1.4 Moment and moment about an axis

When you try to move something by pushing it or by turning it. In mechanics, the measure of your pushing is the net force you apply. The measure of your turning is the net *moment*, also sometimes called the net *torque* or net *couple*.

Although concepts involving moment (and rotation) are often harder for beginners than force (and translation), they were understood first. The ancient 'principle of the lever', which can be viewed as the root of all mechanics, is the basic idea incorporated by moments.

In this section we will define the moment of a force intuitively, geometrically, and finally using vector algebra. We will do this first in 2 dimensions and then in 3. The main mathematical tool here is the vector cross product from sec. 1.3. To start with, however, we can more or less deduce the concept of moment by generalizing from common experience.

Teeter totter mechanics

The two people weighing down on the teeter totter in fig. 1.53 tend to rotate it about its hinge, the right one clockwise and the left one counterclockwise. We will now cook up a measure of the tendency of each force to cause rotation about the hinge and call it *the moment of the force about the hinge*.

As is verified a million times a year by young future engineering students,

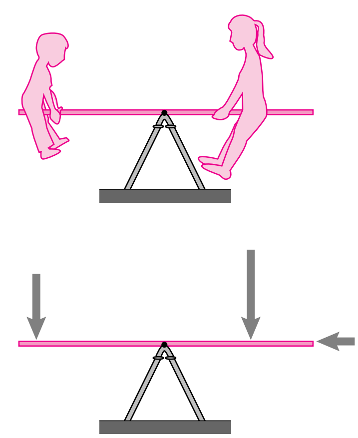
- To balance a teeter-totter the smaller person needs to be further from the hinge.
- If two people are on one side then the teeter totter is balanced by two similar people an equal distance from the hinge on the other side.
- Two people can balance one similar person by scooting twice as close to the hinge.

These proportionalities generalize to this:

> For a force that is perpendicular to the teeter totter, the tendency of the force to cause teeter totter rotation is proportional to the size of the force and to its distance from the hinge.

Further, if someone standing nearby adds a force that is directed towards the hinge (the sideways force in the illustration), that force causes no tendency to rotate. We can deduce more. Because

- any force can be decomposed into a sum of forces, one perpendicular to the teeter totter and the other towards the hinge, and because



(not a free body diagram)

Figure 1.53: On a balanced teeter totter the bigger person gets the short end of the stick. A sideways force directed towards the hinge has no effect on the balance.

Joke: The poet Robert Burns suggested using a Teeter Totter to weigh hogs like this: 1) Get a perfectly symmetric plank and balance it across a saw horse, 2) Put the hog on one end of the plank, 3) pile rocks on the other end of the plank until the plank is again perfectly balanced, 4) Carefully guess the weight of the rocks.

- we assume that the effect of the sum of these forces is the sum of the effects of each separately, and because
- the force towards the hinge has no tendency to rotate,

we can conclude that:

The moment of a force about a hinge is the product of

- its distance from the hinge, and
- the component of the force perpendicular to the line from the hinge to the force.

Thus, by common sense, we have found the formula for tendency to rotate about a hinge C in 2D, the *moment about C* or *moment with respect to C*.①

①The '/' in the subscript of $\vec{\boldsymbol{M}}$ reads as 'relative to' or 'about'. For simplicity we often leave the / out and just write $\vec{\boldsymbol{M}}_C$. So we could say "moment relative to C", or "moment about C", or "M sub C".

$$M_{/C} = |\vec{\boldsymbol{r}}|\,(|\vec{\boldsymbol{F}}|\sin\theta) = (|\vec{\boldsymbol{r}}|\sin\theta)\,|\vec{\boldsymbol{F}}|. \tag{1.25}$$

Here, θ is the angle between $\vec{\boldsymbol{r}}$ (the position of the point of force application relative to the hinge) and $\vec{\boldsymbol{F}}$ (see fig. 1.54). This formula for moment has all of the teeter totter deduced properties. Moment is proportional to r, and to the part of $\vec{\boldsymbol{F}}$ that is perpendicular to $\vec{\boldsymbol{r}}$. The re-grouping as $(|\vec{\boldsymbol{r}}|\sin\theta)$ shows that a force $\vec{\boldsymbol{F}}$ has the same effect if it is applied at a new location that is displaced in the direction of $\vec{\boldsymbol{F}}$. That is, the force $\vec{\boldsymbol{F}}$ can slide along its length without changing its $M_{/C}$ and is *equivalent* in its effect on the teeter totter. The quantity $|\vec{\boldsymbol{r}}|\sin\theta$ is sometimes called the *lever arm* of the force.

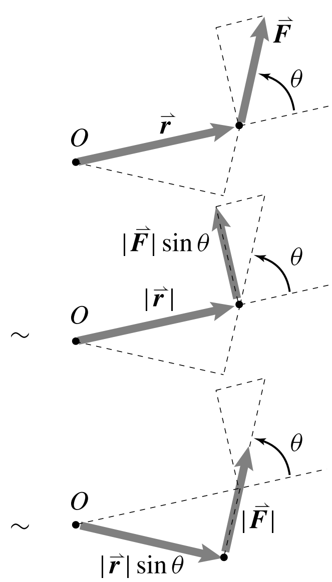


Figure 1.54: The moment of a force $|\vec{\boldsymbol{r}}||\vec{\boldsymbol{F}}|\sin\theta$ is either the product of its distance $|\vec{\boldsymbol{r}}|$ with its perpendicular component $|\vec{\boldsymbol{F}}|\sin\theta$ or of its lever arm (perpendicular distance) $|\vec{\boldsymbol{r}}|\sin\theta$ and the full force $|\vec{\boldsymbol{F}}|$. The $\sim$ means 'same as', indicating that the lower two forces and positions have the same moment as the force at top.

Sign conventions. We, like most others, define as positive a moment that causes a counterclockwise rotation. A moment that causes a clockwise rotation is negative. If we define θ appropriately then eqn. (1.25) obeys this sign convention. We define θ as the angle from the positive vector $\vec{\boldsymbol{r}}$ to the positive vector $\vec{\boldsymbol{F}}$ measured counterclockwise. Point the thumb of your right hand up, out of the paper. Your thumb is the moment direction. That is, the moment for an in-plane system is represented by a vector *out* of the plane. Point the fingers of your right hand along $\vec{\boldsymbol{r}}$ and curl them towards the direction of $\vec{\boldsymbol{F}}$ and see how far you have to rotate them. The force caused by the person on the left of the teeter totter has $\theta = 90^\circ$ so $\sin\theta = 1$ and the formula 1.25 gives a positive counterclockwise M. The force of the person on the right has $\theta = 270^\circ$ (3/4 of a revolution) so $\sin\theta = -1$ and the formula 1.25 gives a negative M.

In two dimensions moment is really a scalar concept, it is either positive or negative. In three dimensions moment is a vector. But even in 2D we find it easier to keep track of signs if we treat moment as a vector. In the xy plane, the 2D moment is a vector in the $\hat{\boldsymbol{k}}$ direction (straight out of the plane). So eqn. 1.25 becomes

$$\vec{\boldsymbol{M}}_{/\mathrm{C}} = |\vec{\boldsymbol{r}}|\;|\vec{\boldsymbol{F}}|\sin\theta\;\hat{\boldsymbol{k}}. \tag{1.26}$$

If you curl the fingers of your *right hand* in the direction of rotation caused by a force your thumb points in the direction of the moment vector.

2D moment by components

We can use the component form of the 2D cross product to find a component form for the moment $\vec{\boldsymbol{M}}_{/\mathrm{C}}$ of eqn. 1.26. Given $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ acting at P, where $\vec{\boldsymbol{r}}_{\mathrm{P/C}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$, the moment of the force about C is

$$\vec{\boldsymbol{M}}_{/\mathrm{C}} = (r_xF_y - r_yF_x)\hat{\boldsymbol{k}}$$

or the moment of $\vec{\boldsymbol{F}}$ about the axis at C is

$$M_C = r_xF_y - r_yF_x. \tag{1.27}$$

We can derive this component formula with the sequence of vector manipulations shown graphically in fig. 1.55.

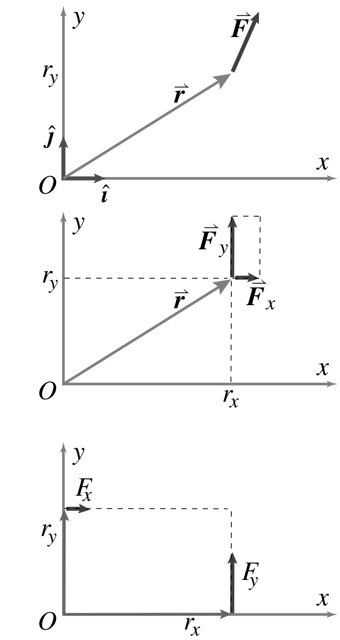


Figure 1.55: The component form of the 2D moment can be found by sequentially breaking the force into components, sliding each component along its line of action to the x and y axis, and adding the moments of the two components.

3D moment about an axis

The concept of moment about an axis is historically, theoretically, and practically important. Moment about an axis describes the principle of the lever, which far precedes Newton's laws. The net moment of a force system about enough different axes determines everything needed in mechanics about a force system. And one can sometimes quickly solve a statics or dynamics problem by considering moment about a judiciously chosen axis.

Let's start by thinking about a teeter totter again. Looking from the side we thought of a teeter totter as a 2D system. But the teeter totter really lives in the 3D world (see fig. 1.56). We now re-interpret the 2D moment M as the moment of the 2D forces about the $\hat{\boldsymbol{k}}$ axis of rotation at the hinge.

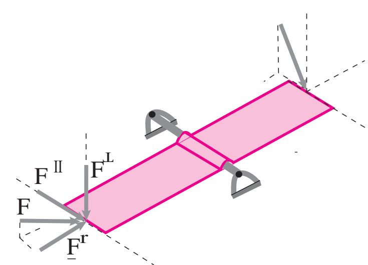


Figure 1.56: Teeter totter with applied forces broken into components parallel to the axis $\vec{\boldsymbol{F}}^{\parallel}$, radial $\vec{\boldsymbol{F}}^{r}$, and perpendicular to the plane containing the axis and the point of force application $\vec{\boldsymbol{F}}^{\perp}$.

It is plain that a force $\vec{\boldsymbol{F}}^{\parallel}$ pushing a teeter totter parallel to the axle causes no tendency to rotate. And we already agreed that a radial force $\vec{\boldsymbol{F}}^{r}$ (towards the hinge) causes no rotation. So, we see that the moment that a force causes about an axis is the distance of the force from the axis times the part of the force that is neither parallel to the axis nor directed towards the axis.

Now look at this in the more 3-dimensional context of fig. 1.57. Here an imagined axis of rotation is defined as the line through C that is in the $\hat{\boldsymbol{\lambda}}$ direction. A force $\vec{\boldsymbol{F}}$ is applied at P. We can break $\vec{\boldsymbol{F}}$ into a sum of three vectors

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}^{\parallel} + \vec{\boldsymbol{F}}^{r} + \vec{\boldsymbol{F}}^{\perp}$$

where $\vec{\boldsymbol{F}}^{\parallel}$ is parallel to the axis, $\vec{\boldsymbol{F}}^{r}$ is directed along the shortest connection between the axis and P (and is thus perpendicular to the axis) and $\vec{\boldsymbol{F}}^{\perp}$ is out of the plane defined by C, P and $\hat{\boldsymbol{\lambda}}$. By analogy with the teeter totter we see that $\vec{\boldsymbol{F}}^{r}$ and $\vec{\boldsymbol{F}}^{\parallel}$ cause no tendency to rotate about the axis. So only the $\vec{\boldsymbol{F}}^{\perp}$ contributes.

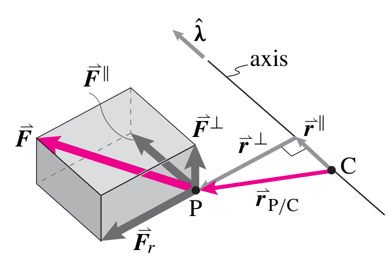


Figure 1.57: Moment about an axis

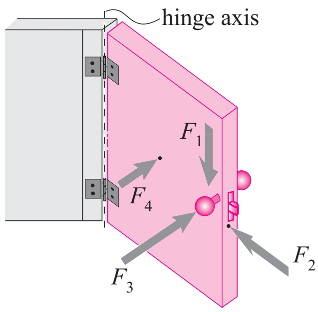


Figure 1.58: Force F_3 has moment about the door's hinge axis. But forces F_1 and F_2 do not. Force F_4, with less moment arm than F_3, rotates the door with less authority.

Example: Try this. Stand facing a partially open door with the front of your body parallel to the plane of the door (a door with no springs is best). Hold the outer edge of the door with one hand (see fig. 1.58).

- Press down (F_1) and note that the door is not opened or closed.
- Push towards the hinge (F_2) and note that the door is not opened or closed.
- Push and pull away and towards your body (F_3) and note how easily you cause the door to rotate.

So, the only force component that tends to rotate the door is perpendicular to the plane of the door (that is the plane defined by the hinge and the line from the hinge to your hand).

- Now, move your hand to the middle of the door, half the distance from the hinge. Note that it takes more force (F_4) to rotate the door with the same authority (push with your pinky if you have trouble feeling the difference).

Now, we know more. For causing rotation, not only is the only potent force perpendicular to the plane defined by the hinge and point of force application, but the potency of that force is increased with distance from the hinge.

We can also decompose $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{P/C}$ into two parts, one parallel to the hinge and one radial, as

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}^{\parallel} + \vec{\boldsymbol{r}}^{\perp}.$$

Clearly $\vec{\boldsymbol{r}}^{\parallel}$ has no effect on how much rotation $\vec{\boldsymbol{F}}$ causes about the axis. If for example the point of force application was moved parallel to the axis a few centimeters, the tendency to rotate would not be changed. Altogether, we have that the moment of the force $\vec{\boldsymbol{F}}$ about the axis $\hat{\boldsymbol{\lambda}}$ through C is given by

$$M_{\lambda \mathrm{C}} = r^{\perp}\, F^{\perp}.$$

The perpendicular distance from the axis to the point of force application is $|\vec{\boldsymbol{r}}^{\perp}|$ and $\vec{\boldsymbol{F}}^{\perp}$ is the part of the force that causes right-handed rotation about the axis. Thus, a moment about an axis is positive if curling the fingers of your right hand gives the sense of rotation when your outstretched thumb is pointing along the axis (as in fig. 1.57). The force of the left person on the teeter totter causes a positive moment about the $\hat{\boldsymbol{k}}$ axis through the hinge.

So long as you interpret the quantities correctly, the freshman physics line

"Moment is distance ($|\vec{\boldsymbol{r}}^{\perp}|$) times force ($|\vec{\boldsymbol{F}}^{\perp}|$)"

perfectly defines moment about an axis, or at least its magnitude.

Three-dimensional geometry is difficult, so a formula for moment about an axis in terms of components would be most useful. The needed formula depends on the 3D moment vector.

The 3D moment vector

We now define the moment of a force $\vec{\boldsymbol{F}}$ applied at P, relative to point C as

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{P/C}} \times \vec{\boldsymbol{F}} \tag{1.28}$$

which we read as 'M is r cross F.' The moment vector is hard to intuit. A look at its components is helpful.

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = (r_y F_z - r_z F_y)\hat{\boldsymbol{\imath}} + (r_z F_x - r_x F_z)\hat{\boldsymbol{\jmath}} + (r_x F_y - r_y F_x)\hat{\boldsymbol{k}}$$

You can recognize the z component of the moment vector as the moment of the force about the $\hat{\boldsymbol{k}}$ axis through C (eqn. 1.27). Similarly the x and y components of $\vec{\boldsymbol{M}}_{\mathrm{C}}$ are the moments about the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ axis through C. So at least the components of $\vec{\boldsymbol{M}}_{\mathrm{C}}$ have intuitive meaning. They are the moments around the positive x, y, and z axes respectively.

Starting with this moment-about-the-coordinate-axes interpretation of the moment vector, each of the three components can be deduced graphically by the moves shown in fig. 1.59. The force is first broken into components. The components are then moved along their lines of action to the coordinate planes. From the resulting picture you can see, say, that the moment about the positive y axis gets a positive contribution from F_x with lever arm r_z and a negative contribution from F_z with lever arm r_x. Thus the y component of $\vec{\boldsymbol{M}}$ is $r_z F_x - r_x F_z$.

Maximum property. Finally, the moment of a force about C is a vector whose magnitude is the product of the distance of the force from C and the magnitude of the force. The direction is the orientation of the axis about which the force has the greatest moment.

More on moment about an axis

We defined moment about an axis geometrically using fig. 1.57 on page 3 as $M_{\hat{\lambda}} = r^r F^{\perp}$. We can now verify that the mixed triple product gives the desired result by guessing the formula and seeing that it agrees with the geometric definition.

$$M_{\lambda\mathrm{C}} = \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{M}}_{/\mathrm{C}} \quad \text{(An inspired guess...)} \tag{1.29}$$

We break both $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ into sums indicated in the figure, use the distributive law, and note that the mixed triple product gives zero if any two of the vectors are parallel. Thus,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{M}}_{/\mathrm{C}} &= \hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}_{\mathrm{P/C}}\times\vec{\boldsymbol{F}}\\
&= \hat{\boldsymbol{\lambda}}\cdot(\vec{\boldsymbol{r}}^{r}+\vec{\boldsymbol{r}}^{\parallel})\times(\vec{\boldsymbol{F}}^{\perp}+\vec{\boldsymbol{F}}^{\parallel}+\vec{\boldsymbol{F}}^{r})\\
&= \hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}^{r}\times\vec{\boldsymbol{F}}^{\perp}+\hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}^{r}\times\vec{\boldsymbol{F}}^{\parallel}+\hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}^{r}\times\vec{\boldsymbol{F}}^{r}\ \ldots\\
&\quad+\hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}^{\parallel}\times\vec{\boldsymbol{F}}^{\perp}+\hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}^{\parallel}\times\vec{\boldsymbol{F}}^{\parallel}+\hat{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{r}}^{\parallel}\times\vec{\boldsymbol{F}}^{r}\\
&= r^{r}F^{\perp}+0+0+0+0+0\\
&= r^{r}F^{\perp}. \quad (\text{... and a good guess too.})
\end{aligned}$$

We can calculate the cross and dot product in any convenient way, say using vector components.

Example: Moment about an axis
Given a force, $\vec{\boldsymbol{F}}_1 = (5\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}})\,\mathrm{N}$ acting at a point P whose position is given by $\vec{\boldsymbol{r}}_{P/O} = (3\hat{\boldsymbol{\imath}}+2\hat{\boldsymbol{\jmath}}-2\hat{\boldsymbol{k}})\,\mathrm{m}$, what is the moment about an axis through the origin O with direction $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}} + \frac{1}{\sqrt{2}}\hat{\boldsymbol{k}}$?

$$\begin{aligned}
M_{\hat{\lambda}} &= (\vec{\boldsymbol{r}}_{P/O}\times\vec{\boldsymbol{F}}_1)\cdot\hat{\boldsymbol{\lambda}}\\
&= [(3\hat{\boldsymbol{\imath}}+2\hat{\boldsymbol{\jmath}}-2\hat{\boldsymbol{k}})\,\mathrm{m}\times(5\hat{\boldsymbol{\imath}}-3\hat{\boldsymbol{\jmath}}+4\hat{\boldsymbol{k}})\,\mathrm{N}]\cdot(\frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}+\frac{1}{\sqrt{2}}\hat{\boldsymbol{k}})\\
&= -\frac{41}{\sqrt{2}}\,\mathrm{m\,N}.
\end{aligned}$$

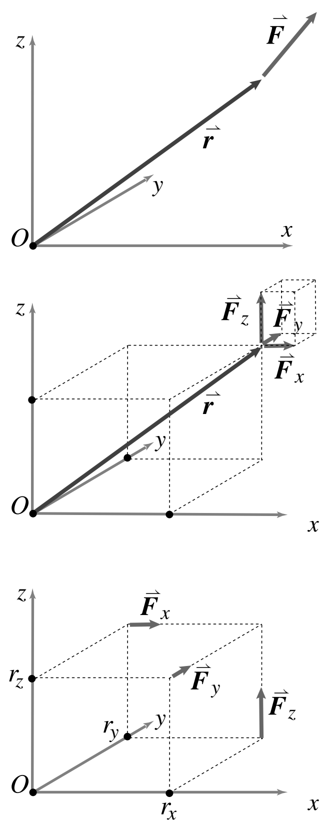


Figure 1.59: The three components of the 3D moment vector are the moments about the three axis. These can be found by sequentially breaking the force into components, sliding each component along its line of action to the coordinate planes, and noting the contribution of each component to moment about each axis. See fig. 1.55 on page 3 for the 2D version of this construction.

The power of our abstract reasoning is apparent when we consider calculating the moment of a force about an axis with two different coordinate systems. Each of the vectors in eqn. 1.4 will have different components in the different systems. Yet the resulting scalar, after all the arithmetic, will be the same no matter what the coordinate system.

Finally, the moment about an axis gives us an interpretation of the moment vector. The direction of the moment vector $\vec{\boldsymbol{M}}_{\mathrm{C}}$ is the direction of the axis through C about which $\vec{\boldsymbol{F}}$ has the greatest moment. The magnitude of $\vec{\boldsymbol{M}}_{\mathrm{C}}$ is the moment of $\vec{\boldsymbol{F}}$ about that axis.

Special optional ways to draw moment vectors

None of the special rotation notations below is needed because moment is a vector like any other. The same is true for the angular velocity vector and the angular momentum vector in dynamics. Nonetheless, sometimes people like to use a notation that suggests the rotational nature of these quantities.

Arced arrow for 2-D moment and angular velocity. In 2D problems in the xy plane, the relevant moment, angular velocity, and angular momentum point straight out or into the plane in the z ($\hat{\boldsymbol{k}}$) direction. A way of drawing this is to use an arced arrow. Wrap the fingers of your right hand in the direction of the arc and your thumb points in the direction of the unit vector that the scalar multiplies. The three representations in fig. 1.60a indicate the same moment vector.

Double headed arrow for 3-D rotations and moments. Two other ways of indicating rotation are to use double-headed arrows or to use an arrow with an arced arrow around it as shown in fig. 1.60b.

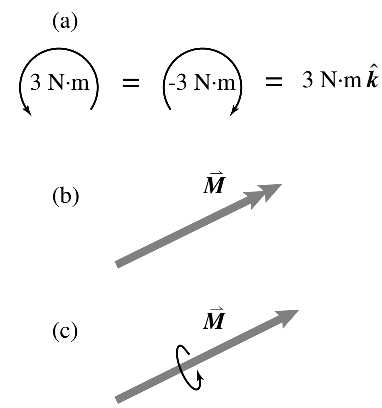


Figure 1.60: Optional drawing methods for moment vectors. (a) shows an arced arrow to represent vectors having to do with rotation in 2 dimensions. Such vectors point directly out of, or into, the page so are indicated with an arc in the direction of the rotation. (b) shows a double-headed arrow for torque or rotation quantity in three dimensions. (c) is another notation for a 3D vector quantity that uses rotations. But a vector is a vector and no special notation is usually used for vectors that concern turning or rotation.