1.3 Vector cross product

The vector cross product[1] is a second way of multiplying vectors together (the first was the dot product).

The cross product of vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A} \times \vec{B}$

and is read 'A cross B' (and is *not* read as 'A times B'). The cross product of two vectors is a third vector. It is geometrically defined to be that vector which has magnitude $AB\sin(\theta_{AB})$, where θ_{AB} is the angle between $\vec{A}$ and $\vec{B}$, and which is perpendicular to $\vec{A}$ and $\vec{B}$ in the direction given by the 'right hand rule' (to be explained below).

The cross product also has a component representation, which is admittedly confusing looking at first:

$$[\vec{A}\times\vec{B}] = \begin{bmatrix} A_yB_z - A_zBy \\ A_zB_x - A_xB_z \\ A_xB_y - A_yB_x \end{bmatrix}.$$

All of the above is a mouthful which you can't understand at a first reading. It is just put here up top for completeness and conciseness. The ideas are introduced more gradually below.

Uses of the cross product

The vector cross product is used to define (and calculate) moment, to solve geometry problems and to calculate various quantities associated with rotations in dynamics. Most uses of the cross product used in this book are listed in box 1.5 on page 2.

In this Mechanics Toolset book we treat the cross product as a mathematical calculation and as a geometrical calculation. Deeper understanding will come when you study statics and think about the cross product of $\vec{r}$ and $\vec{F}$ as a moment vector $\vec{M}$. Comfort with the cross product is a tremendous aid for solving three-dimensional statics problems and for doing all of dynamics. You will probably need to refer back to this section from time to time.

The 2D cross product

Although the cross product is fundamentally a three-dimensional idea, we start with the two-dimensional version. The 2D cross product is defined as :

$$\underbrace{\vec{A}\times\vec{B}}_{\text{'A cross B'}} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{\boldsymbol{k}}. \tag{1.10}$$

where θ is the amount that $\vec{A}$ would need to be rotated counterclockwise to point in the same direction as $\vec{B}$. An equivalent alternative approach is to define the cross product as

$$\vec{A}\times\vec{B} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{\boldsymbol{n}}. \tag{1.11}$$

[1] **Nomenclature.** The vector cross product is sometimes just called 'the vector product'. In this book we usually call it 'the cross product' because of the times symbol (×), a cross, we use for it. The name 'cross' might also be because the component formula uses 'cross' terms.
The angle between the vectors is typically represented with θ (pronounced 'thay-tuh', thay rhymes with hay) or Ψ (pronounced 'sigh').

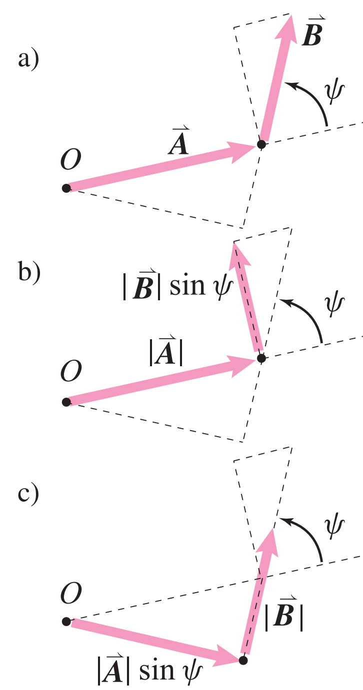


Figure 1.39: **Different interpretations of the 2D cross product.** The cross product of $\vec{A}$ and $\vec{B}$, both in the xy plane, is $|\vec{A}||\vec{B}|\sin\Psi\hat{\boldsymbol{k}}$. Grouping the $\sin\Psi$ with $|\vec{A}|$ or with $|\vec{B}|$ gives two different geometric interpretations of the 2D cross product. First, you can think of the magnitude of the cross product as being the magnitude of $\vec{A}$ times the projection of $\vec{B}$ perpendicular to $\vec{A}$. That's $|\vec{A}|\left(|\vec{B}|\sin\Psi\right)$. Or, you can think of it as the magnitude of $\vec{B}$ times the distance $|\vec{A}|\sin\Psi$. This is marked in the tip-to-tail construction in the third picture above. $\left(|\vec{A}|\sin\Psi\right)|\vec{B}|$.

1.5 Uses of the cross product

Here are the key uses of the cross product in this book. A first time reader is not supposed to understand all of these at the start. They are here for reference and inspiration.

Geometry

1. **3D Normal to a plane.** Find the normal $\vec{N}$ to a plane containing points A,B and C as
$$\vec{N} = \vec{r}_{B/A} \times \vec{r}_{C/A}.$$

2. **3D Unit normal to a plane.** Use $\vec{N}$ above to find the unit normal to a plane containing points A,B and C as
$$\hat{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{\left|\vec{r}_{B/A} \times \vec{r}_{C/A}\right|}$$

3. **3D Distance between a point and a plane.** Use $\hat{n}$ above to find the distance between a point D and the plane containing points A,B and C as
$$d = \quad \vec{r}_{D/A} \cdot \hat{n} \quad = \quad \vec{r}_{D/A} \cdot \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{\left|\vec{r}_{B/A} \times \vec{r}_{C/A}\right|}$$
Replacing $\vec{r}_{D/A}$ with $\vec{r}_{D/B}$ or $\vec{r}_{D/C}$ works too.

4. **3D Distance between two lines.** Find the distance between the line containing the points A and B and the line containing the points C and D as
$$d = \left|\left(\vec{r}_{B/A} \times \vec{r}_{D/C}\right) \cdot \vec{r}_{C/A}\right| / \left|\vec{r}_{B/A} \times \vec{r}_{D/C}\right|.$$
Replacing $\vec{r}_{C/A}$ with $\vec{r}_{D/A}$,$\vec{r}_{C/B}$ or $\vec{r}_{D/B}$ works too.

5. **2D Perpendicular in the xy plane.** Find the in-plane normal $\vec{r}^{\perp}$ to a vector $\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath}$ in the xy plane as
$$\vec{r}^{\perp} = \hat{k} \times \vec{r}.$$
(The vector $\vec{r}^{\perp}$ is also rotation of $\vec{r}$ counterclockwise by 90°.)

6. **2D unit perpendicular in xy plane.** Use $\vec{r}^{\perp}$ to find a unit vector perpendicular to $\vec{r}$ as
$$\hat{n}^{\perp} = \frac{\vec{r}^{\perp}}{|\vec{r}^{\perp}|} = \frac{\hat{k} \times \vec{r}}{\left|\hat{k} \times \vec{r}\right|}.$$

7. **3D distance between point C and a line going through A and B** can with cross product or dot product:
$$d = \left|\vec{r}_{C/A} \times \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}\right| = \left|\vec{r}_{C/A} - \left(\vec{r}_{C/A} \cdot \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}\right)\frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}\right|.$$

8. **2D Distance between a point C and line AB in the plane.** Use $\hat{n}$ to find this distance as
$$d = \left|\hat{n} \cdot \vec{r}_{C/A}\right| = \left|\frac{\left(\hat{k} \times \vec{r}_{B/A}\right) \cdot \vec{r}_{C/A}}{|\hat{k} \times \vec{r}_{B/A}|}\right|.$$
Replacing $\vec{r}_{C/A}$ with $\vec{r}_{C/B}$ works too.

9. **3D Volume V of a parallelepiped** with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$ is
$$V = \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}.$$

Statics

10. **Moment of a force.** Calculate moment $\vec{M}$ of a force $\vec{F}$ at position $\vec{r}$ as
$$\vec{M} = \vec{r} \times \vec{F}$$
.

11. **Moment about an axis.** Calculate moment M of a force $\vec{F}$ at position $\vec{r}$ relative to a point on the axis in the direction of a unit vector $\hat{\lambda}$ as
$$M = \left(\vec{r} \times \vec{F}\right) \cdot \hat{\lambda}.$$

12. **Moment about an axis.** Use the result above to calculate the moment of a force $\vec{F}$ at position C about a line through A and B as
$$M = \left(\vec{r}_{C/A} \times \vec{F}\right) \cdot \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|} = \left(\vec{r}_{C/B} \times \vec{F}\right) \cdot \frac{\vec{r}_{B/A}}{\left|\vec{r}_{B/A}\right|}.$$

Kinematics and Dynamics

13. **Relative velocity of two points on a rigid object.** The velocity of point B relative to point A, where both points are on the same rigid object with angular velocity $\vec{\omega}$ is
$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}.$$

14. **Angular momentum of a particle** with velocity $\vec{v}$, mass m and at position $\vec{r}$ is
$$\vec{H} = m\vec{r} \times \vec{v}.$$

15. **Centripetal acceleration** of point B relative to point A on the same rigid object with angular velocity $\vec{\omega}$ is
$$\text{Centripetal part of } \vec{a}_{B/A} = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right).$$

16. **Relative acceleration due to angular acceleration.** The relative acceleration of points A and B on the same rigid object with angular acceleration $\vec{\alpha}$ is
$$\text{Contribution of angular acceleration to } \vec{a}_{B/A} = \vec{\alpha} \times \vec{r}_{B/A}.$$

17. **Coriolis acceleration** of a particle moving at velocity $\vec{v}_{rel}$ relative to a rotating rigid object is
$$\vec{a}_{Coriolis} = 2\vec{\omega} \times \vec{v}_{rel}.$$

n. Variants and extensions of the kinematics and dynamics formulas are given in the tables at the back of the Dynamics book.

with θ defined to be less than 180° and $\hat{\boldsymbol{n}}$ defined as the unit vector pointing in the direction of the thumb when the fingers are curled from the direction of $\vec{\boldsymbol{A}}$ towards the direction of $\vec{\boldsymbol{B}}$.

Study of fig. 1.39 should convince you that the 2D definition of cross product in eqn.1.10 obeys these standard algebra rules (for any 3 2D vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ and any scalar d):

$$\begin{aligned} d(\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}) &= (d\vec{\boldsymbol{A}})\times\vec{\boldsymbol{B}} = \vec{\boldsymbol{A}}\times(d\vec{\boldsymbol{B}}) \\ \vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}) &= \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}+\vec{\boldsymbol{A}}\times\vec{\boldsymbol{C}}. \end{aligned}$$

A difference between the algebra rules for scalar multiplication and vector cross product multiplication is that for scalar multiplication $AB = BA$ whereas for the cross product $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} \neq \vec{\boldsymbol{B}}\times\vec{\boldsymbol{A}}$ (because the definition of θ in eqn. 1.10 and $\hat{\boldsymbol{n}}$ in 1.11 depends on order). In particular $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}}\times\vec{\boldsymbol{A}}$.

Because the magnitude of the cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is the magnitude of $\vec{\boldsymbol{A}}$ times the magnitude of the projection of $\vec{\boldsymbol{B}}$ in the direction perpendicular to $\vec{\boldsymbol{A}}$ (as shown in the top two illustrations of fig. 1.54) you can think of the cross product as a measure of how much two vectors are perpendicular to each other. In particular,

$$\begin{aligned} \text{if } \vec{\boldsymbol{A}} \text{ is perpendicular to } \vec{\boldsymbol{B}} \quad &\Rightarrow \quad |\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}| = |\vec{\boldsymbol{A}}|\,|\vec{\boldsymbol{B}}|, \text{ and} \\ \text{if } \vec{\boldsymbol{A}} \text{ is parallel to } \vec{\boldsymbol{B}} \quad &\Rightarrow \quad |\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}| = 0. \end{aligned}$$

For example, $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{0}}$, and $\hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}$.

Component form for the 2D cross product

Just like the dot product, the cross product can be expressed using components. As can be verified by writing $\vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{B}} = B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}$ and using the distributive rules:

$$\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = (A_xB_y - B_xA_y)\hat{\boldsymbol{k}}. \tag{1.12}$$

Determinant mnemonic. Some people remember this formula by putting the components of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ into a matrix and calculating the determinant $\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$. If you number the components of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ (*e.g.*, $[\vec{\boldsymbol{A}}]_{x_1x_2} = [A_1, A_2]$), the cross product is $\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = (A_1B_2 - A_2B_1)\hat{\boldsymbol{e}}_3$, or “first times second minus second times first.”

Example: Given that

$$\begin{aligned} & \vec{\boldsymbol{A}} = 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} \\ \text{and} \quad & \vec{\boldsymbol{B}} = 10\hat{\boldsymbol{\imath}} + 20\hat{\boldsymbol{\jmath}} \\ \text{then} \quad & \vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}} = (1\cdot 20 - 2\cdot 10)\hat{\boldsymbol{k}} = 0\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}. \end{aligned}$$

For vectors with just a few components it is often most convenient to use the distributive rule directly.

Example: Given that

$$\begin{aligned}
& & \vec{\boldsymbol{A}} &= 7\hat{\boldsymbol{\imath}} \\
&\text{and} & \vec{\boldsymbol{B}} &= 37.6\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} \\
&\text{then} & \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} &= (7\hat{\boldsymbol{\imath}}) \times (37.6\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}}) = (7\hat{\boldsymbol{\imath}}) \times (37.6\hat{\boldsymbol{\imath}}) + (7\hat{\boldsymbol{\imath}}) \times (10\hat{\boldsymbol{\jmath}}) \\
& & &= \vec{\boldsymbol{0}} + 70\hat{\boldsymbol{k}} = 70\hat{\boldsymbol{k}}.
\end{aligned}$$

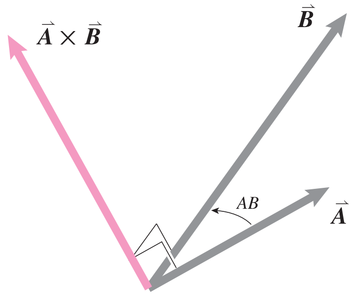


Figure 1.40: The cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is perpendicular to $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ in the direction given by the right hand rule. The magnitude of $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$ is $AB \sin\theta_{AB}$.

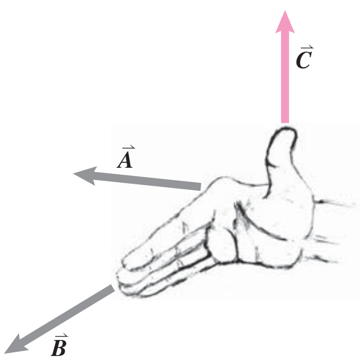


Figure 1.41: The right hand rule for determining the direction of the cross product of two vectors. $\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}$.

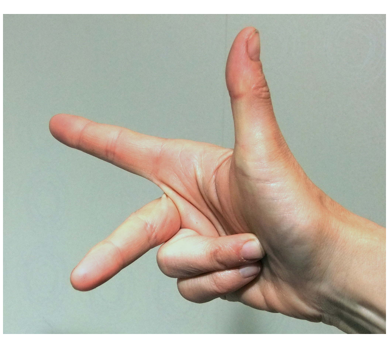


Figure 1.42: Another way to use your right hand for the 'right hand' rule. Set thumb, pointer and middle finger mutually orthogonal to each other. Pointer cross middle finger is thumb. Or, shoot the gun, that is, lower your thumb and the direction of 'thumb cross pointer' is the middle finger.

There are many ways of calculating a 2D cross product

You have several options for calculating the 2D cross product. Which you choose depends on taste and convenience. You can use

- the geometric definition directly,
- the first times the perpendicular part of the second (distance times perpendicular component of force),
- the second times the perpendicular part of the first (lever arm times the force),
- components, or
- break each of the vectors into a sum of vectors and use the distributive rule.

The 3D vector cross product

In contrast to the dot product, which gives a scalar and measures how much two vectors are parallel, the cross product is a vector and measures how much they are perpendicular.

The cross product is defined by:

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \overset{def}{=} |\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}| \sin\theta_{AB}\, \hat{\boldsymbol{n}} \tag{1.13}$$

where $|\hat{\boldsymbol{n}}| = 1$,
$\hat{\boldsymbol{n}} \perp \vec{\boldsymbol{A}}$,
$\hat{\boldsymbol{n}} \perp \vec{\boldsymbol{B}}$,
$0 \le \theta_{AB} \le \pi$, and
$\hat{\boldsymbol{n}}$ is in the direction given by the right hand rule, that is, in the direction of the right thumb when the fingers of the right hand are pointed in the direction of $\vec{\boldsymbol{A}}$ and then wrapped towards the direction of $\vec{\boldsymbol{B}}$.

If $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are perpendicular then θ_{AB} is $\pi/2$, $\sin\theta_{AB} = 1$, and the magnitude of the cross product is AB. If $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are parallel then θ_{AB} is 0, $\sin\theta_{AB} = 0$ and the cross product is $\vec{\boldsymbol{0}}$ (the zero vector). This is why we say the cross product is a measure of the degree of orthogonality of two vectors.

Using the definition above you should be able to verify to your own satisfaction that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$. Applying the definition to the standard base unit vectors you can see that $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, and $\hat{k}\times\hat{\imath} = \hat{\jmath}$ (fig. 1.43).

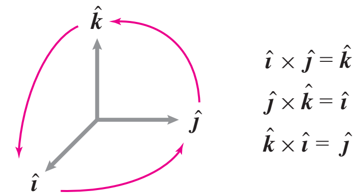


Figure 1.43: Mnemonic device to remember the cross product of the standard base unit vectors.

The geometric definition above and the geometric (tip to tail) definition of vector addition imply that the cross product follows the distributive rule, as explained in box 1.7 on page 9.

$$\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}.$$

Applying the distributive rule to the cross products of $\vec{A} = A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$ leads to the algebraic formula for the Cartesian components of the cross product.

$$\begin{aligned}\vec{A}\times\vec{B} \quad = \quad & [A_yB_z - A_zB_y]\hat{\imath} && (1.14)\\ & +[A_zB_x - A_xB_z]\hat{\jmath} && (1.15)\\ & +[A_xB_y - A_yB_x]\hat{k} && (1.16)\\ & && (1.17)\end{aligned}$$

How the distributive rule is used to derive the component formula for cross product is also described in box 1.7 on page 9.

Memory aids for the component formula for cross product. There are various mnemonics for remembering the component formula for cross products. The most common is to calculate a 'determinant' of the 3×3 matrix with one row given by $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the other two rows the components of $\vec{A}$ and $\vec{B}$.

$$\vec{A}\times\vec{B} = \det\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

Some key facts about cross products. The following identities and special cases of cross products are worth knowing well:

- $(a\vec{A})\times\vec{B} = \vec{A}\times(a\vec{B}) = a(\vec{A}\times\vec{B})$ (a distributive law)
- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (the cross product is not commutative!)
- $\vec{A}\times\vec{B} = \vec{0}$ if $\vec{A}\parallel\vec{B}$ (parallel vectors have zero cross product)
- $|\vec{A}\times\vec{B}| = AB$ if $\vec{A}\perp\vec{B}$
- $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\imath} = \hat{\jmath}$ (assuming the x, y, z coordinate system is right handed — if you use your right hand and point your fingers along the positive x axis, then curl them towards the positive y axis, your thumb will point in the same direction as the positive z axis.)

- $\hat{\boldsymbol{\imath}}' \times \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{k}}', \quad \hat{\boldsymbol{\jmath}}' \times \hat{\boldsymbol{k}}' = \hat{\boldsymbol{\imath}}', \quad \hat{\boldsymbol{k}}' \times \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\jmath}}'$
(assuming the $x'y'z'$ coordinate system is right handed.)
- $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}, \qquad \hat{\boldsymbol{\imath}}' \times \hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\jmath}}' \times \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{k}}' \times \hat{\boldsymbol{k}}' = \vec{\boldsymbol{0}}$

The mixed triple product

The 'mixed triple product' of $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ is so called because it mixes both the dot product and cross product in a single expression. The mixed triple product is also sometimes called the scalar triple product because its value is a scalar. The *mixed triple product* of $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ is defined by and written as

$$\vec{\boldsymbol{A}} \cdot \left(\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}\right)$$

and pronounced 'A dot B cross C.' The parentheses () are sometimes omitted, *i.e.,*

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}},$$

because the wrong grouping $(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}) \times \vec{\boldsymbol{C}}$ is nonsense (you can't take the cross product of a scalar with a vector) . It is apparent that one way of calculating the mixed triple product is to calculate the cross product of $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ and then to take the dot product of that result with $\vec{\boldsymbol{A}}$. Some people use the notation $\left(\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}, \vec{\boldsymbol{C}}\right)$ for the mixed triple product but it will not occur again in this book.

The mixed triple product has the same value if one takes the cross product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ and then the dot product of the result with $\vec{\boldsymbol{C}}$. That is $\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = (\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) \cdot \vec{\boldsymbol{C}}$. This identity can be verified using the geometric description below, or by looking at the (complicated) expression for the mixed triple product of three general vectors $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ in terms of their components as calculated the two different ways. One thus obtains the string of results

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}} = -\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{C}} = -\vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}} \times \vec{\boldsymbol{C}} = \ldots$$

The minus signs in the above expressions follow from the cross product identity that $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$.

The mixed triple product has various geometric interpretations, one of them is that $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}$ is (plus or minus) the volume of the parallelepiped, the crooked shoe box, edged by $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{C}}$ as shown in fig. 1.44.

Another way of calculating the value of the mixed triple product is with the determinant of a matrix whose rows are the components of the vectors.

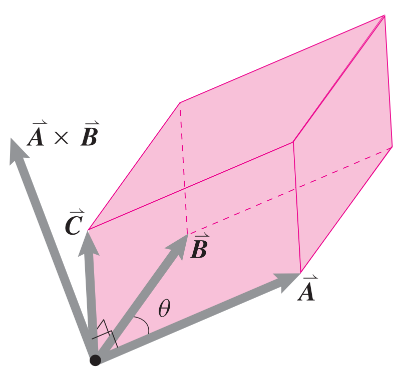


Figure 1.44: One interpretation of the mixed triple product of $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}}$ is as the volume (a scalar) of a parallelepiped with $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ as the three edges emanating from one corner. This interpretation only works if $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ are taken in the appropriate order, otherwise $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}}$ is minus the volume which is calculated.

$$\vec{\boldsymbol{A}} \cdot (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{array}{r} A_x(B_y C_Z - B_z C_y) \\ +A_y(B_z C_x - B_x C_z) \\ +A_z(B_x C_y - B_y C_x) \end{array}$$

The mixed triple product of three vectors is zero if②

- any two of them are parallel, or
- all three of the vectors have one common plane.

②In the language of linear algebra, the mixed triple product of three vectors is zero if the vectors are linearly dependent.

Uses of the mixed triple product. The mixed triple product (or scalar triple product) is useful for calculating the moment of a force about an axis and for related dynamics quantities.

Vector triple product

A different triple product, sometimes called the 'vector triple product', $\vec{\boldsymbol{A}} \times (\vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}})$, is discussed later (in the Dynamics book).

Cross products and computers

The components of the cross product can be calculated with computer code that may look something like this.

```
A = [  1  2  5 ]
B = [ -2  4 19 ]
C = [ ( A(2)*B(3) - A(3)*B(2) ) ...
      ( A(3)*B(1) - A(1)*B(3) ) ...
      ( A(1)*B(2) - A(2)*B(1) ) ]
```

giving the result `C=[18  -29  8]`. Many computer languages have a shorter way to write the cross product like `cross(A,B)`. The mixed triple product might be calculated by assembling a 3×3 matrix of rows and then taking a determinant like this:

```
      A    = [  1  2  5 ]
      B    = [ -2  4 19 ]
      C    = [ 32  4  5 ]
   matrix = [A ;     % Each row under the one above
             B ;
             C   ]
mixedprod = det(matrix)
```

giving the result `mixedprod = 500`. A versatile computer language might allow a command like `dot( A, cross(B,C) )` to calculate the mixed triple product.

1.6 The cross product of vectors as matrix multiplication

For hand calculations in statics, this box is probably not useful. This box is for the theoretically inclined, and for people interested in doing complex dynamics calculations on a computer. We assume here that you know some linear algebra.

The cross product between two vectors $\vec{a}$ and $\vec{b}$ gives a new vector

$$\vec{c} = \vec{a} \times \vec{b}$$

All three of these vectors have components:

$$[\vec{a}]_{xy} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad [\vec{b}]_{xy} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

and

$$[\vec{c}]_{xy} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}.$$

The components of $\vec{c}$ can be calculated from those of $\vec{a}$ and $\vec{b}$ using the component rules from this section as

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}.$$

Because the cross product $\vec{a} \times \vec{b}$ is linear in $\vec{b}$ (meaning $\vec{a} \times (\vec{b}_1 + \vec{b}_2) = \vec{a} \times \vec{b}_1 + \vec{a} \times \vec{b}_2$, *etc.*) we can represent the cross product as some matrix times $\vec{b}$.

We now define a matrix $[A]$, that does the job. We associate the the vector $\vec{a}$ with the matrix $[A]$ as follows:

$$[A] \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \tag{1.18}$$

$[A]$ is an anti-symmetric matrix (also called a skew symmetric matrix), with

$$[A]' = -[A] \qquad \text{(A transpose is minus A).}$$

The terms of $[A]$ on the diagonal are zero and those off the diagonal are negative of their corresponding (transposed) terms. The conversion rule (Equation 1.18) that takes a vector and puts its components in a matrix in the right place we can call $[\mathcal{S}]$ thus

$$[A] = [\mathcal{S}(\vec{a})].$$

$[A]$ is the anti-symmetric matrix defined in terms of the components of $\vec{a}$ by $[\mathcal{S}(\vec{a})]$.

Just multiplying out the terms we see that

$$\mathcal{S}(\vec{a}) \times \vec{b} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{[\mathcal{S}(\vec{a})]} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{1.19}$$

$$= \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}. \tag{1.20}$$

So we get our main result:

$$[\vec{a} \times \vec{b}]_{xyz} = [\mathcal{S}(\vec{a})][\vec{b}]_{xyz} = [A][\vec{b}]_{xyz}.$$

That is, the components of the cross product of $\vec{a}$ and $\vec{b}$ can be found by multiplying the matrix $[A]$ by the components of $\vec{b}$.

Writing the cross product as a matrix multiplication is sometimes useful for dynamics when the first vector $\vec{a}$ is $\vec{\omega}$ (see box 16.5 on page 773). The advantage of the matrix representation over the cross product is that matrix multiplication satisfies the associative rule

$$[A]([B][C]) = ([A][B])[C]$$

whereas the vector cross product does not:

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

Computer calculation

One could write a short program (computer function) that does the conversion from vector $\vec{a}$ to matrix $[A] = \mathcal{S}(\vec{a})$ and call it `skew`. That is, *skew* would calculate eqn. (1.18). Using *skew* we could carry out a cross product like this (see box 3.1 on page 222)

```
a = [1 2 3]'
b = [4 5 6]'
A = skew(a)
c = A*b
```

Define vector $\vec{a}$ with components
Define vector $\vec{b}$ with components
Use the function `skew`' to find A
Calculate the cross product $\vec{c} = \vec{a} \times \vec{b}$ using ordinary matrix multiplication.

For just one cross product this would be silly. But for a long calculation involving various vectors and matrices it often makes things simpler.

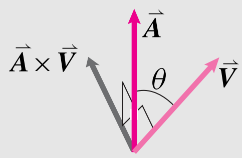
1.7 The cross product: from geometry to components

Why is the 3D vector given by the component formula for the cross product ((1.14) on page 5)

$$[\vec{A}\times\vec{B}]_{xyz} = \begin{bmatrix} (A_yB_z - A_zB_y) \\ (A_zB_x - A_xB_z) \\ (A_xB_y - A_yB_x) \end{bmatrix} \tag{1.21}$$

the same vector as that given by the geometric definition ((1.13) on page 4),

$$\vec{A}\times\vec{V} \equiv |\vec{A}||\vec{V}|\sin\theta_{AV}\hat{\boldsymbol{n}}_{\perp AV}?$$



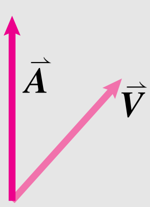
That these different-looking formulas should give the same answer is not obvious. Here we show why. Although the result is often used, the reasoning is not. But for logical completeness, and to entertain the curious, we present it here. We assume we know the geometric definition and want to find the component formula (1.21).

The reasoning has two big steps:

- First we will show that the geometric definitions of the vector cross product 'distributes' over vector addition.
- Then we apply the distributive rule to get our component formula.

A new definition of cross product

Start with $\vec{A}$ and $\vec{V}$:

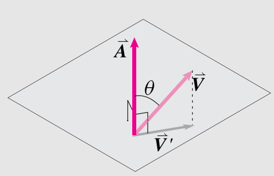


Now we will show that the geometric formula

$$\vec{A}\times\vec{V} \equiv |\vec{A}||\vec{V}|\sin\theta_{AV}\hat{\boldsymbol{n}}_{\perp AV}$$

is equivalent to a sequence of three operations: Project, rotate and stretch.

1. **Project $\vec{V}$ on to the plane P (normal to $\vec{A}$).** The projection of $\vec{V}$ onto the plane orthogonal to $\vec{A}$ is $\vec{V}'$.

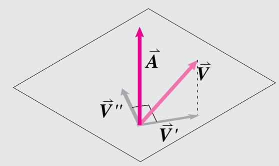


The magnitude of $\vec{V}'$ is

$$|\vec{V}'| = |\vec{V}|\sin\theta_{AV}.$$

$\vec{V}'$ is in the plane defined by $\vec{A}$ and $\vec{V}$ and also in the plane orthogonal to $\vec{A}$.

2. **Then rotate that projection by 90° about $\vec{A}$.**

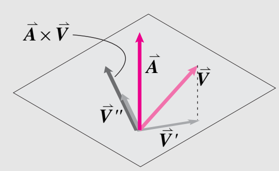


Call the result of this rotation $\vec{V}''$. The magnitude is unchanged by rotation so we still have

$$|\vec{V}''| = |\vec{V}|\sin\theta_{AV}.$$

Note that $\vec{V}''$ is in the $\hat{\boldsymbol{n}}$ direction that is perpendicular to both $\vec{A}$ (it's in the plane $\perp$ to $\vec{A}$) and to $\vec{V}$ (it's rotated 90° from $\vec{V}'$. So $\vec{V}''$ is in the direction of $\vec{A}\times\vec{V}$.

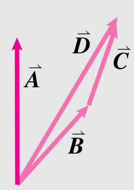
3. **Finally, stretch $\vec{V}''$ by $|\vec{A}|$.**



This result has magnitude $|\vec{A}||\vec{V}|\sin\theta_{AV}$ which is the magnitude of $\vec{A}\times\vec{V}$. This vector is in the direction normal to both $\vec{A}$ and $\vec{V}$ given by the right-hand rule, that's the direction of $\vec{A}\times\vec{V}$. Having the same magnitude and direction as $\vec{A}\times\vec{V}$, it *is* $\vec{A}\times\vec{V}$. As infamously reasoned by Joseph McCarthy, "If it looks like a duck, walks like a duck and quacks like a duck,it's a duck."

Apply the new definition to $\vec{B}+\vec{C}$

Consider $\vec{D} = \vec{B}+\vec{C}$. We are interested in all three cross products: $\vec{A}\times\vec{D}$, $\vec{A}\times\vec{B}$, and $\vec{A}\times\vec{C}$.

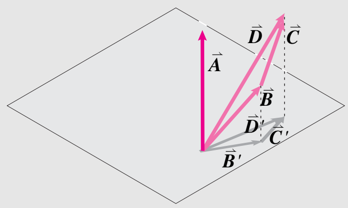


First we will check that each of the operations above (project, rotate, stretch) is distributive.

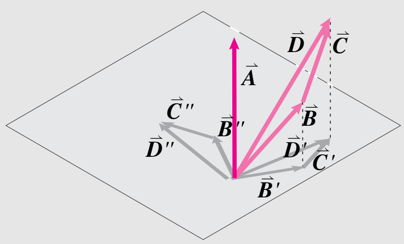
1. **Project.** The projection of a sum is the sum of the projections ($\vec{D}' = \vec{B}' + \vec{C}'$);

(continued...)

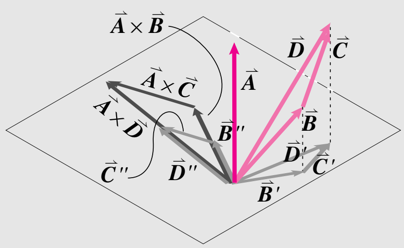
1.7 The cross product: from geometry to components (continued)



2. **Rotate.** The sum of two 90° rotated vectors is the rotation of the sum ($\vec{D}'' = \vec{B}'' + \vec{C}''$);



3. **Stretch.** (stretched $\vec{D}''$) = (stretched $\vec{B}''$) + (stretched $\vec{C}''$), scalar multiplication is distributive.



Thus the act of taking the cross product of $\vec{A}$ with $\vec{B}$ and adding that to the cross product of $\vec{A}$ with $\vec{C}$ gives the same result as taking the cross product of $\vec{A}$ with $\vec{D} \equiv \vec{B}+\vec{C}$. That's the distributive law for vector cross products over vector addition:

$$\vec{A} \quad \times \quad \left(\vec{B}+\vec{C}\right) \quad = \quad \vec{A}\times\vec{B} \quad + \quad \vec{A}\times\vec{C}.$$

The distributive rule also works for a sum of 3 or more vectors (all of the reasoning above would be the same). And that the distributive rule works when the sum is on the left (*i.e.*, that $(\vec{B}+\vec{C})\times\vec{A} = \vec{B}\times\vec{A}+\vec{C}\times\vec{A}$) also follows by very similar reasoning, so we skip those details. In sum, we know now that $(\vec{A}+\vec{B}+\vec{C})\times(\vec{D}+\vec{E}+\vec{F}) = \vec{A}\times\vec{D}+\vec{A}\times\vec{E}+\ldots$ (9 terms in all).

Applying the distributive rule to the vectors' components

A vector is a sum of component parts. First note that

$$\vec{A} = A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k} \qquad \text{and} \qquad \vec{B} = B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$$

From the distributive rule (demonstrated with the 8 pictures above) we know that

$$\vec{A}\times\vec{B} = [A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}]\times[B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}].$$

Now we just apply the distributive law, using what we know about the cross products of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ with each other (*e.g.*, that $\hat{\imath}\times\hat{\imath} = \vec{0}$ and that $\hat{\imath}\times\hat{\jmath} = \hat{k}$).

First in 2D, to better show the patterns in the algebra:

$$\begin{aligned}
\vec{A}\times\vec{B} &= [A_x\hat{\imath}+A_y\hat{\jmath}]\times[B_x\hat{\imath}+B_y\hat{\jmath}]\\
&= [A_x\hat{\imath}+A_y\hat{\jmath}]\times B_x\hat{\imath}+[A_x\hat{\imath}+A_y\hat{\jmath}]\times B_y\hat{\jmath}\\
&= A_xB_x\hat{\imath}\times\hat{\imath}+A_xB_y\hat{\imath}\times\hat{\jmath}\\
&\quad +A_yB_x\hat{\jmath}\times\hat{\imath}+A_yB_y\hat{\jmath}\times\hat{\jmath}\\
&= A_xB_x\vec{0}+A_xB_y\hat{k}-A_yB_x\hat{k}+A_yB_y\vec{0}
\end{aligned}$$

$$\text{So } \vec{A}\times\vec{B} = [A_xB_y-A_yB_x]\hat{k} \qquad \text{(2D)}$$

Now in 3D, applying the distributive rule multiple times,

$$\begin{aligned}
\vec{A}\times\vec{B} &= [A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}]\times[B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}]\\
&= A_xB_x\hat{\imath}\times\hat{\imath}+A_xB_y\hat{\imath}\times\hat{\jmath}+A_xB_z\hat{\imath}\times\hat{k}\\
&\quad +A_yB_x\hat{\jmath}\times\hat{\imath}+A_yB_y\hat{\jmath}\times\hat{\jmath}+A_yB_z\hat{\jmath}\times\hat{k}\\
&\quad +A_zB_x\hat{k}\times\hat{\imath}+A_zB_y\hat{k}\times\hat{\jmath}+A_zB_z\hat{k}\times\hat{k}\\
&= A_xB_x\vec{0}+A_xB_y\hat{k}-A_xB_z\hat{\jmath}\\
&\quad -A_yB_x\hat{k}+A_yB_y\vec{0}+A_yB_z\hat{\imath}\\
&\quad +A_zB_x\hat{\jmath}-A_zB_y\hat{\imath}+A_zB_z\vec{0}
\end{aligned}$$

$$\begin{aligned}
\text{So} \qquad \vec{A}\times\vec{B} = \; & [A_yB_z-A_zB_y]\hat{\imath} && (1.22)\\
& +[A_zB_x-A_xB_z]\hat{\jmath} && (1.23)\\
& +[A_xB_y-A_yB_x]\hat{k} && (1.24)
\end{aligned}$$

from which you can pick out the familiar xyz components of the cross product. As hoped, we have derived the component formula for the cross product from its geometric definition.

[The other way around. On the other hand, if we are *given* the component formula we can (almost) verify that it corresponds to the geometric definition: Use the component formulas for the magnitude, dot product and cross product (1.1,1.3& 1.21) and tediously evaluate $|\vec{A}\times\vec{B}|^2$ and note that it is equal to $|\vec{A}|^2|\vec{B}|^2 - |\vec{A}\cdot\vec{B}|^2$.. Because we already know that $|\vec{A}\cdot\vec{B}| = |\vec{A}||\vec{B}|\cos\theta$ and that $\sin^2\theta = 1-\cos^2\theta$ this gives $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}||\sin\theta|$. To show that $\vec{A}\times\vec{B}$ is orthogonal to $\vec{A}$ and $\vec{B}$ use the component formulas to show that $(\vec{A}\times\vec{B})\cdot\vec{A} = 0$ and $(\vec{A}\times\vec{B})\cdot\vec{B} = 0$. A final tricky point, which we skip here, is that the component formula obeys the right hand, as opposed to the left hand, rule.]

(continued...)

1.7 The cross product: from geometry to components (continued)

SAMPLE 1.17 **Cross product in 2-D:** Two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ of length 10 ft and 6 ft, respectively, are shown in the figure. The angle between the two vectors is $\theta = 60°$. Find the cross product $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$.

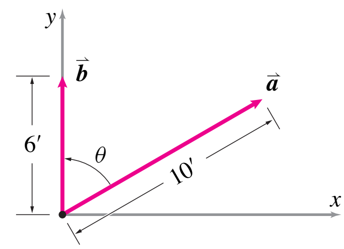


Figure 1.45

Solution Both vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are in the xy plane. Therefore, their cross product will point in either $+\hat{\boldsymbol{k}}$ or $-\hat{\boldsymbol{k}}$ direction (normal to the plane formed by $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$). To determine the direction of the normal, we use the right hand rule and curl our fingers from $\vec{\boldsymbol{a}}$ towards $\vec{\boldsymbol{b}}$, tracing angle θ counterclockwise. The thumb then points in the positive $\hat{\boldsymbol{k}}$ direction, indicating $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$. Thus,

$$\begin{aligned}
\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} &= |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}| \sin\theta \hat{\boldsymbol{n}} \\
&= (10\,\mathrm{ft}) \cdot (6\,\mathrm{ft}) \cdot \sin 60° \hat{\boldsymbol{k}} \\
&= 60\,\mathrm{ft}^2 \cdot \frac{\sqrt{3}}{2} \hat{\boldsymbol{k}} \\
&= 30\sqrt{3}\,\mathrm{ft}^2 \hat{\boldsymbol{k}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = 30\sqrt{3}\,\mathrm{ft}^2 \hat{\boldsymbol{k}}}$$

SAMPLE 1.18 **Computing 2-D cross product in different ways:** The two vectors shown in the figure are $\vec{\boldsymbol{a}} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{b}} = 4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$. The angle between the two vectors turns out to be $\theta = \sin^{-1}(4/5)$. Find the cross product $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$

1. using the angle θ, and
2. using the components of the vectors.

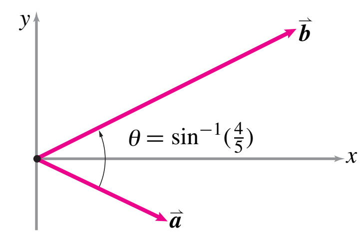


Figure 1.46

Solution

1. Cross product using the angle θ:

$$\begin{aligned}
\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} &= |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}| \sin\theta \hat{\boldsymbol{n}} \\
&= |2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}||4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}| \cdot \sin(\sin^{-1}\frac{4}{5})\hat{\boldsymbol{k}} \\
&= (\sqrt{2^2 + 1^2})(\sqrt{4^2 + 2^2}) \cdot \frac{4}{5}\hat{\boldsymbol{k}} \\
&= \sqrt{5} \cdot \sqrt{20} \cdot \frac{4}{5}\hat{\boldsymbol{k}} = 10 \cdot \frac{4}{5}\hat{\boldsymbol{k}} \\
&= 8\hat{\boldsymbol{k}}.
\end{aligned}$$

2. Cross product using components:

$$\begin{aligned}
\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} &= (2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}) \times (4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \\
&= 2\hat{\boldsymbol{\imath}} \times (4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) - \hat{\boldsymbol{\jmath}} \times (4\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \\
&= 8\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}}_{\vec{0}} + 4\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}}_{\hat{k}} - 4\underbrace{\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}}_{-\hat{k}} - 2\underbrace{\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}}_{\vec{0}} \\
&= 4\hat{\boldsymbol{k}} + 4\hat{\boldsymbol{k}} \\
&= 8\hat{\boldsymbol{k}}.
\end{aligned}$$

The answers obtained from the two methods are, of course, the same as they must be.

$$\boxed{\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = 8\hat{\boldsymbol{k}}}$$

SAMPLE 1.19 Find the shortest distance from a point to a line: A straight line passes through two points, A (-1,1) and B (2,2), in the xy plane. Find the shortest distance from the origin to the line.

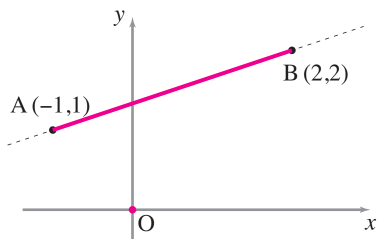


Figure 1.47

Solution There are various ways we can find the required distance. In 2-D, the following two methods can be used for quick calculations.

Method-1: The shortest distance d between a point P and a given line AB is obtained by taking the dot product of a vector from P to any point on the line with a unit normal to the given line. Thus, $d = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{r}}_A = \hat{\boldsymbol{n}} \cdot \vec{\boldsymbol{r}}_B$ where $\hat{\boldsymbol{n}}$ is a unit vector normal to $\vec{\boldsymbol{r}}_{B/A}$. Why?

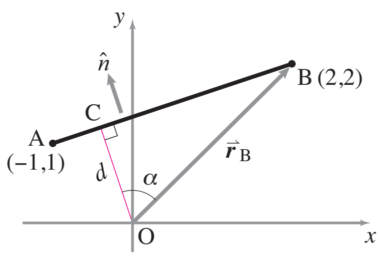


Figure 1.48

From geometry we know that the shortest distance from a given point to a given line is the perpendicular distance from the line to the given point. Thus to find this distance, we need to draw a perpendicular from the given point to the line of interest. In fig. 1.48, this distance d is shown as line OC. From the figure, we also see that $d = \mathrm{OC} = \mathrm{OB}\cos\alpha$, that is, d is the projection of vector $\vec{\boldsymbol{r}}_B$ along OC. But OC is normal to AB, hence, we can find a unit vector $\hat{\boldsymbol{n}}$ along OC by finding a normal to vector $\vec{\boldsymbol{r}}_{B/A}$. Thus,

$$d = \vec{\boldsymbol{r}}_B \cdot \hat{\boldsymbol{n}}, \quad \text{where} \quad \hat{\boldsymbol{n}} \perp \vec{\boldsymbol{r}}_{B/A}.$$

We can use cross product to find $\hat{\boldsymbol{n}}$. Since $\vec{\boldsymbol{r}}_{B/A}$ is in the xy plane, $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}$ gives us a vector perpendicular to AB and in the xy plane. Thus,

$$\hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}}{|\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}|}, \quad \text{and hence,} \quad d = \vec{\boldsymbol{r}}_B \cdot \frac{\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}}{|\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A}|}.$$

From the given coordinates, $\vec{\boldsymbol{r}}_B = 2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{r}}_{B/A} = \vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A = 3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$, we get

$$\begin{aligned} \hat{\boldsymbol{n}} &= \frac{\hat{\boldsymbol{k}} \times (3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}{|\hat{\boldsymbol{k}} \times (3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})|} = \frac{3\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}}{|3\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}|} = -\frac{1}{\sqrt{10}}\hat{\boldsymbol{\imath}} + \frac{3}{\sqrt{10}}\hat{\boldsymbol{\jmath}} \\ d &= \vec{\boldsymbol{r}}_B \cdot \hat{\boldsymbol{n}} = (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \cdot \left(-\frac{1}{\sqrt{10}}\hat{\boldsymbol{\imath}} + \frac{3}{\sqrt{10}}\hat{\boldsymbol{\jmath}}\right) = \frac{4}{\sqrt{10}}. \end{aligned}$$

$$\boxed{d = \frac{4}{\sqrt{10}}}$$

You can easily verify that we get the same result even if we use $\vec{\boldsymbol{r}}_A$ in place of $\vec{\boldsymbol{r}}_B$ (or for that matter, any vector from O to a point on line AB). For example, $\vec{\boldsymbol{r}}_A \cdot \hat{\boldsymbol{n}} = (-ui + \hat{\boldsymbol{\jmath}}) \cdot (-\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})/\sqrt{10} = 4/\sqrt{10}$, the same value as obtained above.

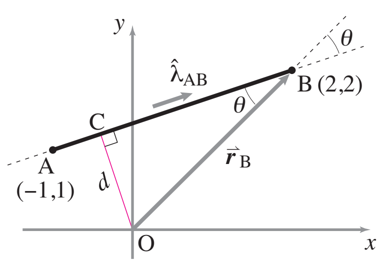


Figure 1.49

Method-2: The shortest distance between a point P and a given line AB can also be found by taking a cross product of a unit vector $\hat{\boldsymbol{\lambda}}_{AB}$ along the line with any vector from P to a point on the line, e.g., $\vec{\boldsymbol{r}}_{B/P}$. For the given case, where P is the origin, we get, $d = \hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B$. Why?

From fig. 1.49, we see that $d \equiv \mathrm{OC} = \mathrm{OB}\sin\theta$ where θ is the angle between $\vec{\boldsymbol{r}}_B$ and $\vec{\boldsymbol{r}}_{B/A}$. Let $\hat{\boldsymbol{\lambda}}_{AB}$ be a unit vector along line AB. Then,

$$\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B = \underbrace{|\hat{\boldsymbol{\lambda}}_{AB}|}_{1} |\vec{\boldsymbol{r}}_B| \sin\theta \hat{\boldsymbol{n}} = |\vec{\boldsymbol{r}}_B| \sin\theta \hat{\boldsymbol{k}}.$$

Thus, $d = OB\sin\theta = |\vec{\boldsymbol{r}}_B|\sin\theta = |\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B|$. We can, therefore, compute d easily by computing the cross product as follows.

$$\begin{aligned} d &= |\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B| = \left|\left(\frac{3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{3^2 + 1^2}}\right) \times (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\right| \\ &= \left|\frac{4}{\sqrt{10}}\hat{\boldsymbol{k}}\right| = \frac{4}{\sqrt{10}}. \end{aligned}$$

which is the same answer as obtained by Method-1. Once again, you can verify that $\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_A$ also gives the same answer.

$d = 4/\sqrt{10}$

SAMPLE 1.20 **Computing cross product in 3-D:** Compute $\vec{a}\times\vec{b}$, where $\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}$.

Solution The calculation of a cross product between two 3-D vectors can be carried out by either using a determinant or the distributive rule. Usually, if the vectors involved have just one or two components, it is easier to use the distributive rule. We show you both methods here and encourage you to learn both. We are given two vectors:

$$
\begin{aligned}
\vec{a} &= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \hat{\imath} + \hat{\jmath} - 2\hat{k},\\
\vec{b} &= b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}.
\end{aligned}
$$

- **Calculation using the determinant formula:** In this method, we first write a 3×3 matrix whose first row has the basis vectors as its elements, the second row has the components of the first vector as its elements, and the third row has the components of the second vector as its elements. Thus,

$$
\begin{aligned}
\vec{a}\times\vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\\
&= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2)\\
&= \hat{\imath}(1-8) + \hat{\jmath}(-6-1) + \hat{k}(-4-3)\\
&= -7(\hat{\imath} + \hat{\jmath} + \hat{k}).
\end{aligned}
$$

- **Calculation using the distributive rule:** In this method, we carry out the cross product by distributing the cross product properly over the three basis vectors. The steps involved are shown below.

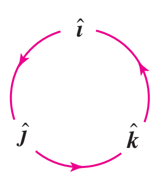


Figure 1.50: The cross product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g. $\hat{\imath}\times\hat{\jmath} = \hat{k}$ but $\hat{\jmath}\times\hat{\imath} = -\hat{k}$.

$$
\begin{aligned}
\vec{a}\times\vec{b} &= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k})\times(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})\\
&= a_1\hat{\imath}\times(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) +\\
&\quad\; a_2\hat{\jmath}\times(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) +\\
&\quad\; a_3\hat{k}\times(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})\\
&= a_1b_1\overbrace{(\hat{\imath}\times\hat{\imath})}^{0} + a_1b_2\overbrace{(\hat{\imath}\times\hat{\jmath})}^{\hat{k}} + a_1b_3\overbrace{(\hat{\imath}\times\hat{k})}^{-\hat{\jmath}} +\\
&\quad\; a_2b_1\overbrace{(\hat{\jmath}\times\hat{\imath})}^{-\hat{k}} + a_2b_2\overbrace{(\hat{\jmath}\times\hat{\jmath})}^{0} + a_2b_3\overbrace{(\hat{\jmath}\times\hat{k})}^{\hat{\imath}} +\\
&\quad\; a_3b_1\overbrace{(\hat{k}\times\hat{\imath})}^{\hat{\jmath}} + a_3b_2\overbrace{(\hat{k}\times\hat{\jmath})}^{-\hat{\imath}} + a_3b_3\overbrace{(\hat{k}\times\hat{k})}^{0}\\
&= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2)\\
&= \hat{\imath}(1-8) + \hat{\jmath}(-6-1) + \hat{k}(-4-3)\\
&= -7(\hat{\imath} + \hat{\jmath} + \hat{k})
\end{aligned}
$$

which, of course, is the same result as obtained above using the determinant. Making a sketch such as Fig. 1.50 is helpful while calculating cross products this way. The product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, *e.g.*, $\hat{\imath}\times\hat{\jmath} = \hat{k}$ but $\hat{\jmath}\times\hat{\imath} = -\hat{k}$.

$$\boxed{\vec{a}\times\vec{b} = -7(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

SAMPLE 1.21 Finding a vector normal to two given vectors: Find a unit vector perpendicular to the vectors $\vec{\boldsymbol{r}}_A = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{r}}_b = 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$.

Solution The cross product between two vectors gives a vector perpendicular to the plane formed by the two vectors. The sense of direction is determined by the right hand rule.

Let $\vec{\boldsymbol{N}} = N\hat{\boldsymbol{\lambda}}$ be the perpendicular vector.

$$\begin{aligned}
\vec{\boldsymbol{N}} &= \vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{r}}_B \\
&= (\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) \times (3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}) \\
&= -7\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}.
\end{aligned}$$

This calculation can be done in either of the two ways shown in the previous sample.

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{N}}}{N} \\
&= \frac{-7\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}}{\sqrt{7^2 + 2^2 + 3^2}} \\
&= -0.89\hat{\boldsymbol{\imath}} - 0.25\hat{\boldsymbol{\jmath}} + 0.38\hat{\boldsymbol{k}}
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} = -0.89\hat{\boldsymbol{\imath}} - 0.25\hat{\boldsymbol{\jmath}} + 0.38\hat{\boldsymbol{k}}}$$

Check:

- $|\hat{\boldsymbol{\lambda}}| = (0.89)^2 + (0.25)^2 + (0.38)^2 \stackrel{\surd}{=} 1.$ (it is a unit vector)
- $\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}_A = 1(-0.89) - 2(-0.25) + 1(0.38) \stackrel{\surd}{=} 0.$ $(\hat{\boldsymbol{\lambda}} \perp \vec{\boldsymbol{r}}_A)$.
- $\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}_B = 3(-0.25) + 2(0.38) \stackrel{\surd}{=} 0.$ $(\hat{\boldsymbol{\lambda}} \perp \vec{\boldsymbol{r}}_B)$.

Comments: If $\hat{\boldsymbol{\lambda}}$ is perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{A}}$ and $\vec{\boldsymbol{r}}_{\mathrm{B}}$, then so is $-\hat{\boldsymbol{\lambda}}$. The perpendicularity does not change by changing the *sense* of direction (from positive to negative) of the vector. In fact, if $\hat{\boldsymbol{\lambda}}$ is perpendicular to a vector $\vec{\boldsymbol{r}}$ then any scalar multiple of $\hat{\boldsymbol{\lambda}}$, *i.e.,* $\alpha\hat{\boldsymbol{\lambda}}$, is also perpendicular to $\vec{\boldsymbol{r}}$. This follows because

$$\alpha\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}} = \alpha(\hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{r}}) = \alpha(0) = 0.$$

The case of $-\hat{\boldsymbol{\lambda}}$ is just a particular instance of this rule with $\alpha = -1$.

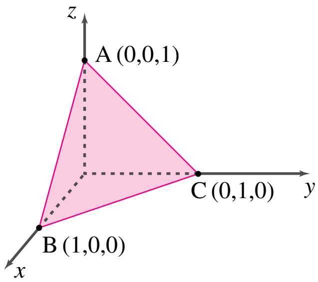


Figure 1.51

SAMPLE 1.22 Finding a vector normal to a plane: Find a unit vector normal to the plane ABC shown in the figure.

Solution A vector normal to the plane ABC would be normal to any vector in that plane. In particular, if we take any two vectors, say $\vec{r}_{AB}$ and $\vec{r}_{AC}$, the normal to the plane would be perpendicular to both $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Since the cross product of two vectors gives a vector perpendicular to both vectors, we can find the desired normal vector by taking the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Thus,

$$\begin{aligned}
\vec{N} &= \vec{r}_{AB} \times \vec{r}_{AC} \\
&= (\hat{\imath} - \hat{k}) \times (\hat{\jmath} - \hat{k}) \\
&= \underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} - \underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} - \underbrace{\hat{k} \times \hat{\jmath}}_{-\hat{\imath}} + \underbrace{\hat{k} \times \hat{k}}_{\vec{0}} \\
&= \hat{\imath} + \hat{\jmath} + \hat{k} \\
\Rightarrow \quad \hat{n} &= \frac{\vec{N}}{|\vec{N}|} \\
&= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}).
\end{aligned}$$

$$\hat{n} = \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$$

Check: Now let us check if $\hat{n}$ is normal to any vector in the plane ABC. It is fairly easy to show that $\hat{n} \cdot \vec{r}_{AB} = \hat{n} \cdot \vec{r}_{AC} = 0$. This is not a surprise because we found $\hat{n}$ from the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Let us check if $\hat{n}$ is normal to $\vec{r}_{BC}$:

$$\begin{aligned}
\hat{n} \cdot \vec{r}_{BC} &= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-\hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-1 + 1) \stackrel{\surd}{=} 0.
\end{aligned}$$

SAMPLE 1.23 **The shortest distance between two lines:** Two lines, AB and CD, in 3-D space are defined by four specified points, A(0,2 m,1 m), B(2 m,1 m,3 m), C(-1 m,0,0), and D(2 m,2 m,2 m) as shown in the figure. Find the shortest distance between the two lines (which are the infinite extensions of the segments shown).

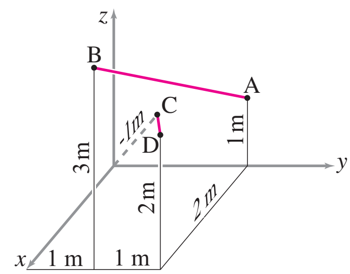


Figure 1.52

Solution The shortest distance between any pair of lines is the length of the line that is perpendicular to both the lines. We can find the shortest distance in three steps:

1. First find a vector that is perpendicular to both the lines. This is easy. Take two vectors $\vec{r}_1$ and $\vec{r}_2$, one along each of the two given lines. Take the cross product of the two vectors and normalize the result to get unit vector $\hat{n}$.
2. Find a vector parallel to $\hat{n}$ that connects the two lines. This is a little tricky. We don't know where to start on any of the two lines. However, we can take any vector from one line to the other and then, take its component along $\hat{n}$.
3. Find the length (magnitude) of the vector just found (in the direction of $\hat{n}$). This is simply the component we find in step (b) devoid of its sign.

Now let us carry out these steps on the given problem.

1. Step-1: Find a unit vector $\hat{n}$ that is perpendicular to both the lines.

$$
\begin{aligned}
\vec{r}_{AB} &= 2\,\text{m}\hat{\imath} - 1\,\text{m}\hat{\jmath} + 2\,\text{m}\hat{k} \\
\vec{r}_{CD} &= 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath} + 2\,\text{m}\hat{k} \\
\Rightarrow \quad \vec{r}_{AB} \times \vec{r}_{CD} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} \text{m}^2 \\
&= \hat{\imath}(-2-4)\,\text{m}^2 + \hat{\jmath}(6-4)\,\text{m}^2 + \hat{k}(4+3)\,\text{m}^2 \\
&= (-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k})\,\text{m}^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{n} &= \frac{\vec{r}_{AB} \times \vec{r}_{CD}}{|\vec{r}_{AB} \times \vec{r}_{CD}|} \\
&= \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}).
\end{aligned}
$$

2. Step-2: Find any vector from one line to the other line and find its component along $\hat{n}$.

$$
\begin{aligned}
\vec{r}_{AC} &= -1\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} - 1\,\text{m}\hat{k} \\
\vec{r}_{AC} \cdot \hat{n} &= -(\hat{\imath} + 2\hat{\jmath} + \hat{k})\,\text{m} \cdot \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{k}) \\
&= \frac{1}{\sqrt{89}}(6-4-7)\,\text{m} = -\frac{5}{\sqrt{89}}\,\text{m}.
\end{aligned}
$$

3. Step-3: Find the required distance d by taking the magnitude of the component along $\hat{n}$.

$$
d = \left|\vec{r}_{AC} \cdot \hat{n}\right| = \left|-\frac{5}{\sqrt{89}}\,\text{m}\right| = 0.53\,\text{m}
$$

$$\boxed{d = 0.53\,\text{m}}$$

SAMPLE 1.24 The mixed triple product: Calculate the mixed triple product $\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})$ for $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$, $\vec{\boldsymbol{a}} = 3\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{b}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$.

Solution We compute the given mixed triple product in two ways here:

- **Method-1:** Straight calculation using cross product and dot product.

$$\begin{aligned}
\text{Let} \quad \vec{\boldsymbol{c}} &= \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} \\
&= (3\hat{\boldsymbol{\imath}}) \times (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}) \\
&= 3\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}}_{\vec{0}} + 3\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}}_{\hat{\boldsymbol{k}}} + 9\underbrace{\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}}}_{-\hat{\boldsymbol{\jmath}}} \\
&= -9\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}} \\
\text{So,} \quad \hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) &= \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{c}} \\
&= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \cdot (-9\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}) \\
&= \frac{1}{\sqrt{2}}(-9\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}}_{0} + 3\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}}}_{0} - 9\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}}_{1} + 3\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}}}_{0}) \\
&= -\frac{9}{\sqrt{2}}.
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) = -\tfrac{9}{\sqrt{2}}}$$

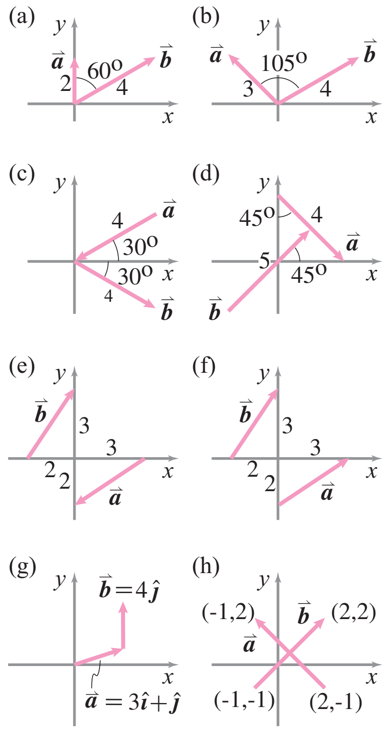
- **Method-2:** Using the determinant formula for mixed product.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\
&= \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix} \\
&= \frac{1}{\sqrt{2}}(0-0) + \frac{1}{\sqrt{2}}(0-9) + 0 \\
&= -\frac{9}{\sqrt{2}}.
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) = -\tfrac{9}{\sqrt{2}}}$$

1.3 Vector Cross Product

1.3.1 Find the cross product of the two vectors shown in the figures below from the information given in the figures.



Problem 1.3.1

1.3.2 Vector algebra. For each equation below state whether:

1. The equation is nonsense. If so, why?
2. Is always true. Why? Give an example.
3. Is never true. Why? Give an example.
4. Is sometimes true. Give examples both ways.

You may use trivial examples.

a) $\vec{B}\times\vec{C}=\vec{C}\times\vec{B}$

b) $\vec{B}\times\vec{C}=\vec{C}\cdot\vec{B}$

c) $\vec{C}\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{C}\times\vec{A})$

d) $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$

1.3.3 What do you get when you cross a vector and a scalar? *

1.3.4 Carry out the following cross products in different ways and determine which method takes the least amount of time for you.

a) $\vec{r}=2.0\,\mathrm{ft}\hat{\imath}+3.0\,\mathrm{ft}\hat{\jmath}-1.5\,\mathrm{ft}\hat{k}$; $\vec{F}=-0.3\,\mathrm{lbf}\hat{\imath}-1.0\,\mathrm{lbf}\hat{k}$; $\vec{r}\times\vec{F}=?$

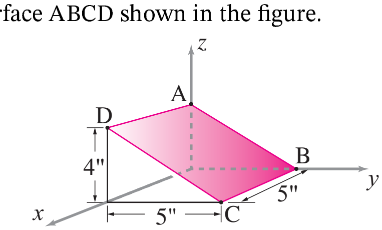
b) $\vec{r}=(-\hat{\imath}+2.0\hat{\jmath}+0.4\hat{k})\,\mathrm{m}$; $\vec{L}=(3.5\hat{\jmath}-2.0\hat{k})\,\mathrm{kg\,m/s}$; $\vec{r}\times\vec{L}=?$

c) $\vec{\omega}=(\hat{\imath}-1.5\hat{\jmath})\,\mathrm{rad/s}$; $\vec{r}=(10\hat{\imath}-2\hat{\jmath}+3\hat{k})\,\in$; $\vec{\omega}\times\vec{r}=?$

1.3.5 Cross Product program Write a program that will calculate cross products. The input to the function should be the components of the two vectors and the output should be the components of the cross product. As a model, here is a function file that calculates dot products in pseudo code.

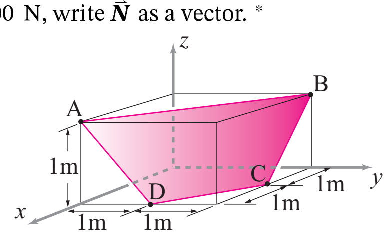
```
%program definition
z(1)=a(1)*b(1);
z(2)=a(2)*b(2);
z(3)=a(3)*b(3);
w=z(1)+z(2)+z(3);
```

1.3.6 Find a unit vector normal to the surface ABCD shown in the figure.



Problem 1.3.6

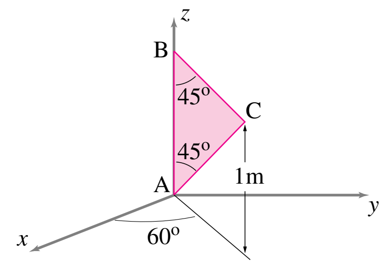
1.3.7 If the magnitude of a force $\vec{N}$ normal to the surface ABCD in the figure is 1000 N, write $\vec{N}$ as a vector. *



Problem 1.3.7

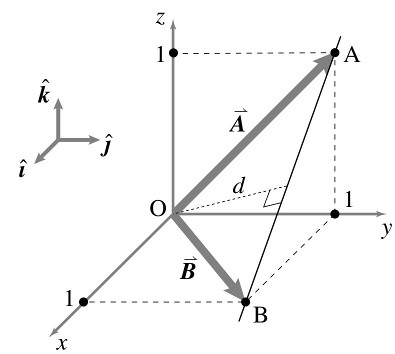
1.3.8 The equation of a surface is given as $z=2x-y$. Find a unit vector $\hat{n}$ normal to the surface.

1.3.9 In the figure, a triangular plate ACB, attached to rod AB, rotates about the z-axis. At the instant shown, the plate makes an angle of 60° with the x-axis. Find and draw a vector normal to the surface ACB.



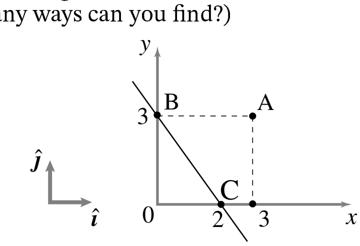
Problem 1.3.9

1.3.10 What is the distance d between the origin and the line AB shown? (You may write your solution in terms of $\vec{A}$ and $\vec{B}$ before doing any arithmetic). *



Problem 1.3.10

1.3.11 What is the perpendicular distance between point A and line BC shown? (There are at least 3 ways to do this using various vector products, how many ways can you find?)



Problem 1.3.11

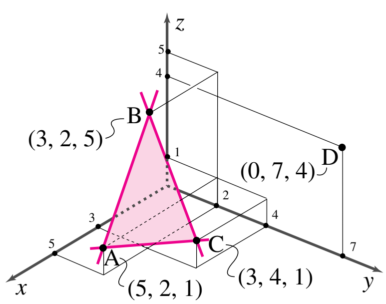
1.3.12 Given a force, $\vec{F}_1=(-3\hat{\imath}+2\hat{\jmath}+5\hat{k})\,\mathrm{N}$ acting at a point P whose position is given by $\vec{r}_{P/O}=(4\hat{\imath}-2\hat{\jmath}+7\hat{k})\,\mathrm{m}$, what is the moment about an axis through the origin O with direction $\hat{\lambda}=\frac{2}{\sqrt{5}}\hat{\imath}+\frac{1}{\sqrt{5}}\hat{\jmath}$?

1.3.13 A, B, and C are located by position vectors $\vec{r}_A = (1, 2, 3)$, $\vec{r}_B = (4, 5, 6)$, and $\vec{r}_C = (7, 8, 10)$.

a) Use the vector dot product to find the angle BAC (A is at the vertex of this angle).
b) Use the vector cross product to find the angle BCA (C is at the vertex of this angle).
c) Find a unit vector perpendicular to the plane ABC.
d) How far is the infinite line defined by AB from the origin? (That is, how close is the closest point on this line to the origin?)
e) Is the origin co-planar with the points A, B, and C?

1.3.14 Points A, B, and C in the figure define an infinite plane.

a) Find a unit normal vector to the plane. *
b) Find the perpendicular distance from point D to this infinite plane (not necessarily inside the triangle ABC). *
c) What are the coordinates of the point on the plane closest to point D? *
d) Is this point on or off the triangle used to define the plane?



Problem 1.3.14

1.3.15 What point on the line that goes through the points (1,2,3) and (7,12,15) is closest to the origin?

1.3.16 A regular tetrahedron is a triangular-based pyramid where all 6 edges have the same length ℓ. What is the perpendicular distance between a pair of non-touching edges? (There are many ways to solve this problem). *

1.3.17 Why did the chicken cross the road? *