1.5 Solving vector equations

Often you know less than you like. So you try to figure out more. One way to do this is with reasoning. In mechanics you reason with the laws of mechanics (including geometry and kinematics). You find some things you want to know from other things that you already do know. Because many of the laws of mechanics (and geometry) are vector equations,

> Engineering analysis often involves solving vector equations.

How you calculate with vectors is the same whether the problems are in geometry, kinematics, statics, dynamics or a combination of these. In this section we will show a few methods for solving some of the more common vector equations. In a sense there are no new concepts here; it's just a matter of guiding the rules of vector algebra that you already know.

Vector algebra

We want to manipulate equations that involve vectors (like $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{0}$) and scalars (like a, b, c, and 0). Without knowing anything about mechanics, or even geometry, you can learn to do correct vector algebra by just following the manipulation rules in box 1.8. These are elaborations of elementary scalar algebra to accommodate vectors and the three new kinds of multiplication: scalar times vector, dot product (or scalar product), and cross product (or vector product). Just looking out for the rules and exceptions is enough to make your manipulations correct. That's enough to keep the car on the road. And if you just follow the traffic rules you might by chance get to your goal. But to get to your goal efficiently you need to steer correctly. For vector algebra this means using the simplification rules to your advantage.

Example. Say you know $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ and you know that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

but you don't know a, b, and c. How could you find a? First dot both sides with $\vec{B} \times \vec{C}$ and then blindly follow the rules:

$$\begin{aligned}
\left\{a\vec{A} + b\vec{B} + c\vec{C}\right. &= \left.\vec{D}\right\} \cdot \left(\vec{B} \times \vec{C}\right) \\
a\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) + b\,\underbrace{\vec{B} \cdot \left(\vec{B} \times \vec{C}\right)}_{0} + c\,\underbrace{\vec{C} \cdot \left(\vec{B} \times \vec{C}\right)}_{0} &= \vec{D} \cdot \left(\vec{B} \times \vec{C}\right) \\
\Rightarrow \qquad a = \frac{\vec{D} \cdot \left(\vec{B} \times \vec{C}\right)}{\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)}. & \qquad (1.30)
\end{aligned}$$

The two zeros followed from the general rules that $\vec{D} \cdot (\vec{V} \times \vec{W}) = (\vec{D} \times \vec{V}) \cdot \vec{W}$ and $\vec{D} \times \vec{D} = \vec{0}$. ①.

① The linear-algebra savvy reader may recognize the manipulation leading to eqn. (1.30) as a derivation of Cramer's rule for a 3×3 matrix whose columns are the components of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, respectively.
Note, if the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar then the last line of the calculation would have $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = 0$ in the denominator. Bad. If $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar the original problem is either nonsense (if $\vec{D}$ is off that plane) or has non-unique solutions (if $\vec{D}$ is on that plane). So the failure of the derivation in that case is sensible. See box 1.10 on page 17 for more discussion of when equations do and do not have solutions.)

The point of the example above was to show the vector algebra rules at work. However, to get to the end took the first 'move' of dotting the equation with the appropriate vector. That move could be motivated this way. We are trying to find a and not b or c. We can get rid of the terms in the equation that contain b and c if we can dot $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ with a vector perpendicular to both of them. $\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}$ is perpendicular to both $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ so can be used to kill them off with a dot product. The 0s (the terms that dropped

1.8 The rules of vector algebra.

You can do good vector calculations by just following the rules in this box. On the other hand, even if you know vector geometry and mechanics you are stuck with these rules.

Vector algebra

In the expressions below a, b and c are any scalars. And $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ are any vectors.

You can add vectors

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$$

And you can multiply them three ways

1. **Scalar multiplication:** $a\vec{\boldsymbol{A}}$.
2. **Dot product, inner product or scalar product:** $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}$.
3. **Cross product or vector product:** $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$.

And you can combine expressions using usual rules you are used to in scalar arithmetic (with some exceptions) and the extra vector simplification rules.

The usual rules.

Vector addition and all three kinds of multiplication (scalar multiplication, dot product, cross product) all follow many of the usual commutative, associative, and distributive laws (usual, meaning that which you know from regular scalar algebra).

Associative rules. These have to do with how you group terms.

$$(\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}) + \vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + (\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}})$$
$$a(b\vec{\boldsymbol{C}}) = (ab)\vec{\boldsymbol{C}}$$
$$(a\vec{\boldsymbol{B}}) \times \vec{\boldsymbol{C}} = a(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}})$$

Commutative rules. These have to do with the order of terms in a calculation.

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}$$
$$ab\vec{\boldsymbol{C}} = ba\vec{\boldsymbol{C}} = \vec{\boldsymbol{C}}ab = \vec{\boldsymbol{C}}ba$$
$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}}$$

Distributive rules. These have to do with multiplying individual terms that are added, rather than their sum.

$$a(\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}) = a\vec{\boldsymbol{B}} + a\vec{\boldsymbol{C}}$$
$$\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}) = \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}}$$
$$\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}) = \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}} \times \vec{\boldsymbol{C}}$$

Extra simplification rules

As you proceed with using the rules above you can simplify using the following extra vector simplification rules.

- $a\vec{\boldsymbol{A}}$ is a vector,
- $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}$ is a scalar,
- $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$ is a vector,
- $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$ (so $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{A}} = \vec{\boldsymbol{0}}$),
- $\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{C}}$; in particular, setting $\vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}$ gives $\vec{\boldsymbol{B}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = 0$ (since $\vec{\boldsymbol{B}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{0}}$),
- $\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}})\vec{\boldsymbol{B}} - (\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}})\vec{\boldsymbol{C}}$, 'BAC -CAB' (see box 17.4 on page 865),

Exceptions

In the list of usual rules, above, we did not include all of the rules from scalar algebra, they don't all work. So look out for these exceptions.

- $a + \vec{\boldsymbol{A}}$ is nonsense,
- $a/\vec{\boldsymbol{A}}$ is nonsense,
- $\vec{\boldsymbol{A}}/\vec{\boldsymbol{B}}$ is nonsense (a common beginner's error),
- $a \cdot \vec{\boldsymbol{A}}$ is nonsense (unless you mean by it $a\vec{\boldsymbol{A}}$),
- $a \times \vec{\boldsymbol{A}}$ is nonsense,
- $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \neq \vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$,
- $\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) \neq (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) \times \vec{\boldsymbol{C}}$.

Substitutions

In all of the expressions in this box, the scalars and vectors can be the result of other calculations. For example, all of the expressions above would be equally valid if every place you see a you substituted $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}$ and everyplace you see $\vec{\boldsymbol{A}}$ you substituted $\vec{\boldsymbol{B}} + \vec{\boldsymbol{C}}$ or $\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}$.

out) in the example calculation were thus expected for geometric reasons.

A simpler two-dimensional example using a judiciously chosen dot product, in the same spirit as the example above, is on page 8.

Count equations and unknowns.

One cannot (usually) find more unknowns than one has scalar equations

There are famous counter-examples where you can solve for more variables than you have equations.

Example: Finding more unknowns than you have equations
The simplest example is

$$x^2 + y^2 = 0.$$

This is one equation which can be solved for both x and y to get $x = 0$ and $y = 0$.

Although such examples seem to be mathematical trickery, they do show up sometimes, but they are always nonlinear. The simultaneous equations in mechanics are most often linear equations. In short, the previous example is not generic. That is, the previous exceptional example withstanding, before you do lots of algebra, you should check that you have as many equations as unknowns. If not, you probably can't find all the unknowns.

How do you count vector equations and vector unknowns? A two-dimensional vector is fully described by two numbers. For example, a $2D$ vector is described by its x and y components or its magnitude and the angle it makes with the positive x axis. A three-dimensional vector is described by three numbers. So a vector equation counts as 2 or 3 equations in 2 or 3 dimensional problems, respectively. And an unknown vector counts as 2 or 3 unknowns in 2 or 3 dimensions, respectively. If the direction of a vector is known but its magnitude is not, then the magnitude is the only unknown. Magnitude is a scalar, so it counts as one unknown.

Example: Counting equations
Say you are doing a 2-D problem where you already know the vector $\hat{\boldsymbol{\lambda}} = \sqrt{2}\hat{\boldsymbol{\imath}} + \sqrt{2}\hat{\boldsymbol{\jmath}}$ and you are given the vector equation

$$C\hat{\boldsymbol{\lambda}} = \vec{\boldsymbol{a}}.$$

You then have two equations (a vector equation in 2-D) and three unknowns (the scalar C and the vector $\vec{\boldsymbol{a}}$). There are more unknowns than equations so this vector equation is not sufficient for finding C and $\vec{\boldsymbol{a}}$.

Most often:

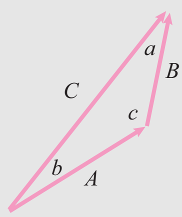
- When you have as many equations as unknowns the equations have a unique solution;
- When you have more equations than unknowns there is no solution to the equations; and

- When you have more unknowns than equations, you have a whole family of solutions.

However, these are only guidelines, although they usually work. In practice, no matter how many equations and unknowns you have, you could have no solutions, many solutions or a unique solution. The geometric significance of some cases that satisfy and that don't satisfy these guidelines is given in box 1.10 on page 17.

1.9 Vector triangles and the laws of sines and cosines

The tip to tail rule of vector addition defines a triangle. Knowing something about the vectors in this triangle how can we find more? One approach is to use the laws of sines and cosines.

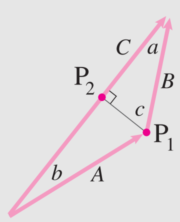


Consider the vector sum $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{C}}$ represented by the triangle shown with traditionally labeled sides A, B, and C and internal angles a, b, and c. The sides and angles are related by

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C} \qquad \text{the law of sines, and} \tag{1.31}$$

$$C^2 = A^2 + B^2 - 2AB\cos c \qquad \text{the law of cosines.} \tag{1.32}$$

Proof of the law of sines The first equality in the law of sines can be proved by calculating the altitude from c two ways.

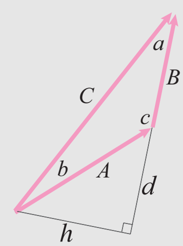


On the one hand length P_1P_2 is given by $P_1P_2 = B\sin a$ and on the other hand by $P_1P_2 = A\sin b$

so $B\sin a = A\sin b \quad \Rightarrow \quad \dfrac{\sin a}{A} = \dfrac{\sin b}{B}$.

We can do likewise with all three altitudes thus proving the triple equality.

Proof of the law of cosines. Look at altitude h of the triangle.



This is the base of two different right triangles. So by the pythagorean theorem we have on the one hand that

$h^2 = A^2 - d^2$ and on the other that $h^2 = C^2 - (B+d)^2$.

Equating these expressions and expanding the square we get

$$\begin{aligned} A^2 - d^2 &= C^2 - (B^2 + d^2 - 2dB) \\ \Rightarrow \quad A^2 + B^2 + 2dB &= C^2 \end{aligned} \tag{1.33}$$

But $d = -A\cos c$ so

$$C^2 = A^2 + B^2 - 2AB\cos c.$$

Sometimes the angle we call c is called θ.

Applications. These laws are useful when you want to figure out the shape and size of a triangle and when, of the six triangle quantities (three sides and three angles), only 3 are given. At least one of these three has to be a length.

As noted, it is possible to give problems of this type that have no solutions. And it is possible to give problems that have either 1 or 2 solutions.

In this era where vector algebra is popular, as is the representation of vectors in terms of their components, the laws of sines and cosines are used little. But sometimes they are the easiest approach.

Vector triangles

In 2D one often wants to know all three vectors in a triangle where the triangle comes from the diagram for expressions like

$$\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}=\vec{\boldsymbol{C}} \quad \text{or} \quad \vec{\boldsymbol{A}}-\vec{\boldsymbol{C}}=\vec{\boldsymbol{B}} \quad \text{or} \quad \vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}=\vec{\boldsymbol{0}} \quad \textit{etc.}$$

Usually at least one vector is given and some information is given about the others. The situation is much like the geometry problem of drawing a triangle given various bits of information about the lengths of its sides and its interior angles. If enough information is given to prove triangle congruence, then enough information is given to determine all angles and sides. A difference between vector triangles and proofs of triangle congruence is that triangle congruence does not depend on the overall orientation, whereas vector triangles need to have the correct orientation. Nonetheless, the tools used to solve triangles are useful for solving vector equations.

Vector addition

We start with a problem that is in some sense solved at the start. Say $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are known and you want to find $\vec{\boldsymbol{C}}$ given that

$$\vec{\boldsymbol{C}}=\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}.$$

The obvious and correct answer is that you find $\vec{\boldsymbol{C}}$ by vector addition. You could do this addition graphically by drawing a scale picture, or by adding corresponding vector components. Suppose now that $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are given to you in terms of magnitude and direction and that you are interested in the direction of $\vec{\boldsymbol{C}}$.

Example: adding vectors defined by magnitude and direction
Say direction is indicated by angle measured counterclockwise form the positive x axis and that $A=5\sqrt{2}$, $\theta_A=\pi/4$, $B=4$, and $\theta_B=2\pi/3$. So

$$\begin{aligned}
\vec{\boldsymbol{A}} &= A\left(\cos\theta_A\hat{\boldsymbol{\imath}}+\sin\theta_A\hat{\boldsymbol{\jmath}}\right)\\
&= 5\sqrt{2}\,(\cos(\pi/4)\hat{\boldsymbol{\imath}}+\sin(\pi/4)\hat{\boldsymbol{\jmath}}) = 5\hat{\boldsymbol{\imath}}+5\hat{\boldsymbol{\jmath}}\\
\vec{\boldsymbol{B}} &= B\left(\cos\theta_B\hat{\boldsymbol{\imath}}+\sin\theta_B\hat{\boldsymbol{\jmath}}\right)\\
&= 4\,(\cos(2\pi/3)\hat{\boldsymbol{\imath}}+\sin(2\pi/3)\hat{\boldsymbol{\jmath}}) = -2\hat{\boldsymbol{\imath}}+2\sqrt{3}\hat{\boldsymbol{\jmath}}\\
\vec{\boldsymbol{C}} &= \vec{\boldsymbol{A}}+\vec{\boldsymbol{B}} = (5\hat{\boldsymbol{\imath}}+5\hat{\boldsymbol{\jmath}})+\left(-2\hat{\boldsymbol{\imath}}+2\sqrt{3}\hat{\boldsymbol{\jmath}}\right)\\
&= 3\hat{\boldsymbol{\imath}}+(5+2\sqrt{3})\hat{\boldsymbol{\jmath}}\\
\Rightarrow \quad \theta_C &= \tan^{-1}\left(C_y/C_x\right) = \tan^{-1}\left(\left(5+2\sqrt{3}\right)/3\right) \approx 1.23 \approx 70.5^\circ\\
\text{and} \quad C &= \sqrt{3^2+(5+2\sqrt{3})^2} \approx 8.98
\end{aligned}$$

To find θ_C we used the arctan (or $\tan^{-1}$) function which can be off by π②.To find the angle of $\vec{\boldsymbol{C}}$ we had to convert $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ to coordinate form, add components, and then convert back to find the angle of $\vec{\boldsymbol{C}}$. That is, even though the desired answer is given by a sum, carrying out the sum takes a bit of effort. An alternative approach avoids some work.

②The problem is that, measuring angles between 0 and 2π (or equivalently between $-\pi$ and π) there are always two different angles that have the same tangent. The inverse tangent function picks one. Some computers or calculators always pick an angle between 0 and π, and some always pick a value between $-\pi/2$ and $\pi/2$. Both of these could be the wrong answer. So you need to check and possibly add π to your answer. Alternatively, we recommend that you use one of these two commands: 1) the two-argument inverse tangent ($\arctan(y,x)$) or 2) rectangular-to-polar coordinate conversion, using the angle as the desired arctangent.

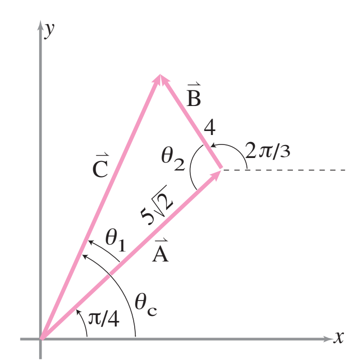


Figure 1.66: Using trig to solve vector triangles

Example: Same as above, different method
Start with picture of the situation, fig. 1.66. By adding angles,

$$\theta_2 = \pi/4 + \pi/3 = 7\pi/12.$$

From the law of cosines (see box 1.9 on page 4),

$$\begin{aligned} C^2 &= A^2 + B^2 - 2AB\cos\theta_2 \\ \Rightarrow \quad C &= \sqrt{(5\sqrt{2})^2 + 4^2 - 2(5\sqrt{2})\cdot 4\cdot \cos(7\pi/12)} \\ &\approx 8.98 \quad \text{(as before)} \end{aligned}$$

And from the law of sines (see box 1.9),

$$\begin{aligned} \frac{\sin\theta_1}{B} &= \frac{\sin\theta_2}{C} \\ \Rightarrow \quad \theta_1 &= \sin^{-1}\left(\frac{B\sin\theta_2}{C}\right) \approx \sin^{-1}\left(\frac{4\sin(7\pi/12)}{8.98}\right) \\ &\approx .445 \\ \Rightarrow \quad \theta_C &= \theta_A + \theta_2 \approx \pi/4 + .445 \approx 1.23 \quad \text{(as before).} \end{aligned}$$

This second approach is somewhat more direct in some situations.

The determination of a third vector by vector addition is analogous to the determination of a triangle in geometry by “side-angle-side”.

Vector subtraction

Say you want to find $\vec{\boldsymbol{C}}$ given $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ and that $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ add to zero. So, subtracting $\vec{\boldsymbol{C}}$ from both sides and multiplying through by -1 we get

$$\begin{aligned} \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \vec{\boldsymbol{C}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad \vec{\boldsymbol{C}} &= -\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}. \end{aligned}$$

The problem has now been reduced to one of addition which can be done by drawing, components, or trig as shown above.

Find the magnitude of two vectors given their directions and their sum (2D)

Often one knows that 2 vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ add to a given third vector $\vec{\boldsymbol{C}}$. The directions of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are known but not their magnitudes. That is, given $\hat{\boldsymbol{\lambda}}_A, \hat{\boldsymbol{\lambda}}_B$ and $\vec{\boldsymbol{C}}$ and that

$$\begin{aligned} \vec{\boldsymbol{A}} \quad + \quad \vec{\boldsymbol{B}} &= \vec{\boldsymbol{C}} \\ A\hat{\boldsymbol{\lambda}}_A \quad + \quad B\hat{\boldsymbol{\lambda}}_B &= \vec{\boldsymbol{C}} \end{aligned} \tag{1.34}$$

you would like to find $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ (which you will know if you find A and B).

Example: A walk on a flat earth
You walked SE (half way between South and East) for a while and NNW (half way between North and NorthWest, 22.5° West of North) for a while and ended up going a net distance of 200 m East. $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are your displacements on the first and second parts of your walk.

So, taking xy axes aligned with East and North, the directions are

$$\hat{\boldsymbol{\lambda}}_A = \frac{\sqrt{2}}{2}\hat{\boldsymbol{\imath}} - \frac{\sqrt{2}}{2}\hat{\boldsymbol{\jmath}} \text{ and } \hat{\boldsymbol{\lambda}}_B = -\sin(\frac{\pi}{8})\hat{\boldsymbol{\imath}} + \cos(\frac{\pi}{8})\hat{\boldsymbol{\jmath}}$$

and the given sum is $\vec{\boldsymbol{C}} = 200\,\text{m}\hat{\boldsymbol{\imath}}$. Still unknown are the distances A and B.

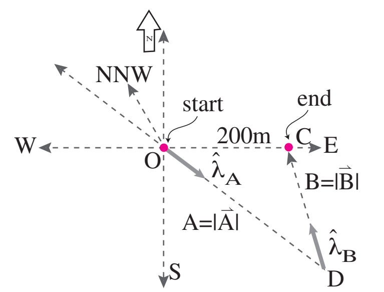


Figure 1.67: An indirect walk from O to C via D.

Statics problems of this type have A and B representing the unknown magnitudes of forces $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ and $\hat{\boldsymbol{\lambda}}_A$ and $B\hat{\boldsymbol{\lambda}}_B$ their known directions. Here are four ways to solve eqn. (1.34) which will be illustrated with "**a walk**".

Method I: Use dot products with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$

If we take the dot product of both sides of eqn. (1.34) with $\hat{\boldsymbol{\imath}}$ and then again with $\hat{\boldsymbol{\jmath}}$ we get:

$$\begin{aligned} \hat{\boldsymbol{\imath}}\cdot\{\text{eqn. (1.34)}\} \quad &\Rightarrow \quad A\lambda_{Ax} + B\lambda_{Bx} = C_x, \text{ and} \\ \hat{\boldsymbol{\jmath}}\cdot\{\text{eqn. (1.34)}\} \quad &\Rightarrow \quad A\lambda_{Ay} + B\lambda_{By} = C_y \end{aligned} \tag{1.35}$$

where the components of the vectors $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$, and $\vec{\boldsymbol{C}}$ are known, or easily determined, because the vectors are known (however they are represented). Eqns. 1.35 are two scalar equations in the unknowns A and B. You can solve these any way that pleases you. One method would be to write the equations in matrix form

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} \\ \lambda_{Ay} & \lambda_{By} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix} \tag{1.36}$$

Example: Solving "A walk": method I, simultaneous equations
For the walk example above we would have

$$\begin{bmatrix} \sqrt{2}/2 & -\sin(\frac{\pi}{8}) \\ -\sqrt{2}/2 & \cos(\frac{\pi}{8}) \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 200\,\text{m} \\ 0 \end{bmatrix}$$

which solves (on a computer or calculator) to $A \approx 483$ m and $B \approx 370$ (with the total walked distance being about 852 m).

Taking dot products of a vector equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ is equivalent to extracting the x and y components of the equation. But we use the dot product notation to highlight that you could dot both sides of the vector equation with any vector that pleases you and you would get a legitimate scalar equation. Use any other vector that pleases you (not parallel with the first) and you will get a second independent equation. And the two resulting equations will have the same solution for A and B as the x and y (or "$\hat{\boldsymbol{\imath}}$" and "$\hat{\boldsymbol{\jmath}}$") equations above.

Method II: pick a vector for a dot product that gets rid of terms you don't know.

Pretend for a paragraph that you only want to find A in eqn. (1.34), for example that you only wanted to know the distance walked on the first leg of the indirect walk in the example above. It would be nice to reduce eqn. (1.34) to a single scalar equation in the single unknown A. We'd like to get rid of the term with B, a quantity that we do not know. Suppose we knew a vector $\hat{\boldsymbol{n}}_B$ that was perpendicular to $\hat{\boldsymbol{\lambda}}_B$. If we dotted both sides of eqn. (1.34) we'd get:

$$\begin{array}{rcccl}
\hat{\boldsymbol{n}}\cdot\{\text{eqn. (1.34)}\} & \Rightarrow & \hat{\boldsymbol{n}}_B\cdot\left(A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B\right) & = & \hat{\boldsymbol{n}}_B\cdot\vec{\boldsymbol{C}}\\
 & \Rightarrow & \hat{\boldsymbol{n}}_B\cdot\left(A\hat{\boldsymbol{\lambda}}_A\right) + \hat{\boldsymbol{n}}_B\cdot\left(B\hat{\boldsymbol{\lambda}}_B\right) & = & \hat{\boldsymbol{n}}_B\cdot\vec{\boldsymbol{C}}\\
\hat{\boldsymbol{n}}_B\cdot\hat{\boldsymbol{\lambda}}_B = 0 & \Rightarrow & \left(\hat{\boldsymbol{n}}_B\cdot\hat{\boldsymbol{\lambda}}_A\right)A & = & \hat{\boldsymbol{n}}_B\cdot\vec{\boldsymbol{C}}\\
 & \Rightarrow & A & = & \frac{\hat{\boldsymbol{n}}_B\cdot\vec{\boldsymbol{C}}}{\hat{\boldsymbol{n}}_B\cdot\hat{\boldsymbol{\lambda}}_A}.
\end{array}$$

To make use of this method we have to cook up a vector $\hat{\boldsymbol{n}}_B$ that is perpendicular to $\hat{\boldsymbol{\lambda}}_B$ ③.Crossing $\hat{\boldsymbol{\lambda}}_B$ with $\hat{\boldsymbol{k}}$ serves the purpose:

$$\hat{\boldsymbol{n}}_B = \hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B = \hat{\boldsymbol{k}}\times\left(\lambda_{Bx}\hat{\boldsymbol{\imath}} + \lambda_{By}\hat{\boldsymbol{\jmath}}\right) = -\lambda_{By}\hat{\boldsymbol{\imath}} + \lambda_{Bx}\hat{\boldsymbol{\jmath}}.$$

Without doing the cross product explicitly you can remember that a vector orthogonal to a 2D vector $\hat{\boldsymbol{\lambda}}_B$ has the x and y components switched and the sign of first component then changed. So we get

$$A = \frac{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B)\cdot\vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B)\cdot\hat{\boldsymbol{\lambda}}_A} = \frac{\lambda_{By}C_x - \lambda_{Bx}C_y}{\lambda_{By}\lambda_{Ax} - \lambda_{Bx}\lambda_{Ay}}$$

which is a direct formula for the desired answer ④. You could use this formula by substituting in numbers, but that requires memorization or look up. Rather, if you like this short cut, you should remember the idea and reproduce the steps with the symbols or numbers in your problem. Summarizing,

> To reduce eqn. (1.34) to one scalar equation in the one unknown A, use a judiciously chosen dot product. For example, get rid of the $\hat{\boldsymbol{\lambda}}_B$ or $\vec{\boldsymbol{B}}$ term by dotting both sides of with $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B$ (or, to save the trouble of finding the unit vector $\hat{\boldsymbol{\lambda}}_B$ just dot with $\hat{\boldsymbol{k}}\times\vec{\boldsymbol{B}}$). ⑤.

.

Altogether you can think of this method as something like the "component" method. But we are taking components of the vectors in the direction perpendicular to $\vec{\boldsymbol{B}}$. Alternatively you can think of this method as taking the projection of the vector equation onto a line perpendicular to $\vec{\boldsymbol{B}}$.

Similarly dotting both sides of eqn. (1.34) with $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_A$ gives

$$B = \frac{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_A)\cdot\vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_A)\cdot\hat{\boldsymbol{\lambda}}_B}.$$

③The vector $\hat{\boldsymbol{k}}$ (the unit vector out of the page) is perpendicular to $\hat{\boldsymbol{\lambda}}_B$ but is unfortunately not suitable because it is also perpendicular to $\hat{\boldsymbol{\lambda}}_A$ and $\vec{\boldsymbol{C}}$ so only yields the equation $0 + 0 = 0$ or the nonsense that $A = 0/0$.

④This solution is identical to the Cramer's rule solution of eqn. (1.42) on page 12. That is, we have used dot products to derive Cramer's rule for 2×2 matrices.

⑤**Judicious choice of vectors for dot products.** The basic idea is this: *Get rid of things you don't know and don't care about.*
Say a and b are unknown and appear in your equations in terms like $a\vec{\boldsymbol{A}}$ and $b\vec{\boldsymbol{B}}$. First take an interest in a. You don't know anything about b, and for a moment you don't care about it. So get rid of it. The two ways to get rid of a vector term you don't know and don't care about are 1) dotting the force-balance equation with a vector orthogonal to the vector of disinterest, and 2) using moment balance about a point or axis about which the force of disinterest has no moment. So, for example, dotting an equation containing $a\vec{\boldsymbol{A}}$ and $b\vec{\boldsymbol{B}}$ with a vector orthogonal to $\vec{\boldsymbol{B}}$ eliminates the unknown b from your equation.

Example: Solving "A walk": method II, judicious dot products
You should be able to derive the formulas above as needed. Dotting, for example, both sides of eqn. (1.34) with $\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B$ and plugging in the known components yields

$$\begin{aligned} A = \frac{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B)\cdot\vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\lambda}}_B)\cdot\hat{\boldsymbol{\lambda}}_A} &= \frac{\lambda_{By}C_x - \lambda_{Bx}C_y}{\lambda_{By}\lambda_{Ax} - \lambda_{Bx}\lambda_{Ay}} \\ &= \frac{\cos(\pi/8)\cdot 200\,\mathrm{m} - (-\sin(\pi/8))\cdot 0}{\cos(\pi/8)\cdot(\sqrt{2}/2) - (-\sin(\pi/8))\cdot(-\sqrt{2}/2)} \\ &\approx 483\,\mathrm{m} \qquad \text{(as before)} \end{aligned}$$

Method III, graphical solution

On the vector triangle defined by $\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}}=\vec{\boldsymbol{C}}$ we call O the tail end of $\vec{\boldsymbol{A}}$. The location of the tip of $\vec{\boldsymbol{C}}$ at G can be drawn to scale. Then the point H can be located as at the intersection of two lines: one emanating from O and in the direction of $\hat{\boldsymbol{\lambda}}_A$ and one emanating from H and in the direction of $\hat{\boldsymbol{\lambda}}_B$. Once the point H is located, the lengths A and B can be measured.

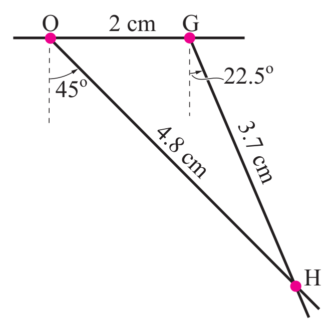


Figure 1.68: Method III: Find point H as the intersection of two known lines.

Example: Solving "A walk": method III, graphing
Taking 100 m as drawn to scale as, say 1 cm, point G is drawn 2 cm to the right of O. The location of the point H is found as the intersection of two lines: one emanating from O and pointing 45° counterclockwise from the $-\hat{\boldsymbol{\jmath}}$ axis, and the other emanating from G and pointing 22.5° counterclockwise from the $-\hat{\boldsymbol{\jmath}}$ axis. The distance from O to H can be measured as about 4.8 cm yielding $A\approx 480$ m.

This construction can be done with pencil and paper or with a computer drawing program.

Method IV, trigonometry

The final method, the classical method used predominantly before vector notation was well accepted, is to treat the vector triangle as a triangle with some known sides and some known angles, and to use the *law of sines* (discussed in box 1.9).

Because $\vec{\boldsymbol{C}}$ and the directions of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are assumed known, the angles a (opposite side A) and b (opposite side B) are known. Because the sum of interior angles in a triangle is π we know the angle $c=\pi-a-b$. The law of sines tells us that

$$\frac{\sin a}{A}=\frac{\sin c}{C} \quad \text{and} \quad \frac{\sin b}{B}=\frac{\sin c}{C}$$

which we can rewrite as

$$A=\frac{C\sin a}{\sin c} \quad \text{and} \quad B=\frac{C\sin b}{\sin c}.$$

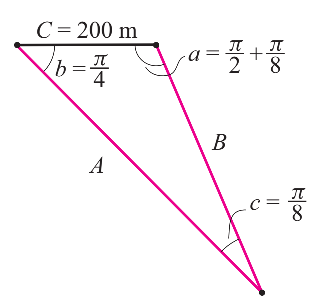


Figure 1.69: Solving "A walk" using the law of sines.

Example: Solving "A walk": method IV, the law of sines
Referring to fig. 1.69 we get

$$A = \frac{C \sin a}{\sin c} = \frac{200\,\text{m} \cdot \sin(5\pi/8)}{\sin(\pi/8)} \approx 483\,\text{m}$$

and

$$B = \frac{C \sin b}{\sin c} = \frac{200\,\text{m} \cdot \sin(\pi/4)}{\sin(\pi/8)} \approx 370\,\text{m}$$

as we have found three times already.

The determination of two vectors by knowing their directions and their sum is analogous to determination of a triangle by "angle-side-angle".

The magnitudes and sum of two vectors are known (2D)

Two vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ in the plane have known magnitudes A and B but unknown directions $\hat{\boldsymbol{\lambda}}_A$ and $\hat{\boldsymbol{\lambda}}_B$. Their sum $\vec{\boldsymbol{C}}$ is known. So, measuring angles counterclockwise relative to the positive x axis, we have:

$$\begin{aligned} \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} &= \vec{\boldsymbol{C}} \\ A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B &= \vec{\boldsymbol{C}} \\ A\left(\cos\theta_A \hat{\boldsymbol{\imath}} + \sin\theta_A \hat{\boldsymbol{\jmath}}\right) + B\left(\cos\theta_B \hat{\boldsymbol{\imath}} + \sin\theta_B \hat{\boldsymbol{\jmath}}\right) &= \vec{\boldsymbol{C}} \qquad (1.37) \end{aligned}$$

where eqn. (1.37) is one 2D vector equation in 2 unknowns: θ_A and θ_B.

Method 1: using an appropriate dot product

This problem is really best solved with trig (see below) and getting it right with component method is a matter of hindsight. Eqn. 1.37 can be rewritten as

$$C\left(\cos\theta_C \hat{\boldsymbol{\imath}} + \sin\theta_C \hat{\boldsymbol{\jmath}}\right) - A\left(\cos\theta_A \hat{\boldsymbol{\imath}} + \sin\theta_A \hat{\boldsymbol{\jmath}}\right) = B\left(\cos\theta_B \hat{\boldsymbol{\imath}} + \sin\theta_B \hat{\boldsymbol{\jmath}}\right)$$

Taking the dot product of each side with itself gives

$$C^2 + A^2 - 2AC\underbrace{\left(\cos\theta_C \cos\theta_A + \sin\theta_C \sin\theta_A\right)}_{\cos(\theta_C - \theta_A)} = B^2$$

so

$$\theta_A = \theta_C - \arccos\left(\frac{C^2 + A^2 - B^2}{2AC}\right).$$

Now $\vec{\boldsymbol{A}}$ is fully determined and $\vec{\boldsymbol{B}}$ can be found by vector subtraction. Note that the arccos function is always double valued (the negative of any arccos is also a legitimate arccos), so that the solution of this problem is not unique. Also, if the argument of the arccos function is greater than 1 in magnitude, there is no solution; this happens if any two of A, B, and C are greater than the third (that is, if the so-called "triangle inequality" is violated) and there is no way of making a triangle with the given lengths.)

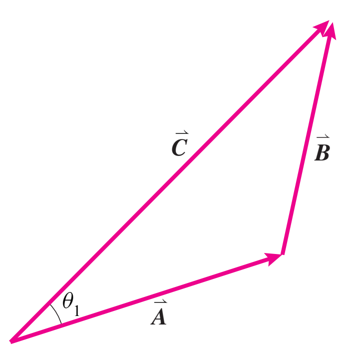


Figure 1.70: For use in using the law of cosines to solve a vector triangle.

Method II: The law of cosines

Referring to fig. 1.70, we can apply the law of cosines directly to get

$$B^2 = A^2 + C^2 - 2AC\cos\theta_B \quad (1.38)$$

which we can solve to get $$\theta_1 = \arccos\left(\frac{C^2 + A^2 - B^2}{2AC}\right) \quad (1.39)$$

Thus the orientation of $\vec{\boldsymbol{A}}$ is determined in relation to $\vec{\boldsymbol{C}}$. This method is a bit quicker than the component method above because it skips the steps where, in effect, the component method derives the law of cosines.

Method III: graphical construction

From the tail of $\vec{\boldsymbol{C}}$ draw a circle with radius A (see fig. 1.71). From the tip of $\vec{\boldsymbol{C}}$ draw a circle with radius B. For each of the two points of intersection, P_1 and P_2, a solution has been found. Vector $\vec{\boldsymbol{A}}$ goes from the tail of $\vec{\boldsymbol{C}}$ to, say, P_1, and $\vec{\boldsymbol{B}}$ goes from P_1 to the tip of $\vec{\boldsymbol{C}}$. An $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ based on P_2 is also a legitimate solution. Each pair is a legitimate solution to the problem. To get a unique solution set other information would have to be provided.

Determining a vector triangle when one vector is known and only the magnitudes of the other two are known is analogous to determining a triangle from "side-side-side" in geometry. It is interesting that this, the most elementary of all geometric constructions does not have an equally simple analytic representation.

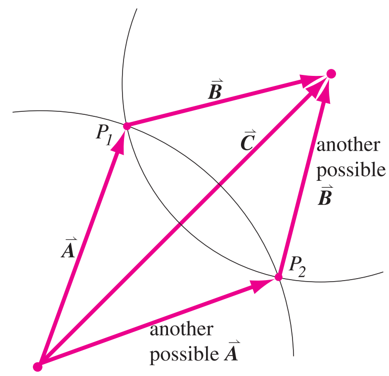


Figure 1.71: Solving a vector triangle where vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ have known magnitude but unknown direction.

Find the magnitude of three vectors given their directions and their sum (3D)

This problem is close in approach to its junior 2D cousin on page 6 and to the example on page 1. It is the most common of the 3D vector equation problems. Assume that you know the directions of three vectors $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ (given, say, as the unit vectors $\hat{\boldsymbol{\lambda}}_A, \hat{\boldsymbol{\lambda}}_B$, and $\hat{\boldsymbol{\lambda}}_C$), as well as their sum $\vec{\boldsymbol{D}}$. So we have

$$\begin{aligned} \vec{\boldsymbol{A}} \;&+\; \vec{\boldsymbol{B}} \;+\; \vec{\boldsymbol{C}} \;=\; \vec{\boldsymbol{D}} \\ A\hat{\boldsymbol{\lambda}}_A \;&+\; B\hat{\boldsymbol{\lambda}}_B \;+\; C\hat{\boldsymbol{\lambda}}_C \;=\; \vec{\boldsymbol{D}} \end{aligned} \quad (1.40)$$

and we want to find A, B, and C from which we can find $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ (*e.g.*, $\vec{\boldsymbol{A}} = A\hat{\boldsymbol{\lambda}}_A$). We can think of the last of eqn. (1.40) as one 3D vector equation in three unknowns.

In three dimensions the graphical approach is essentially impossible. And the trigonometric approach is awkward to say the least, and probably only generally practical for people with British accents who are long dead. The general ideas in the first two methods still stand, however. Thus the use of vector concepts is basically unavoidable in 3D problems.

Method I: dotting with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

We can dot the left and right sides of eqn. (1.40) with $\hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{\jmath}}$ or $\hat{\boldsymbol{k}}$. This is equivalent to taking the x, y and z components of the equation. We get then

$$\begin{array}{lcl} \hat{\boldsymbol{\imath}} \cdot \{\text{eqn. (1.40)}\} & \Rightarrow & A\lambda_{Ax} + B\lambda_{Bx} + C\lambda_{Cx} = D_x, \\ \hat{\boldsymbol{\jmath}} \cdot \{\text{eqn. (1.40)}\} & \Rightarrow & A\lambda_{Ay} + B\lambda_{By} + C\lambda_{Cy} = D_y, \text{and} \\ \hat{\boldsymbol{k}} \cdot \{\text{eqn. (1.40)}\} & \Rightarrow & A\lambda_{Az} + B\lambda_{Bz} + C\lambda_{Cz} = D_z \end{array} \tag{1.41}$$

which can be written in matrix form as

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} & \lambda_{Cx} \\ \lambda_{Ay} & \lambda_{By} & \lambda_{Cy} \\ \lambda_{Az} & \lambda_{Bz} & \lambda_{Cz} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}. \tag{1.42}$$

Unless the matrix is sparse (has a lot of zeros as entries) it is probably best to solve such a set of equations for A, B and C on a computer or calculator.

Method II: pick a vector for dot product that kills terms you don't know.

The philosophy here is the same as for method II in 2D (page 8). Pretend for a paragraph that you only want to find A in eqn. (1.40). We can kill the terms involving the unknowns B and C by dotting both sides of the equation with a vector perpendicular to $\hat{\boldsymbol{\lambda}}_B$ and $\hat{\boldsymbol{\lambda}}_C$. Such a vector is $\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C$. Thus

$$\begin{array}{lrcl} (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \{\text{eqn. (1.34)}\} & & & \\ \Rightarrow & (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot (A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B + C\hat{\boldsymbol{\lambda}}_C) & = & (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \vec{\boldsymbol{D}} \\ \Rightarrow & (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot (A\hat{\boldsymbol{\lambda}}_A) + \vec{\boldsymbol{0}} + \vec{\boldsymbol{0}} & = & (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \vec{\boldsymbol{D}} \\ \Rightarrow & A & = & \dfrac{\vec{\boldsymbol{D}} \cdot (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)}{\hat{\boldsymbol{\lambda}}_A \cdot (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C).} \end{array}$$

If you use a matrix determinant to evaluate the mixed triple product you can recognize this formula (like the formula solving the example on 1) as Cramer's rule. By a judicious dot product we have reduced the vector equation to a scalar equation in one unknown. Similarly we could get one equation in one unknown for B or for C by doting eqn. (1.40) with $\hat{\boldsymbol{\lambda}}_A \times \hat{\boldsymbol{\lambda}}_C$ and $\hat{\boldsymbol{\lambda}}_A \times \hat{\boldsymbol{\lambda}}_B$, respectively(6).

(6)Note that the key to the method was dotting with a vector in an appropriate direction, the magnitude of the vector did not matter. So if, for example, you knew any vector $\vec{\boldsymbol{v}}_B$ in the direction of $\hat{\boldsymbol{\lambda}}_B$ and any vector $\vec{\boldsymbol{v}}_C$ in the direction of $\hat{\boldsymbol{\lambda}}_C$ you could dot both sides of eqn. (1.40) with $\vec{\boldsymbol{v}}_B \times \vec{\boldsymbol{v}}_C$ to get one scalar equation for A. This can simplify calculations by avoiding the square roots (which cancel in the end) that you calculate to find unit vectors.

Parametric equations for lines and planes

A line in 2D

In geometry a line on a plane is often described as the set of x and y points that satisfy an equation like

$$Ax + By = D \qquad \text{or} \qquad y = mx + b$$

for given A, B, and D or m and b. However a line is a "one dimensional" object and it is nice to describe it that way. The parametric form that is often useful is:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\mathrm{A}} + s\vec{\boldsymbol{v}} \tag{1.43}$$

where $\vec{\boldsymbol{r}}$ are the position vectors of set of points on the line, one point for each value of the scalar parameter s. $\vec{\boldsymbol{r}}_{\mathrm{A}}$ is the position vector of one given reference point on the line and $\vec{\boldsymbol{v}}$ is a vector parallel to the line. In the special case that $\vec{\boldsymbol{v}}$ is a unit vector, s is the distance from the point at $\vec{\boldsymbol{r}}_{\mathrm{A}}$ to the point at $\vec{\boldsymbol{r}}$. If the vector $\vec{\boldsymbol{v}}$ was the velocity of a point moving on the line then $s|\vec{\boldsymbol{v}}|$ would be the distance of the point from the point at $\vec{\boldsymbol{r}}_{\mathrm{A}}$.

Example: Parametric equation of a line

A parametric equation for the line going through the points with position vectors $\vec{\boldsymbol{r}}_{\mathrm{A}}$ and $\vec{\boldsymbol{r}}_{\mathrm{B}}$ is

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\mathrm{A}} + s\left(\underbrace{\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}}_{\vec{\boldsymbol{v}}}\right) \qquad \text{or better} \qquad \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\mathrm{A}} + s\hat{\boldsymbol{\lambda}}_{A}$$

where $\hat{\boldsymbol{\lambda}}_{A} = (\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}) \,/\, |\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}|$

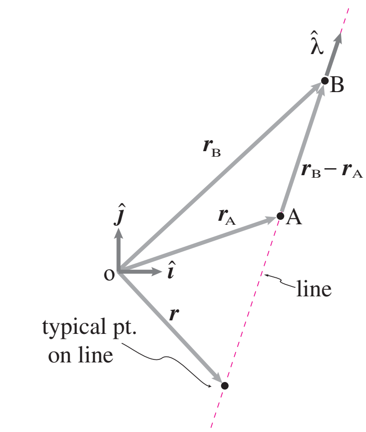


Figure 1.72: Parametric description of a line using vectors.

A line in 3D

In three dimensions a line is often described geometrically as the intersection of two planes. But a line in three dimensions is still a one dimensional object so the parametric form eqn. (1.43), applicable in three dimensions as well as two, is nice.

A plane

A plane in three dimensions can be described as the set of points x,y, and z that satisfy an equation like:

$$Ax + By + Cz = D$$

for a given A,B,C, and D. The parametric description of a plane uses two parameters s_1 and s_2 and is

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{\mathrm{O}} + s_1\vec{\boldsymbol{v}}_1 + s_2\vec{\boldsymbol{v}}_2 \tag{1.44}$$

where $\vec{\boldsymbol{r}}$ is a typical point on the plane, $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are any two non-parallel vectors that lie in the plane and s_1 and s_2 are any two real numbers. Each pair (s_1, s_2) corresponds to one point in the plane and vice versa. The numbers s_1 and s_2 can be thought of as in-plane distance coordinates if the vectors $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are mutually orthogonal unit vectors.

Example: A plane

A parametric equation for the plane going through the three points with position vectors $\vec{\boldsymbol{r}}_{\mathrm{A}}$, $\vec{\boldsymbol{r}}_{\mathrm{B}}$, and $\vec{\boldsymbol{r}}_{\mathrm{C}}$ is

$$\vec{\boldsymbol{r}} = \underbrace{\vec{\boldsymbol{r}}_{\mathrm{A}}}_{\vec{\boldsymbol{r}}_0} + s_1\underbrace{(\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}})}_{\vec{\boldsymbol{v}}_1} + s_2\underbrace{(\vec{\boldsymbol{r}}_{\mathrm{C}} - \vec{\boldsymbol{r}}_{\mathrm{A}})}_{\vec{\boldsymbol{v}}_2}$$

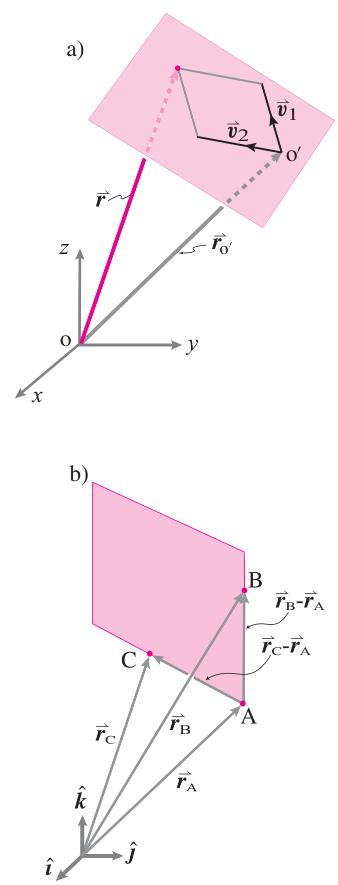


Figure 1.73: a) Parametric equation of a plane. b) the plane through the points A,B, and C

You can check that when $s_1 = s_2 = 0$ the point on the plane $\vec{\boldsymbol{r}}_{\mathrm{A}}$ is given. And when one of the s values is one and the other zero the points $\vec{\boldsymbol{r}}_{\mathrm{B}}$ and $\vec{\boldsymbol{r}}_{\mathrm{C}}$ are given.

Vectors, matrices, and linear algebraic equations

Once one has drawn a free-body diagram and written the force and moment balance equations one is left with vector equations to solve for various unknowns. The vector equations of mechanics can be reduced to scalar equations by using dot products. The simplest dot product to use is with the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. This use of dot products is equivalent to taking the x, y, and z components of the vector equation. The two vector equations

$$\begin{aligned} a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} &= (c-5)\hat{\boldsymbol{\imath}} + (d+7)\hat{\boldsymbol{\jmath}} \\ (a-c)\hat{\boldsymbol{\imath}} + (a+b)\hat{\boldsymbol{\jmath}} &= (c+b)\hat{\boldsymbol{\imath}} + (2a+c)\hat{\boldsymbol{\jmath}} \end{aligned}$$

with four scalar unknowns a, b, c, and d, can be rewritten as four scalar equations, two from each two-dimensional vector equation. Taking the dot product of the first equation with $\hat{\boldsymbol{\imath}}$ gives $a = c - 5$. Similarly dotting with $\hat{\boldsymbol{\jmath}}$ gives $b = d + 7$. Repeating the procedure with the second equation gives 4 scalar equations:

$$\begin{aligned} a &= c-5 \\ b &= d+7 \\ a-c &= c+b \\ a+b &= 2a+c. \end{aligned}$$

These equations can be re-arranged putting unknowns on the left side and knowns on the right side:

$$\begin{array}{rcrcrcrcr} 1a &+& 0b &+& -1c &+& 0d &=& -5 \\ 0a &+& 1b &+& 0c &+& -1d &=& 7 \\ 1a &+& -1b &+& -2c &+& 0d &=& 0 \\ -1a &+& 1b &+& -1c &+& 0d &=& 0 \end{array}$$

These equations can in turn be written in standard matrix form. The standard matrix form is a short hand notation for writing (linear) equations, such as the equations above:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}}_{[\boldsymbol{A}]} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[\boldsymbol{x}]} = \underbrace{\begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{[\boldsymbol{y}]}$$

$$\Rightarrow \quad [\boldsymbol{A}] \cdot [\boldsymbol{x}] = [\boldsymbol{y}].$$

The matrix equation $[\boldsymbol{A}] \cdot [\boldsymbol{x}] = [\boldsymbol{y}]$ is in a form that is easy to input to any of several programs that solve linear equations. The computer (or a do-able but probably untrustworthy hand calculation) should return the following solution for $[\boldsymbol{x}]$ (a, b, c, and d).

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix}.$$

That is, $a = -5$, $b = -5$, $c = 0$, and $d = -12$. If you doubt the solution, check it. To check the answer, plug it back into the original matrix equation and note the equality (or lack thereof!). In this case, we have done our calculations correctly and

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

Going back to the original vector equations we can also check that

$$\begin{aligned} -5\hat{\boldsymbol{\imath}} + -5\hat{\boldsymbol{\jmath}} &= (0-5)\hat{\boldsymbol{\imath}} + (-12+7)\hat{\boldsymbol{\jmath}} \\ (-5-0)\hat{\boldsymbol{\imath}} + (-5+-5)\hat{\boldsymbol{\jmath}} &= (0+-5)\hat{\boldsymbol{\imath}} + (2\cdot -5+0)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Computer solution of simultaneous equations

Depending on your computer package you might solve the equations above like this

```
eqset  =  {  a      -  c       = -5
                b         - d  =  7
             a - b - 2c        =  0
            -a + b -  c        =  0}
Solve eqset for a,b,c,d.
```

Or, if your computer package is set up especially for linear algebra then you could write something analogous to this:

```
 M =  [  1  0 -1  0
         0  1  0 -1
         1 -1 -2  0
        -1  1 -1  0]
 w = [-5 7 0 0]'
Solve M*z = w for z
             % the elements of z are a,b,c,d
```

‘Physical’ vectors and row or column vectors

The word ‘vector’ has two related but subtly different meanings. One is a physical vector like $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$, a quantity with magnitude and direction. The other meaning is a list of numbers like the row vector

$$[\boldsymbol{x}] = [x_1, x_2, x_3]$$

or the column vector

$$[\boldsymbol{y}] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Once you have picked a basis, like $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, you can represent a physical vector $\vec{\boldsymbol{F}}$ as a row vector $\left[F_x, F_y, F_z\right]$ or a column vector $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$. But the components of a given vector depend on the base coordinate system (or base vectors) that are used. For clarity it is best to distinguish a physical vector from a list of components using a notation like the following:

$$[\vec{\boldsymbol{F}}]_{XYZ} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

The square brackets around $\vec{\boldsymbol{F}}$ indicate that we are looking at its components. The subscript XYZ identifies what coordinate system or base vectors are being used. The right side is a list of three numbers (in this case arranged as a column, the default arrangement in linear algebra).

1.10 Existence, uniqueness, and geometry

No homework problem solutions depend on your understanding this theoretical aside.

Sometimes there is a *unique* solution set to a set of simultaneous solutions. Sometimes it's impossible to solve a set of vector equations; no solutions *exist*. And, sometimes there are lots of solutions; solutions *exist* but are not *unique*. These cases sometimes have simple geometric interpretations.

Example 1. Consider a very simple equation

$$a\vec{v}_1 = \vec{w}$$

where $\vec{v}_1$ and $\vec{w}$ are given and you are to find a. The left hand side is a parametric expression for points on a line through the origin in the direction $\vec{v}_1$.

- if $\vec{w}$ is parallel to $\vec{v}$ then the equation has exactly one solution for a;
- if $\vec{w}$ is not parallel to $\vec{v}$ then there is no possible a that could make the equation true. The equation has no solutions.

This vector equation is equivalent to 2 scalar equations (3 in 3D) with one scalar unknown and we expect generally to find no solution. That is, two random vectors $\vec{v}_1$ and $\vec{w}$ are unlikely to be parallel either in 2D or 3D.

Example 2. Now consider this 2D vector equation in two unknown scalars a and b:

$$a\vec{v}_1 + b\vec{v}_2 = \vec{w}.$$

- If $\vec{v}_1$ and $\vec{v}_2$ are not parallel $a\vec{v}_1 + b\vec{v}_2$ could be, with appropriate choice of a and b, any 2D vector. There would be a unique solution for every possible $\vec{w}$.
- But if $\vec{v}_1$ and $\vec{v}_2$ are parallel then the expression $a\vec{v}_1 + b\vec{v}_2$ describes a line.
  - If $\vec{w}$ is on this line there are many solutions for a and b because the two vectors $a\vec{v}_1$ and $\vec{v}_2$ can be added various ways that partially cancel.
  - If $\vec{w}$ is off the line then there are no combinations of a and b that get vectors off the line, there are no solutions.

In 2D a test to see if two vectors are parallel is to take their cross product. So, if

$$(\vec{v}_1 \times \vec{v}_2) \cdot \hat{\boldsymbol{k}} = v_{1x}v_{2y} - v_{1y}v_{2x} = \det \begin{bmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{bmatrix} = 0$$

then $\vec{v}_1$ and $\vec{v}_2$ are parallel and there are either many solutions or no solutions depending on whether or not $\vec{w}$ is also parallel to $\vec{v}_1$ and $\vec{v}_2$.

Example 3. Consider the same example as above but now in 3D.

$$a\vec{v}_1 + b\vec{v}_2 = \vec{w}.$$

Now the question is whether the vector $\vec{w}$ is in the plane described parametrically by $a\vec{v}_1 + b\vec{v}_2$. We have more equations than unknowns, $3 > 2$ so solution should be unlikely. Given 3 random vectors in 3D $\vec{v}_1$, $\vec{v}_2$ and $\vec{w}$, it is unlikely that $\vec{w}$ would be in the plane determined by $\vec{v}_1$ and $\vec{v}_2$. If $\vec{w}$ is in that plane, we get again the three possibilities from the previous example.

Example 4. Finally consider this common equation in 3D.

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{w}. \tag{1.45}$$

where $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{w}$ are given vectors and a, b and c are unknowns.

- If $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are not co-planar, then by imagining flying in through space in each of three directions, you can see that you can get to any point in space $\vec{w}$ by using one and only one set of multiples a, b and c of the three vectors.
- On the other hand, if $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are co-planar, they are redundant, and
  - there can only be a solution if $\vec{w}$ is on the plane and, assuming the three vectors are not also collinear, there are many solutions. There are various ways for combinations of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ to cancel each other out.
  - if $\vec{w}$ is off this plane there are no solutions.

If the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are coplanar then there are either no solutions for a,b and c or many solutions. We can test for coplanarity of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ with geometric reasoning and cross products. The vector $\vec{v}_1 \times \vec{v}_2$ is orthogonal to the plane of $\vec{v}_1$ and $\vec{v}_2$. So, if $\vec{v}_3$ is in the plane defined by $\vec{v}_1$ and $\vec{v}_2$ it will be orthogonal to $\vec{v}_1 \times \vec{v}_2$. Thus if

$$(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = 0$$

the three vectors are co-planar. This test can also be written as

$$\det \begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} = 0$$

which is what we would expect from considering the matrix form of eqn. (1.45)

$$\begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

and checking to see if the 3×3 matrix is "singular" (a linear algebra word meaning that the determinant is zero).

Relation to more general linear algebra. For systems of equations in 4 or more dimensions we can't use our geometric intuition quite so directly. But the cases above are analogous to what one always finds. The geometric interpretations are helpful for gaining an intuition, even in higher than 3 dimensions when they don't strictly hold. Consider the matrix equation

$$M * v = b$$

with the square matrix M and the column vector b given.

- If the columns of M are not redundant (e.g., they are linearly independent) then there exists a unique v for any b. This is like having $\vec{v}_1, \vec{v}_2, \vec{v}_3$ not coplanar in 3D.
- If the columns of M are redundant (e.g., they are linearly dependent) this is like having coplanar $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and
  - if b is in the span of the columns of M, like $\vec{w}$ being in the plane, there are many solutions, and
  - if b is not in the span of the columns of M, like $\vec{W}$ being off the plane, there are no solutions.

