1.2 The dot product of two vectors

The *dot product* of two vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is written $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}$ (pronounced 'A dot B'). The dot product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is the product of the magnitudes of the two vectors times a number that expresses the degree to which $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are parallel: $\cos\theta_{AB}$, where θ_{AB} is the angle between $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$. That is,

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \overset{def}{=} |\vec{\boldsymbol{A}}|\,|\vec{\boldsymbol{B}}| \cos\theta_{AB}$$

which is sometimes written more concisely as

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB\cos\theta.$$

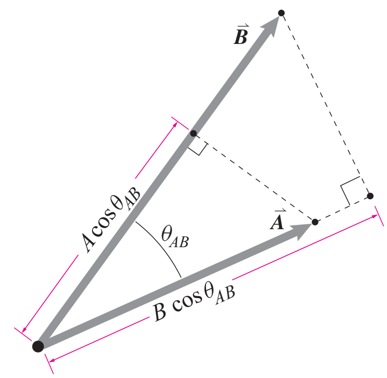


Figure 1.31: The dot product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is a scalar and so is not easily drawn. It is given by $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB\cos\theta_{AB}$ which is A times the projection of $\vec{\boldsymbol{B}}$ in the A direction and also B times the projection of $\vec{\boldsymbol{A}}$ in the B direction.

Key special cases. Here are two special cases that come up again and again.

Example: Parallel vectors
If $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are parallel, then $\cos\theta_{AB} = 1$, and

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = AB.$$

Example: Perpendicular vectors
If $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are perpendicular, then $\cos\theta_{AB} = 0$, and

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = 0.$$

It is helpful to know, almost without a thought, that ①

$$\begin{aligned} \cos(0) &= 1, \\ \sin(0) &= 0, \\ \cos(\pi/2) &= 0, \text{ and} \\ \sin(\pi/2) &= 1. \end{aligned}$$

①**Advice:** Draw as many triangles and unit circles as it takes to get these 4 special cases into your head so they can be used with only a quick thought.

The dot product of two vectors is a scalar. So the dot product is sometimes called the *scalar product*.

What is the dot product good for?

The dot product is used

- To project a vector in a given direction;
- To reduce a vector to components;
- To reduce vector equations to scalar equations;
- To define work and power; and

- To help solve geometry problems.

Most fundamental is the third item 'To reduce vector equations to scalar equations'. Mechanics equations are most often vector equations. Yet, when you solve problems, in the end, you are always solving scalar equations. Thus, you need, somehow or other, to use dot products for most mechanics problems. In some sense, the dot product is the *only* way to get from vector equations to scalar equations.

Using the geometric definition of dot product, and the rules for vector addition we have already discussed, you should be able to convince yourself of the features of the dot products in box 1.3. The identities in box 1.3 lead to the following equivalent ways of expressing the dot product of $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$.

$$\begin{aligned}
\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} &= |\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}|\cos\theta_{AB} \\
&= A_xB_x + A_yB_y + A_zB_z \quad \text{(component form)} \qquad (1.3)\\
&= A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'} \\
&= |\vec{\boldsymbol{A}}|\cdot[\text{scalar projection of } \vec{\boldsymbol{B}} \text{ in the } \vec{\boldsymbol{A}} \text{ direction}] \\
&= |\vec{\boldsymbol{B}}|\cdot[\text{scalar projection of } \vec{\boldsymbol{A}} \text{ in the } \vec{\boldsymbol{B}} \text{ direction}]
\end{aligned}$$

Mechanics solutions make use of all of these relations. The most famous, of course, is the second, sometimes written as

1.3 Useful basic features of the vector dot product

Here are some features of the dot product that you will use again and again. They all follow naturally from the definition

$$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = AB\cos\theta.$$

Commutative law. Order doesn't matter with dot products. $AB\cos\theta = BA\cos\theta$

$$\Rightarrow \quad \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}\cdot\vec{\boldsymbol{A}}.$$

A distributive law. Scalars slide through vector products like mercury through a chicken. $(aA)B\cos\theta = A(aB)\cos\theta$

$$\Rightarrow \quad (a\vec{\boldsymbol{A}})\cdot\vec{\boldsymbol{B}} = \vec{\boldsymbol{A}}\cdot(a\vec{\boldsymbol{B}}) = a(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}).$$

Another distributive law. Vector dot products distribute like regular multiplication. the projection of $\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}$ onto $\vec{\boldsymbol{A}}$ is the sum of the two separate projections, so

$$\vec{\boldsymbol{A}}\cdot(\vec{\boldsymbol{B}}+\vec{\boldsymbol{C}}) = \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{C}}.$$

Perpendicular vectors have zero for a dot product. If $\vec{\boldsymbol{A}}\perp\vec{\boldsymbol{B}}$ then the angle between them is $\pi/2$. Because $AB\cos\pi/2 = 0$

$$\Rightarrow \quad \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = 0 \quad \text{if} \quad \vec{\boldsymbol{A}}\perp\vec{\boldsymbol{B}}.$$

The dot product of parallel vectors is the product of their magnitudes. The angle between parallel vectors is zero and $AB\cos 0 = AB$. In particular, $\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{A}} = A^2$ or $|\vec{\boldsymbol{A}}| = \sqrt{\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{A}}}$

$$\Rightarrow \quad \vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = |\vec{\boldsymbol{A}}||\vec{\boldsymbol{B}}| \quad \text{if} \quad \vec{\boldsymbol{A}}\parallel\vec{\boldsymbol{B}}.$$

The standard unit base vectors are orthonormal. They are unit vectors (in this case 'normal' means normalized, meaning taken down to size) and they are perpendicular (ortho) to each other.

$$\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} = 1 \quad \text{and} \quad \hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\imath}} = 0$$

Also, the standard tilted base vectors are orthonormal.

$$\hat{\boldsymbol{\imath}}'\cdot\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\jmath}}'\cdot\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}' = 1 \quad \text{and} \quad \hat{\boldsymbol{\imath}}'\cdot\hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}'\cdot\hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{\imath}}' = 0$$

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = A_1 B_1 + A_2 B_2 + A_3 B_3.$$

To see how this component formula for the dot product follows from the geometric definition above see box 1.4 on page 7.

Using the dot product to find components

To find the x component of a vector (or vector expression) one can use the dot product (fig. 1.32),

$$v_x = \text{projection of } \vec{\boldsymbol{v}} \text{ in the } \hat{\boldsymbol{\imath}} \text{ direction} = \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{\imath}}. \qquad (1.4)$$

This idea can be used for finding components in any direction.

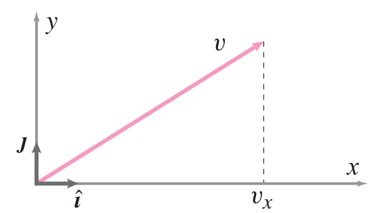


Figure 1.32: The dot product with unit vectors gives projection. For example, $v_x = \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{\imath}}$.

Tilted base vectors. If one knows the orientation of the tilted unit vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, $\hat{\boldsymbol{k}}'$ relative to the standard bases $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$ then you can find the dot products between the standard base vectors and the tilted base vectors.

In 2D, assume that (fig. 1.33) $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}' = -\sin\theta$ and $\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}' = \cos\theta$. One can then use the dot product to find the $x'y'$ components $(A_{x'}, A_{y'})$ from the xy components (A_x, A_y). For example, We start with the absurdly obvious (and ridiculously useful) equation

$$\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}$$

and dot both sides with $\hat{\boldsymbol{\jmath}}'$ to get:

$$\begin{aligned}
\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\jmath}}' &= \vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\jmath}}' \\
\underbrace{(A_{x'}\hat{\boldsymbol{\imath}}' + A_{y'}\hat{\boldsymbol{\jmath}}')}_{\vec{\boldsymbol{A}}} \cdot \hat{\boldsymbol{\jmath}}' &= \underbrace{(A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{A}}} \cdot \hat{\boldsymbol{\jmath}}' \\
A_{x'} \underbrace{\hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}}'}_{0} + A_{y'} \underbrace{\hat{\boldsymbol{\jmath}}' \cdot \hat{\boldsymbol{\jmath}}'}_{1} &= A_x \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}' + A_y \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}' \\
\Rightarrow \quad A_{y'} &= A_x \underbrace{(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}')}_{-\sin\theta} + A_y \underbrace{(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}')}_{\cos\theta}.
\end{aligned}$$

Similarly, one could find the component $A_{x'}$ using a dot product with $\hat{\boldsymbol{\imath}}'$ as

$$A_{x'} = A_x \underbrace{(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}')}_{\cos\theta} + A_y \underbrace{(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}}')}_{\sin\theta}.$$

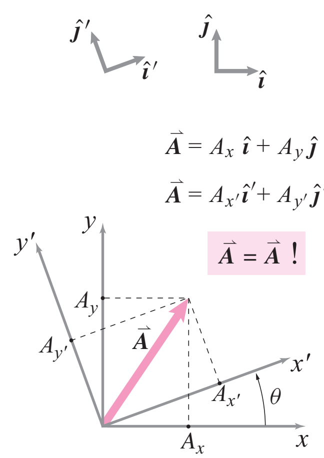


Figure 1.33: The dot product helps find components in terms of tilted unit vectors. Note, the components of $\vec{\boldsymbol{A}}$ depend on which coordinate system you use. Nonetheless, $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}$, no matter how $\vec{\boldsymbol{A}}$ is represented.

Using dot products to find components of vectors is particularly useful when more than one base vector system is used.

Matrix form. For those comfortable with linear algebra, note that the results above can be condensed into the matrix equations

$$[\vec{\boldsymbol{A}}]_{x'y'} = [R][\vec{\boldsymbol{A}}]_{xy}, \text{ or} \tag{1.5}$$

$$\begin{bmatrix} A_{x'} \\ A_{y'} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}}_{R} \begin{bmatrix} A_x \\ A_y \end{bmatrix}. \tag{1.6}$$

Using dot products to get scalar equations

Again note that,

> Dot products are *the* way to get scalar equations from vector equations.

Example: Force balance
The statics vector force-balance equation

$$\vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{F}}_3 = \vec{\boldsymbol{0}} \tag{1.7}$$

can be reduced to two scalar equations by taking the dot product of both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$

$$\{(1.7)\} \cdot \hat{\boldsymbol{\imath}} \quad \Rightarrow \quad F_{1x} + F_{2x} + F_{3x} = 0 \tag{1.8}$$

$$\{(1.7)\} \cdot \hat{\boldsymbol{\jmath}} \quad \Rightarrow \quad F_{1y} + F_{2y} + F_{3y} = 0. \tag{1.9}$$

This approach is more general than the common (informal) approach of 'taking x and y components'.

Dotting with vectors other than $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, or $\hat{\boldsymbol{k}}$ We will often use unit vectors other than $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. We can use dot products with these 'crooked' vectors to get scalar equations.

Example: Getting scalar equations without dotting with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, or $\hat{\boldsymbol{k}}$.
Given the vector equation

$$-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}} = ma\hat{\boldsymbol{\lambda}}$$

where it is known that the unit vector $\hat{\boldsymbol{n}}$ is perpendicular to the unit vector $\hat{\boldsymbol{\lambda}}$, we can get a scalar equation and eliminate an unknown at the same time by dotting both sides with $\hat{\boldsymbol{\lambda}}$,

$$\begin{aligned}
\{\ (-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}}) &= (ma\hat{\boldsymbol{\lambda}})\,\}\cdot\hat{\boldsymbol{\lambda}} \\
(-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{\lambda}} &= (ma\hat{\boldsymbol{\lambda}})\cdot\hat{\boldsymbol{\lambda}} \\
-mg\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\lambda}} + N\underbrace{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}}}_{0} &= ma\underbrace{\hat{\boldsymbol{\lambda}}\cdot\hat{\boldsymbol{\lambda}}}_{1} \\
-mg\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\lambda}} &= ma \quad \Rightarrow \quad a = -g\cos\theta
\end{aligned}$$

with $\hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\lambda}}$ being $\cos\theta$, with θ being the angle between $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\lambda}}$.

Using dot products to solve geometry problems

A big part of solving many mechanics problems is the solving of geometry problems. Vector math, including the dot product, is useful for solving geometry problems.

Perpendicular and parallel parts. Given any vector $\vec{\boldsymbol{A}}$ and a unit vector $\hat{\boldsymbol{\lambda}}$, vector $\vec{\boldsymbol{A}}$ can be written as the sum of two parts,

$$\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}^{\parallel} + \vec{\boldsymbol{A}}^{\perp}$$

where $\vec{\boldsymbol{A}}^{\parallel}$ ('A parallel') is parallel to $\hat{\lambda}$ and $\vec{\boldsymbol{A}}^{\perp}$ ('A perp') is perpendicular to $\hat{\boldsymbol{\lambda}}$ (see fig. 1.34). The part parallel to $\hat{\boldsymbol{\lambda}}$ is

$$\vec{\boldsymbol{A}}^{\parallel} = (\text{projection of } \vec{\boldsymbol{A}} \text{ in } \hat{\lambda} \text{ direction})\hat{\boldsymbol{\lambda}} = (\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}}.$$

The perpendicular part of $\vec{\boldsymbol{A}}$ is just what you get when you subtract out the parallel part, namely,

$$\vec{\boldsymbol{A}}^{\perp} = \vec{\boldsymbol{A}} - \vec{\boldsymbol{A}}^{\parallel} = \vec{\boldsymbol{A}} - (\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}}$$

The claimed properties of the decomposition can now be checked, namely that $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}^{\parallel} + \vec{\boldsymbol{A}}^{\perp}$ (just add the 2 equations above and see), that $\vec{\boldsymbol{A}}^{\parallel}$ is in the direction of $\hat{\boldsymbol{\lambda}}$ (it's a scalar multiple), and that $\vec{\boldsymbol{A}}^{\perp}$ is perpendicular to $\hat{\boldsymbol{\lambda}}$ ($\vec{\boldsymbol{A}}^{\perp} \cdot \hat{\boldsymbol{\lambda}} = 0$).

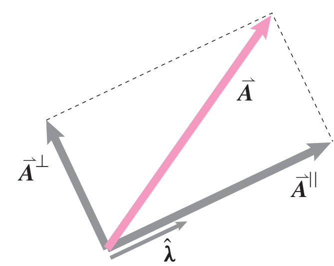
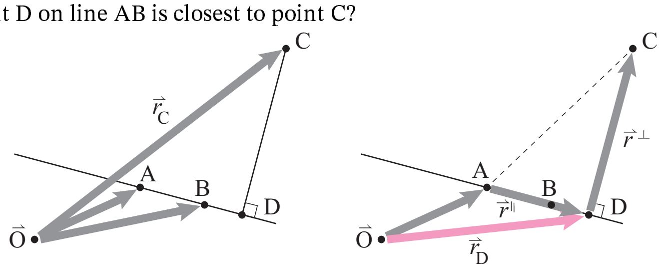


Figure 1.34: For any $\vec{\boldsymbol{A}}$ and $\hat{\boldsymbol{\lambda}}$, $\vec{\boldsymbol{A}}$ can be decomposed into a part parallel to $\hat{\boldsymbol{\lambda}}$ and a part perpendicular to $\hat{\boldsymbol{\lambda}}$.

Example: Closest point on a given line, to a given point off of the line

What point D on line AB is closest to point C?



In other words, a line is defined by the two given points A and B. A given point C is off of the line. What point D, on the line, is closest to C. Or, in vector language, Given vectors $\vec{\boldsymbol{r}}_A$ and $\vec{\boldsymbol{r}}_B$ defining a line, and $\vec{\boldsymbol{r}}_C$, defining a point off of the line, find $\vec{\boldsymbol{r}}_D$, the position of the point on the line that is closest to C. What is point $\vec{\boldsymbol{r}}_D$, the point on the line AB that is closest to C? The answer is,

$$\vec{\boldsymbol{r}}_D = \vec{\boldsymbol{r}}_A + \vec{\boldsymbol{r}}_{C/A}{}^{\parallel}$$

where $\vec{\boldsymbol{r}}_{C/A}{}^{\parallel}$ is the part of $\vec{\boldsymbol{r}}_{C/A}$ that is parallel to the line segment AB. Thus,

$$\vec{\boldsymbol{r}}_D = \vec{\boldsymbol{r}}_A + \left[(\vec{\boldsymbol{r}}_C - \vec{\boldsymbol{r}}_A) \cdot \frac{\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A}{|\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A|}\right] \frac{\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A}{|\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A|}.$$

Example: Gram-Schmidt orthogonalization

A plane is defined in 3D by two vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ that lie in a plane. The goal of 'Gram-Schmidt orthogonalization' is to find a pair of orthonormal vectors② that lie in the plane. First find a unit vector $\hat{\boldsymbol{\lambda}}_A$ in the $\vec{\boldsymbol{A}}$ direction:

$$\hat{\boldsymbol{\lambda}}_A = \vec{\boldsymbol{A}}/|\vec{\boldsymbol{A}}|.$$

②A set of *orthonormal* vectors are a set of unit vectors that are all perpendicular to each other. The 'ortho' is for orthogonal, meaning perpendicular. And, for no good reason, vectors are called *normalized* if you replace them with unit vectors in their directions.

Then find the part $\vec{B}^{\perp}$ of $\vec{B}$ that is orthogonal to $\hat{\lambda}_A$:

$$\vec{B}^{\perp} = \vec{B} - \vec{B}^{\parallel} = \vec{B} - (\vec{B}\cdot\hat{\lambda}_A)\hat{\lambda}_A.$$

Finally, normalize:

$$\hat{\lambda}_B = \vec{B}^{\perp}/|\vec{B}^{\perp}|$$

From two vectors in a plane, $\vec{A}$ and $\vec{B}$, we have found a pair of vectors $(\hat{\lambda}_A, \hat{\lambda}_B)$ that are orthonormal (unit vectors that are orthogonal to each other) and in the same plane.

Components of a vector perpendicular and parallel to a given plane. The ideas above also apply to planes. What parts of a vector $\vec{C}$ are in ($\vec{C}^{\parallel}$) and orthogonal ($\vec{C}^{\perp}$) to a given plane?

Method 1. The plane is defined by two vectors $\vec{A}$ and $\vec{B}$ (two vectors that are in the plane). But $\vec{A}$ and $\vec{B}$ are not necessarily orthogonal. So, we first use Gram-Schmidt orthogonalization (sample above) to find orthonormal vectors $\hat{\lambda}_A$ and $\hat{\lambda}_B$ (see the example above). Then, we have that

$$\vec{C}^{\parallel} = (\vec{C}\cdot\hat{\lambda}_A)\hat{\lambda}_A + (\vec{C}\cdot\hat{\lambda}_B)\hat{\lambda}_B \qquad \text{and} \qquad \vec{C}^{\perp} = \vec{C} - \vec{C}^{\parallel}$$

Method 2. If the plane has known normal $\hat{n}$ then

$$\vec{C}^{\perp} = (\vec{C}\cdot\hat{n})\hat{n} \qquad \text{and} \qquad \vec{C}^{\parallel} = \vec{C} - \vec{C}^{\perp}.$$

Vector algebra

Vectors are algebraic quantities. So you use algebra to manipulate them in equations. The rules for vector algebra are similar to the rules for ordinary (scalar) algebra. For example, if vector $\vec{A}$ is the same as vector $\vec{B}$, $\vec{A} = \vec{B}$, then, for any scalar a and any vector $\vec{C}$, we have,

$$\begin{aligned} \vec{A} + \vec{C} &= \vec{B} + \vec{C}, \\ a\vec{A} &= a\vec{B}, \text{ and} \\ \vec{A}\cdot\vec{C} &= \vec{B}\cdot\vec{C}. \end{aligned}$$

Why? Because performing the same operation on equal quantities gives equal results. The vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ might themselves be expressions involving other vectors.

The equations above show all③of the allowable manipulations of vector equations:

- Adding a common term to both sides,
- Multiplying both sides by a common scalar, and
- Taking the dot product of both sides with a common vector.

All the distributive, associative, and commutative laws of ordinary addition and multiplication hold *except* for when there is no sensible meaning to the expressions④.

More about vector algebra and the use of vector algebra to 'solve triangles' is discussed in sec. 1.5 starting on page 106.

③All for now, that is. Later we will add 'taking the cross product of both sides with a common vector'.

④ **Beware**. It does not make sense to add a vector and a scalar; $7 + \vec{A}$ is a *nonsense* expression. And you cannot divide a vector by a vector or a scalar by a vector; $7/\hat{\imath}$ and $\vec{A}/\vec{C}$ are *nonsense* expressions.

Dot products on the computer

Using computers for vector addition was discussed on page 56. Most computer languages will agree to calculating a dot product for you in response to commands something like this:

```
A   = [  1  2  5 ]
B   = [ -2  4 19 ]
D   = A(1)*B(1) + A(2)*B(2) + A(3)*B(3).
```

In pseudo code we could write `D = A dot B`. Many computer languages have a shorter way to write the dot product like `dot(A,B)`. In a language built for linear algebra `D = A*B'`⑤ will work because the rules of matrix multiplication are then the same as the component formula for the dot product.

⑤ Recall that `B'` means the transpose of `B`, in this case turning the row of numbers `B` into a column of numbers.

1.4 Using the geometric definition of the dot product to find the components formula

This theoretical aside answers the question: 'Why does

$$AB\cos(\theta) = A_x B_x + A_y B_y + A_z B_z?$$

' Vectors are essentially a geometric concept and we have consequently defined the dot product geometrically as $\vec{A} \cdot \vec{B} = AB cos\theta$. Almost 400 years ago René Descartes discovered that you could do geometry by doing algebra on the coordinates of points.

So we should be able to figure out the dot product of two vectors by knowing their components. The central key to finding this component formula is the distributive law

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

which we derived geometrically. If we write $\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}$ and $\vec{B} = B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}$ then we just repeatedly use the distributive law to derive the component formulae, as follows.

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= (A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}) \cdot (B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}) \\
&= (A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}) \cdot B_x \hat{\imath} \; + \\
&\quad\ (A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}) \cdot B_y \hat{\jmath} \; + \\
&\quad\ (A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}) \cdot B_z \hat{k} \\
&= A_x B_x \hat{\imath} \cdot \hat{\imath} + A_y B_x \hat{\jmath} \cdot \hat{\imath} + A_z B_x \hat{k} \cdot \hat{\imath} \; + \\
&\quad\ A_x B_y \hat{\imath} \cdot \hat{\jmath} + A_y B_y \hat{\jmath} \cdot \hat{\jmath} + A_z B_y \hat{k} \cdot \hat{\jmath} \; + \\
&\quad\ A_x B_z \hat{\imath} \cdot \hat{k} + A_y B_z \hat{\jmath} \cdot \hat{k} + A_z B_z \hat{k} \cdot \hat{k} \\
&= A_x B_x (1) + A_y B_x (0) + A_z B_x (0) \; + \\
&\quad\ A_x B_y (0) + A_y B_y (1) + A_z B_y (0) \; + \\
&\quad\ A_x B_z (0) + A_y B_z (0) + A_z B_z (1)
\end{aligned}$$

$$\Rightarrow \quad \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \qquad \text{(3D).}$$

$$\Rightarrow \quad \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y \qquad \text{(2D).}$$

If we call our coordinate x_1, x_2, and x_3; and our unit base vectors $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ we would have $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ and $\vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3$ and the dot product has the familiar tidy form:

$$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = \sum_{i=1}^{3} A_i B_i.$$

Tilted coordinates: xyz *vs* $x'y'z'$.

The demonstration above could have been carried out using a different orthogonal coordinate system $x'y'z'$ that was tilted with respect to the xyz system. By identical reasoning we would find that $\vec{A} \cdot \vec{B} = A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'}$. Even though all of the numbers in the list $[A_x, A_y, A_z]$ might be different from the numbers in the list $[A_{x'}, A_{y'}, A_{z'}]$ and similarly all the list $[\vec{B}]_{xyz}$ might be different than the list $[\vec{B}]_{x'y'z'}$, so (remarkably, luckily and necessarily),

$$A_x B_x + A_y B_y + A_z B_z = A_{x'} B_{x'} + A_{y'} B_{y'} + A_{z'} B_{z'}.$$

The formula for the dot product is the same in the different coordinate systems. And the value of the dot product is the same in the different coordinate systems. Yet all the numbers on the two sides of the formula above are likely different from each other.