Dynamic model derivation and controller design for an autonomous bicycle

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Mechanical Engineering

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1. Work Summary

This report describes what I have done in the whole semester. In summary, I have derived the equations of motion of the bicycle, designed the controller and observer for a self-stabilizing bicycle, and simulated the controller and observer in MATLAB. The derivation of the equation of motion of the bicycle is based on a point mass model. Two controllers were then designed using this model. One is for bicycle's self-stabilization and another observer based controller is for bicycle stability and steering reference tracking. These controllers were simulated in MATLAB by using the ode45 solver and they were shown to meet the design requirements.

2. Motivation

My academic interest is in Dynamics & Control. Autonomous Bicycle project is a combination of Dynamics and Controls, which definitely attracts me a lot. Even though I learned how to ride a bike when I was 5 years old, I have no idea about why we can ride bicycle in that way. With a curious heart and energetic personality, I want to know the equation of motion of bicycle analytically and design a controller that can stable and steer this bicycle. After a semester's work, I have a new understanding about the bicycle and also lay a foundation for our future project. Our goal next semester is to create a steer-by-wire bicycle.

3. Dynamics Model Derivation
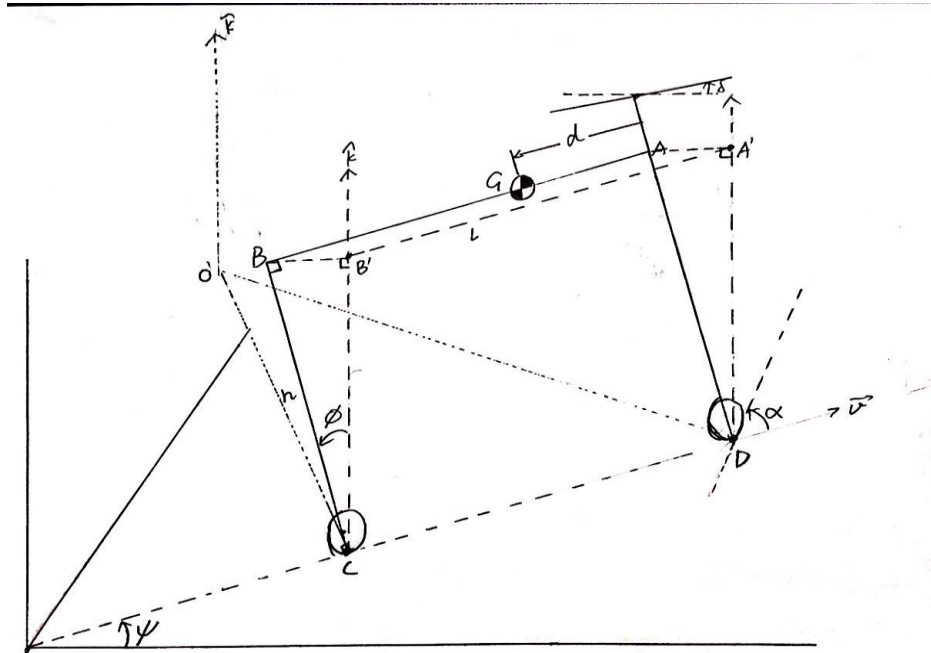
3.1 Point mass model

Figure 1 Point mass model

Figure 1 is the point mass model.

Where

G is the center of mass. AD is the front handlebar.

$BG = b$ $\qquad\qquad AG = d$ $\qquad l = wheelbase\ AB$

$\phi = lean\ angle$ $\qquad \psi = yaw\ angle\ (heading)$

$\delta = steer\ angle$ $\qquad a = the\ angle\ between\ front\ wheel\ and\ line\ CD$

$xG = x\ position\ of\ the\ center\ of\ mass$

$yG = y\ position\ of\ the\ center\ of\ mass$

$zG = z\ position\ of\ the\ center\ of\ mass$

$v = forward\ velocity\ of\ bicycle\ (assumed\ constant)$

$C = the\ contact\ point\ between\ the\ rear\ wheel\ and\ ground$

$D = the\ contact\ point\ between\ the\ front\ wheel\ and\ ground$

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This model satisfies the following assumptions:

- The handlebar has no moment of inertia
- A vertical head angle ($\lambda = 0$)
- No trail ($c = 0$)
- Rear wheel has no inertia
- Rear frame is a point mass
- Front wheel has no mass or inertia

3.2 Find the acceleration of the center of mass

To build the relationship between the lean angle, the yaw angle and the steer angle, we should first know the acceleration of the center of mass.
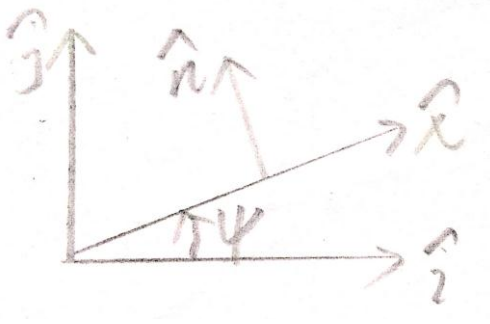

Figure 2 yaw angle in x-y plane

First, find the position of the center of mass (COM).

$$\vec{r}_{B/F} = \vec{r}_{C/F} + \vec{r}_{B/C}$$
$$\vec{r}_{G/F} = \vec{r}_{B/F} + \vec{r}_{G/B}$$
$$\vec{r}_{G/F} = \vec{r}_{C/F} + \vec{r}_{B/C} + \vec{r}_{G/B}$$

Second, take derivative of the position vector with respect to time.

$$\vec{v}_G = \dot{\vec{r}}_{G/F} = \dot{\vec{r}}_{C/F} + \dot{\vec{r}}_{B/C} + \dot{\vec{r}}_{G/B}$$

$$\vec{v}_c = \dot{\vec{r}}_{C/F} = v\hat{\lambda}$$

$$\dot{\vec{r}}_{B/C} = (r_{CB'} + r_{B'B})' = \dot{\vec{r}}_{CB'} + \dot{\vec{r}}_{B'B} = -h\sin(\phi)\dot{\phi}\hat{k} + h\cos(\phi)\dot{\phi}\hat{n} - h\sin(\phi)\dot{\psi}\hat{\lambda}$$

$$\dot{\vec{r}}_{G/B} = (b\hat{\lambda})' = b\dot{\psi}\hat{n}$$

$$\vec{v}_G = (v - h\sin(\phi)\dot{\psi})\hat{\lambda} + (h\cos(\phi)\dot{\phi} + b\dot{\psi})\hat{n} - h\sin(\phi)\dot{\phi}\hat{k}$$

Third, take derivative of the velocity vector with respect to time.

$$\vec{a}_G = \dot{\vec{v}}_G = ((v - h\sin(\phi)\dot{\psi})\hat{\lambda})' + ((h\cos(\phi)\dot{\phi} + b\dot{\psi})\hat{n})' - (h\sin(\phi)\dot{\phi}\hat{k})'$$

$$((v - h\sin(\phi)\dot{\psi})\hat{\lambda})' = (\dot{v} - h\cos(\phi)\dot{\phi}\dot{\psi} - h\sin(\phi)\ddot{\psi})\hat{\lambda} + (v - h\sin(\phi)\dot{\psi})\dot{\psi}\hat{n}$$

$$((h\cos(\phi)\dot{\phi} + b\dot{\psi})\hat{n})' = (b\ddot{\psi} - h\sin(\phi)\dot{\phi}^2 + h\cos(\phi)\ddot{\phi})\hat{n} - (h\cos(\phi)\dot{\phi} + b\dot{\psi})\dot{\psi}\hat{\lambda}$$

$$-(h\sin(\phi)\dot{\phi}\hat{k})' = -(h\cos(\phi)\dot{\phi}^2 + h\sin(\phi)\ddot{\phi})\hat{k}$$

By adding these three components together, we could get the expression for $\vec{a}_G$.

$$\vec{a}_G = (\dot{v} - h\cos(\phi)\dot{\phi}\dot{\psi} - h\sin(\phi)\ddot{\psi} - h\cos(\phi)\dot{\phi}\dot{\psi} - b\dot{\psi}^2)\hat{\lambda} + ...$$
$$(b\ddot{\psi} - h\sin(\phi)\dot{\phi}^2 + h\cos(\phi)\ddot{\phi} + v\dot{\psi} - h\sin(\phi)\dot{\psi}^2)\hat{n} - (h\cos(\phi)\dot{\phi}^2 + h\sin(\phi)\ddot{\phi})\hat{k}$$

However, $\vec{a}_G$ contains terms like $\dot{\psi}$ and $\ddot{\psi}$. We do not like these terms because we cannot control the yaw angle. So we should find other relations that can help us kill these terms.

3.3 Kill the yaw angle term

To eliminate the yaw angle term, we could build connection between this yaw angle and α trough the relative motion.
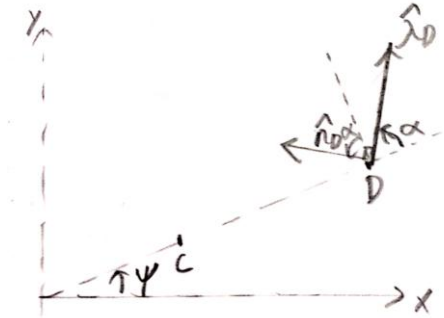

Figure 3 relations between a and ψ

$$\hat{\lambda}_D = \cos(\alpha)\hat{\lambda} + \sin(\alpha)\hat{n}, \vec{n}_D = \cos(\alpha)\hat{n} - \sin(\alpha)\hat{\lambda}$$
$$\vec{r}_{D/F} = \vec{r}_{C/F} + \vec{r}_{D/C}, \vec{v}_D = \vec{v}_{C/F} + (l\hat{\lambda}) = v\hat{\lambda} + l\dot{\psi}\hat{n}$$

It is known that the velocity of point D could only go along $\hat{\lambda}_D$ so $\vec{v}_D \cdot \vec{n}_D = 0$. That is

$$-v\sin\alpha + l\dot{\psi}\cos\alpha = 0$$

$$\dot{\psi} = \frac{v\tan\alpha}{l}$$

Now we can use v and α to replace $\dot{\psi}$. But we do not like α term either. How could we do to replace this α? We could use geometry relationship to replace α with lean angle and steer angle.

3.4 Geometrical relationships

For the geometrical relationships of angle α, δ and ϕ, I have done this part before and confirmed that their relationships should be
$$\tan\alpha\cos\phi = \tan\delta$$
However, because the 3-D pictures I drew are not so beautiful, I put them in the appendix. Here, I use the picture Professor Ruina sent to me on Oct 18.
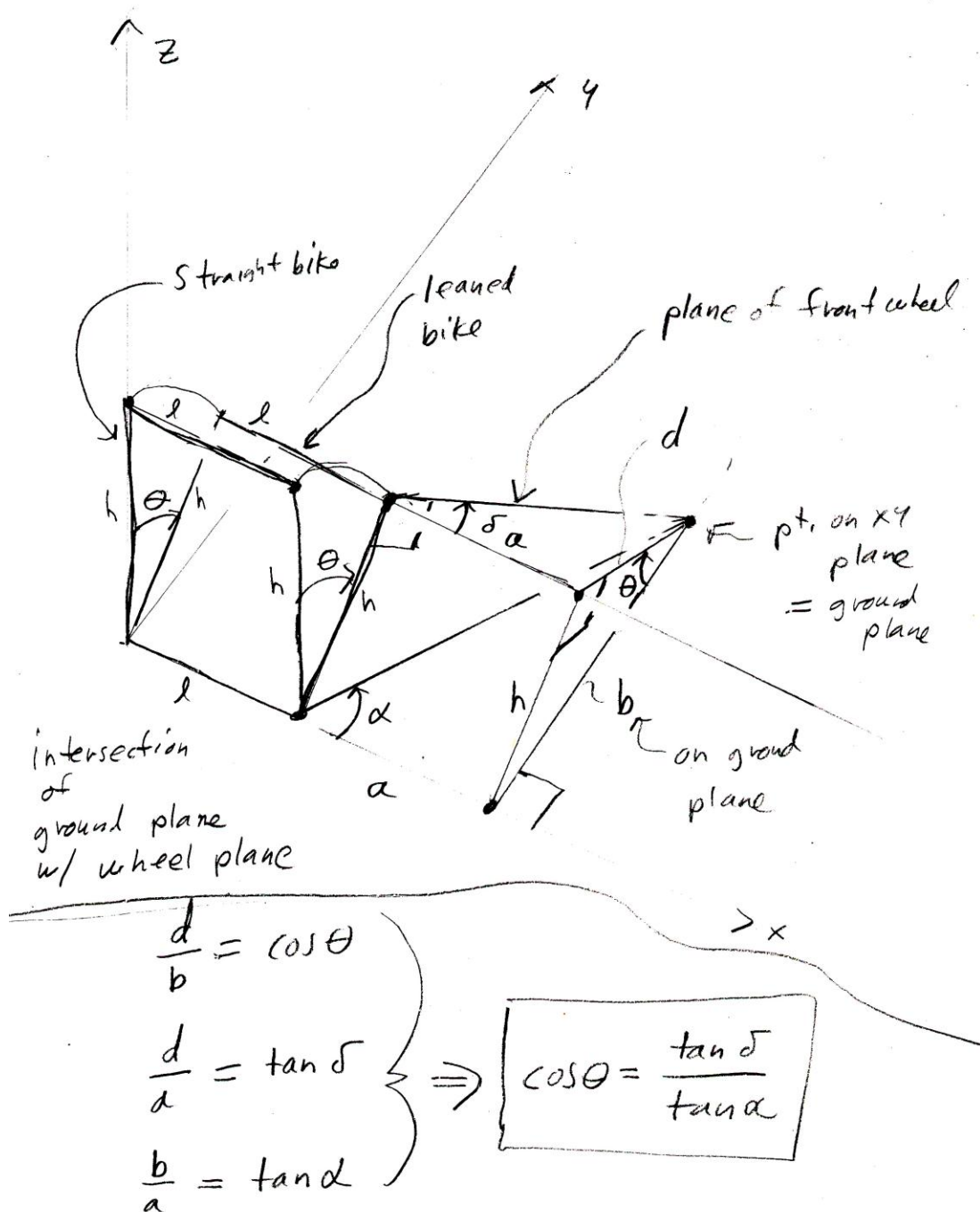
Figure 4 the 3D plot to find the relationships

We just replace $\tan\alpha$ with $\dfrac{\tan\delta}{\cos\phi}$. That is

$$\dot\psi = \frac{v\tan\alpha}{l} = \frac{v\tan\delta}{l\cos\phi}$$

We could also find $\ddot\psi$ by taking derivative of $\dot\psi$.

$$\ddot\psi = \frac{(\dot v\tan\delta + \dfrac{v\dot\delta}{\cos^2\delta})\cos\phi + v\tan\delta\sin(\phi)\dot\phi}{l\cos^2\phi}$$

By substituting $\dot\psi$ and $\ddot\psi$, the expression of acceleration of center of mass only has steer angle δ and lean angle ϕ. We could measure those two angles and could use steer angle δ to control the bicycle. That is what we want.

3.5 Angular Momentum Balance to point C

$$AMB_{/C}:\ \vec r_{G/C}\times mg(-\hat k) + \vec r_{D/C}\times\vec F_D = \vec r_{G/C}\times ma_G$$

Dot product with $\hat\lambda$:

$$\vec r_{G/C}\times mg(-\hat k)\cdot\hat\lambda = \vec r_{G/C}\times ma_G\cdot\hat\lambda$$

$$\frac{mh}{l^2}(lv^2\tan\delta - gl^2\sin\phi + \ddot\phi hl^2 + bl\dot v\tan\delta + ...$$

$$-hv^2\tan^2\delta\tan\phi + b\dot\phi lv\tan\delta\tan\phi) + \frac{b\dot\delta hmv}{l\cos^2\delta} = 0$$

That is

$$\frac{mhv^2\tan\delta}{l} - mgh\sin\phi + m\ddot\phi h + \frac{mb\dot v\tan\delta}{l} + ...$$

$$-\frac{mh^2}{l^2}v^2\tan^2\delta\tan\phi + \frac{mhb\dot\phi v\tan\delta\tan\phi}{l} + \frac{b\dot\delta hmv}{l\cos^2\delta} = 0$$

In this project, we could control the velocity v to be a constant value so $\dot v = 0$. Plug in

$$\frac{mh}{l^2}(lv^2 \tan\delta + \ddot{\phi}hl^2 - hv^2 \tan^2\delta \tan\phi + b\dot{\phi}lv \tan\delta \tan\phi) + \frac{b\dot{\delta}hmv}{l\cos^2\delta} - mgh\sin\phi = 0$$

This equation is very important in our control task because it contains the angle that we care most. However, in order to use this differential equation, we should linearize and simplify it.

3.6 Linearization of the differential equation

The linearization strategy I use is

$$\sin\theta \approx \theta, \cos\delta \approx 1, \tan\delta \approx \delta, \tan\phi \approx \phi$$

Substituting them in

$$mhv^2\delta - mghl\phi + mhl\ddot{\phi} - \frac{mh^2v^2\delta^2\phi}{l} + mh\dot{\phi}v\delta\phi + b\dot{\delta}hmv = 0$$

Simplify

$$v^2\delta - gl\phi + hl\ddot{\phi} - \frac{hv^2\delta^2\phi}{l} + \dot{\phi}v\delta\phi + b\dot{\delta}v = 0$$

The next thing is to find the equilibrium state: ϕ^*, δ^* where

$$(\phi^*)' = (\phi^*)'' = (\phi^*)''' = \cdots\cdots = (\phi^*)^{(n)} = 0$$
$$(\delta^*)' = (\delta^*)'' = (\delta^*)''' = \cdots\cdots = (\delta^*)^{(n)} = 0$$

So the equilibrium equation is

$$v^2\delta^* = (gl + \frac{hv^2\delta^2}{l})\phi^*$$

To make sure that this bicycle is not falling down, the lean angle should be zero so ϕ^* should be zero. According to this equilibrium equation, δ^* will also be zero.

Let $\delta = \delta^* + \partial\delta, \phi = \phi^* + \partial\phi$ where $\partial\delta, \partial\phi$ represent the small change of δ, ϕ. Leaving the high power or order product terms like $\partial\delta^2, \partial\delta \cdot \partial\phi, \partial\delta \cdot \partial\phi \cdot (\partial\phi)'$:

$$v^2\partial\delta - gl\partial\phi + hl(\partial\phi)'' + bv(\partial\delta)' = 0$$

Rewrite this equation

$$v^2\delta + bv\delta' = hl\phi'' - gl\phi$$

At this point, I have finished the derivation of the equation of motion of bicycle☺. But how could I make sure that the equation I get is right? What if I made mistakes in the derivation? So I compare this equation to the whole complex model, cross the terms that are not contained in this point mass model and linearize the equation. The result matches this point mass model.

4. Controller design for Self Stabilization

4.1 State space form

Now we have the differential equation. Then next task is to design a controller. For a controller, the first thing is to know about the state, input and output. Then I could write them in state space form and design the controller .After discussing with Marvin and Rany, I know that they could measure the lean angle ϕ, the lean angular velocity $\dot{\phi}$ and the steer angle δ. So these three angles should be our states. The differential equation is

$$\ddot{\phi} = \frac{g}{h}\phi - \frac{v^2}{hl}\delta - \frac{bv}{hl}\dot{\delta}$$

The state space form of the equation is

$$\begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{g}{h} & 0 & -\frac{v^2}{hl} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \delta \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{bv}{hl} \\ 1 \end{bmatrix} \dot{\delta}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \dot{\phi} \\ \delta \end{bmatrix}$$

Where   The state $x = \begin{bmatrix} \phi & \dot{\phi} & \delta \end{bmatrix}'$ input u(t) is $\dot{\delta}$
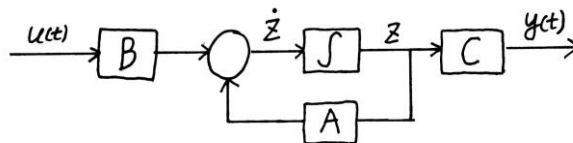
The system open loop block diagram is

Figure 5 the system open loop block diagram

Where $A = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{g}{h} & 0 & -\dfrac{v^2}{hl} \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ -\dfrac{bv}{hl} \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$

4.2 Controllability

Controllability is an important property of a control system, and the controllability property plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, or optimal control. Just like what we did in the single input-single output system, if we want to control this system realization totally, we should first test the controllability of this system.

For complete state controllable, the rank of the n*n matrix Mc should be full rank.

$$M_c = [B \vdots AB \vdots \cdots \cdots \vdots A^{n-1}B]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{g}{h} & 0 & -\dfrac{v^2}{hl} \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -\dfrac{bv}{hl} \\ 1 \end{bmatrix}$$

Where $g = 9.8m/s^2, h = 0.4m, l = 0.98m, b = 0.5m, v = 3.81m/s$

Using the MATLAB, rank (Mc) =3. So we could move any pole of this system to achieve our design requirements because this system is of complete control.

4.3 Design requirements

Current behavior of this system is as follows. If we give a unit step input, the output is
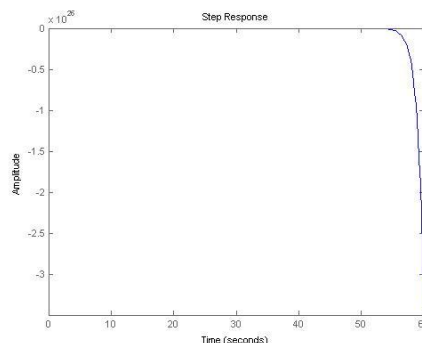

Figure 6 Current behavior of this system

It is obvious that when given a step input, this system will have a bad behavior. So we should change the system by changing the place the poles of the system. In order to guide the control design process, a set of desired specifications are listed for the overall system dynamics. Firstly,

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the bicycle is required to have a rise time smaller than 0.1 second to a step input, ensuring a fast response. The overshoot to a step response should be less than 5% in order to keep our bicycle from vacillating.

Requirements: 1. Rise time tr<0.1s 2.overshoot Mp <5%.

1. Settling time ts<0.1s

$$\frac{1.8}{\omega_n} < 0.1 \Rightarrow \omega_n > 18$$

2. Overshoot Mp <5%.

$$M_P = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.05 \Rightarrow \zeta > 0.69$$

So the desired region is as follows.
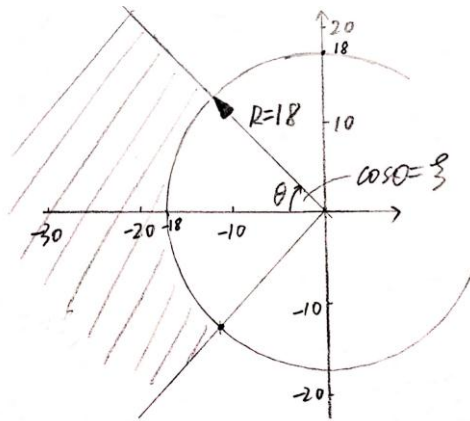
Figure 7 the desired design region

The shaded region is the desired place where we want the poles of the system to be.

4.4 Pole placement method

Our system can be expressed in this form

$$\dot{x} = Ax + Bu$$
$$y = Cz$$

We shall choose the control signal u to be

$$u = -Kx$$

Where K is a row vector $K = [k_1, k_2, k_3]$. Plug in u into $\dot{x}$

$$\dot{x} = (A - BK)x$$

The solution of this equation is given by

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$$x(t) = e^{(A-BK)}x(0)$$

Based on the knowledge of linear algebra, the eigenvalues of matrix A-BK are the poles of our system. By choosing the matrix K, we could change the poles of the system. If we want our system to have a good behavior, we only need to place the poles (eigenvalues) of A-BK in the desired region. This gives us a closed loop system. The system block diagram is as follows.
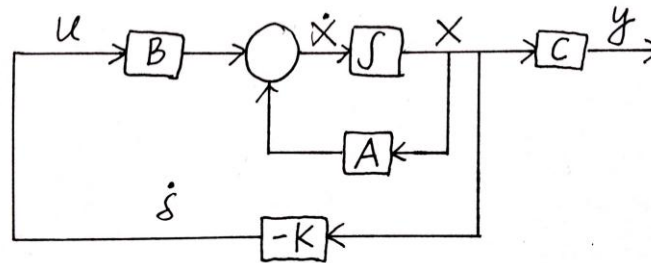

Figure 8 the closed loop system block diagram

Because the number of components in the state is three, we have three poles in our system. We want them to be in the shaded region of Figure 6. To balance the acceptable response and the amount of control energy required, I choose three poles near the right boundary of the desired region. If writing in a row vectors,

$$P = [-20 \quad -25 \quad -30]$$

The formal by hand is to plug in these poles in the A-BK matrix and solve for k1,k2 and k3.

Figure 9 Solve for K by hand

Then we calculate the determinant of matrix $A - BK - \lambda I_{3\times 3}$. The three roots of that polynomials are the components of $P = [-20 \quad -25 \quad -30]$.

The other method is to use MATLAB command place(A,B,P). Then we could get the K vector directly.

$$K=place(A,B,[-20 \ -25 \ -30])$$

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$$K=[\ -843.1298\ -119.7419\ \ 8.2379]$$

Now I have finished the design of controller for bicycle's stabilization.

4.5 Simulation

For the simulation part, I set the initial conditions to be

Initial roll angle=-pi/4; Initial steer angle=pi/18; Initial yaw angle=pi/6;
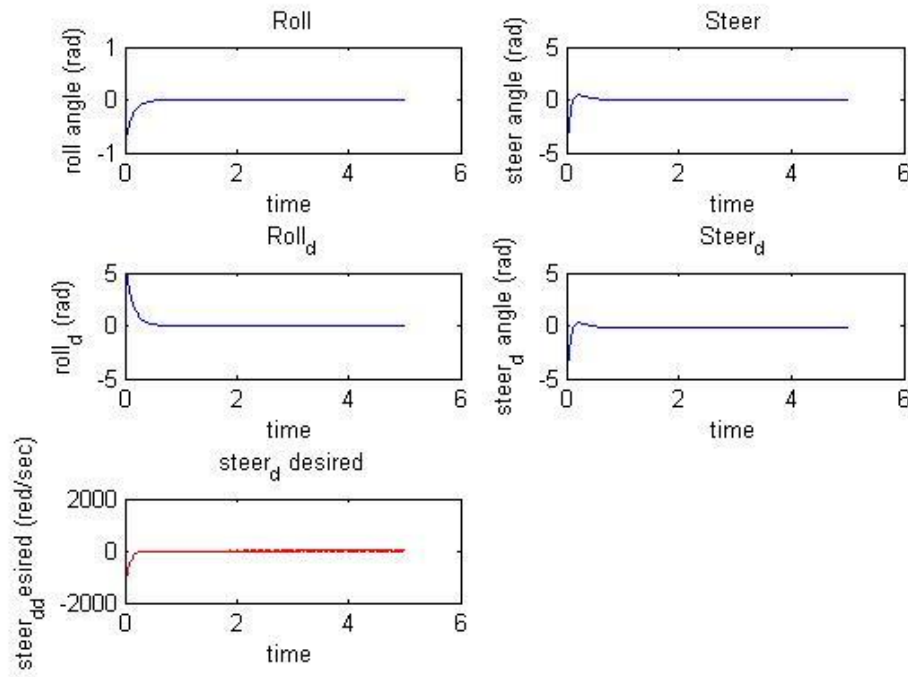
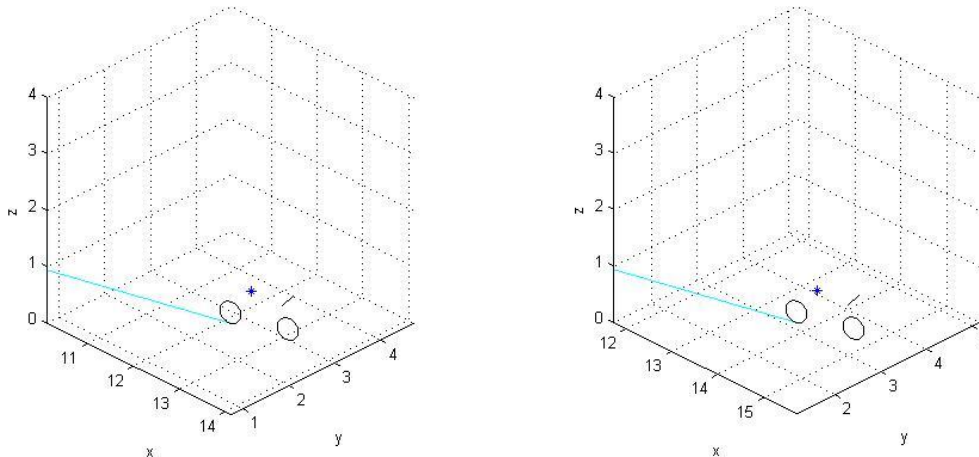Initial position x0=0, y0=0;

Figure 10 ODE Simulation of Controller

Figure 11 Animation of simulation

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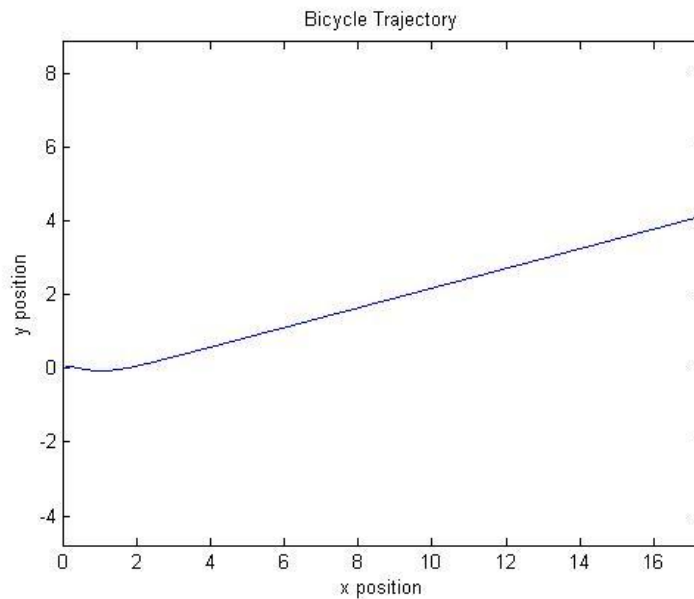
Figure 12 the trajectory of the motion of bicycle

From the simulation, we could see that this controller works well and could satisfy our design requirements. And this is also the controller that our team is using now.

5. Observer Based Controller for Reference Tracking

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables. A state observer estimates the state variables based on the measurements of the output and control variables.

5.1 Observability

In order to design the observer, we should first check about the observability of this system realization. As discussed in the controllability chapter, the completer state controllability condition for this system is that the rank of the n*n matrix Mo should be full rank.

$$M_o = \begin{bmatrix} C \\ CA \\ \cdots\cdots \\ CA^{n-2} \\ CA^{n-1} \end{bmatrix}$$

Where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{g}{h} & 0 & -\dfrac{v^2}{hl} \\ 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$$

So

$$M_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1.04 & 0 & -1.58 \end{bmatrix}$$

From the matrix M_o, we could know that its rank is 3 so it is of full observability.

5.2 State Observer

The observer is a subsystem to reconstruct the state vector of the plant. The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices A and B and the lack of the initial error. We define the observer to be

$$\dot{\tilde{x}} = A\tilde{x} + Bu + H(y - C\tilde{x})$$

And

$$u = -K\tilde{x}$$

So the observer could be expressed as

$$\dot{\tilde{x}} = (A - BK)\tilde{x} + H(Cx - C\tilde{x})$$

To make sure that our observer would truly estimate the real state, let us subtract the above term by the real state and discuss about the error.

$$\dot{x} - \dot{\tilde{x}} = (A - HC)(x - \tilde{x})$$

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Define the difference between $\dot{x}$ and $\dot{\tilde{x}}$ as the error vector

$$\dot{e} = \dot{x} - \dot{\tilde{x}}$$
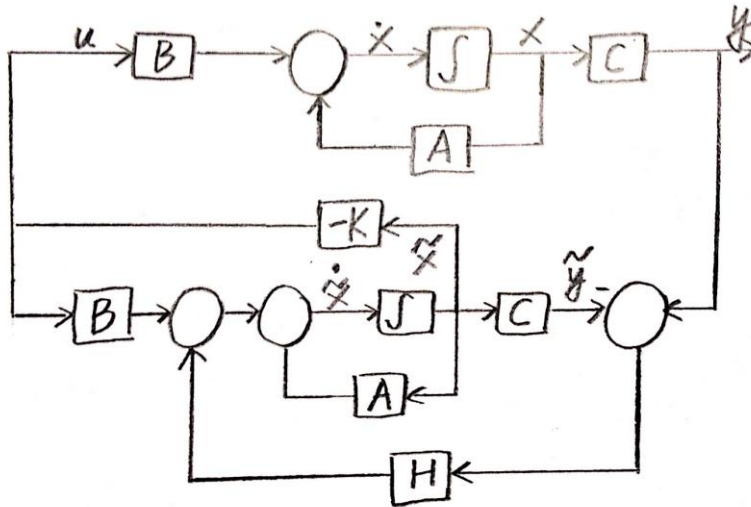
Then we could get

$$\dot{e} = (A - HC)e$$

Figure 13 the observer on a closed-loop block diagram

The dynamic behavior of the error vector is determined by the A-HC matrix. What we want is that this error approaches zero quickly. The eigenvalues of this A-HC matrix determine the behavior of our observer. Because this realization is completely observable, then we could choose a proper H to make the observer have better behavior.

5.3 Pole place method to design the state observer

The method used to design the observer is the same as we design the controller. We would like the observer dynamics to be sufficiently faster than the dynamics of the overall system. Thus we would put the roots of the observer matrix five times or ten times faster than the poles we design in our controller.

The fastest pole in our controller is p3 = -30. Here the poles of the observer should satisfy

$$[P] > 5 * |{-30}| = 150$$

So I choose the poles of the observer to be

$$P = [-155 \quad -156 \quad -157]$$

By using the MATLAB's place () command, we could find the arrow H vector is

$$H = place(A',C',[-155\ -156\ -157])$$

$$H=[78\ 2028\ -11137]'$$

Here, we finished the design of the observer based controller for reference tracking.

5.4 Simulation

To simulate the effect of this controller, I use the classic control method in MATLAB. Based on the state space model, we could treat them as two parts. One is Plant and the other is Controller.
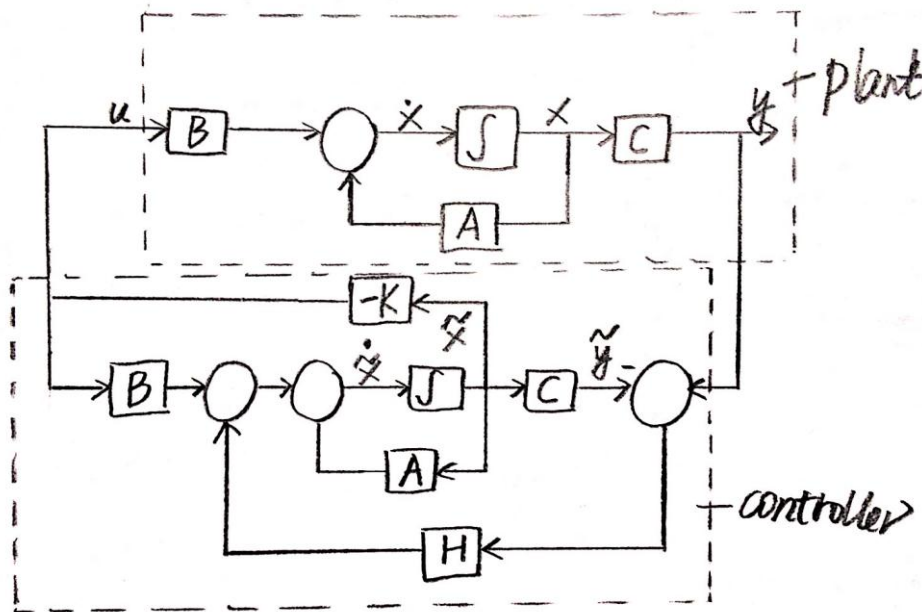

Figure 14 Plant and Controller of the state space form

Then I could rewrite it in the block diagram form.
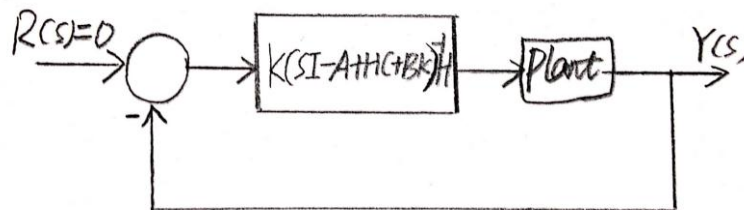
Figure 15 Block diagram of State space model

By using MATLAB, the transfer function of Plant is

$$P(s) = \frac{-0.2068s - 1.576}{s^3 - 2.22 \cdot 10^{-16} s^2 - 1.043s}$$

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Also, the transfer function of the Controller is

$$C(s) = \frac{-2.056 \cdot 10^5 s^2 - 1.153 \cdot 10^6 s - 2.16 \cdot 10^6}{s^3 + 99s^2 + 3715s + 23160}$$

So the transfer function of the closed-loop system is

$$L(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}$$

$$L(s) = \frac{42530s^3 + 56260s^2 + 2.264e^6 s + 3.403e^6}{s^6 + 99s^5 + 3714s^4 + 65590s^3 + 5.587e^5 s^2 + 2.24e^6 s + 3.403e^6}, e^n means 10^n$$

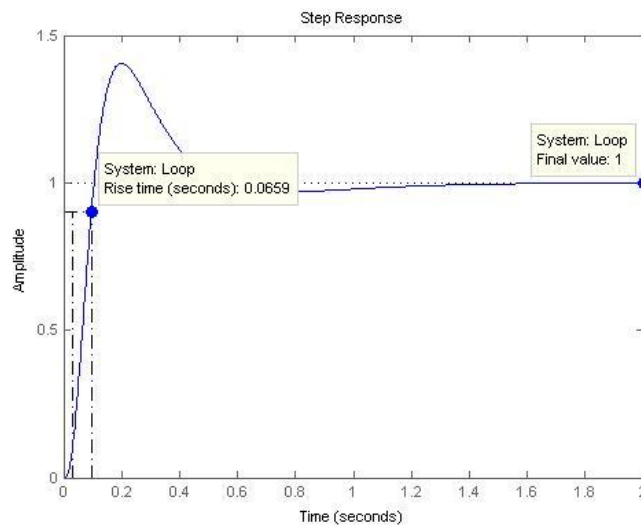Given a step input the response of the closed-loop system is

Figure 16 the step response of the closed-loop system

From this step response, the rise time is 0.0629s <0.1s and the steady state value is 1 which means that this controller has a good behavior. We will use this controller next semester.

Up to here, I have finished all my subtasks related to dynamics and control this semester. ☺

Conclusion

This report has detailed the approach taken towards designing controllers for bicycle. The derivation of the equation of motion of the bicycle is introduced first. This derivation is based the point mass model. Then, in order to use that differential equation, the methods to linearize and change it into state space form were shown. The state space form was used to create two different kinds of controllers: one for stability and another for stability and steering reference tracking. The controllers were then simulated and shown to meet the required specifications.

Appendix A:
My derivation of the relationship of the angle alpha and steer angle

The derivation procedure is as follows:

1. We can abstract our bicycle model to the geometry model (Figure 17).
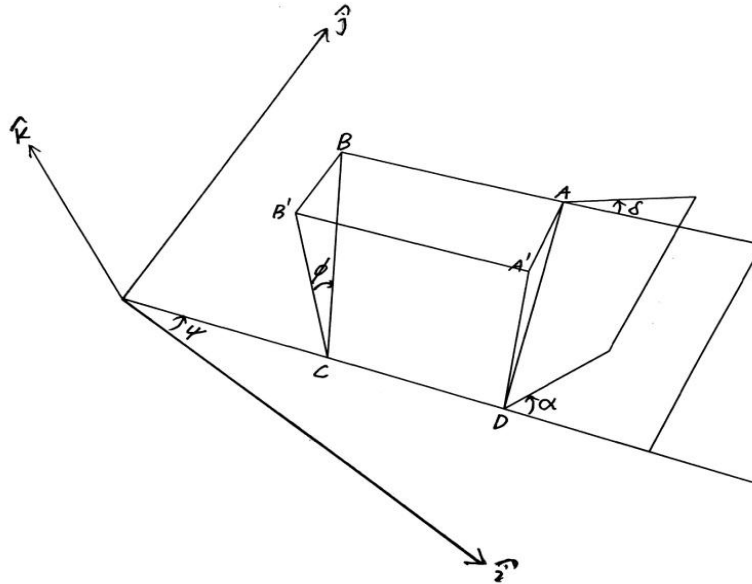
<div align="center">Figure 17</div>

(1) The yaw angle (CD direction) is φ. The lean angle $\angle BCB'$ or $\angle ADA'$ is ϕ.

(2) The steer angle is δ. $\angle BB'C = \dfrac{\pi}{2}$, $\angle AA'D = \dfrac{\pi}{2}$.

The three direction vectors are $\hat{i}, \hat{j}, \hat{k}$.

2. Find point H along the line BA and let the length of AH be 1.

3. In the plane where the steer angle δ is in, make $\angle AHE = \dfrac{\pi}{2}$ and cross one side of the angle δ with point E.

4. Crossing point H , make a line parallel to line AD and cross the line CD with point G.(Let AHGD be parallelogram AH = DG = 1) and make EF parallel to AD and cross the XoY plane with point F.
5. Connect the FG.

Then, we can get this graph. See Figure 18
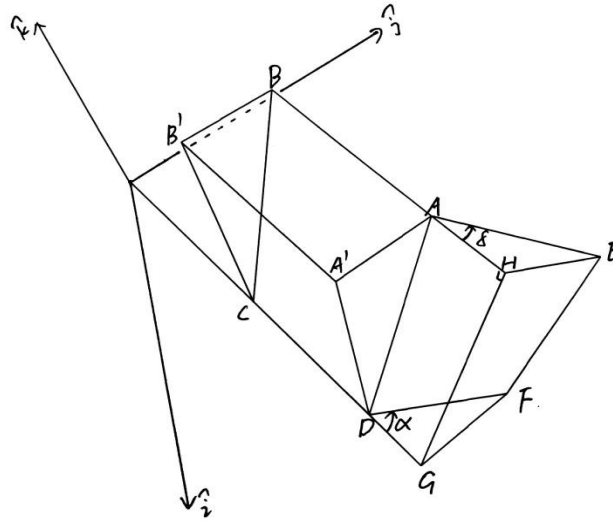
In $\triangle AHE$, $\angle AHE = \dfrac{\pi}{2}$, $\angle HAE = \delta$, $AH = 1$ so $HE = \tan\delta$.

$\because AD \perp AB \therefore AD \perp AH$ and also $AD \perp AE$ so $AD \perp \Box AHE$

$\because AD \perp \Box AHE \therefore AD \perp HE$, $\because HG \parallel AD \therefore HG \perp \Box AHE \therefore \angle EHG = \dfrac{\pi}{2}$ (EH $\perp$ HG)

$\because AH \perp HE, AH \perp HG \therefore AH \perp \Box HEFG \therefore AH \perp GF$

$\because DG \parallel AH \therefore DG \perp \Box HEFG \therefore \angle DGF = \dfrac{\pi}{2}$ (DG $\perp$ GF)

In $\triangle DGF$, $\angle DGF = \dfrac{\pi}{2}$, $\angle FDG = \alpha$, $DG = AH = 1$ so $GF = \tan\alpha$

Crossing point F, make line $FH' \parallel EH$, crossing line HG with point H' and crossing $\Box A'B'CD$ with point I. See Figure 19.

$\because EH \perp HG \therefore FH' \perp H'G$

$\because \angle IGH' = \phi \therefore \angle H'FG = \phi$

In $\triangle CFH'$, $H'F = HE = \tan\delta$, $CF = \tan\alpha$, $H'F = GF \cdot \cos\angle H'FG = GF \cdot \cos\phi$

So $\tan\alpha\cos\phi = \tan\delta$.

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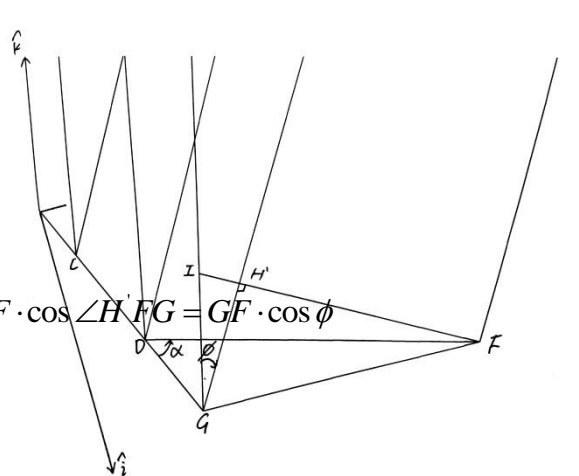
Appendix B: Scanning copies of my hand script

$\vec{v}_{BA} = \vec{v}_{B} + \vec{v}_{A}$

$\vec{n}_{B} = \cos\delta \cdot \hat{i} - \sin\delta \cdot \hat{j}$

$\vec{v}_{B} = v_{B}^2 + L\dot{S}_{B}^2$

$= v_{B}^2 + \dot{y}\hat{j} \cdot \hat{n}$

$\vec{v}_{B}^2 : \vec{v}_{B}^2 = 0$

Áp dụng:

$\vec{v}_{BA} \times m g(-\hat{j}) + \vec{v}_{BA} \times \vec{F}_{ND} = \vec{v}_{A} \times m\vec{a}$

$\vec{v}_{BA} \times m g(-\hat{j}) \times \vec{n} = \vec{v}_{BA} \times m\vec{a}_c \cdot \hat{S}$

$\frac{mh}{L}(Lv_c^2 \tan\delta - gL^2 \sin\phi + \phi \dot{h}L^2 + bL\dot{v}_c \tan\delta - hv_c^2 \sin^2\phi)$

$b\dot{\phi}(Lv_c \tan\delta \sin\phi) + \frac{b\dot{\phi}hmv_c}{L \cos^2\delta} = 0$

$\boxed{\hat{v}_c \cos\phi \cdot \tan\delta}$

$\dfrac{mhv_c^2 \tan\delta}{L} - mgh \sin\phi + mh\phi^2 + \dfrac{mhb\dot{\phi}}{L}v_c \tan\delta \tan\phi -$

$-v_c \sin\alpha + L\dot{\psi}\cos\alpha = 0$

$\dfrac{mh}{L^2}v_c^2 \tan^2\delta \tan\phi + \dfrac{mhb\dot{\phi}}{L}v_c \tan\delta \tan\phi +$

$\dot{\psi} = \dfrac{v_c \tan\delta}{L}$

$\dfrac{v_c \tan\delta}{\cos\phi}$

$\dfrac{b\dot{\phi}hmv_c}{\cos^2 L}$

$\dot{\psi} = \dfrac{\hat{v}_c \tan\delta}{L \cos^2\phi}$

$\boxed{\hat{\psi} = \dfrac{(\dot{v}_c \tan\delta + \frac{v_c\dot{\delta}}{\cos^2\delta})\cos\phi + v_c\cos\delta \tan\delta \sin\phi\dot\phi}{L\cos^2\phi}}$

$-mgh\sin\phi + \dfrac{b\dot{\phi}hmv_c}{\cos^2\delta} + \ldots$

$\boxed{\hat{v}_c = 0}$

$\dfrac{mh}{L}(Lv_c^2 \tan\delta + \dot{\phi}hL^2 - hv_c^2 \tan^2\delta \tan\phi +$

$b\dot{\phi}L v_c \tan\delta \tan\phi) = 0$

Linearization:

for small θ, $\cos\theta \approx 1$, $\tan\theta = \theta$

$$\phi = \rho\phi, \quad \tan\theta = \theta$$

$$\frac{q b}{l}(\phi l v^2 S - g l^2 \phi + \phi \ddot{h} l^2 - h v^2 c^2 \phi + b l \phi \dot{v} c \phi) + \frac{b \dot{h} m v_c}{l} = 0$$

$$m h v_c^2 S - m g h l \phi + m h l^2 \dot\phi - \frac{m h^2 v_c^2 S^2 \phi}{l} + m h \phi v_c S \phi + b \dot{h} m v_c = 0$$

$$v_c^2 S - g l \phi + h l \dot\phi - \frac{h v_c^2 S^2 \phi}{l} + \phi v_c S \phi = 0$$

Equilibrium: $v_c^2 S - g l \phi - \frac{h v_c^2 S^2 \phi}{l} = 0$

$$\phi = \phi^*, \quad \phi = S = \dot\phi = \ddot\phi$$

$$v_c^2 S \phi - g l \phi + h l \dot\phi + \phi v_c S \phi = 0$$

$$\Downarrow$$

$$v_c^2 S - g l \phi + b \dot{h} v_c = 0$$

$$v_c^2 S + b v_c \dot{h} = g l \phi - h l \dot\phi$$

$$v_c^2 S + b v_c \dot{h} - g l \phi + h l \dot\phi = 0$$

$$v_c^2 S \phi - g l \phi + \frac{h v_c^2 S^2 \phi}{l} = 0 \quad \text{let} \quad S = \dot\phi = 0$$