

1.4 Moment and moment about an axis

When you try to move something by pushing it or by turning it. In mechanics, the measure of your pushing is the net force you apply. The measure of your turning is the net *moment*, also sometimes called the net *torque* or net *couple*.

Although concepts involving moment (and rotation) are often harder for beginners than force (and translation), they were understood first. The ancient ‘principle of the lever’, which can be viewed as the root of all mechanics, is the basic idea incorporated by moments.

In this section we will define the moment of a force intuitively, geometrically, and finally using vector algebra. We will do this first in 2 dimensions and then in 3. The main mathematical tool here is the vector cross product from sec. 1.3. To start with, however, we can more or less deduce the concept of moment by generalizing from common experience.

Teeter totter mechanics

The two people weighing down on the teeter totter in fig. 1.53 tend to rotate it about its hinge, the right one clockwise and the left one counter-clockwise. We will now cook up a measure of the tendency of each force to cause rotation about the hinge and call it *the moment of the force about the hinge*.

As is verified a million times a year by young future engineering students,

- To balance a teeter-totter the smaller person needs to be further from the hinge.
- If two people are on one side then the teeter totter is balanced by two similar people an equal distance from the hinge on the other side.
- Two people can balance one similar person by scooting twice as close to the hinge.

These proportionalities generalize to this:

For a force that is perpendicular to the teeter totter, the tendency of the force to cause teeter totter rotation is proportional to the size of the force and to its distance from the hinge.

Further, if someone standing nearby adds a force that is directed towards the hinge (the sideways force in the illustration), that force causes no tendency to rotate. We can deduce more. Because

- any force can be decomposed into a sum of forces, one perpendicular to the teeter totter and the other towards the hinge, and because

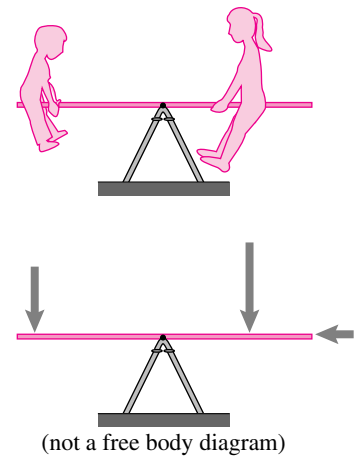


Figure 1.53: On a balanced teeter totter the bigger person gets the short end of the stick. A sideways force directed towards the hinge has no effect on the balance.

Joke: The poet Robert Burns suggested using a Teeter Totter to weigh hogs like this: 1) Get a perfectly symmetric plank and balance it across a saw horse, 2) Put the hog on one end of the plank, 3) pile rocks on the other end of the plank until the plank is again perfectly balanced, 4) Carefully guess the weight of the rocks.

- we assume that the effect of the sum of these forces is the sum of the effects of each separately, and because
- the force towards the hinge has no tendency to rotate,

we can conclude that:

The moment of a force about a hinge is the product of

- its distance from the hinge, and
- the component of the force perpendicular to the line from the hinge to the force.

①The ‘/’ in the subscript of $\vec{M}$ reads as ‘relative to’ or ‘about’. For simplicity we often leave the / out and just write $\vec{M}_C$. So we could say “moment relative to C”, or “moment about C”, or “M sub C”.

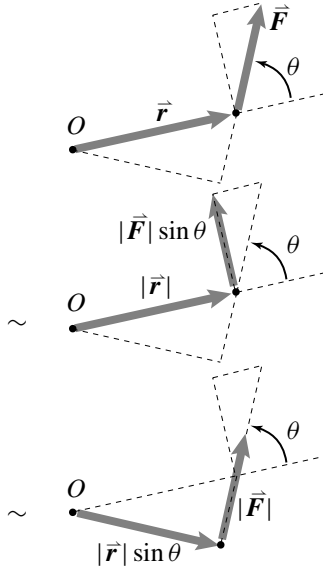


Figure 1.54: The moment of a force $|\vec{r}||\vec{F}|\sin\theta$ is either the product of its distance $|\vec{r}|$ with its perpendicular component $|\vec{F}|\sin\theta$ or of its lever arm (perpendicular distance) $|\vec{r}|\sin\theta$ and the full force $|\vec{F}|$. The $\sim$ means ‘same as’, indicating that the lower two forces and positions have the same moment as the force at top.

Thus, by common sense, we have found the formula for tendency to rotate about a hinge C in 2D, the *moment about C* or *moment with respect to C*.^①

$$M_{/C} = |\vec{r}| (|\vec{F}|\sin\theta) = (|\vec{r}|\sin\theta) |\vec{F}|. \quad (1.25)$$

Here, θ is the angle between $\vec{r}$ (the position of the point of force application relative to the hinge) and $\vec{F}$ (see fig. 1.54). This formula for moment has all of the teeter totter deduced properties. Moment is proportional to r , and to the part of $\vec{F}$ that is perpendicular to $\vec{r}$. The re-grouping as $(|\vec{r}|\sin\theta)$ shows that a force $\vec{F}$ has the same effect if it is applied at a new location that is displaced in the direction of $\vec{F}$. That is, the force $\vec{F}$ can slide along its length without changing its $M_{/C}$ and is *equivalent* in its effect on the teeter totter. The quantity $|\vec{r}|\sin\theta$ is sometimes called the *lever arm* of the force.

Sign conventions. We, like most others, define as positive a moment that causes a counterclockwise rotation. A moment that causes a clockwise rotation is negative. If we define θ appropriately then eqn. (1.25) obeys this sign convention. We define θ as the angle from the positive vector $\vec{r}$ to the positive vector $\vec{F}$ measured counterclockwise. Point the thumb of your right hand up, out of the paper. Your thumb is the moment direction. That is, the moment for an in-plane system is represented by a vector *out* of the plane. Point the fingers of your right hand along $\vec{r}$ and curl them towards the direction of $\vec{F}$ and see how far you have to rotate them. The force caused by the person on the left of the teeter totter has $\theta = 90^\circ$ so $\sin\theta = 1$ and the formula 1.25 gives a positive counterclockwise M . The force of the person on the right has $\theta = 270^\circ$ ($3/4$ of a revolution) so $\sin\theta = -1$ and the formula 1.25 gives a negative M .

In two dimensions moment is really a scalar concept, it is either positive or negative. In three dimensions moment is a vector. But even in 2D we find it easier to keep track of signs if we treat moment as a vector. In the xy plane, the 2D moment is a vector in the $\hat{k}$ direction (straight out of the plane). So eqn. 1.25 becomes

$$\vec{M}_{/C} = |\vec{r}| |\vec{F}| \sin\theta \hat{k}. \quad (1.26)$$

If you curl the fingers of your *right hand* in the direction of rotation caused by a force your thumb points in the direction of the moment vector.

2D moment by components

We can use the component form of the 2D cross product to find a component form for the moment $\vec{M}_{/C}$ of eqn. 1.26. Given $\vec{F} = F_x\hat{i} + F_y\hat{j}$ acting at P, where $\vec{r}_{P/C} = r_x\hat{i} + r_y\hat{j}$, the moment of the force about C is

$$\vec{M}_{/C} = (r_x F_y - r_y F_x)\hat{k}$$

or the moment of $\vec{F}$ about the axis at C is

$$M_C = r_x F_y - r_y F_x. \quad (1.27)$$

We can derive this component formula with the sequence of vector manipulations shown graphically in fig. 1.55.

3D moment about an axis

The concept of moment about an axis is historically, theoretically, and practically important. Moment about an axis describes the principle of the lever, which far precedes Newton's laws. The net moment of a force system about enough different axes determines everything needed in mechanics about a force system. And one can sometimes quickly solve a statics or dynamics problem by considering moment about a judiciously chosen axis.

Let's start by thinking about a teeter totter again. Looking from the side we thought of a teeter totter as a 2D system. But the teeter totter really lives in the 3D world (see fig. 1.56). We now re-interpret the 2D moment M as the moment of the 2D forces about the $\hat{k}$ axis of rotation at the hinge.

It is plain that a force $\vec{F}^{\parallel}$ pushing a teeter totter parallel to the axle causes no tendency to rotate. And we already agreed that a radial force $\vec{F}^r$ (towards the hinge) causes no rotation. So, we see that the moment that a force causes about an axis is the distance of the force from the axis times the part of the force that is neither parallel to the axis nor directed towards the axis.

Now look at this in the more 3-dimensional context of fig. 1.57. Here an imagined axis of rotation is defined as the line through C that is in the $\hat{\lambda}$ direction. A force $\vec{F}$ is applied at P. We can break $\vec{F}$ into a sum of three vectors

$$\vec{F} = \vec{F}^{\parallel} + \vec{F}^r + \vec{F}^{\perp}$$

where $\vec{F}^{\parallel}$ is parallel to the axis, $\vec{F}^r$ is directed along the shortest connection between the axis and P (and is thus perpendicular to the axis) and $\vec{F}^{\perp}$ is out of the plane defined by C, P and $\hat{\lambda}$. By analogy with the teeter totter we see that $\vec{F}^r$ and $\vec{F}^{\parallel}$ cause no tendency to rotate about the axis. So only the $\vec{F}^{\perp}$ contributes.

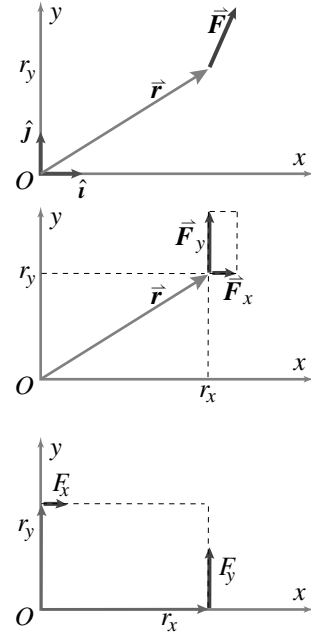


Figure 1.55: The component form of the 2D moment can be found by sequentially breaking the force into components, sliding each component along its line of action to the x and y axis, and adding the moments of the two components.

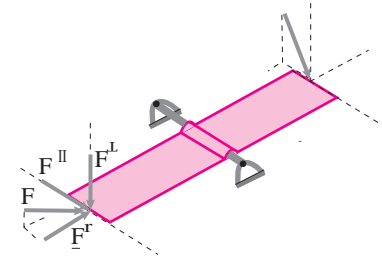


Figure 1.56: Teeter totter with applied forces broken into components parallel to the axis $\vec{F}^{\parallel}$, radial $\vec{F}^r$, and perpendicular to the plane containing the axis and the point of force application $\vec{F}^{\perp}$.

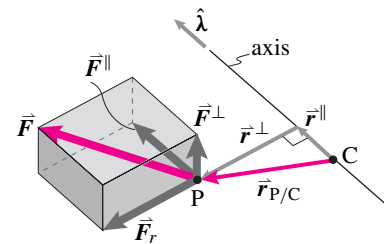


Figure 1.57: Moment about an axis

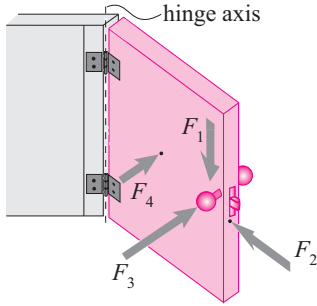


Figure 1.58: Force F_3 has moment about the door's hinge axis. But forces F_1 and F_2 do not. Force F_4 , with less moment arm than F_3 , rotates the door with less authority.

Example: Try this. Stand facing a partially open door with the front of your body parallel to the plane of the door (a door with no springs is best). Hold the outer edge of the door with one hand (see fig. 1.58).

- Press down (F_1) and note that the door is not opened or closed.
- Push towards the hinge (F_2) and note that the door is not opened or closed.
- Push and pull away and towards your body (F_3) and note how easily you cause the door to rotate.

So, the only force component that tends to rotate the door is perpendicular to the plane of the door (that is the plane defined by the hinge and the line from the hinge to your hand).

- Now, move your hand to the middle of the door, half the distance from the hinge. Note that it takes more force (F_4) to rotate the door with the same authority (push with your pinky if you have trouble feeling the difference).

Now, we know more. For causing rotation, not only is the only potent force perpendicular to the plane defined by the hinge and point of force application, but the potency of that force is increased with distance from the hinge.

We can also decompose $\vec{r} = \vec{r}_{P/C}$ into two parts, one parallel to the hinge and one radial, as

$$\vec{r} = \vec{r}^{\parallel} + \vec{r}^{\perp}.$$

Clearly $\vec{r}^{\parallel}$ has no effect on how much rotation $\vec{F}$ causes about the axis. If for example the point of force application was moved parallel to the axis a few centimeters, the tendency to rotate would not be changed. Altogether, we have that the moment of the force $\vec{F}$ about the axis $\hat{\lambda}$ through C is given by

$$M_{\lambda C} = r^{\perp} F^{\perp}.$$

The perpendicular distance from the axis to the point of force application is $|\vec{r}^{\perp}|$ and $\vec{F}^{\perp}$ is the part of the force that causes right-handed rotation about the axis. Thus, a moment about an axis is positive if curling the fingers of your right hand gives the sense of rotation when your outstretched thumb is pointing along the axis (as in fig. 1.57). The force of the left person on the teeter totter causes a positive moment about the $\hat{k}$ axis through the hinge.

So long as you interpret the quantities correctly, the freshman physics line

$$\text{“Moment is distance } (|\vec{r}^{\perp}|) \text{ times force } (|\vec{F}^{\perp}|)\text{”}$$

perfectly defines moment about an axis, or at least its magnitude.

Three-dimensional geometry is difficult, so a formula for moment about an axis in terms of components would be most useful. The needed formula depends on the 3D moment vector.

The 3D moment vector

We now define the moment of a force $\vec{F}$ applied at P, relative to point C as

$$\vec{\mathbf{M}}_C = \vec{\mathbf{r}}_{P/C} \times \vec{\mathbf{F}} \quad (1.28)$$

which we read as ‘M is r cross F.’ The moment vector is hard to intuit. A look at its components is helpful.

$$\vec{\mathbf{M}}_C = (r_y F_z - r_z F_y) \hat{\mathbf{i}} + (r_z F_x - r_x F_z) \hat{\mathbf{j}} + (r_x F_y - r_y F_x) \hat{\mathbf{k}}$$

You can recognize the z component of the moment vector as the moment of the force about the $\hat{\mathbf{k}}$ axis through C (eqn. 1.27). Similarly the x and y components of $\vec{\mathbf{M}}_C$ are the moments about the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ axis through C. So at least the components of $\vec{\mathbf{M}}_C$ have intuitive meaning. They are the moments around the positive x, y, and z axes respectively.

Starting with this moment-about-the-coordinate-axes interpretation of the moment vector, each of the three components can be deduced graphically by the moves shown in fig. 1.59. The force is first broken into components. The components are then moved along their lines of action to the coordinate planes. From the resulting picture you can see, say, that the moment about the positive y axis gets a positive contribution from F_x with lever arm r_z and a negative contribution from F_z with lever arm r_x . Thus the y component of $\vec{\mathbf{M}}$ is $r_z F_x - r_x F_z$.

Maximum property. Finally, the moment of a force about C is a vector whose magnitude is the product of the distance of the force from C and the magnitude of the force. The direction is the orientation of the axis about which the force has the greatest moment.

More on moment about an axis

We defined moment about an axis geometrically using fig. 1.57 on page 3 as $M_{\hat{\lambda}} = r F^{\perp}$. We can now verify that the mixed triple product gives the desired result by guessing the formula and seeing that it agrees with the geometric definition.

$$M_{\lambda C} = \hat{\lambda} \cdot \vec{\mathbf{M}}_{/C} \quad (\text{An inspired guess...}) \quad (1.29)$$

We break both $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ into sums indicated in the figure, use the distributive law, and note that the mixed triple product gives zero if any two of

the vectors are parallel. Thus,

$$\begin{aligned}
 \hat{\lambda} \cdot \vec{M}_{/C} &= \hat{\lambda} \cdot \vec{r}_{P/C} \times \vec{F} \\
 &= \hat{\lambda} \cdot (\vec{r}^r + \vec{r}^{\parallel}) \times (\vec{F}^{\perp} + \vec{F}^{\parallel} + \vec{F}^r) \\
 &= \hat{\lambda} \cdot \vec{r}^r \times \vec{F}^{\perp} + \hat{\lambda} \cdot \vec{r}^r \times \vec{F}^{\parallel} + \hat{\lambda} \cdot \vec{r}^r \times \vec{F}^r \dots \\
 &\quad + \hat{\lambda} \cdot \vec{r}^{\parallel} \times \vec{F}^{\perp} + \hat{\lambda} \cdot \vec{r}^{\parallel} \times \vec{F}^{\parallel} + \hat{\lambda} \cdot \vec{r}^{\parallel} \times \vec{F}^r \\
 &= r^r F^{\perp} + 0 + 0 + 0 + 0 + 0 \\
 &= r^r F^{\perp}. \quad (\dots \text{ and a good guess too.})
 \end{aligned}$$

We can calculate the cross and dot product in any convenient way, say using vector components.

Example: Moment about an axis

Given a force, $\vec{F}_1 = (5\hat{i} - 3\hat{j} + 4\hat{k})$ N acting at a point P whose position is given by $\vec{r}_{P/O} = (3\hat{i} + 2\hat{j} - 2\hat{k})$ m, what is the moment about an axis through the origin O with direction $\hat{\lambda} = \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$?

$$\begin{aligned}
 M_{\hat{\lambda}} &= (\vec{r}_{P/O} \times \vec{F}_1) \cdot \hat{\lambda} \\
 &= [(3\hat{i} + 2\hat{j} - 2\hat{k}) \text{ m} \times (5\hat{i} - 3\hat{j} + 4\hat{k}) \text{ N}] \cdot \left(\frac{1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) \\
 &= -\frac{41}{\sqrt{2}} \text{ m N}.
 \end{aligned}$$

The power of our abstract reasoning is apparent when we consider calculating the moment of a force about an axis with two different coordinate systems. Each of the vectors in eqn. 1.4 will have different components in the different systems. Yet the resulting scalar, after all the arithmetic, will be the same no matter what the coordinate system.

Finally, the moment about an axis gives us an interpretation of the moment vector. The direction of the moment vector $\vec{M}_C$ is the direction of the axis through C about which $\vec{F}$ has the greatest moment. The magnitude of $\vec{M}_C$ is the moment of $\vec{F}$ about that axis.

Special optional ways to draw moment vectors

None of the special rotation notations below is needed because moment is a vector like any other. The same is true for the angular velocity vector and the angular momentum vector in dynamics. Nonetheless, sometimes people like to use a notation that suggests the rotational nature of these quantities.

Arched arrow for 2-D moment and angular velocity. In 2D problems in the xy plane, the relevant moment, angular velocity, and angular momentum point straight out or into the plane in the z ($\hat{k}$) direction. A way of drawing this is to use an arched arrow. Wrap the fingers of your right hand in the direction of the arc and your thumb points in the direction of the unit vector that the scalar multiplies. The three representations in fig. 1.60a indicate the same moment vector.

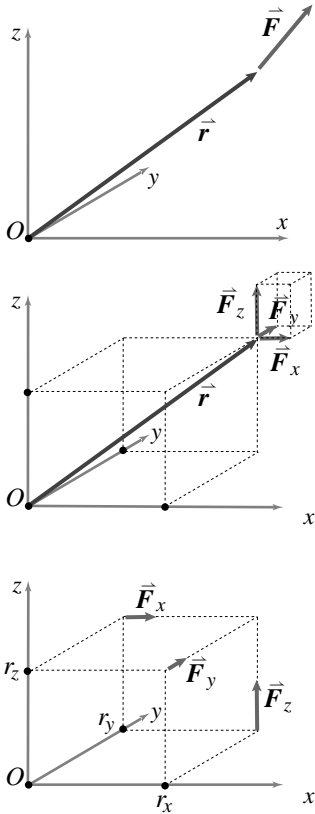


Figure 1.59: The three components of the 3D moment vector are the moments about the three axis. These can be found by sequentially breaking the force into components, sliding each component along its line of action to the coordinate planes, and noting the contribution of each component to moment about each axis. See fig. 1.55 on page 3 for the 2D version of this construction.

Double headed arrow for 3-D rotations and moments. Two other ways of indicating rotation are to use double-headed arrows or to use an arrow with an arced arrow around it as shown in fig. 1.60b.

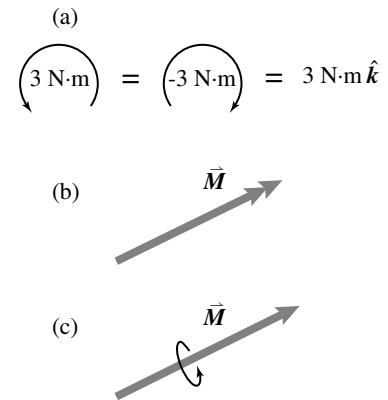


Figure 1.60: Optional drawing methods for moment vectors. (a) shows an arced arrow to represent vectors having to do with rotation in 2 dimensions. Such vectors point directly out of, or into, the page so are indicated with an arc in the direction of the rotation. (b) shows a double-headed arrow for torque or rotation quantity in three dimensions. (c) is another notation for a 3D vector quantity that uses rotations. But a vector is a vector and no special notation is usually used for vectors that concern turning or rotation.

SAMPLE 1.25 Moment of a force: A force $\vec{F} = 1\text{ N}\hat{i} + 20\text{ N}\hat{j}$ acts at point A of an object pinned at O as shown in the figure. The distance OA = 2 m. Find the moment of the force about the pin at point O.

Solution The given force acts through point A on the body. Therefore, we can compute its moment about O as follows.

$$\begin{aligned}
 \vec{M}_O &= \vec{r}_{OA} \times \vec{F} \\
 &= \underbrace{(-2\text{ m} \cdot \cos 60^\circ \hat{i} - 2\text{ m} \cdot \sin 60^\circ \hat{j})}_{\vec{r}_{OA}} \times \underbrace{(1\text{ N}\hat{i} + 20\text{ N}\hat{j})}_{\vec{F}} \\
 &= (-1\text{ m}\hat{i} - \sqrt{3}\text{ m}\hat{j}) \times (1\text{ N}\hat{i} + 20\text{ N}\hat{j}) \\
 &= -20\text{ N}\cdot\text{m}\hat{k} + 1.73\text{ N}\cdot\text{m}\hat{k} \\
 &= -18.27\text{ N}\cdot\text{m}\hat{k}.
 \end{aligned}$$

$$\vec{M}_O = -18.27\text{ N}\cdot\text{m}\hat{k}$$

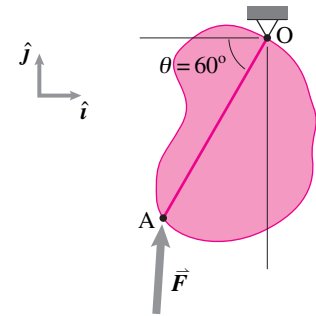


Figure 1.61

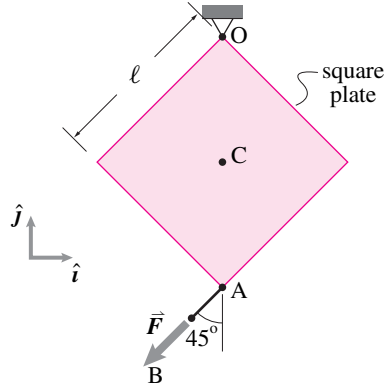


Figure 1.62

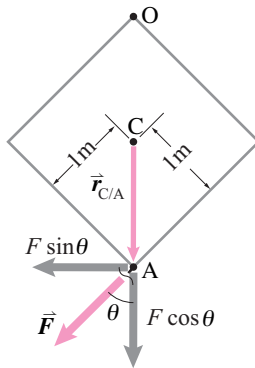


Figure 1.63: The moment $M_{/C} = F \sin \theta \cdot |\vec{r}_{A/C}|$ and acts clockwise. [Note: this is *not* a free-body diagram. It is just an illustration of the force at A. Not shown are the reaction at O and the weight.]

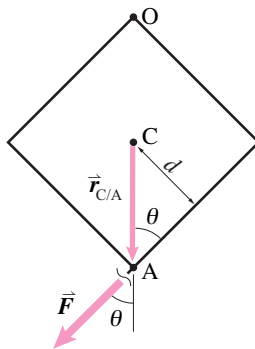


Figure 1.64: The moment $M_{/C} = Fd$ and acts clockwise.

SAMPLE 1.26 A 2 m×2 m square plate hangs from one of its corners as shown in the figure. At the diagonally opposite end, a force of 50 N is applied by pulling on the string AB. Find the moment of the applied force about the center C of the plate using

1. The component of the force perpendicular to $\vec{r}_{A/C}$,
2. The lever arm (the perpendicular distance from C to the force vector), and
3. The vectors $\vec{F}$ and $\vec{r}_{A/C}$.

Solution

1. To find the moment about point C, we need to find the component of $\vec{F}$ perpendicular to CA. From the figure, we see that the desired component is $F \sin \theta$ where $\theta = 45^\circ$. Therefore,

$$M_{/C} = |\vec{F}| \sin \theta |\vec{r}_{A/C}| = 50 \text{ N} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{2} \text{ m} = 50 \text{ N} \cdot \text{m}.$$

The direction of this moment is obtained by curling the fingers of the right hand from $\vec{r}_{A/C}$ towards $\vec{F}$ which points into the page, i.e., $-\hat{k}$ direction. Thus, $\vec{M}_{/C} = -50 \text{ N} \cdot \text{m} \hat{k}$.

$$\vec{M}_{/C} = -50 \text{ N} \cdot \text{m} \hat{k}$$

2. The lever arm is the perpendicular distance from point C to the line of action of $\vec{F}$. This perpendicular distance is $d = |\vec{r}_{A/C}| \sin \theta = \sqrt{2} \text{ m} \cdot (1/\sqrt{2}) = 1 \text{ m}$ (see fig. 1.63). Therefore, the moment of $\vec{F}$ about point C is

$$\begin{aligned} \vec{M}_{/C} &= Fd(-\hat{k}) \\ &= -(50 \text{ N}) \cdot (1 \text{ m}) \hat{k} = -50 \text{ N} \cdot \text{m} \hat{k}. \end{aligned}$$

where the direction of the moment is evident from the right hand rule as pointed out above.

$$\vec{M}_{/C} = -50 \text{ N} \cdot \text{m} \hat{k}$$

3. The moment $\vec{M}_C$ is calculated from $\vec{F}$ and $\vec{r}_{A/C}$ by carrying out the cross product $\vec{r}_{A/C} \times \vec{F}$ in a straightforward manner. For this calculation, we first need to find the vectors $\vec{r}_{A/C}$ and $\vec{F}$:

$$\begin{aligned} \vec{r}_{A/C} &= -CA \hat{j} = -\frac{\ell}{\sqrt{2}} \hat{j} \quad (\text{since } OA = 2 CA = \sqrt{2} \ell) \\ \vec{F} &= F(-\sin \theta \hat{i} - \cos \theta \hat{j}) = -F(\sin \theta \hat{i} + \cos \theta \hat{j}). \end{aligned}$$

Hence,

$$\begin{aligned} \vec{M}_{/C} &= \vec{r}_{A/C} \times \vec{F} \\ &= -\frac{\ell}{\sqrt{2}} \hat{j} \times [-F(\cos \theta \hat{i} + \sin \theta \hat{j})] \\ &= -\frac{\ell}{\sqrt{2}} F \cos \theta \hat{k} \\ &= -\frac{2 \text{ m}}{\sqrt{2}} \cdot 50 \text{ N} \cdot \cos 45^\circ \hat{k} = -50 \text{ N} \cdot \text{m} \hat{k}. \end{aligned}$$

$$\vec{M}_{/C} = -50 \text{ N} \cdot \text{m} \hat{k}$$

SAMPLE 1.27 Moment about an axis: A vertical force of unknown magnitude F acts at point B of a triangular plate ABC shown in the figure. Find the moment of the force about the edge CA of the plate.

Solution The moment of a force $\vec{F}$ about an axis x-x is given by

$$M_{xx} = \hat{\lambda}_{xx} \cdot (\vec{r} \times \vec{F})$$

where $\hat{\lambda}_{xx}$ is a unit vector along the axis x-x, $\vec{r}$ is a position vector from any point on the axis to the applied force. In this problem, the given axis is CA. Therefore, we can take $\vec{r}$ to be $\vec{r}_{AB}$ or $\vec{r}_{CB}$. Here,

$$\hat{\lambda}_{CA} = \frac{\vec{r}_{CA}}{|\vec{r}_{CA}|} = \frac{3(-\hat{i} + \hat{j})}{\sqrt{9+9}} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}.$$

Now, moment about point A is

$$\begin{aligned}\vec{M}_A &= \vec{r}_{AB} \times \vec{F} \\ &= (-2\hat{i} - 3\hat{j}) \times F\hat{k} = 2F\hat{j} - 3F\hat{i}.\end{aligned}$$

Therefore, the moment about CA is

$$\begin{aligned}M_{CA} &= \hat{\lambda}_{CA} \cdot (\vec{r}_{AB} \times \vec{F}) = \hat{\lambda}_{CA} \cdot \vec{M}_A \\ &= \left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) \cdot (-3F\hat{i} + 2F\hat{j}) \\ &= \left(\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right)F = \frac{5}{\sqrt{2}}F.\end{aligned}$$

$$M_{CA} = \frac{5}{\sqrt{2}}F$$

Comments: Note that the sign of M_{CA} depends on whether we use $\hat{\lambda}_{CA}$ or $\hat{\lambda}_{AC}$ in our calculation. Because $\hat{\lambda}_{CA} = -\hat{\lambda}_{AC}$, M_{AC} will turn out to be $-\frac{5}{\sqrt{2}}F$. Since moment is a vector, it has a definite direction. Naturally, $\vec{M}_{CA} = M_{CA}\hat{\lambda}_{CA} = -M_{CA}\hat{\lambda}_{AC}$. Thus, $M_{AC} = -M_{CA}$.

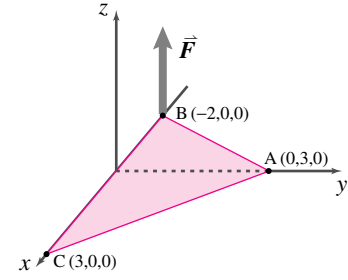
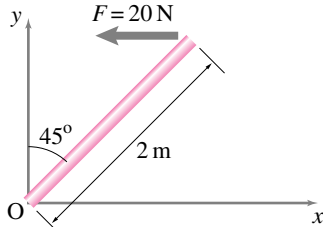


Figure 1.65

1.4 Moment and Moment about an Axis

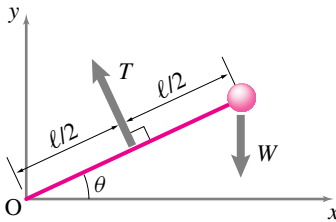
1.4.1 What is the moment $\vec{M}$ produced by a 20 N force $\vec{F}$ acting in the x direction with a lever arm of $\vec{r} = (16 \text{ mm})\hat{j}$?

1.4.2 Find the moment of the force shown on the rod about point O.



Problem 1.4.2

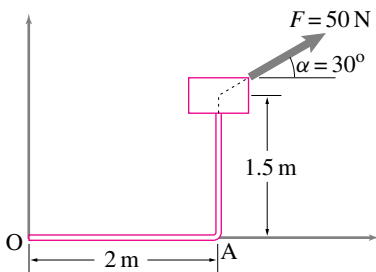
1.4.3 Find the sum of moments of forces $\vec{W}$ and $\vec{T}$ about the origin, given that $W = 100 \text{ N}$, $T = 120 \text{ N}$, $\ell = 4 \text{ m}$, and $\theta = 30^\circ$.



Problem 1.4.3

1.4.4 Find the moment of the force

- about point A
- about point O.



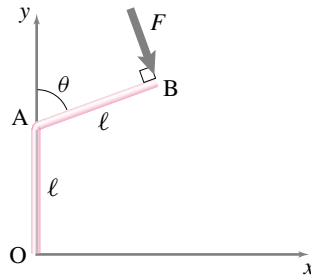
Problem 1.4.4

1.4.5 The line of action of a force $\vec{F} = 20 \text{ N}\hat{j} - 5 \text{ N}\hat{k}$ passes through a point A with coordinates (200 mm, 300 mm, -100 mm). What is the moment $\vec{M}(\vec{r} \times \vec{F})$ of the force about the origin?

1.4.6 Drawing vectors and computing with vectors. In an xyz coordinate system, let point O be the origin. Point A has xyz coordinates $(0\text{m}, 5\text{m}, 12\text{m})$ and point B has xyz coordinates $(4\text{m}, 5\text{m}, 12\text{m})$.

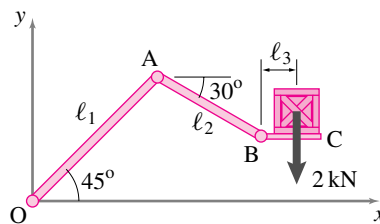
- Make a neat sketch of the vectors OA, OB, and AB.
- Find a unit vector in the direction of OA, call it $\hat{\lambda}_{OA}$.
- Find the force $\vec{F}$ which is 5 N in size and is in the direction of OA.
- What is the angle between OA and OB?
- What is $\vec{r}_{BO} \times \vec{F}$?
- Assume $\vec{F}$ acts at O. What is the moment of $\vec{F}$ about a line parallel to the z axis that goes through point B?

1.4.7 In the figure shown, OA = AB = 2 m. The force $F = 40 \text{ N}$ acts perpendicular to the arm AB. Find the moment of $\vec{F}$ about O, given that $\theta = 45^\circ$. If $\vec{F}$ always acts normal to the arm AB, would increasing θ increase the magnitude of the moment? In particular, what value of θ will give the largest moment?



Problem 1.4.7

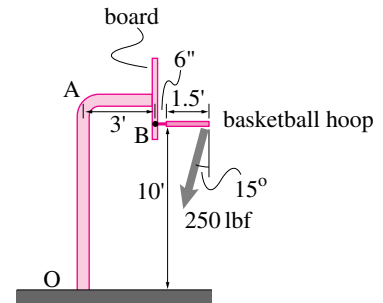
1.4.8 Calculate the moment of the 2 kN payload on the robot arm about (i) joint A, and (ii) joint B, if $\ell_1 = 0.8 \text{ m}$, $\ell_2 = 0.4 \text{ m}$, and $\ell_3 = 0.1 \text{ m}$.



Problem 1.4.8

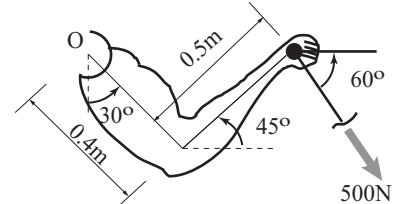
1.4.9 During a slam-dunk, a basketball player pulls on the hoop with a 250 lbf force at point C of the ring as shown in the figure. Find the moment of the force about

- the point of the ring attachment to the board (point B), and
- the root of the pole, point O.



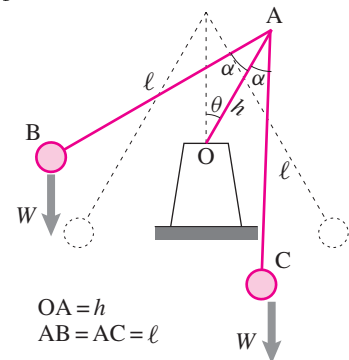
Problem 1.4.9

1.4.10 During weight training, an athlete pulls a weight of 500 N with his arms pulling on a handlebar connected to a universal machine by a cable. Find the moment of the force about the shoulder joint O in the configuration shown.



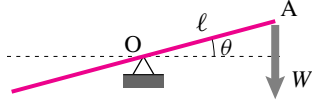
Problem 1.4.10

1.4.11 Find the sum of moments due to the two weights of the teeter-totter when the teeter-totter is tipped at an angle θ from its vertical position. Give your answer in terms of the variables shown in the figure. [Note that AC is not vertical.]



Problem 1.4.11

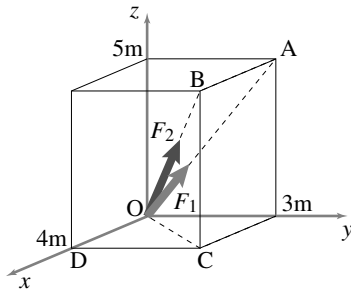
1.4.12 Find the percentage error in computing the moment of $\vec{W}$ about the pivot point O as a function of θ , if the weight is assumed to act normal to the arm OA (a good approximation when θ is very small).



Problem 1.4.12

1.4.13 Vector Calculations and Geometry. The 5 N force $\vec{F}_1$ is along the line OA. The 7 N force $\vec{F}_2$ is along the line OB.

- Find a unit vector in the direction OB. *
- Find a unit vector in the direction OA. *
- Write both $\vec{F}_1$ and $\vec{F}_2$ as the product of their magnitudes and unit vectors in their directions. *
- What is the angle AOB? *
- What is the component of $\vec{F}_1$ in the x-direction? *
- What is $\vec{r}_{DO} \times \vec{F}_1$? ($\vec{r}_{DO} \equiv \vec{r}_{O/D}$ is the position of O relative to D.) *
- What is the moment of $\vec{F}_2$ about the axis DC? *
- Again find the moment of $\vec{F}_2$ about the axis DC. This time use either a different reference point on the axis DC or a different point on the line of action OB. Does the solution agree? [Hint: it should.] *



Problem 1.4.13

1.4.1 What is the moment $\vec{M}$ produced by a 20 N force F acting in the x direction with a lever arm of $\vec{r} = (16 \text{ mm})\hat{j}$?

$$\vec{F} = 20 \text{ N} \hat{i}$$

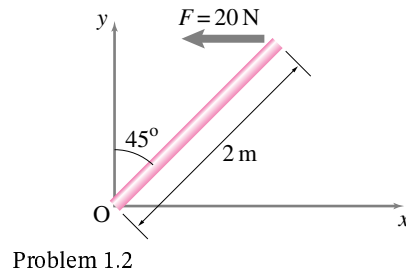
$$\vec{r} = (16 \text{ mm}) \hat{j}$$

$$\vec{M} = \vec{r} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 16 \text{ mm} & 0 \\ 20 \text{ N} & 0 & 0 \end{vmatrix}$$

$$= -32 \text{ N cm}$$

1.4.2 Find the moment of the force shown on the rod about point O.



$$\vec{r} = (2\text{ m} \sin 45^\circ)\hat{i} + (2\text{ m} \cos 45^\circ)\hat{j}$$

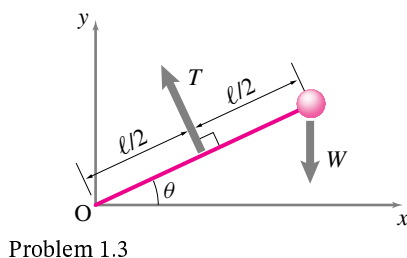
$$= \sqrt{2}\text{ m} \hat{i} + \sqrt{2}\text{ m} \hat{j}$$

$$\vec{F} = -20\text{ N} \hat{i}$$

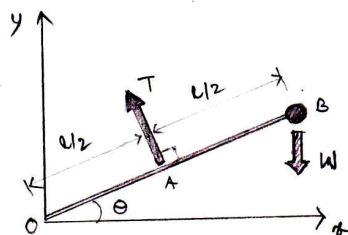
$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2}\text{ m} & \sqrt{2}\text{ m} & 0 \\ -20\text{ N} & 0 & 0 \end{vmatrix}$$

$$= -20\sqrt{2}\text{ Nm} \hat{k}$$

1.4.3 Find the sum of moments of forces $\vec{W}$ and $\vec{T}$ about the origin, given that $W = 100\text{ N}$, $T = 120\text{ N}$, $\ell = 4\text{ m}$, and $\theta = 30^\circ$.



2.4.3



$$\begin{aligned} W &= 100\text{ N} \\ T &= 120\text{ N} \\ \theta &= 30^\circ \\ \ell &= 4\text{ m} \end{aligned}$$

Forces $\vec{W}$ and $\vec{T}$ are acting at points B and A respectively.

Moment due to forces $\vec{W}$ & $\vec{T}$ about O are given by

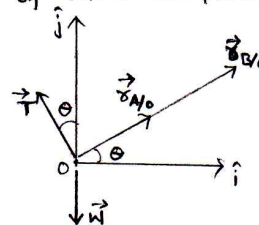
$$\vec{M}_{W/O} = \vec{r}_{B/O} \times \vec{W}$$

$$\vec{M}_{T/O} = \vec{r}_{A/O} \times \vec{T}$$

Writing the forces and vectors in terms of their components:

$$\vec{W} = -100\text{ N } \hat{j}$$

$$\vec{T} = (-T \sin \theta) \hat{i} + (T \cos \theta) \hat{j}$$



$$\vec{r}_{A/O} = (\ell/2 \cos \theta) \hat{i} + (\ell/2 \sin \theta) \hat{j}$$

$$\vec{r}_{B/O} = (\ell \cos \theta) \hat{i} + (\ell \sin \theta) \hat{j}$$

2.4.3 continued

Now finding the moments

$$\begin{aligned}
 \vec{M}_{W/O} &= \vec{r}_{B/O} \times \vec{W} \\
 &= [(L \cos \theta) \hat{i} + (L \sin \theta) \hat{j}] \times (-100 \text{ N}) \hat{j} \\
 &= [4 \cos 30^\circ \hat{i} + 4 \sin 30^\circ \hat{j}] \times (-100 \text{ N}) \hat{j} \\
 &= [3.464 \text{ m} \hat{i} + 2 \text{ m} \hat{j}] \times [-100 \text{ N}] \hat{j} \\
 &= -346.4 \text{ Nm} \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \vec{M}_{T/O} &= \vec{r}_{A/O} \times \vec{T} \\
 &= \left[\left(\frac{L}{2} \cos \theta \right) \hat{i} + \left(\frac{L}{2} \sin \theta \right) \hat{j} \right] \times [(-T \sin \theta) \hat{i} + (T \cos \theta) \hat{j}] \\
 &= \left[\frac{L}{2} \cdot T \cdot (\cos \theta)^2 \hat{k} + \frac{L}{2} \cdot T \cdot (\sin \theta)^2 \hat{k} \right] \\
 &= \left[\frac{4}{2} \cdot 120 \cdot (\cos 30^\circ)^2 \text{ Nm} \hat{k} + \frac{4}{2} \cdot 120 \cdot (\sin 30^\circ)^2 \text{ Nm} \hat{k} \right] \\
 &= 180 \text{ Nm} \hat{k} + 60 \text{ Nm} \hat{k} \\
 &= 240 \text{ Nm} \hat{k}
 \end{aligned}$$

Sum of the moments

$$\begin{aligned}
 \vec{M}_{W/O} + \vec{M}_{T/O} &= -346.4 \text{ Nm} \hat{k} + 240 \text{ Nm} \hat{k} \\
 &= -106.4 \text{ Nm} \hat{k}
 \end{aligned}$$

$$|\vec{W}| = 100 \text{ N}$$

$$|\vec{T}| = 120 \text{ N}$$

$$\theta = 30^\circ$$

$$l = 4 \text{ m}$$

$$\vec{W} = -100 \text{ N } \hat{j}$$

$$\begin{aligned} \vec{T} &= -120 \text{ N } \cos \theta \hat{i} + 120 \text{ N } \sin \theta \hat{j} \\ &= -60\sqrt{3} \text{ N } \hat{i} + 60 \text{ N } \hat{j} \end{aligned}$$

$$\vec{r}_T = \frac{l}{2} \cos \theta \hat{i} + \frac{l}{2} \sin \theta \hat{j}$$

$$= \sqrt{3} \text{ m } \hat{i} + 1 \text{ m } \hat{j}$$

$$\vec{r}_W = l \cos \theta \hat{i} + l \sin \theta \hat{j}$$

$$= 2\sqrt{3} \text{ m } \hat{i} + 2 \text{ m } \hat{j}$$

Find Moments,

$$\vec{r}_W \times \vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sqrt{3} \text{ m} & 2 \text{ m} & 0 \\ 0 & -100 \text{ N} & 0 \end{vmatrix}$$

$$= -200\sqrt{3} \text{ N m } \hat{k}$$

$$\vec{r}_T \times \vec{T} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{3} \text{ m} & 1 \text{ m} & 0 \\ -60\sqrt{3} \text{ N} & 60 \text{ N} & 0 \end{vmatrix}$$

$$= 120\sqrt{3} \text{ N m } \hat{k}$$

$$\text{Sum of Moments} = (-200\sqrt{3} + 120\sqrt{3}) \text{ N m } \hat{k}$$

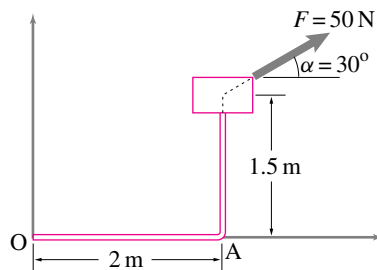
$$= -80\sqrt{3} \text{ N m } \hat{k}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

1.4.4 Find the moment of the force

a) about point A

b) about point O.



Problem 1.4

$$\vec{F} = 50 \text{ N} \cos 30^\circ \hat{i} + 50 \text{ N} \sin 30^\circ \hat{j}$$

$$= 25\sqrt{3} \text{ N} \hat{i} + 25 \text{ N} \hat{j}$$

$$\vec{r}_{r/A} = 1.5 \text{ m} \hat{j}$$

Find Moment Across A and O,

$$\vec{M}_A = \vec{r}_{r/A} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1.5 \text{ m} & 0 \\ 25\sqrt{3} \text{ N} & 25 \text{ N} & 0 \end{vmatrix}$$

$$= -37.5\sqrt{3} \text{ Nm} \hat{k}$$

$$\vec{r}_{r/O} = 2 \text{ m} \hat{i} + 1.5 \text{ m} \hat{j}$$

$$\vec{M}_O = \vec{r}_{r/O} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 \text{ m} & 1.5 \text{ m} & 0 \\ 25\sqrt{3} \text{ N} & 25 \text{ N} & 0 \end{vmatrix}$$

$$= (50 - 37.5\sqrt{3}) \text{ Nm} \hat{k}$$

$$= -14.95 \text{ Nm} \hat{k}$$

1.4.5 The line of action of a force $\vec{F} = 20\text{ N}\hat{j} - 5\text{ N}\hat{k}$ passes through a point A with coordinates (200 mm, 300 mm, -100 mm). What is the moment $\vec{M}$ ($= \vec{r} \times \vec{F}$) of the force about the origin?

$$\vec{r} = 20\text{ cm}\hat{i} + 30\text{ cm}\hat{j} - 10\text{ cm}\hat{k}$$

$$\vec{F} = 20\text{ N}\hat{j} - 5\text{ N}\hat{k}$$

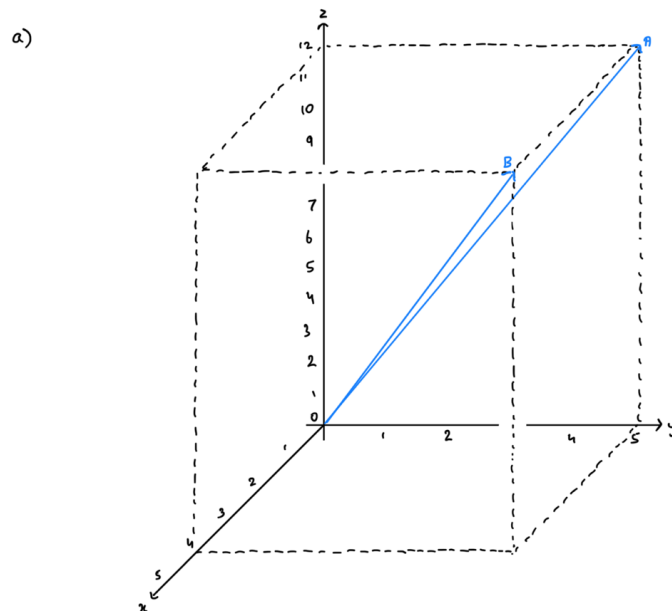
$$\vec{M} = \vec{r} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 20\text{ cm} & 30\text{ cm} & -10\text{ cm} \\ 0 & 20\text{ N} & -5\text{ N} \end{vmatrix}$$

$$= 50\text{ N cm}\hat{i} + 100\text{ N cm}\hat{j} + 400\text{ N cm}\hat{k}$$

1.4.6 Drawing vectors and computing with vectors. In an xyz coordinate system, let point O be the origin. Point A has xyz coordinates $(0\text{m}, 5\text{m}, 12\text{m})$ and point B has xyz coordinates $(4\text{m}, 5\text{m}, 12\text{m})$.

- Make a neat sketch of the vectors $\vec{OA}$, $\vec{OB}$, and $\vec{AB}$.
- Find a unit vector in the direction of $\vec{OA}$, call it $\hat{\lambda}_{OA}$.
- Find the force $\vec{F}$ which is 5 N in size and is in the direction of $\vec{OA}$.
- What is the angle between $\vec{OA}$ and $\vec{OB}$?
- What is $\vec{r}_{BO} \times \vec{F}$?
- Assume $\vec{F}$ acts at O . What is the moment of $\vec{F}$ about a line parallel to the z axis that goes through point B ?



$$\begin{aligned} \vec{r}_{OA} &= 5\text{m}\hat{j} + 12\text{m}\hat{k} \\ |\vec{r}_{OA}| &= \sqrt{(5\text{m})^2 + (12\text{m})^2} \\ &= 13\text{m} \end{aligned}$$

$$\hat{\lambda}_{OA} = \frac{5}{13}\hat{j} + \frac{12}{13}\hat{k}$$

$$\begin{aligned} \vec{F} &= 5\text{N}\hat{\lambda}_{OA} \\ &= \frac{25}{13}\text{N}\hat{j} + \frac{60}{13}\text{N}\hat{k} \end{aligned}$$

$$\begin{aligned} \vec{r}_{OB} &= 4\text{m}\hat{i} + 5\text{m}\hat{j} + 12\text{m}\hat{k} \\ |\vec{r}_{OB}| &= \sqrt{(4\text{m})^2 + (5\text{m})^2 + (12\text{m})^2} \\ &= \sqrt{185}\text{m} \end{aligned}$$

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

Find unit vector in direction of OB,

$$\hat{\lambda}_{OB} = \frac{4}{\sqrt{185}} \hat{i} + \frac{5}{\sqrt{185}} \hat{j} + \frac{12}{\sqrt{185}} \hat{k}$$

$$\theta = \cos^{-1} \left(\frac{\hat{\lambda}_{OA} \cdot \hat{\lambda}_{OB}}{|\hat{\lambda}_{OA}| |\hat{\lambda}_{OB}|} \right)$$

$$= \cos^{-1}(0.955)$$

$$= 17.1^\circ$$

$$c) \vec{r}_{BO} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4\text{ m} & -5\text{ m} & -12\text{ m} \\ 0 & \frac{25}{13}\text{ N} & \frac{60}{13}\text{ N} \end{vmatrix}$$

$$= (167.79 \hat{i} + 18.46 \hat{j} - 7.69 \hat{k}) \text{ Nm}$$

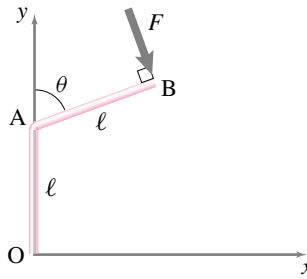
d) Line parallel to z axis,

$$\vec{r} = 12\text{ m} \hat{k}$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 12\text{ m} \\ 0 & \frac{25}{13}\text{ N} & \frac{60}{13}\text{ N} \end{vmatrix}$$

$$= 23.07 \text{ Nm} \hat{i}$$

1.4.7 In the figure shown, $OA = AB = 2\text{ m}$. The force $F = 40\text{ N}$ acts perpendicular to the arm AB . Find the moment of $\vec{F}$ about O , given that $\theta = 45^\circ$. If $\vec{F}$ always acts normal to the arm AB , would increasing θ increase the magnitude of the moment? In particular, what value of θ will give the largest moment?



Problem 1.7

$$\begin{aligned}
 \vec{F} &= 40\text{ N} \cos(360-\theta) \hat{i} + 40\text{ N} \sin(360-\theta) \hat{j} \\
 &= -40\text{ N} \cos\theta \hat{i} - 40\text{ N} \sin\theta \hat{j} \\
 &= -20\sqrt{2}\text{ N} \hat{i} - 20\sqrt{2}\text{ N} \hat{j} \\
 \vec{r}_B &= 2\text{ m} \hat{j} + 2\text{ m} \cos(90-\theta) \hat{i} + 2\text{ m} \sin(90-\theta) \hat{j} \\
 &= 2\text{ m} \hat{j} + 2\text{ m} \sin\theta \hat{i} + 2\text{ m} \cos\theta \hat{j} \\
 &= 2\text{ m} (1 + \cos\theta) \hat{j} + 2\text{ m} \sin\theta \hat{i} \\
 &= 2\text{ m} (1 + 1/\sqrt{2}) \hat{j} + \sqrt{2}\text{ m} \hat{i}
 \end{aligned}$$

Find Moment across O ,

$$\begin{aligned}
 \vec{M}_O &= \vec{r}_B \times \vec{F} \\
 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3.41\text{ m} & 1.41\text{ m} \\ -20\sqrt{2}\text{ N} & -20\sqrt{2}\text{ N} & 0 \end{vmatrix} \\
 &= 40\text{ N} \hat{i} - 40\text{ N} \hat{j} + 96.56\text{ N}\hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \vec{M}_O &= \vec{r}_B \times \vec{F} \\
 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2(1+\cos\theta) & 2\sin\theta \\ -40\cos\theta & -40\sin\theta & 0 \end{vmatrix} \\
 &= +80\sin^2\theta + 80\cos^2\theta + 80\cos\theta + 80\sin^2\theta \\
 &= 80 + 80\cos^2\theta + 80\cos\theta
 \end{aligned}$$

This is maximised when $\cos\theta = 1$, or $\theta = 0$.

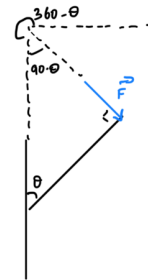
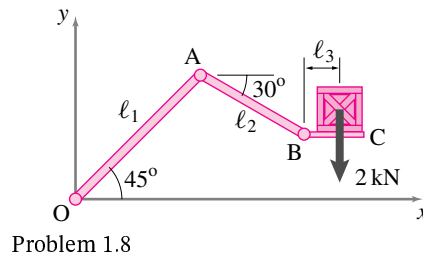


Figure 1:
To determine
angle between
 $\vec{F}$ and horizontal

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1.4.8 Calculate the moment of the 2 kN payload on the robot arm about (i) joint A, and (ii) joint B, if $\ell_1 = 0.8 \text{ m}$, $\ell_2 = 0.4 \text{ m}$, and $\ell_3 = 0.1 \text{ m}$.



$$(i) \vec{r}_{AC} = (\ell_2 \cos 30 + \ell_3) \hat{i} + (\ell_2 \sin 30) \hat{j}$$

$$= (0.2\sqrt{3} + 0.1) \text{ m} \hat{i} + 0.2 \text{ m} \hat{j}$$

$$\vec{F} = -2 \text{ kN} \hat{j}$$

Moment across A,

$$\vec{M}_A = \vec{r}_{AC} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (0.2\sqrt{3}+0.1) \text{ m} & 0.2 \text{ m} & 0 \\ 0 & -2 \text{ kN} & 0 \end{vmatrix}$$

$$= -0.89 \text{ kNm}$$

$$= -890 \text{ Nm}$$

$$(ii) \vec{r}_{BC} = \ell_3 \hat{i}$$

$$= 0.1 \text{ m} \hat{i}$$

Moment across B,

$$\vec{M}_B = \vec{r}_{BC} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.1 \text{ m} & 0 & 0 \\ 0 & -2 \text{ kN} & 0 \end{vmatrix}$$

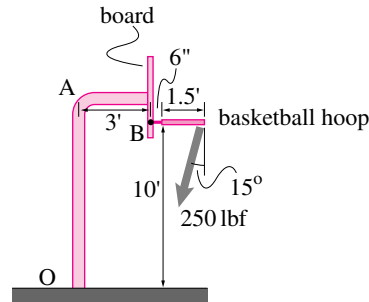
$$= -0.2 \text{ kNm}$$

$$= -200 \text{ Nm}$$

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1.4.9 During a slam-dunk, a basketball player pulls on the hoop with a 250 lbf force at point C of the ring as shown in the figure. Find the moment of the force about

- the point of the ring attachment to the board (point B), and
- the root of the pole, point O.



Problem 1.9

Let C be the point at which the force acts.

$$a) \vec{r}_{BC} = 1.5 \hat{i}$$

$$\begin{aligned} \vec{F} &= (250 \text{ lbf} \times \cos 255^\circ) \hat{i} + (250 \text{ lbf} \times \sin 255^\circ) \hat{j} \\ &= -64.7 \text{ lbf} \hat{i} - 241.48 \text{ lbf} \hat{j} \end{aligned}$$

Moment across B,

$$\begin{aligned} \vec{M}_B &= \vec{r}_{BC} \times \vec{F} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1.5 \text{ in} & 0 & 0 \\ -64.7 \text{ lbf} & -241.48 \text{ lbf} & 0 \end{vmatrix} \\ &= -362.22 \text{ lbf in } \hat{k} \end{aligned}$$

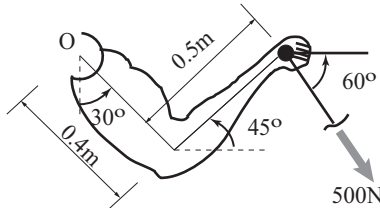
$$\begin{aligned} b) \vec{r}_C &= (3 + 1.5) \text{ in } \hat{i} + 10 \text{ in } \hat{j} \\ &= 4.5 \text{ in } \hat{i} + 10 \text{ in } \hat{j} \end{aligned}$$

Moment across O,

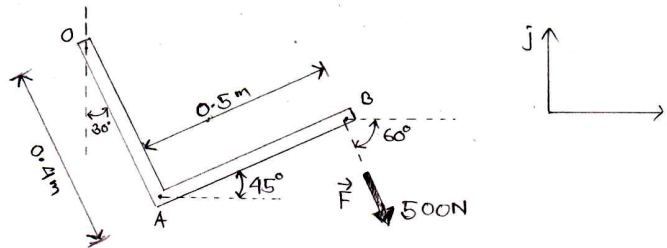
$$\begin{aligned} \vec{M}_O &= \vec{r}_C \times \vec{F} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4.5 \text{ in} & 10 \text{ in} & 0 \\ -64.7 \text{ lbf} & -241.48 \text{ lbf} & 0 \end{vmatrix} \\ &= -1733.66 \text{ lbf in} \end{aligned}$$

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1.4.10 During weight training, an athlete pulls a weight of 500 N with his arms pulling on a handlebar connected to a universal machine by a cable. Find the moment of the force about the shoulder joint O in the configuration shown.



Problem 1.10

2.4.10

The figure above depicts the arm of the athlete

Moment due to force $\vec{F}$ about O is

$$\vec{M}_O = \vec{r}_{B/O} \times \vec{F}$$

$$\vec{r}_{B/O} = \vec{r}_{A/O} + \vec{r}_{B/A}$$

$$\vec{r}_{A/O} = (OA \sin 30^\circ) \hat{i} - (OA \cos 30^\circ) \hat{j}$$

$$\vec{r}_{B/A} = (AB \cos 45^\circ) \hat{i} + (AB \sin 45^\circ) \hat{j}$$

$$\begin{aligned} \text{Now } \vec{r}_{B/O} &= \vec{r}_{A/O} + \vec{r}_{B/A} \\ &= (OA \sin 30^\circ) \hat{i} - (OA \cos 30^\circ) \hat{j} \\ &\quad + (AB \cos 45^\circ) \hat{i} + (AB \sin 45^\circ) \hat{j} \\ &= \frac{OA}{2} \hat{i} - \frac{\sqrt{3} OA}{2} \hat{j} + \frac{AB}{\sqrt{2}} \hat{i} + \frac{AB}{\sqrt{2}} \hat{j} \\ &= \left(\frac{OA}{2} + \frac{AB}{\sqrt{2}} \right) \hat{i} + \left(-\frac{\sqrt{3}}{2} OA + \frac{AB}{\sqrt{2}} \right) \hat{j} \end{aligned}$$

Writing force $\vec{F}$ in terms of components along both axes.

$$\begin{aligned} \vec{F} &= (F \cos 60^\circ) \hat{i} - (F \sin 60^\circ) \hat{j} \\ &= \left(\frac{F}{2} \right) \hat{i} - \left(\frac{\sqrt{3}}{2} F \right) \hat{j} \end{aligned}$$

Finding the moment

$$\begin{aligned} \vec{M}_O &= \vec{r}_{B/O} \times \vec{F} \\ &= \left[\left(\frac{OA}{2} + \frac{AB}{\sqrt{2}} \right) \hat{i} + \left(-\frac{\sqrt{3}}{2} OA + \frac{AB}{\sqrt{2}} \right) \hat{j} \right] \times \left[\frac{F}{2} \hat{i} - \frac{\sqrt{3}}{2} F \hat{j} \right] \end{aligned}$$

2.4.10 continued

$$= \left[\left(\frac{OA}{2} + \frac{AB}{\sqrt{2}} \right) \left(-\frac{\sqrt{3}}{2} F \right) \hat{k} + \left(-\frac{\sqrt{3}}{2} OA + \frac{AB}{\sqrt{2}} \right) \left(\frac{F}{2} \right) (-\hat{k}) \right]$$

$$= \left[\left(\frac{0.4}{2} + \frac{0.5}{\sqrt{2}} \right) \left(-\frac{\sqrt{3}}{2} \times 500 \right) \hat{k} + \left[\left(-\frac{\sqrt{3}}{2} \times 0.4 + \frac{0.5}{\sqrt{2}} \right) \left(\frac{500}{2} \right) \right] (-\hat{k}) \right]$$

$$= [0.5535 \times -433.01] \hat{k} + [(-0.3464 + 0.3535) \times 250] (-\hat{k})$$

$$= -239.67 \hat{k} - 1.775 \hat{k}$$

$$= -241.445 (Nm) \hat{k}$$

Let A be the point where the athlete holds the weight.

$$\vec{r}_A = (0.4 \cos 30 + 0.5 \cos 45) \hat{i} + (-0.4 \sin 30 + 0.5 \sin 45) \hat{j}$$

$$= 0.699 \hat{i} + 0.153 \hat{j}$$

$$\vec{F} = 500 \cos 60 \hat{i} - 500 \sin 60 \hat{j}$$

$$= 250 \hat{i} - 433.01 \hat{j}$$

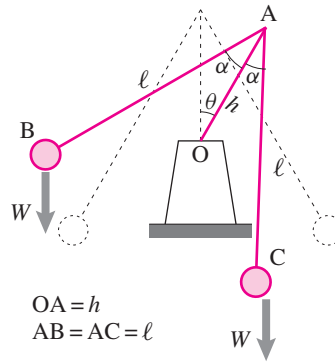
Moment across O ,

$$\vec{M}_O = \vec{r}_A \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.699 \text{ m} & 0.153 \text{ m} & 0 \\ 250 \text{ N} & -433.01 \text{ N} & 0 \end{vmatrix}$$

$$= -340.92 \text{ N m } \hat{k}$$

1.4.11 Find the sum of moments due to the two weights of the teeter-totter when the teeter-totter is tipped at an angle θ from its vertical position. Give your answer in terms of the variables shown in the figure. [Note that AC is not vertical.]



Problem 1.11

$$\vec{r}_A = (h \sin \theta) \hat{i} + (h \cos \theta) \hat{j}$$

Angle of AB with horizontal is determined from Figure 1.

$$\begin{aligned} \vec{r}_B &= \vec{r}_A + \vec{r}_{AB} \\ &= \vec{r}_A + (\ell \cos(270^\circ - \theta - \alpha)) \hat{i} + (\ell \sin(270^\circ - \theta - \alpha)) \hat{j} \\ &= (h \sin \theta + \ell \cos(270^\circ - \theta - \alpha)) \hat{i} + (h \cos \theta + \ell \sin(270^\circ - \theta - \alpha)) \hat{j} \end{aligned}$$

Angle of AC with horizontal is determined from Figure 2.

$$\begin{aligned} \vec{r}_C &= \vec{r}_A + \vec{r}_{AC} \\ &= \vec{r}_A + (\ell \cos(270^\circ + \alpha - \theta)) \hat{i} + (\ell \sin(270^\circ + \alpha - \theta)) \hat{j} \\ &= (h \sin \theta + \ell \cos(270^\circ + \alpha - \theta)) \hat{i} + (h \cos \theta + \ell \sin(270^\circ + \alpha - \theta)) \hat{j} \end{aligned}$$

$$\begin{aligned} \vec{M}_{B/O} &= \vec{r}_B \times \vec{W}_B \\ &= W (h \sin \theta + \ell \cos(270^\circ - \theta - \alpha)) \hat{k} \end{aligned}$$

$$\begin{aligned} \vec{M}_{C/O} &= \vec{r}_C \times \vec{W}_C \\ &= W (h \sin \theta + \ell \cos(270^\circ + \alpha - \theta)) \hat{k} \end{aligned}$$

Find Sum of Moments,

$$\begin{aligned} \vec{M}_O &= \vec{M}_{B/O} + \vec{M}_{C/O} \\ &= W (2h \sin \theta + \ell [\cos(270^\circ - \theta - \alpha) + \cos(270^\circ + \alpha - \theta)]) \hat{k} \end{aligned}$$

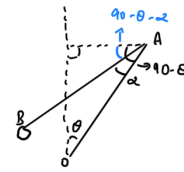


Figure 1:
To find angle of
AB to horizontal

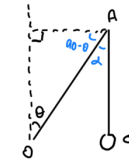
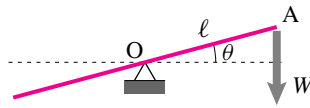


Figure 2:
To find angle of
AC to horizontal

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

1.4.12 Find the percentage error in computing the moment of $\vec{W}$ about the pivot point O as a function of θ , if the weight is assumed to act normal to the arm OA (a good approximation when θ is very

small).



Problem 1.12

$$\vec{M}_O = \vec{r}_A \times \vec{W}$$

$$= |\vec{r}_A| |\vec{W}| \cos \theta \hat{k}$$

$$= l W \cos \theta \hat{k}$$

For small θ , $\cos \theta \approx 1$

$$= l W \hat{k}$$

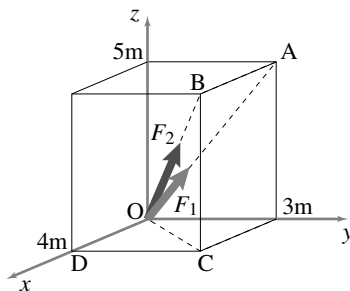
$$\text{Error} = \frac{l W \cos \theta - l W}{l W \cos \theta} \times 100\%$$

$$= (1 - \sec \theta) \times 100\%$$

1.4.13 Vector Calculations and Geometry. The 5 N force $\vec{F}_1$ is along the line OA. The 7 N force $\vec{F}_2$ is along the line OB.

- Find a unit vector in the direction OB.
- Find a unit vector in the direction OA.
- Write both $\vec{F}_1$ and $\vec{F}_2$ as the product of their magnitudes and unit vectors in their directions.
- What is the angle AOB?
- What is the component of $\vec{F}_1$ in the x-direction?
- What is $\vec{r}_{DO} \times \vec{F}_1$? ($\vec{r}_{DO} \equiv \vec{r}_{O/D}$ is the position of O relative to D.)
- What is the moment of $\vec{F}_2$ about the axis DC?
- Again find the moment of $\vec{F}_2$ about the axis DC. This time use either a different reference point

on the axis DC or a different point on the line of action OB. Does the solution agree? [Hint: it should.]



Problem 1.13

Answer:

- $\hat{\lambda}_{OB} = \frac{1}{\sqrt{50}}(4\hat{i} + 3\hat{j} + 5\hat{k})$.
- $\hat{\lambda}_{OA} = \frac{1}{\sqrt{34}}(3\hat{j} + 5\hat{k})$.
- $\vec{F}_1 = \frac{5\text{ N}}{\sqrt{34}}(3\hat{j} + 5\hat{k})$, $\vec{F}_2 = \frac{7\text{ N}}{\sqrt{50}}(4\hat{i} + 3\hat{j} + 5\hat{k})$.
- $\angle AOB = 34.45^\circ$.
- $F_{1x} = 0$.
- $\vec{r}_{DO} \times \vec{F}_1 = \left(\frac{100}{\sqrt{34}}\hat{j} - \frac{60}{\sqrt{34}}\hat{k} \right) \text{ N}\cdot\text{m}$.
- $M_\lambda = \frac{140}{\sqrt{50}} \text{ N}\cdot\text{m}$.
- $M_\lambda = \frac{140}{\sqrt{50}} \text{ N}\cdot\text{m}$ (same as (7)).

Page 2

1.32 Vector Calculations and Geometry. The 5 N force $\vec{F}_1$ is along the line OA. The 7 N force $\vec{F}_2$ is along the line OB.

- Find a unit vector in the direction OB.
- Find a unit vector in the direction OA.
- Write both $\vec{F}_1$ and $\vec{F}_2$ as the product of their magnitudes and unit vectors in their directions.
- What is the angle AOB?
- What is the component of $\vec{F}_1$ in the x-direction?
- What is $\vec{r}_{DO} \times \vec{F}_1$? ($\vec{r}_{DO} \equiv \vec{r}_{O/D}$ is the position of O relative to D.)
- What is the moment of $\vec{F}_2$ about the axis DC? (The moment of a force about an axis parallel to the unit vector $\hat{\lambda}$ is defined as $M_\lambda = \hat{\lambda} \cdot (\vec{r} \times \vec{F})$ where $\vec{r}$ is the position of the point of application of the force relative to some point on the axis. The result does not depend on which point on the axis is used or which point on the line of action of $\vec{F}$ is used.)
- Repeat the last problem using either a different reference point on the axis DC or the line of action OB. Does the solution agree? [Hint: it should.]

problem 1.32

(a) $\hat{\lambda}_{OB} = \frac{\vec{r}_{OB}}{|\vec{r}_{OB}|} = \frac{4\hat{i} + 3\hat{j} + 5\hat{k}}{\sqrt{4^2 + 3^2 + 5^2}} = \frac{1}{\sqrt{50}}(4\hat{i} + 3\hat{j} + 5\hat{k})$

(b) $\hat{\lambda}_{OA} = \frac{\vec{r}_{OA}}{|\vec{r}_{OA}|} = \frac{3\hat{j} + 5\hat{k}}{\sqrt{3^2 + 5^2}} = \frac{1}{\sqrt{34}}(3\hat{j} + 5\hat{k})$

(c) $\vec{F}_1 = \frac{5\text{ N}}{\sqrt{34}}(3\hat{j} + 5\hat{k})$, $\vec{F}_2 = \frac{7\text{ N}}{\sqrt{50}}(4\hat{i} + 3\hat{j} + 5\hat{k})$

(d) $\angle AOB = \cos^{-1} \left[\frac{\vec{r}_{OB} \cdot \vec{r}_{OA}}{|\vec{r}_{OB}| |\vec{r}_{OA}|} \right] = \cos^{-1} \left[\frac{1}{\sqrt{1700}} (9 + 25) \right] = 34.4^\circ$

(e) $\vec{F}_1 \cdot \hat{i} = 0$

(f) $\vec{r}_{DO} \times \vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 0 \\ 0 & 0 & \frac{35}{\sqrt{34}} \end{vmatrix} = \left(\frac{100}{\sqrt{34}}\hat{j} - \frac{60}{\sqrt{34}}\hat{k} \right) \text{ N}\cdot\text{m}$

(g) $M_\lambda = \hat{\lambda} \cdot (\vec{r}_{DO} \times \vec{F}_2) = \begin{vmatrix} 0 & \hat{j} & 0 \\ 0 & 0 & 5\text{m} \\ \frac{28}{\sqrt{50}} & \frac{21}{\sqrt{50}} & \frac{35}{\sqrt{50}} \end{vmatrix} = \frac{140}{\sqrt{50}} \text{ N}\cdot\text{m}$

(h) $M_\lambda = \hat{\lambda} \cdot (\vec{r}_{CB} \times \vec{F}_2) = \begin{vmatrix} 0 & \hat{j} & 0 \\ 0 & 0 & 5\text{m} \\ \frac{28}{\sqrt{50}} & \frac{21}{\sqrt{50}} & \frac{35}{\sqrt{50}} \end{vmatrix} = \frac{140}{\sqrt{50}} \text{ N}\cdot\text{m}$

NOTE THAT SCALAR TRIPLE PRODUCT SAME AS DETERMINANT

Disclaimer: We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.

$$a) \vec{r}_B = (4\hat{i} + 3\hat{j} + 5\hat{k})\text{m}$$

$$|\vec{r}_B| = \sqrt{(4\text{m})^2 + (3\text{m})^2 + (5\text{m})^2}$$

$$= 5\sqrt{2} \text{ m}$$

$$\hat{\lambda}_{B/O} = \frac{1}{5\sqrt{2}} (4\hat{i} + 3\hat{j} + 5\hat{k})$$

$$b) \vec{r}_A = (3\hat{j} + 5\hat{k})\text{m}$$

$$|\vec{r}_A| = \sqrt{(3\text{m})^2 + (5\text{m})^2}$$

$$= \sqrt{34} \text{ m}$$

$$\hat{\lambda}_{A/O} = \frac{1}{\sqrt{34}} (3\hat{j} + 5\hat{k})$$

$$c) \vec{r}_1 = 5\text{N} \hat{\lambda}_{A/O}$$

$$= \frac{5\text{N}}{\sqrt{34}} (3\hat{j} + 5\hat{k})$$

$$\vec{F}_2 = \frac{7\text{N}}{5\sqrt{2}} (4\hat{i} + 3\hat{j} + 5\hat{k})$$

$$d) \theta_{AB} = \cos^{-1} \left[\frac{\hat{\lambda}_{A/O} \cdot \hat{\lambda}_{B/O}}{|\hat{\lambda}_{A/O}| |\hat{\lambda}_{B/O}|} \right]$$

$$= \cos^{-1} \left[\frac{4}{\sqrt{17}} \right]$$

$$= 14.03^\circ$$

$$e) \vec{F}_1 \cdot \hat{c} = \frac{15}{\sqrt{34}} \text{ N}$$

$$= 2.57 \text{ N}$$

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$$f) \vec{r}_{p0} = 4m \hat{i}$$

$$\vec{r}_{p0} \times \vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4m & 0 & 0 \\ 0 & \frac{15}{\sqrt{34}} N & \frac{25}{\sqrt{34}} N \end{vmatrix}$$

$$= -\frac{100}{\sqrt{34}} Nm \hat{j} + \frac{60}{\sqrt{34}} Nm \hat{k}$$

$$= \frac{20}{\sqrt{34}} Nm (-5 \hat{j} + 3 \hat{k})$$

$$g) \vec{r}_{p0} \times \vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4m & 0 & 0 \\ \frac{28N}{5\sqrt{2}} & \frac{21N}{5\sqrt{2}} & \frac{7N}{\sqrt{2}} \end{vmatrix}$$

$$= -14\sqrt{2} N \hat{j} + \frac{42\sqrt{2}}{5} N \hat{k}$$

$$\vec{r}_{oc} = 3 \hat{j}$$

$$M_{loc} = \vec{r}_{oc} \cdot (\vec{r}_{p0} \times \vec{F}_2)$$

$$= -42\sqrt{2} N$$

$$h) \vec{r}_c = 4 \hat{i} + 3 \hat{j}$$

$$M_{loc} = \vec{r}_c \cdot (\vec{r}_{p0} \times \vec{F}_2)$$

$$= -42\sqrt{2} N$$