SAMPLE 1.12 **Calculating dot products:** Find the dot product of the two vectors $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{r} = 5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\,\hat{\jmath}$.

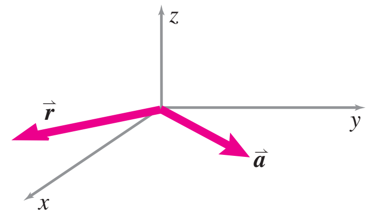

Figure 1.35

Solution The dot product of the two vectors is

$$\begin{aligned}
\vec{a}\cdot\vec{r} &= (2\hat{\imath} + 3\hat{\jmath} - 2\hat{k})\cdot(5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\,\hat{\jmath}) \\
&= (2\cdot 5\,\mathrm{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (2\cdot 2\,\mathrm{m})\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} \\
&\quad + (3\cdot 5\,\mathrm{m})\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - (3\cdot 2\,\mathrm{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} \\
&\quad - (2\cdot 5\,\mathrm{m})\underbrace{\hat{k}\cdot\hat{\imath}}_{0} + (2\cdot 2\,\mathrm{m})\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} \\
&= 10\,\mathrm{m} - 6\,\mathrm{m} = 4\,\mathrm{m}.
\end{aligned}$$

$$\boxed{\vec{a}\cdot\vec{r} = 4\,\mathrm{m}}$$

Comments: Note that with just a little bit of foresight, we could totally ignore the $\hat{k}$ component of $\vec{a}$ since $\vec{r}$ has no $\hat{k}$ component, *i.e.,* $\hat{k}\cdot\vec{r} = 0$. Also, if we keep in mind that $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = 0$, we could compute the above dot product in one line:

$$\vec{a}\cdot\vec{r} = (2\hat{\imath} + 3\hat{\jmath})\cdot(5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\,\hat{\jmath}) = (2\cdot 5\,\mathrm{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (3\cdot 2\,\mathrm{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} = 4\,\mathrm{m}.$$

SAMPLE 1.13 **Component of a vector in a given direction:** A force acting at some point is given as $\vec{F} = 5\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\,\hat{\jmath} + 2\,\mathrm{N}\hat{k}$.

1. Find the y-component of $\vec{F}$.
2. Find the component of $\vec{F}$ along the vector $\vec{r} = 3\,\mathrm{m}\hat{\imath} - 4\,\mathrm{m}\,\hat{\jmath}$.

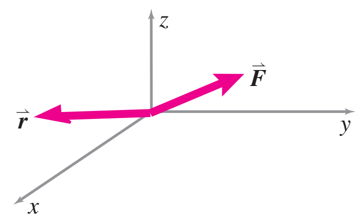

Figure 1.36

Solution

1. *Component along the y-direction:* The y-component of $\vec{F}$ is the scalar quantity multiplying the unit vector $\hat{\jmath}$, that is, 3 N. Although the y-component of $\vec{F}$ is obvious here (and hence the problem is trivial), we can find it in a formal way using the dot product between $\vec{F}$ and $\hat{\jmath}$.

$$\begin{aligned}
F_y &= \vec{F}\cdot(\text{a unit vector along } y\text{-axis}) \\
&= (5\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\,\hat{\jmath} + 2\,\mathrm{N}\hat{k})\cdot\hat{\jmath} \\
&= 5\,\mathrm{N}\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} + 3\,\mathrm{N}\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} + 2\,\mathrm{N}\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} = 3\,\mathrm{N}.
\end{aligned}$$

$$\boxed{F_y = \vec{F}\cdot\hat{\jmath} = 3\,\mathrm{N}.}$$

2. *Component along the $\vec{r}$-direction:* The component of $\vec{F}$ along $\vec{r}$ is obtained from the dot product of $\vec{F}$ with a unit vector along $\vec{r}$. Therefore, we first need to find a unit vector $\hat{\lambda}_r$ along $\vec{r}$ and then dot it with $\vec{F}$.

$$\begin{aligned}
\hat{\lambda}_r &= \frac{\vec{r}}{|\vec{r}|} = \frac{3\,\mathrm{m}\hat{\imath} - 4\,\mathrm{m}\,\hat{\jmath}}{\sqrt{3^2 + 4^2}\,\mathrm{m}} = 0.6\hat{\imath} - 0.8\hat{\jmath} \\
F_r &= \vec{F}\cdot\hat{\lambda}_r \\
&= (5\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\,\hat{\jmath} + 2\,\mathrm{N}\hat{k})\cdot(0.6\hat{\imath} - 0.8\hat{\jmath}) \\
&= 3.0\,\mathrm{N} - 2.4\,\mathrm{N} = 0.6\,\mathrm{N}.
\end{aligned}$$

$$\boxed{F_r = \vec{F}\cdot\hat{\lambda}_r = 0.6\,\mathrm{N}}$$

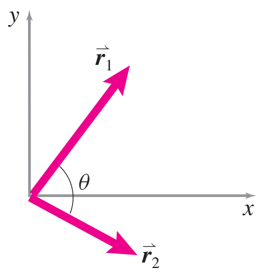

Figure 1.37

SAMPLE 1.14 Finding angle between two vectors using dot product: Find the angle between the vectors $\vec{r}_1 = 2\hat{\imath} + 3\hat{\jmath}$ and $\vec{r}_2 = 2\hat{\imath} - \hat{\jmath}$.

Solution From the definition of dot product between two vectors

$$\begin{aligned} \vec{r}_1 \cdot \vec{r}_2 &= |\vec{r}_1||\vec{r}_2| \cos\theta \\ \text{or} \quad \cos\theta &= \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} \\ &= \frac{(2\hat{\imath} + 3\hat{\jmath}) \cdot (2\hat{\imath} - \hat{\jmath})}{(\sqrt{2^2+3^2})(\sqrt{2^2+1^2})} \\ &= \frac{4-3}{\sqrt{13}\sqrt{5}} = 0.124 \\ \text{Therefore,} \quad \theta &= \cos^{-1}(0.124) = 82.87^\circ. \end{aligned}$$

$\boxed{\theta = 83^\circ}$

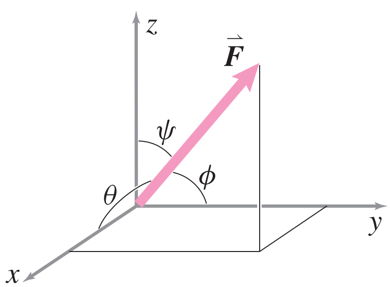

Figure 1.38

SAMPLE 1.15 Finding direction cosines from unit vectors: Find the angles between $\vec{F} = 4\,\mathrm{N}\hat{\imath} + 6\,\mathrm{N}\hat{\jmath} + 7\,\mathrm{N}\hat{k}$ and each of the three axes.

Solution

$$\begin{aligned} \vec{F} &= F\hat{\lambda} \\ \hat{\lambda} &= \frac{\vec{F}}{F} \\ &= \frac{4\,\mathrm{N}\hat{\imath} + 6\,\mathrm{N}\hat{\jmath} + 7\,\mathrm{N}\hat{k}}{\sqrt{4^2+6^2+7^2}\,\mathrm{N}} \\ &= 0.4\hat{\imath} + 0.6\hat{\jmath} + 0.7\hat{k}. \end{aligned}$$

Let the angles between $\hat{\lambda}$ and the x, y, and z axes be θ, ϕ and ψ respectively. Then

$$\begin{aligned} \cos\theta &= \frac{\hat{\imath}\cdot\hat{\lambda}}{|\hat{\imath}||\hat{\lambda}|} = \frac{0.4}{|1||1|} = 0.4. \\ \Rightarrow \quad \theta &= \cos^{-1}(0.4) = 66.4^\circ. \end{aligned}$$

Similarly,

$$\begin{aligned} \cos\phi &= 0.6 \quad \text{or} \quad \phi = 53.1^\circ \\ \cos\psi &= 0.7 \quad \text{or} \quad \psi = 45.6^\circ. \end{aligned}$$

$\boxed{\theta = 66.4^\circ,\ \phi = 53.1^\circ,\ \psi = 45.6^\circ}$

Comments: The components of a unit vector give the direction cosines with the respective axes. That is, if the angle between the unit vector and the x, y, and z axes are θ, ϕ and ψ, respectively (as above), then

$$\hat{\lambda} = \underbrace{\cos\theta}_{\lambda_x}\,\hat{\imath} + \underbrace{\cos\phi}_{\lambda_y}\,\hat{\jmath} + \underbrace{\cos\psi}_{\lambda_z}\,\hat{k}.$$

SAMPLE 1.16 Separating a vector equation into scalar equations:

Assume that after writing the equilibrium equation as $\sum \vec{F} = mg\hat{\jmath}$ in a particular problem, a student finds that $\sum \vec{F} = (20\,\mathrm{N} - P_1)\hat{\imath} + 7\,\mathrm{N}\hat{\jmath} - P_2\hat{k}$. Separate the scalar equations in the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ directions.

Solution From a vector equation, separating the scalar equations is trivial as long as both sides of a vector equation are in the same basis, *e.g.*, , $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. We just equate the individual components on both sides. That is, if

$$\begin{aligned}
\overbrace{(20\,\mathrm{N} - P_1)\hat{\imath} + 7\,\mathrm{N}\hat{\jmath} - P_2\hat{k}}^{\sum \vec{F}} &= mg\hat{\jmath} & \\
\text{then,} \quad 20\,\mathrm{N} - P_1 &= 0 & (\hat{\imath}\text{ component}) \\
7\,\mathrm{N} &= mg & (\hat{\jmath}\text{ component}) \\
-P_2 &= 0. & (\hat{k}\text{ component})
\end{aligned}$$

These are the three independent scalar equations in the $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ directions.

$$\boxed{20\,\mathrm{N} - P_1 = 0, \quad 7\,\mathrm{N} = mg, P_2 = 0}$$

Comments: The results obtained by equating individual components on both sides of the vector equation are based on the general technique of taking the dot product of both sides of an equation with a vector. It gives a scalar equation valid in any direction that one desires. For the example at hand, the long but easily readable and illustrative calculation is as follows. Taking the dot product of both sides of $\sum \vec{F} = mg\hat{\jmath}$ equation with $\hat{\imath}$, we write

$$\begin{aligned}
\hat{\imath} \cdot \left[(20\,\mathrm{N} - P_1)\hat{\imath} + 7\,\mathrm{N}\hat{\jmath} - P_2\hat{k}\right] &= \hat{\imath} \cdot [mg\hat{\jmath}] \\
\Rightarrow \quad (20\,\mathrm{N} - P_1)\underbrace{\hat{\imath} \cdot \hat{\imath}}_{1} + 7\,\mathrm{N}\underbrace{\hat{\jmath} \cdot \hat{\imath}}_{0} - P_2\underbrace{\hat{k} \cdot \hat{\imath}}_{0} &= mg\underbrace{\hat{\jmath} \cdot \hat{\imath}}_{0}) \\
\Rightarrow \quad 20\,\mathrm{N} - P_1 &= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{\jmath} \cdot \left[\sum \vec{F}\right] = \hat{\jmath} \cdot [mg\hat{\jmath}] \quad &\Rightarrow \quad 7\,\mathrm{N} = mg \\
\hat{k} \cdot \left[\sum \vec{F}\right] = \hat{k} \cdot [mg\hat{\jmath}] \quad &\Rightarrow \quad -P_2 = 0.
\end{aligned}$$