1.3 Vector cross product

The vector cross product① is a second way of multiplying vectors together (the first was the dot product).

The cross product of vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is written $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$

and is read 'A cross B' (and is *not* read as 'A times B'). The cross product of two vectors is a third vector. It is geometrically defined to be that vector which has magnitude $AB\sin(\theta_{AB})$, where θ_{AB} is the angle between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$, and which is perpendicular to $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ in the direction given by the 'right hand rule' (to be explained below).

The cross product also has a component representation, which is admittedly confusing looking at first:

$$[\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}] = \begin{bmatrix} A_yB_z - A_zBy \\ A_zB_x - A_xB_z \\ A_xB_y - A_yB_x \end{bmatrix}.$$

All of the above is a mouthful which you can't understand at a first reading. It is just put here up top for completeness and conciseness. The ideas are introduced more gradually below.

①**Nomenclature.** The vector cross product is sometimes just called 'the vector product'. In this book we usually call it 'the cross product' because of the times symbol (×), a cross, we use for it. The name 'cross' might also be because the component formula uses 'cross' terms.
The angle between the vectors is typically represented with θ (pronounced 'thay-tuh', thay rhymes with hay) or Ψ (pronounced 'sigh').

Uses of the cross product

The vector cross product is used to define (and calculate) moment, to solve geometry problems and to calculate various quantities associated with rotations in dynamics. Most uses of the cross product used in this book are listed in box 1.5 on page 2.

In this Mechanics Toolset book we treat the cross product as a mathematical calculation and as a geometrical calculation. Deeper understanding will come when you study statics and think about the cross product of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ as a moment vector $\vec{\boldsymbol{M}}$. Comfort with the cross product is a tremendous aid for solving three-dimensional statics problems and for doing all of dynamics. You will probably need to refer back to this section from time to time.

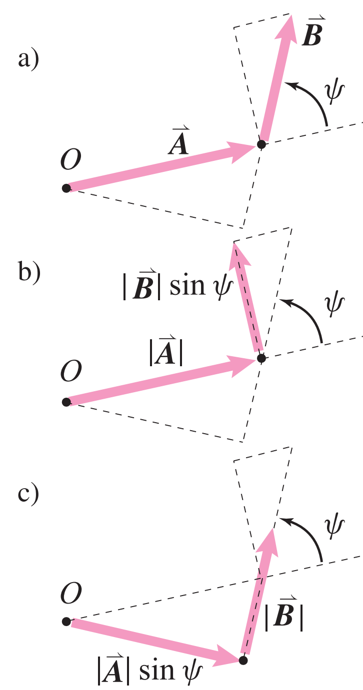


Figure 1.39: **Different interpretations of the 2D cross product.** The cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$, both in the xy plane, is $|\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}|\sin\Psi\hat{\boldsymbol{k}}$. Grouping the $\sin\Psi$ with $|\vec{\boldsymbol{A}}|$ or with $|\vec{\boldsymbol{B}}|$ gives two different geometric interpretations of the 2D cross product. First, you can think of the magnitude of the cross product as being the magnitude of $\vec{\boldsymbol{A}}$ times the projection of $\vec{\boldsymbol{B}}$ perpendicular to $\vec{\boldsymbol{A}}$. That's $|\vec{\boldsymbol{A}}|\,(|\vec{\boldsymbol{B}}|\sin\Psi)$. Or, you can think of it as the magnitude of $\vec{\boldsymbol{B}}$ times the distance $|\vec{\boldsymbol{A}}|\sin\Psi$. This is marked in the tip-to-tail construction in the third picture above. $(|\vec{\boldsymbol{A}}|\sin\Psi)\,|\vec{\boldsymbol{B}}|$.

The 2D cross product

Although the cross product is fundamentally a three-dimensional idea, we start with the two-dimensional version. The 2D cross product is defined as :

$$\underbrace{\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}}_{\text{'A cross B'}} \stackrel{def}{=} |\vec{\boldsymbol{A}}|\,|\vec{\boldsymbol{B}}|\,\sin\theta\,\hat{\boldsymbol{k}}. \tag{1.10}$$

where θ is the amount that $\vec{\boldsymbol{A}}$ would need to be rotated counterclockwise to point in the same direction as $\vec{\boldsymbol{B}}$. An equivalent alternative approach is to define the cross product as

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \stackrel{def}{=} |\vec{\boldsymbol{A}}|\,|\vec{\boldsymbol{B}}|\,\sin\theta\,\hat{\boldsymbol{n}}. \tag{1.11}$$

1.5 Uses of the cross product

Here are the key uses of the cross product in this book. A first time reader is not supposed to understand all of these at the start. They are here for reference and inspiration.

Geometry

1. **3D Normal to a plane.** Find the normal $\vec{N}$ to a plane containing points A,B and C as
$$\vec{N} = \vec{r}_{B/A} \times \vec{r}_{C/A}.$$

2. **3D Unit normal to a plane.** Use $\vec{N}$ above to find the unit normal to a plane containing points A,B and C as
$$\hat{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{\left|\vec{r}_{B/A} \times \vec{r}_{C/A}\right|}$$

3. **3D Distance between a point and a plane.** Use $\hat{n}$ above to find the distance between a point D and the plane containing points A,B and C as
$$d = \quad \vec{r}_{D/A} \cdot \hat{n} \quad = \quad \vec{r}_{D/A} \cdot \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{\left|\vec{r}_{B/A} \times \vec{r}_{C/A}\right|}$$
Replacing $\vec{r}_{D/A}$ with $\vec{r}_{D/B}$ or $\vec{r}_{D/C}$ works too.

4. **3D Distance between two lines.** Find the distance between the line containing the points A and B and the line containing the points C and D as
$$d = \left|\left(\vec{r}_{B/A} \times \vec{r}_{D/C}\right) \cdot \vec{r}_{C/A}\right| / \left|\vec{r}_{B/A} \times \vec{r}_{D/C}\right|.$$
Replacing $\vec{r}_{C/A}$ with $\vec{r}_{D/A}$,$\vec{r}_{C/B}$ or $\vec{r}_{D/B}$ works too.

5. **2D Perpendicular in the xy plane.** Find the in-plane normal $\vec{r}^{\perp}$ to a vector $\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath}$ in the xy plane as
$$\vec{r}^{\perp} = \hat{k} \times \vec{r}.$$
(The vector $\vec{r}^{\perp}$ is also rotation of $\vec{r}$ counterclockwise by 90°.)

6. **2D unit perpendicular in xy plane.** Use $\vec{r}^{\perp}$ to find a unit vector perpendicular to $\vec{r}$ as
$$\hat{n}^{\perp} = \frac{\vec{r}^{\perp}}{|\vec{r}^{\perp}|} = \frac{\hat{k} \times \vec{r}}{\left|\hat{k} \times \vec{r}\right|}.$$

7. **3D distance between point C and a line going through A and B** can with cross product or dot product:
$$d = \left|\vec{r}_{C/A} \times \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}\right| = \left|\vec{r}_{C/A} - \left(\vec{r}_{C/A} \cdot \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}\right)\frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}\right|.$$

8. **2D Distance between a point C and line AB in the plane.** Use $\hat{n}$ to find this distance as
$$d = \left|\hat{n} \cdot \vec{r}_{C/A}\right| = \left|\frac{\left(\hat{k} \times \vec{r}_{B/A}\right) \cdot \vec{r}_{C/A}}{|\hat{k} \times \vec{r}_{B/A}|}\right|.$$
Replacing $\vec{r}_{C/A}$ with $\vec{r}_{C/B}$ works too.

9. **3D Volume V of a parallelepiped** with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$ is
$$V = \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}.$$

Statics

10. **Moment of a force.** Calculate moment $\vec{M}$ of a force $\vec{F}$ at position $\vec{r}$ as
$$\vec{M} = \vec{r} \times \vec{F}$$
.

11. **Moment about an axis.** Calculate moment M of a force $\vec{F}$ at position $\vec{r}$ relative to a point on the axis in the direction of a unit vector $\hat{\lambda}$ as
$$M = \left(\vec{r} \times \vec{F}\right) \cdot \hat{\lambda}.$$

12. **Moment about an axis.** Use the result above to calculate the moment of a force $\vec{F}$ at position C about a line through A and B as
$$M = \left(\vec{r}_{C/A} \times \vec{F}\right) \cdot \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|} = \left(\vec{r}_{C/B} \times \vec{F}\right) \cdot \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}.$$

Kinematics and Dynamics

13. **Relative velocity of two points on a rigid object.** The velocity of point B relative to point A, where both points are on the same rigid object with angular velocity $\vec{\omega}$ is
$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}.$$

14. **Angular momentum of a particle** with velocity $\vec{v}$, mass m and at position $\vec{r}$ is
$$\vec{H} = m\vec{r} \times \vec{v}.$$

15. **Centripetal acceleration** of point B relative to point A on the same rigid object with angular velocity $\vec{\omega}$ is
$$\text{Centripetal part of } \vec{a}_{B/A} = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right).$$

16. **Relative acceleration due to angular acceleration.** The relative acceleration of points A and B on the same rigid object with angular acceleration $\vec{\alpha}$ is
$$\text{Contribution of angular acceleration to } \vec{a}_{B/A} = \vec{\alpha} \times \vec{r}_{B/A}.$$

17. **Coriolis acceleration** of a particle moving at velocity $\vec{v}_{rel}$ relative to a rotating rigid object is
$$\vec{a}_{Coriolis} = 2\vec{\omega} \times \vec{v}_{rel}.$$

n. Variants and extensions of the kinematics and dynamics formulas are given in the tables at the back of the Dynamics book.

with θ defined to be less than 180° and $\hat{\boldsymbol{n}}$ defined as the unit vector pointing in the direction of the thumb when the fingers are curled from the direction of $\vec{\boldsymbol{A}}$ towards the direction of $\vec{\boldsymbol{B}}$.

Study of fig. 1.39 should convince you that the 2D definition of cross product in eqn.1.10 obeys these standard algebra rules (for any 3 2D vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ and any scalar d):

$$\begin{aligned} d(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) &= (d\vec{\boldsymbol{A}})\times\vec{\boldsymbol{B}} = \vec{\boldsymbol{A}}\times(d\vec{\boldsymbol{B}}) \\ \vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}) &= \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}+\vec{\boldsymbol{A}}\times\vec{\boldsymbol{C}}. \end{aligned}$$

A difference between the algebra rules for scalar multiplication and vector cross product multiplication is that for scalar multiplication $AB = BA$ whereas for the cross product $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} \neq \vec{\boldsymbol{B}}\times\vec{\boldsymbol{A}}$ (because the definition of θ in eqn. 1.10 and $\hat{\boldsymbol{n}}$ in 1.11 depends on order). In particular $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}}\times\vec{\boldsymbol{A}}$.

Because the magnitude of the cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is the magnitude of $\vec{\boldsymbol{A}}$ times the magnitude of the projection of $\vec{\boldsymbol{B}}$ in the direction perpendicular to $\vec{\boldsymbol{A}}$ (as shown in the top two illustrations of fig. 1.54) you can think of the cross product as a measure of how much two vectors are perpendicular to each other. In particular,

$$\begin{aligned} \text{if } \vec{\boldsymbol{A}} \text{ is perpendicular to } \vec{\boldsymbol{B}} \quad \Rightarrow \quad |\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}| &= |\vec{\boldsymbol{A}}|\,|\vec{\boldsymbol{B}}|, \text{and} \\ \text{if } \vec{\boldsymbol{A}} \text{ is parallel to } \vec{\boldsymbol{B}} \quad \Rightarrow \quad |\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}| &= 0. \end{aligned}$$

For example, $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{0}}$, and $\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}$.

Component form for the 2D cross product

Just like the dot product, the cross product can be expressed using components. As can be verified by writing $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{B}} = B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}$ and using the distributive rules:

$$\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = (A_xB_y - B_xA_y)\hat{\boldsymbol{k}}. \tag{1.12}$$

Determinant mnemonic. Some people remember this formula by putting the components of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ into a matrix and calculating the determinant $\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$. If you number the components of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ (*e.g.*, $[\vec{\boldsymbol{A}}]_{x_1x_2} = [A_1, A_2]$), the cross product is $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = (A_1B_2 - A_2B_1)\hat{\boldsymbol{e}}_3$, or "first times second minus second times first."

Example: Given that

$$\begin{aligned} & & \vec{\boldsymbol{A}} &= 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} \\ &\text{and} & \vec{\boldsymbol{B}} &= 10\hat{\boldsymbol{\imath}} + 20\hat{\boldsymbol{\jmath}} \\ &\text{then} & \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} &= (1\cdot 20 - 2\cdot 10)\hat{\boldsymbol{k}} = 0\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}. \end{aligned}$$

For vectors with just a few components it is often most convenient to use the distributive rule directly.

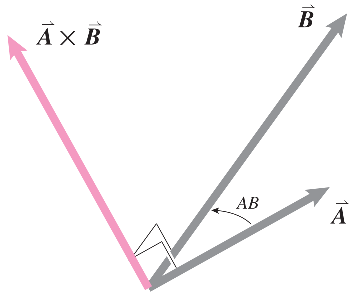


Figure 1.40: The cross product of $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$ in the direction given by the right hand rule. The magnitude of $\vec{A} \times \vec{B}$ is $AB \sin\theta_{AB}$.

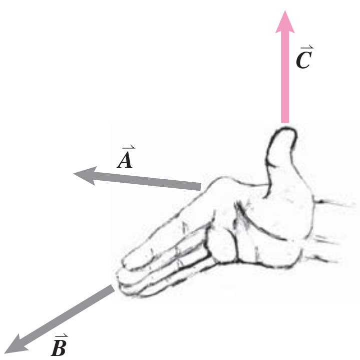


Figure 1.41: The right hand rule for determining the direction of the cross product of two vectors. $\vec{C} = \vec{A} \times \vec{B}$.

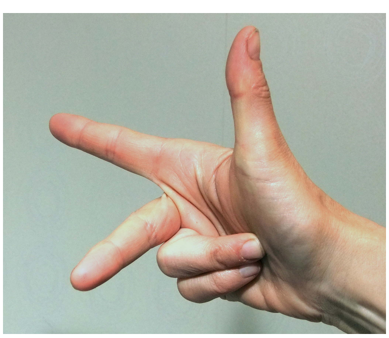


Figure 1.42: Another way to use your right hand for the 'right hand' rule. Set thumb, pointer and middle finger mutually orthogonal to each other. Pointer cross middle finger is thumb. Or, shoot the gun, that is, lower your thumb and the direction of 'thumb cross pointer' is the middle finger.

Example: Given that

$$
\begin{aligned}
& & \vec{A} &= 7\hat{\imath} \\
&\text{and} & \vec{B} &= 37.6\hat{\imath} + 10\hat{\jmath} \\
&\text{then} & \vec{A} \times \vec{B} &= (7\hat{\imath}) \times (37.6\hat{\imath} + 10\hat{\jmath}) = (7\hat{\imath}) \times (37.6\hat{\imath}) + (7\hat{\imath}) \times (10\hat{\jmath}) \\
& & &= \vec{0} + 70\hat{k} = 70\hat{k}.
\end{aligned}
$$

There are many ways of calculating a 2D cross product

You have several options for calculating the 2D cross product. Which you choose depends on taste and convenience. You can use

- the geometric definition directly,
- the first times the perpendicular part of the second (distance times perpendicular component of force),
- the second times the perpendicular part of the first (lever arm times the force),
- components, or
- break each of the vectors into a sum of vectors and use the distributive rule.

The 3D vector cross product

In contrast to the dot product, which gives a scalar and measures how much two vectors are parallel, the cross product is a vector and measures how much they are perpendicular.

The cross product is defined by:

$$
\vec{A} \times \vec{B} \overset{def}{=} |\vec{A}||\vec{B}| \sin\theta_{AB}\, \hat{n} \tag{1.13}
$$

where $|\hat{n}| = 1$,
$\hat{n} \perp \vec{A}$,
$\hat{n} \perp \vec{B}$,
$0 \le \theta_{AB} \le \pi$, and
$\hat{n}$ is in the direction given by the right hand rule, that is, in the direction of the right thumb when the fingers of the right hand are pointed in the direction of $\vec{A}$ and then wrapped towards the direction of $\vec{B}$.

If $\vec{A}$ and $\vec{B}$ are perpendicular then θ_{AB} is $\pi/2$, $\sin\theta_{AB} = 1$, and the magnitude of the cross product is AB. If $\vec{A}$ and $\vec{B}$ are parallel then θ_{AB} is 0, $\sin\theta_{AB} = 0$ and the cross product is $\vec{0}$ (the zero vector). This is why we say the cross product is a measure of the degree of orthogonality of two vectors.

Using the definition above you should be able to verify to your own satisfaction that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$. Applying the definition to the standard base unit vectors you can see that $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, and $\hat{k}\times\hat{\imath} = \hat{\jmath}$ (fig. 1.43).

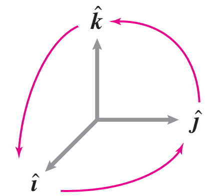


Figure 1.43: Mnemonic device to remember the cross product of the standard base unit vectors.

The geometric definition above and the geometric (tip to tail) definition of vector addition imply that the cross product follows the distributive rule, as explained in box 1.7 on page 9.

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}.$$

Applying the distributive rule to the cross products of $\vec{A} = A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$ leads to the algebraic formula for the Cartesian components of the cross product.

$$\begin{aligned}
\vec{A}\times\vec{B} \quad = \quad & [A_yB_z - A_zB_y]\hat{\imath} && (1.14)\\
& +[A_zB_x - A_xB_z]\hat{\jmath} && (1.15)\\
& +[A_xB_y - A_yB_x]\hat{k} && (1.16)\\
& && (1.17)
\end{aligned}$$

How the distributive rule is used to derive the component formula for cross product is also described in box 1.7 on page 9.

Memory aids for the component formula for cross product. There are various mnemonics for remembering the component formula for cross products. The most common is to calculate a 'determinant' of the 3×3 matrix with one row given by $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the other two rows the components of $\vec{A}$ and $\vec{B}$.

$$\vec{A}\times\vec{B} = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z \end{vmatrix}$$

Some key facts about cross products. The following identities and special cases of cross products are worth knowing well:

- $(a\vec{A})\times\vec{B} = \vec{A}\times(a\vec{B}) = a(\vec{A}\times\vec{B})$ (a distributive law)
- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (the cross product is not commutative!)
- $\vec{A}\times\vec{B} = \vec{0}$ if $\vec{A}\parallel\vec{B}$ (parallel vectors have zero cross product)
- $|\vec{A}\times\vec{B}| = AB$ if $\vec{A}\perp\vec{B}$
- $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\imath} = \hat{\jmath}$ (assuming the x, y, z coordinate system is right handed — if you use your right hand and point your fingers along the positive x axis, then curl them towards the positive y axis, your thumb will point in the same direction as the positive z axis.)

- $\hat{\imath}' \times \hat{\jmath}' = \hat{k}'$, $\quad \hat{\jmath}' \times \hat{k}' = \hat{\imath}'$, $\quad \hat{k}' \times \hat{\imath}' = \hat{\jmath}'$
  (assuming the $x'y'z'$ coordinate system is right handed.)
- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$, $\qquad \hat{\imath}' \times \hat{\imath}' = \hat{\jmath}' \times \hat{\jmath}' = \hat{k}' \times \hat{k}' = \vec{0}$

The mixed triple product

The 'mixed triple product' of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is so called because it mixes both the dot product and cross product in a single expression. The mixed triple product is also sometimes called the scalar triple product because its value is a scalar. The *mixed triple product* of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is defined by and written as

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

and pronounced 'A dot B cross C.' The parentheses () are sometimes omitted, *i.e.,*

$$\vec{A} \cdot \vec{B} \times \vec{C},$$

because the wrong grouping $(\vec{A} \cdot \vec{B}) \times \vec{C}$ is nonsense (you can't take the cross product of a scalar with a vector) . It is apparent that one way of calculating the mixed triple product is to calculate the cross product of $\vec{B}$ and $\vec{C}$ and then to take the dot product of that result with $\vec{A}$. Some people use the notation $\left(\vec{A}, \vec{B}, \vec{C}\right)$ for the mixed triple product but it will not occur again in this book.

The mixed triple product has the same value if one takes the cross product of $\vec{A}$ and $\vec{B}$ and then the dot product of the result with $\vec{C}$. That is $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$. This identity can be verified using the geometric description below, or by looking at the (complicated) expression for the mixed triple product of three general vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in terms of their components as calculated the two different ways. One thus obtains the string of results

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = -\vec{B} \times \vec{A} \cdot \vec{C} = -\vec{B} \cdot \vec{A} \times \vec{C} = \ldots$$

The minus signs in the above expressions follow from the cross product identity that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

The mixed triple product has various geometric interpretations, one of them is that $\vec{A} \cdot \vec{B} \times \vec{C}$ is (plus or minus) the volume of the parallelepiped, the crooked shoe box, edged by $\vec{A}$, $\vec{B}$ and $\vec{C}$ as shown in fig. 1.44.

Another way of calculating the value of the mixed triple product is with the determinant of a matrix whose rows are the components of the vectors.

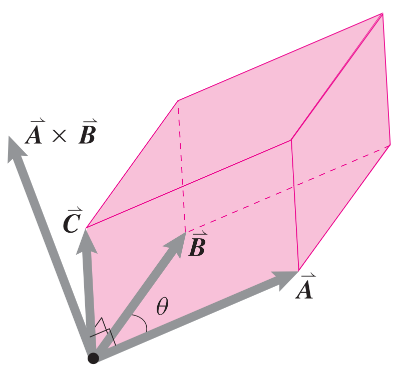


Figure 1.44: One interpretation of the mixed triple product of $\vec{A} \times \vec{B} \cdot \vec{C}$ is as the volume (a scalar) of a parallelepiped with $\vec{A}$, $\vec{B}$, and $\vec{C}$ as the three edges emanating from one corner. This interpretation only works if $\vec{A}$, $\vec{B}$, and $\vec{C}$ are taken in the appropriate order, otherwise $\vec{A} \times \vec{B} \cdot \vec{C}$ is minus the volume which is calculated.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{array}{r} A_x(B_yC_Z - B_zC_y) \\ +A_y(B_zC_x - B_xC_z) \\ +A_z(B_xC_y - B_yC_x) \end{array}$$

The mixed triple product of three vectors is zero if[2]

- any two of them are parallel, or
- all three of the vectors have one common plane.

[2] In the language of linear algebra, the mixed triple product of three vectors is zero if the vectors are linearly dependent.

Uses of the mixed triple product. The mixed triple product (or scalar triple product) is useful for calculating the moment of a force about an axis and for related dynamics quantities.

Vector triple product

A different triple product, sometimes called the 'vector triple product', $\vec{A}\times(\vec{B}\times\vec{C})$, is discussed later (in the Dynamics book).

Cross products and computers

The components of the cross product can be calculated with computer code that may look something like this.

```
A = [  1  2  5 ]
B = [ -2  4 19 ]
C = [ ( A(2)*B(3) - A(3)*B(2) ) ...
      ( A(3)*B(1) - A(1)*B(3) ) ...
      ( A(1)*B(2) - A(2)*B(1) ) ]
```

giving the result `C=[18  -29  8]`. Many computer languages have a shorter way to write the cross product like `cross(A,B)`. The mixed triple product might be calculated by assembling a 3×3 matrix of rows and then taking a determinant like this:

```
      A   = [  1  2  5 ]
      B   = [ -2  4 19 ]
      C   = [ 32  4  5 ]
   matrix = [A ;     % Each row under the one above
             B ;
             C   ]
mixedprod = det(matrix)
```

giving the result `mixedprod = 500`. A versatile computer language might allow a command like `dot( A, cross(B,C) )` to calculate the mixed triple product.

1.6 The cross product of vectors as matrix multiplication

For hand calculations in statics, this box is probably not useful. This box is for the theoretically inclined, and for people interested in doing complex dynamics calculations on a computer. We assume here that you know some linear algebra.

The cross product between two vectors $\vec{a}$ and $\vec{b}$ gives a new vector

$$\vec{c} = \vec{a} \times \vec{b}$$

All three of these vectors have components:

$$[\vec{a}]_{xy} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad [\vec{b}]_{xy} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

and

$$[\vec{c}]_{xy} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}.$$

The components of $\vec{c}$ can be calculated from those of $\vec{a}$ and $\vec{b}$ using the component rules from this section as

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}.$$

Because the cross product $\vec{a} \times \vec{b}$ is linear in $\vec{b}$ (meaning $\vec{a} \times (\vec{b}_1 + \vec{b}_2) = \vec{a} \times \vec{b}_1 + \vec{a} \times \vec{b}_2$, *etc.*) we can represent the cross product as some matrix times $\vec{b}$.

We now define a matrix $[A]$, that does the job. We associate the the vector $\vec{a}$ with the matrix $[A]$ as follows:

$$[A] \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \tag{1.18}$$

$[A]$ is an anti-symmetric matrix (also called a skew symmetric matrix), with

$$[A]' = -[A] \qquad \text{(A transpose is minus A).}$$

The terms of $[A]$ on the diagonal are zero and those off the diagonal are negative of their corresponding (transposed) terms. The conversion rule (Equation 1.18) that takes a vector and puts its components in a matrix in the right place we can call $[\mathcal{S}]$ thus

$$[A] = [\mathcal{S}(\vec{a})].$$

$[A]$ is the anti-symmetric matrix defined in terms of the components of $\vec{a}$ by $[\mathcal{S}(\vec{a})]$.

Just multiplying out the terms we see that

$$\mathcal{S}(\vec{a}) \times \vec{b} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{[\mathcal{S}(\vec{a})]} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{1.19}$$

$$= \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}. \tag{1.20}$$

So we get our main result:

$$[\vec{a} \times \vec{b}]_{xyz} = [\mathcal{S}(\vec{a})][\vec{b}]_{xyz} = [A][\vec{b}]_{xyz}.$$

That is, the components of the cross product of $\vec{a}$ and $\vec{b}$ can be found by multiplying the matrix $[A]$ by the components of $\vec{b}$.

Writing the cross product as a matrix multiplication is sometimes useful for dynamics when the first vector $\vec{a}$ is $\vec{\omega}$ (see box 16.5 on page 773). The advantage of the matrix representation over the cross product is that matrix multiplication satisfies the associative rule

$$[A]([B][C]) = ([A][B])[C]$$

whereas the vector cross product does not:

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

Computer calculation

One could write a short program (computer function) that does the conversion from vector $\vec{a}$ to matrix $[A] = \mathcal{S}(\vec{a})$ and call it `skew`. That is, *skew* would calculate eqn. (1.18). Using *skew* we could carry out a cross product like this (see box 3.1 on page 222)

`a = [1 2 3]'`	Define vector $\vec{a}$ with components
`b = [4 5 6]'`	Define vector $\vec{b}$ with components
`A = skew(a)`	Use the function `skew`' to find A
`c = A*b`	Calculate the cross product $\vec{c} = \vec{a} \times \vec{b}$ using ordinary matrix multiplication.

For just one cross product this would be silly. But for a long calculation involving various vectors and matrices it often makes things simpler.

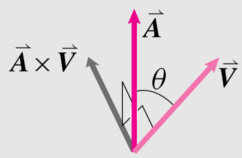
1.7 The cross product: from geometry to components

Why is the 3D vector given by the component formula for the cross product ((1.14) on page 5)

$$[\vec{A}\times\vec{B}]_{xyz} = \begin{bmatrix} (A_yB_z - A_zB_y) \\ (A_zB_x - A_xB_z) \\ (A_xB_y - A_yB_x) \end{bmatrix} \tag{1.21}$$

the same vector as that given by the geometric definition ((1.13) on page 4),

$$\vec{A}\times\vec{V} \equiv |\vec{A}||\vec{V}|\sin\theta_{AV}\hat{\boldsymbol{n}}_{\perp AV}?$$



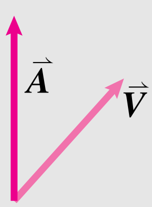
That these different-looking formulas should give the same answer is not obvious. Here we show why. Although the result is often used, the reasoning is not. But for logical completeness, and to entertain the curious, we present it here. We assume we know the geometric definition and want to find the component formula (1.21).

The reasoning has two big steps:

- First we will show that the geometric definitions of the vector cross product 'distributes' over vector addition.
- Then we apply the distributive rule to get our component formula.

A new definition of cross product

Start with $\vec{A}$ and $\vec{V}$:

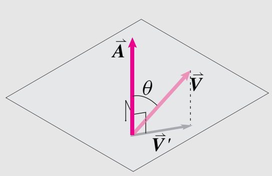


Now we will show that the geometric formula

$$\vec{A}\times\vec{V} \equiv |\vec{A}||\vec{V}|\sin\theta_{AV}\hat{\boldsymbol{n}}_{\perp AV}$$

is equivalent to a sequence of three operations: Project, rotate and stretch.

1. **Project $\vec{V}$ on to the plane P (normal to $\vec{A}$).** The projection of $\vec{V}$ onto the plane orthogonal to $\vec{A}$ is $\vec{V}'$.

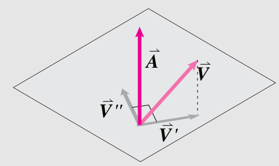


The magnitude of $\vec{V}'$ is

$$|\vec{V}'| = |\vec{V}|\sin\theta_{AV}.$$

$\vec{V}'$ is in the plane defined by $\vec{A}$ and $\vec{V}$ and also in the plane orthogonal to $\vec{A}$.

2. **Then rotate that projection by 90° about $\vec{A}$.**

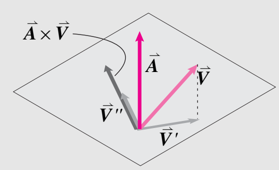


Call the result of this rotation $\vec{V}''$. The magnitude is unchanged by rotation so we still have

$$|\vec{V}''| = |\vec{V}|\sin\theta_{AV}.$$

Note that $\vec{V}''$ is in the $\hat{\boldsymbol{n}}$ direction that is perpendicular to both $\vec{A}$ (it's in the plane $\perp$ to $\vec{A}$) and to $\vec{V}$ (it's rotated 90° from $\vec{V}'$. So $\vec{V}''$ is in the direction of $\vec{A}\times\vec{V}$.

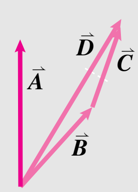
3. **Finally, stretch $\vec{V}''$ by $|\vec{A}|$.**



This result has magnitude $|\vec{A}||\vec{V}|\sin\theta_{AV}$ which is the magnitude of $\vec{A}\times\vec{V}$. This vector is in the direction normal to both $\vec{A}$ and $\vec{V}$ given by the right-hand rule, that's the direction of $\vec{A}\times\vec{V}$. Having the same magnitude and direction as $\vec{A}\times\vec{V}$, it *is* $\vec{A}\times\vec{V}$. As infamously reasoned by Joseph McCarthy, "If it looks like a duck, walks like a duck and quacks like a duck,it's a duck."

Apply the new definition to $\vec{B}+\vec{C}$

Consider $\vec{D} = \vec{B}+\vec{C}$. We are interested in all three cross products: $\vec{A}\times\vec{D}$, $\vec{A}\times\vec{B}$, and $\vec{A}\times\vec{C}$.

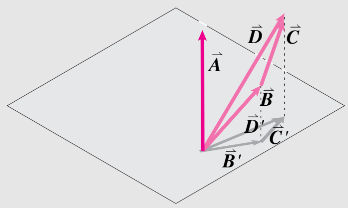


First we will check that each of the operations above (project, rotate, stretch) is distributive.

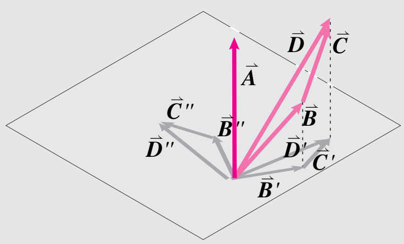
1. **Project.** The projection of a sum is the sum of the projections ($\vec{D}' = \vec{B}' + \vec{C}'$);

(continued...)

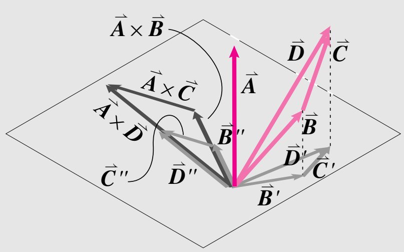
1.7 The cross product: from geometry to components (continued)



2. **Rotate.** The sum of two 90° rotated vectors is the rotation of the sum ($\vec{D}'' = \vec{B}'' + \vec{C}''$);



3. **Stretch.** (stretched $\vec{D}''$) = (stretched $\vec{B}''$) + (stretched $\vec{C}''$), scalar multiplication is distributive.



Thus the act of taking the cross product of $\vec{A}$ with $\vec{B}$ and adding that to the cross product of $\vec{A}$ with $\vec{C}$ gives the same result as taking the cross product of $\vec{A}$ with $\vec{D} \equiv \vec{B}+\vec{C}$. That's the distributive law for vector cross products over vector addition:

$$\vec{A} \quad \times \quad \left(\vec{B}+\vec{C}\right) \quad = \quad \vec{A}\times\vec{B} \quad + \quad \vec{A}\times\vec{C}.$$

The distributive rule also works for a sum of 3 or more vectors (all of the reasoning above would be the same). And that the distributive rule works when the sum is on the left (*i.e.*, that $(\vec{B}+\vec{C})\times\vec{A} = \vec{B}\times\vec{A}+\vec{C}\times\vec{A}$) also follows by very similar reasoning, so we skip those details. In sum, we know now that $(\vec{A}+\vec{B}+\vec{C})\times(\vec{D}+\vec{E}+\vec{F}) = \vec{A}\times\vec{D}+\vec{A}\times\vec{E}+\ldots$ (9 terms in all).

Applying the distributive rule to the vectors' components

A vector is a sum of component parts. First note that

$$\vec{A} = A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k} \qquad \text{and} \qquad \vec{B} = B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$$

From the distributive rule (demonstrated with the 8 pictures above) we know that

$$\vec{A}\times\vec{B} = [A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}]\times[B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}].$$

Now we just apply the distributive law, using what we know about the cross products of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ with each other (*e.g.*, that $\hat{\imath}\times\hat{\imath}=\vec{0}$ and that $\hat{\imath}\times\hat{\jmath}=\hat{k}$).

First in 2D, to better show the patterns in the algebra:

$$\begin{aligned}
\vec{A}\times\vec{B} &= [A_x\hat{\imath}+A_y\hat{\jmath}]\times[B_x\hat{\imath}+B_y\hat{\jmath}] \\
&= [A_x\hat{\imath}+A_y\hat{\jmath}]\times B_x\hat{\imath}+[A_x\hat{\imath}+A_y\hat{\jmath}]\times B_y\hat{\jmath} \\
&= A_xB_x\hat{\imath}\times\hat{\imath}+A_xB_y\hat{\imath}\times\hat{\jmath} \\
&\quad +A_yB_x\hat{\jmath}\times\hat{\imath}+A_yB_y\hat{\jmath}\times\hat{\jmath} \\
&= A_xB_x\vec{0}+A_xB_y\hat{k}-A_yB_x\hat{k}+A_yB_y\vec{0}
\end{aligned}$$

$$\text{So } \vec{A}\times\vec{B} = [A_xB_y-A_yB_x]\hat{k} \qquad \text{(2D)}$$

Now in 3D, applying the distributive rule multiple times,

$$\begin{aligned}
\vec{A}\times\vec{B} &= [A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}]\times[B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}] \\
&= A_xB_x\hat{\imath}\times\hat{\imath}+A_xB_y\hat{\imath}\times\hat{\jmath}+A_xB_z\hat{\imath}\times\hat{k} \\
&\quad +A_yB_x\hat{\jmath}\times\hat{\imath}+A_yB_y\hat{\jmath}\times\hat{\jmath}+A_yB_z\hat{\jmath}\times\hat{k} \\
&\quad +A_zB_x\hat{k}\times\hat{\imath}+A_zB_y\hat{k}\times\hat{\jmath}+A_zB_z\hat{k}\times\hat{k} \\
&= A_xB_x\vec{0}+A_xB_y\hat{k}-A_xB_z\hat{\jmath} \\
&\quad -A_yB_x\hat{k}+A_yB_y\vec{0}+A_yB_z\hat{\imath} \\
&\quad +A_zB_x\hat{\jmath}-A_zB_y\hat{\imath}+A_zB_z\vec{0}
\end{aligned}$$

$$\text{So} \quad \vec{A}\times\vec{B} = [A_yB_z-A_zB_y]\hat{\imath} \tag{1.22}$$
$$+[A_zB_x-A_xB_z]\hat{\jmath} \tag{1.23}$$
$$+[A_xB_y-A_yB_x]\hat{k} \tag{1.24}$$

from which you can pick out the familiar xyz components of the cross product. As hoped, we have derived the component formula for the cross product from its geometric definition.

[The other way around. On the other hand, if we are *given* the component formula we can (almost) verify that it corresponds to the geometric definition: Use the component formulas for the magnitude, dot product and cross product (1.1, 1.3& 1.21) and tediously evaluate $|\vec{A}\times\vec{B}|^2$ and note that it is equal to $|\vec{A}|^2|\vec{B}|^2 - |\vec{A}\cdot\vec{B}|^2$.. Because we already know that $|\vec{A}\cdot\vec{B}| = |\vec{A}||\vec{B}|\cos\theta$ and that $\sin^2\theta = 1-\cos^2\theta$ this gives $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}||\sin\theta|$. To show that $\vec{A}\times\vec{B}$ is orthogonal to $\vec{A}$ and $\vec{B}$ use the component formulas to show that $(\vec{A}\times\vec{B})\cdot\vec{A} = 0$ and $(\vec{A}\times\vec{B})\cdot\vec{B} = 0$. A final tricky point, which we skip here, is that the component formula obeys the right hand, as opposed to the left hand, rule.]

(continued...)

1.7 The cross product: from geometry to components (continued)