan to take an **equal length** and **equal mass** square cross section steel beam and load it.

a) **(15 points)** Estimate the failure load of the steel beam? (A numerical answer is desired.) Make the standard strength-of-materials assumptions for both beams.

b) (15 points) How much stiffer is the steel beam than the balsa beam? (stiffness = force/deflection)

c) (5 points) In words, give a qualitative explanation for the solutions you found, or should have found, above.

Some key assumptions you should use:

- The failure stress of mild steel in tension is about 100 times that of balsa wood.
- The "Young's modulus" (the elastic modulus in tension) of steel is about 100 times that of Balsa.
- The density (mass per unit volume) of steel is about 36 times that of balsa wood.

⇐ *Please put scrap work for problem 2 on the page to the left* ⇐.
⇓ *Put neat, clear work to be graded for problem 2 below.* ⇓
(If you need the space, clearly mark work to be graded on the scrap page.)

(a) $P_{f_B} = 100\,\text{lb}$ $\quad L_B = L_S \quad M_B = M_S \quad \sigma_{f_S} = 100\,\sigma_{f_B} \quad D_S = 36\,D_B$

Square cross section

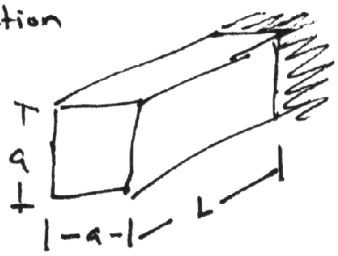
$M_B = M_S \Rightarrow D = \dfrac{M}{V}$

$D_B V_B = D_S V_S$

$D_B[\cancel{L_B}\,a_B^2] = D_S[\cancel{L_S}\,a_S^2]$

$a_S^2 = \dfrac{D_B}{D_S} a_B^2 = \dfrac{1}{36} a_B^2$

$\boxed{a_S = \dfrac{a_B}{6} \quad \text{or} \quad a_B = 6a_S}$ 5pts.



$I = \frac{1}{12} b h^3$

$I = \frac{1}{12} a^4$

$\sigma_f = \dfrac{M_f c}{I} = \dfrac{P_f L c}{\frac{1}{12} a^4}$

$\sigma_f = \dfrac{12 P_f L (a/2)}{a^4}$

$\boxed{\sigma_f = \dfrac{6 P_f L}{a^3}}$ 2.5 pts

$\boxed{\sigma_{f_S} = 100\,\sigma_{f_B}}$ 2.5pts.

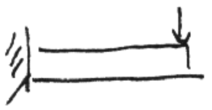
$\dfrac{6 P_{f_S} \cancel{L_S}}{a_S^3} = 100 \dfrac{6 P_{f_B} \cancel{L_B}}{a_B^3}$

$P_{f_S} = \dfrac{100}{\cancel{6}} \dfrac{\cancel{6} P_{f_B} a_S^3}{a_B^3} = 100 P_{f_B} \dfrac{(a_B/6)^3}{\cancel{a_B^3}} = \dfrac{(100) P_{f_B}}{216}$

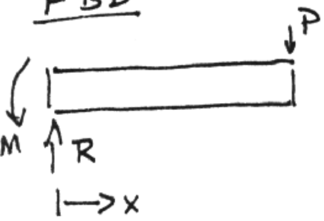
6× 6×6=216

$\boxed{P_{f_S} = \dfrac{10{,}000}{216}\ \text{lbs}}$ 5pts

(b) Beam deflection.



FBD



BCs:
(a) $V(0^-)=0$
(b) $M(0^-)=0$
(c) $V(L^+)=0$
(d) $M(L^+)=0$
(e) $u'(0)=0$
(f) $u(0)=0$

$q=0$

$V=\int q\,dx = R\langle x\rangle^0 - P\langle x-L\rangle^0 + c_1$

from (a) $c_1=0$, from (c) $R=P$

$M=\int V\,dx = P\langle x\rangle^1 - P\langle x-L\rangle^1 - M\langle x\rangle^0 + c_2$

from (b) $c_2=0$, from (d) $M=PL$

$EIu' = \int M\,dx = \frac{P}{2}\langle x\rangle^2 - \frac{P}{2}\langle x-L\rangle^2 - PL\langle x\rangle^1 + c_3$

from (e) $c_3=0$

$EIu = \int u'\,dx = \frac{P}{6}\langle x\rangle^3 - \frac{P}{6}\langle x-L\rangle^3 - \frac{PL}{2}\langle x\rangle^2 + c_4$

from (f) $c_4=0$

@ $x=L$

$u=\frac{P}{EI}\left(\frac{L^3}{6}-\frac{L^3}{2}\right) = -\frac{PL^3}{3EI}$

$\boxed{\delta = \frac{PL^3}{3EI}}$ 5pts.

so $0<x<L$

$u(x) = \frac{P}{EI}\left(\frac{x^3}{6} - \frac{L}{2}x^2\right)$

$K=\frac{F}{\delta} = \frac{P}{\frac{PL^3}{3EI}} \Rightarrow \boxed{K=\frac{3EI}{L^3}}$ 5pts

$L_s = L_B \quad E_s = 100\,E_B$

$\frac{E_s}{E_B}=100$

$\frac{K_s}{K_B} = \frac{\frac{3E_sI_s}{L_s^3}}{\frac{3E_BI_B}{L_B^3}} = \frac{E_sI_s}{E_BI_B} = (100)\frac{I_s}{I_B}$

$6\times6\times6\times6 = 1296$

$= 100\,\frac{\frac{1}{12}(a_s^4)}{\frac{1}{12}(a_B)^4} = 100\,\frac{(a_B/6)^4}{a_B^4} = \frac{100}{6^4} = \frac{100}{1296}$

$\boxed{\frac{K_s}{K_B} = \frac{100}{1296}}$ 5pts.

(c) Steel beam fails at a lower load ≈ 46.3 lb!
Steel beam is also less stiff! } 2.5pts

$\Longrightarrow$ Geometry overrides material properties } 2.5pts.
$I \propto a^4$

Alternatively for part (a):

$$I = \frac{1}{12} a^4 \qquad V = \frac{M}{D} = L a^2$$

$$a^2 = \frac{M}{LD} \qquad a = \sqrt{\frac{M}{LD}}$$

$$\frac{\sigma_{fs}}{\sigma_{fB}} = \frac{\dfrac{P_{fs} \cancel{L} \left[\cancel{\tfrac{1}{2}} \sqrt{\left(\tfrac{\cancel{M}}{\cancel{L}D}\right)_s} \right]}{\cancel{\tfrac{1}{12}} \left(\tfrac{\cancel{M}}{\cancel{L}D}\right)_s^2}}{\dfrac{P_{fB} \cancel{L} \left[\cancel{\tfrac{1}{2}} \sqrt{\left(\tfrac{\cancel{M}}{\cancel{L}D}\right)_B} \right]}{\cancel{\tfrac{1}{12}} \left(\tfrac{\cancel{M}}{\cancel{L}D}\right)_B^2}} \Rightarrow 100 = \frac{P_{fs} D_B^2 \sqrt{D_s}}{P_{fB} D_B^2 \sqrt{D_B}}$$

$$P_{fs} = 100 \, P_{fB} \left(\frac{D_B}{D_s} \right)^{3/2} = \frac{(100)(100)}{(36)^{3/2}} = \frac{10000}{216}$$

part (b):

$$\frac{K_s}{K_B} = \frac{E_s I_s}{E_B I_B} = 100 \, \frac{\cancel{\tfrac{1}{12}} \left(\tfrac{\cancel{M}}{\cancel{L}D}\right)_s^2}{\cancel{\tfrac{1}{12}} \left(\tfrac{\cancel{M}}{\cancel{L}D}\right)_B^2} = 100 \left(\frac{D_B}{D_s} \right)^2$$

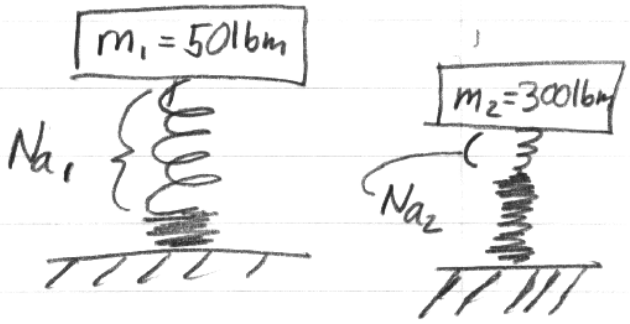
$$= \frac{100}{36^2} = \frac{100}{1296}$$

MAE 325, Fall 99, Prelim 2, Problem 3 "Soln."

— A. Ruina

prob 3 page 1

3 a) Given $K = \dfrac{Gd^4}{8D^3 N_a}$



$$\omega_n^* = \sqrt{\frac{K}{m}} \Rightarrow K = m\,\omega_n^{*2}$$

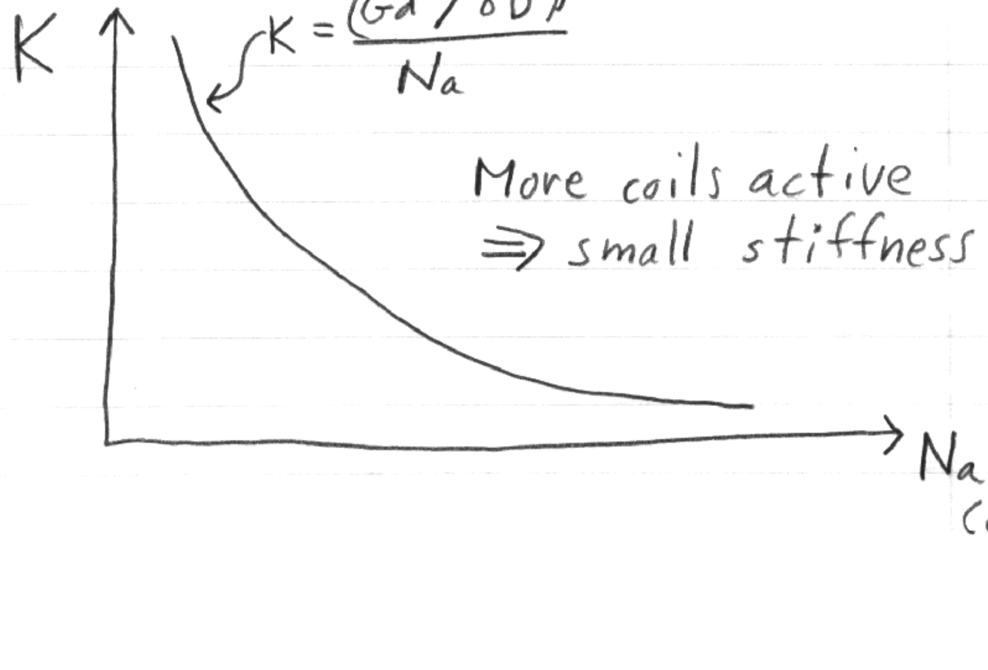
$$\Rightarrow \frac{Gd^4}{8D^3 Na} = \frac{w}{g}\,\omega_n^{*2}$$

$$\Rightarrow \quad Na = \underbrace{\left(\frac{gGd^4}{8D^3\omega_n^{*2}}\right)}_{C \equiv \frac{gGd^4}{8D^3\omega_n^{*2}}}\frac{1}{w} = \frac{C}{w} \quad (1)$$

$$\boxed{\frac{Na_1}{Na_2} = \frac{C/w_1}{C/w_2} = \frac{w_2}{w_1} = \frac{300\,lbf}{50\,lbf} = 6} \quad (a)$$

6 times as many coils needed for 50 lbf person as for 300 lbf person.

3 b)

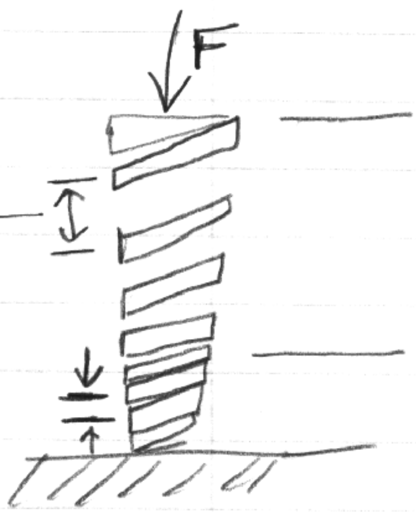


More coils active
$\Rightarrow$ small stiffness

(cont'd)

3c) How to vary the pitch.

Define position on spring from top down by y.



y = # of coils down from top

$y = \begin{cases} 0, & \text{at top} \\ 1, & \text{at first coil} \\ \vdots & \\ n & \text{at nth coil} \\ 7.5 & \text{half way betwen 7th \& 8th coil.} \\ N_a & \text{at boundary between active \& locked portions of coil} \end{cases}$

$P(y)$ = pitch of spring at coil y

$= P_0(y) - \Delta(\text{pitch due to load})$

$P_0(y)$ ← pitch when there is no load

$$= P_0(y) - \underbrace{\frac{8D^3}{Gd^4}F}_{\frac{1}{N_a} * \text{deflection of a const. pitch spring}} \quad \text{(small angle approx.)} \quad (2)$$

prob 3, page 3

At the boundary between active & solid spring

at boundary

$$P(y) = d$$

(pitch is pitch of a a solid spring)

(2) ⇒ $$P_0(y) - \frac{8D^3}{Gd^4} F = d$$

$F = \frac{c}{N_a}$ (from (1))

$N_a = y$

force on spring, also equal weight w

$$\Rightarrow P_0(y) - \frac{8D^3}{Gd^4}\frac{c}{y} = d$$

$$\Rightarrow P_0(y) = d + \frac{8D^3}{Gd^4}\,\frac{\frac{gGd^4}{8D^3\omega_n^{*2}}}{y}$$

$$\boxed{P_0(y) = d + \frac{g}{\omega_n^{*2}\, y}}$$

Pitch of (unstretched) spring at coil y, measured from top.
Need not apply all the way to $y = 0$. Need only apply down to $y = c/W_{max}$

say, 300lbf

dist. from ground

[To give $P_0(x)$, need to integrate to calc. x

$x = l_0 - \int_0^y P_0(y')\,dy'$]