1.1 Vector notation and vector addition

Facility with vectors has several aspects.

1. You must recognize which quantities are vectors (such as relative position) and which are scalars (such as length).
2. You have to use a notation that distinguishes between vectors and scalars using, for example, $\vec{\boldsymbol{a}}$, or $\underline{\boldsymbol{a}}$ for acceleration and a for a scalar with the same magnitude so that $|a| = |\vec{\boldsymbol{a}}|$.
3. You need skills in vector arithmetic, perhaps more than you learned in your previous math and physics courses.

In this first section (1.1) we start with notation. Mastery of the notation, perhaps a seemingly petty topic, is surprisingly well-correlated with mastery of mechanics. We go on to find the relative position vector from a picture, to multiplication of a vector by a scalar, and to vector addition and vector subtraction.

How to write vectors

A scalar is written as a single English or Greek letter. This book uses italics for scalars (*e.g.,* m for mass) but ordinary printing is fine for hand work (*e.g.,* m for mass). A vector is also represented by a single letter of the alphabet, either English or Greek, but ornamented to indicate that it is a vector and not a scalar. The common ornamentations are described below.

*Use one of these vector notations in all of your work*①.

① **Be careful to distinguish vectors from scalars** in all of your written work. Clear notation helps clear thinking and will help you solve problems. If you notice that you are not using clear vector notation, *stop*, determine which quantities are vectors and which scalars, and fix your notation. Rare is the student who consistently gets correct answers to exam questions without clear vector notation. And almost as rare is the student who has clear vector usage and can't do problems. For some students, accepting this vector language and syntax is a bitter pill. Just swallow it. You'll feel much better.

$\vec{\boldsymbol{F}}$ Putting a harpoon (or arrow) over the letter $\boldsymbol{F}$. The arrow is suggestive of the idea of a vector. This is the notation used in this book.

$\boldsymbol{F}$ In most texts a bold $\boldsymbol{F}$ represents the vector $\vec{\boldsymbol{F}}$. But, bold face is inconvenient for hand written work. The lack of bold face pens and pencils tempts students to transcribe a bold $\boldsymbol{F}$ as F. But F with no adornment represents a scalar and not a vector. So, when using other books, take care to transcribe $\boldsymbol{F}$ as $\vec{\boldsymbol{F}}$ or as $\underline{F}$.

$\underline{\boldsymbol{F}}$ Underlining or under-squiggling ($\underset{\sim}{\boldsymbol{F}}$) is an easy and unambiguous notation for hand writing vectors. A recent Gallup poll survey found that 117 out of 168 mechanics professors use this notation. These professors would copy a $\vec{\boldsymbol{F}}$ from this book by writing $\underline{F}$. The origin of the underline notation seems to be from typesetting. In the pre-computer days, a book author would indicate that a letter should be printed in bold by underlining it.

$\bar{\boldsymbol{F}}$ It is a stroke simpler to put a bar rather than a harpoon over a symbol. But the saved effort causes ambiguity. Why? An over-bar is often used to indicate average. There could then be confusion, say, between the velocity $\bar{v}$ and the average speed $\bar{v}$.

$\hat{\boldsymbol{\imath}}$ Over-hat. Putting a hat on top is like an over-arrow or over-bar. In this book we reserve the hat for unit vectors. For example, we use $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, or $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ for unit vectors parallel to the x, y, and z axes, respectively. The same poll of 168 mechanics professors found that 32 of them used no special notation for unit vectors and just wrote them like, *e.g.*, $\underline{\boldsymbol{\imath}}$.

Drawing vectors

In fig. 1.1 on page 45, the magnitude of $\vec{\boldsymbol{A}}$ was represented by the drawing length. But drawing a vector using its magnitude as length would be awkward if, say, we were interested in vector $\vec{\boldsymbol{B}}$ that points Northwest and has a magnitude of 2 meters. To fit $\vec{\boldsymbol{B}}$ in a drawing would require a piece of paper about 2 meters square (each edge the length of a basketball player). This situation moves from difficult to ridiculous if the magnitude of the vector of interest is 2 km and it would take half an hour to draw the vector by walking from tail to tip dragging a purple crayon. Thus, in pictures, we merely make scale drawings of vectors with, for example, one centimeter of graph paper representing 1 kilometer of vector magnitude.

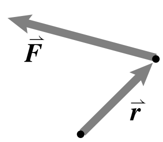

Figure 1.2: Position and force vectors are drawn with different scales.

The necessity for using scale drawings to represent vectors is especially apparent for a vector whose magnitude is *not* length. Force is a vector since it has magnitude and direction. Say $\vec{\boldsymbol{F}}_{\text{gr}}$ is the 700 N force that the ground pushes up on your chair as you sit reading. We can't draw a line segment with length 700 N for $\vec{\boldsymbol{F}}_{\text{gr}}$ because a Newton is a unit of force not a unit of length. So, a scale drawing is the only choice.

1.1 The scalars in mechanics

Most of the scalars in this book are listed below. Dimensions (units) are given in brackets [ ], (M for mass, L for length, T for time, F for force, and E for energy).

- mass m, $[M]$;
- length or distance ℓ, w, x, r, ρ, d, or s, $[L]$;
- time t, $[T]$;
- pressure p, $[F/L^2] = [M/(L \cdot T^2)]$;
- angles θ 'theta', ϕ 'phi', γ 'gamma', and ψ 'psi', [dimensionless];
- energy E, kinetic energy E_{K}, potential energy E_{P}, $[E]$=$[F \cdot L]$=$[M \cdot L^2/T^2]$;
- work W, $[E]$=$[F \cdot L]$=$[M \cdot L^2/T^2]$;
- tension T, $[M \cdot L/T^2] = [F]$;
- power P, $[E/T]$=$[M \cdot L^2/T^3]$;
- the magnitudes of all the vector quantities are also scalars, for example
  - speed $|\vec{\boldsymbol{v}}|$, $[L/T]$;
  - magnitude of acceleration $|\vec{\boldsymbol{a}}|$, $[L/T^2]$;
  - magnitude of angular momentum $|\vec{\boldsymbol{H}}|$, $[M \cdot L^2/T]$;
- the components of vectors, for example
  - r_x (where $\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$), or
  - $L_{x'}$ (where $\vec{\boldsymbol{L}} = L_{x'}\hat{\boldsymbol{\imath}}' + L_{y'}\hat{\boldsymbol{\jmath}}'$);
- coefficient of friction μ 'mu', or friction angle ϕ 'phi';
- coefficient of restitution e;
- mass per unit length, area, or volume ρ;
- oscillation frequency β or λ.

One often needs to draw vectors with different units on the same picture, as for showing the position $\vec{\boldsymbol{r}}$ at which a force $\vec{\boldsymbol{F}}$ is applied (see fig. 1.2). In this case, even though the vectors are on the same drawing, different scale factors are used for the drawing of the vectors that have different units.

Drawing and measuring are tedious, and also not very accurate. And drawing in 3 dimensions is particularly hard (given the poor quality of 3D graph paper nowadays). So, the magnitudes and directions of vectors are usually defined with numbers and units rather than scale drawings. Despite that we will ultimately calculate using numbers and lists of numbers, the drawing rules and geometric descriptions define all the vector concepts.

Adding vectors

Tip to tail rule. The sum of two vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ is defined by the *tip to tail* rule of vector addition shown in fig. 1.3a for the sum $\vec{\boldsymbol{C}} = \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$. Vector $\vec{\boldsymbol{A}}$ is drawn. Then, vector $\vec{\boldsymbol{B}}$ is drawn with its tail at the tip (or head) of $\vec{\boldsymbol{A}}$. The sum $\vec{\boldsymbol{C}}$ is the vector from the tail of $\vec{\boldsymbol{A}}$ to the tip of $\vec{\boldsymbol{B}}$.

1.2 The Vectors in Mechanics

The vector quantities used in mechanics, and the notations for them used in this book, are shown below. The dimensional symbols of each are shown in brackets [ ].

- position $\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{x}}$, $[L]$;
- velocity $\vec{\boldsymbol{v}}$ or $\dot{\vec{\boldsymbol{x}}}$ or $\dot{\vec{\boldsymbol{r}}}$, $[L/T]$;
- acceleration $\vec{\boldsymbol{a}}$ or $\dot{\vec{\boldsymbol{v}}}$ or $\ddot{\vec{\boldsymbol{r}}}$, $[L/T^2]$;
- angular velocity $\vec{\boldsymbol{\omega}}$ 'omega' (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\dot{\theta}\hat{\boldsymbol{k}}$), $[1/T]$;
- rate of change of angular velocity $\vec{\boldsymbol{\alpha}}$ 'alpha' or $\dot{\vec{\boldsymbol{\omega}}}$ (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\ddot{\theta}\hat{\boldsymbol{k}}$), $[1/T^2]$;
- force $\vec{\boldsymbol{F}}$ or $\vec{\boldsymbol{N}}$, $[m \cdot L/T^2]$=$[F]$;
- moment or torque $\vec{\boldsymbol{M}}$, $[m \cdot L^2/T^2]$=$[F \cdot L]$;
- linear momentum $\vec{\boldsymbol{L}}$, $[m \cdot L/T]$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$, $[m \cdot L/T^2]$;
- angular momentum $\vec{\boldsymbol{H}}$, $[m \cdot L^2/T]$; and its rate of change $\dot{\vec{\boldsymbol{H}}}$, $[m \cdot L^2/T^2]$.
- unit vectors to help write other vectors [dimensionless]:
  - $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ for cartesian coordinates,
  - $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, and $\hat{\boldsymbol{k}}'$ for crooked cartesian coordinates,
  - $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ for polar coordinates,
  - $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$ for path coordinates, and
  - $\hat{\boldsymbol{\lambda}}$ 'lambda' and $\hat{\boldsymbol{n}}$ as miscellaneous unit vectors.

Ornamentation of vectors. Subscripts and superscripts are often added to indicate the point, points, object, or objects the vectors are describing. Upper case letters (O, A, B, C,...) are used to denote points. Upper case calligraphic (or script if you are writing by hand) letters ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$...$\mathcal{F}$...) are for labeling rigid objects or reference frames. $\mathcal{F}$, or $\mathcal{N}$, is the fixed, Newtonian, or 'absolute' reference frame (think of $\mathcal{F}$ as the ground if you are a first-time reader). For example, $\vec{\boldsymbol{r}}_{\mathrm{AB}}$ or $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ is the position of the point B relative to point A. $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is the absolute angular velocity of the object called $\mathcal{B}$ ($\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is short hand for $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$). And $\vec{\boldsymbol{H}}_{\mathcal{A}/C}$ is the angular momentum of object $\mathcal{A}$ relative to point C (We will explain the meaning of the words 'angular velocity' and 'angular momentum' in the dynamics book. Here, we are just using them to show the subscript ornamentation.).

The notation is further complicated when we want to take derivatives with respect to moving frames, a topic which comes up later in the Dynamics book. For completeness here: ${}^{\mathcal{B}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{E}}$ is the time derivative with respect to reference frame $\mathcal{B}$ of the angular velocity of object $\mathcal{D}$ with respect to object (or frame) $\mathcal{E}$. (If this paragraph doesn't read like gibberish to you, you have already studied dynamics. This paragraph is for the experts who are looking back.)

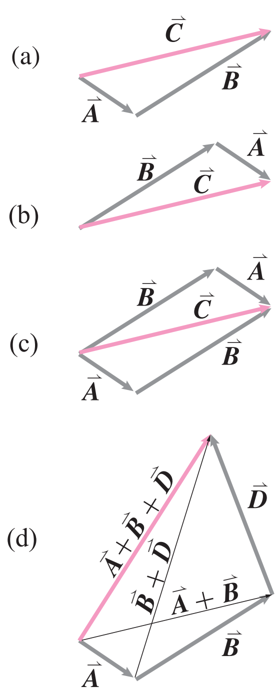

Figure 1.3: (a) tip to tail addition of $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}$, (b) tip to tail addition of $\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}$, (c) the parallelogram interpretation of vector addition which shows the commutative law of vector addition: $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}$, and (d) The associative law of vector addition: $(\vec{\boldsymbol{A}}+\vec{\boldsymbol{B}})+\vec{\boldsymbol{D}} = \vec{\boldsymbol{A}}+(\vec{\boldsymbol{B}}+\vec{\boldsymbol{D}})$.

This figure also makes sense in 3D. Drawings (a), (b) and (c) are all on a tilted planes. And the 6 vectors drawn in (d) lie on the edges of a tetrahedron.

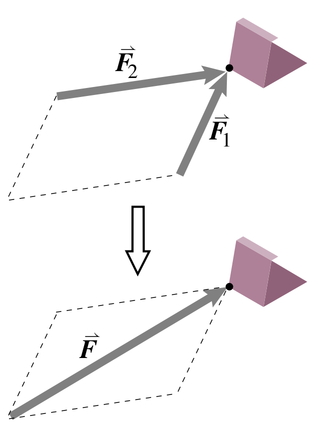

Figure 1.4: Two forces acting at a point may be replaced by their sum for all mechanics purposes.

Parallelogram rule. The same sum is achieved if $\vec{\boldsymbol{B}}$ is drawn first, as shown in fig. 1.3b. Putting both ways of adding $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ on the same picture draws a parallelogram as shown in fig. 1.3c. Hence the tip to tail rule of vector addition is also called the *parallelogram* rule. The parallelogram construction shows the *commutative* property of vector addition, namely that $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}$.

3D. Note that you can view fig. 1.3a-c as 3D pictures. In 3D, the parallelogram will be on a plane but that may well be tilted relative to the x, y and z axes.

Adding many vectors. Three vectors are added by the same tip to tail rule. The construction shown in fig. 1.3d shows that $(\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}) + \vec{\boldsymbol{D}} = \vec{\boldsymbol{A}} + (\vec{\boldsymbol{B}} + \vec{\boldsymbol{D}})$ so that the expression $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \vec{\boldsymbol{D}}$ is unambiguous. This is the *associative* property of vector addition. Showing this triple sum in all different orders and groupings draws a parallelepiped (with some extra lines). Thus, one might call the geometric addition of three vectors in 3D the 'parallelepiped rule' for vector addition (but no-one does).

With these two properties, the commutative and associative laws of vector addition, we see that the sum $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \vec{\boldsymbol{D}} + \ldots$ can be permuted to $\vec{\boldsymbol{D}} + \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \ldots$, or any which way, without changing the result. That is, vector addition shares the associativity and commutativity of scalar addition that you are used to *e.g.*, that $3 + (7 + \pi) = (\pi + 3) + 7$.

Concurrent forces. We can reconsider the statement 'force is a vector' and see that it hides (or contains) one of the basic assumptions in mechanics, namely:

> If forces $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ are applied to a point on a structure they can be replaced, for all mechanics considerations, with a single force $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2$ applied to that point

as illustrated in fig. 1.4. The force $\vec{\boldsymbol{F}}$ is said to be *equivalent* to the *concurrent* (acting at one point) force system consisting of $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ acting at the same point.

Apples and oranges. Note that two vectors with different dimensions cannot be added. Figure 1.2 on page 2 can no more sensibly be taken to represent meaningful vector addition than can the scalar sum of a length and a weight, "2 ft + 3 N", be taken as meaningful.

Subtraction, negation, and the zero vector

Subtraction is most simply defined by inverse addition. The expression $\vec{\boldsymbol{C}} - \vec{\boldsymbol{A}}$ means: find the vector which, when added to $\vec{\boldsymbol{A}}$, gives $\vec{\boldsymbol{C}}$[2]. We can draw $\vec{\boldsymbol{C}}$, draw $\vec{\boldsymbol{A}}$ and then find the vector which, when added tip to tail to $\vec{\boldsymbol{A}}$, gives $\vec{\boldsymbol{C}}$. Figure 1.3a shows that $\vec{\boldsymbol{B}}$ answers the question. Another interpretation comes from defining the negative of a vector $-\vec{\boldsymbol{A}}$ as $\vec{\boldsymbol{A}}$ with the head and tail switched. Again you can see from fig. 1.3b by imagining that the head and tail on $\vec{\boldsymbol{A}}$ were switched to make $-\vec{\boldsymbol{A}}$, that $\vec{\boldsymbol{C}} + (-\vec{\boldsymbol{A}}) = \vec{\boldsymbol{B}}$. The negative of a vector evidently has the expected property that $\vec{\boldsymbol{A}} + (-\vec{\boldsymbol{A}}) = \vec{\boldsymbol{0}}$, where $\vec{\boldsymbol{0}}$ is the vector with no magnitude so that $\vec{\boldsymbol{C}} + \vec{\boldsymbol{0}} = \vec{\boldsymbol{C}}$ for all vectors $\vec{\boldsymbol{C}}$.

(2) Just like for money. \$100 - \$70 means 'find how much money you have to add to \$70 to get \$100'.

Relative-position vectors

The concept of relative position is used in most mechanics equations. The position of point B relative to point A is represented by the vector

$$\vec{\boldsymbol{r}}_{\mathrm{B/A}} \text{ (pronounced 'r of B relative to A')}$$

drawn from A and to B (as shown in fig. 1.5). An alternate notation for the relative position vector $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ is

$$\vec{\boldsymbol{r}}_{B/A} \equiv \vec{\boldsymbol{r}}_{\mathrm{AB}} \text{ (pronounced 'r A B' or 'r from A to B').}$$

You can think of the position of B relative to A as being the position of B relative to you if you were standing on A. Similarly $\vec{\boldsymbol{r}}_{\mathrm{C/B}} = \vec{\boldsymbol{r}}_{\mathrm{BC}}$ is the position of C relative to B.

Figure 1.5a shows that relative positions add by the tip to tail rule. That is,

$$\vec{\boldsymbol{r}}_{\mathrm{C/A}} = \vec{\boldsymbol{r}}_{\mathrm{B/A}} + \vec{\boldsymbol{r}}_{\mathrm{C/B}} \qquad \text{or} \qquad \vec{\boldsymbol{r}}_{\mathrm{AC}} = \vec{\boldsymbol{r}}_{\mathrm{AB}} + \vec{\boldsymbol{r}}_{\mathrm{BC}}$$

so, as needed to get the description 'vector', for relative-position vectors, vector addition has a sensible meaning.

Note that the position of B relative to A is the opposite (negative vector) of the position of A relative to B, so

$$\vec{\boldsymbol{r}}_{\mathrm{B/A}} = -\vec{\boldsymbol{r}}_{\mathrm{A/B}} \qquad \text{or} \qquad \vec{\boldsymbol{r}}_{\mathrm{AB}} = -\vec{\boldsymbol{r}}_{\mathrm{BA}}.$$

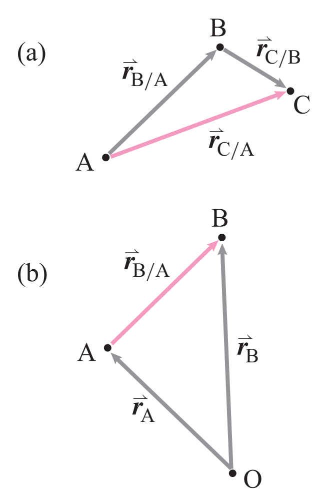

Figure 1.5: a) Relative position of points A, B, and C; b) Relative position of points O, A, and B.

Position relative to the origin. Often when doing problems we pick a distinguished point in space, say a prominent point or corner of a machine or structure, and use it as the origin of a coordinate system O. The position of point A relative to O is $\vec{\boldsymbol{r}}_{\mathrm{A/0}}$ or $\vec{\boldsymbol{r}}_{\mathrm{OA}}$ but we often adopt the shorthand notation $\vec{\boldsymbol{r}}_{\mathrm{A}}$ (pronounced 'r A') leaving the reference point O as implied,

$$\vec{\boldsymbol{r}}_{\mathrm{A}} \quad \text{means} \quad \vec{\boldsymbol{r}}_{\mathrm{A/0}}.$$

Figure 1.5b shows that

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

which rolls off the tongue more easily than $\vec{r}_{B/A} = \vec{r}_{B/0} - \vec{r}_{A/0}$, and also makes the concept of relative position easier to remember. ③

③Until we get to rotating reference frames (in the Dynamics book), you can just interpret 'relative to' to mean 'minus', as in english. 'How much money does Rudra have *relative to* Andy?' means what is Rudra's wealth *minus* Andy's wealth? What is the position of B *relative to* A? It is the position of B *minus* the position of A.

Multiplication by a scalar stretches a vector

Naturally enough $2\vec{F}$ means $\vec{F}+\vec{F}$ (see fig. 1.6) and $127\vec{A}$ means 127 copies of $\vec{A}$ added up. . Similarly $\vec{A}/7$ or $\frac{1}{7}\vec{A}$ means a vector in the direction of $\vec{A}$ that when added to 6 copies of itself gives $\vec{A}$. By combining these two ideas we can define any rational multiple of $\vec{A}$. For example $\frac{29}{13}\vec{A}$ means add 29 copies of that vector which, when 13 copies of itself are added up gives $\vec{A}$ ④.

④It is a mathematical fine point to extend the definition to $c\vec{A}$ for c that is irrational (*i.e.,* pick a sequence of rational numbers that approach c in the limit).

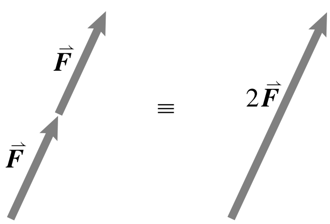

Figure 1.6: Multiplying a vector by a scalar stretches it.

Combining our abilities to negate a vector and multiply it by a positive scalar, we define $-17\vec{A}$ as $17(-\vec{A})$. In general, for any scalar c we define $c\vec{A}$ as the vector that is in the same direction as $\vec{A}$, or opposite if c is negative, and whose magnitude is multiplied by $|c|$. Five times a 5 N force pointed NorthEast is a 25 N force pointed NorthEast. Minus 5 times a 5 N force pointed NorthEast is a 25 N force pointed SouthWest.

Distributive rule for scalar multiplication of vectors. If you imagine stretching a whole vector addition diagram (*e.g.,* fig. 1.3a on page 4) equally in all directions the distributive rule for scalar multiplication is apparent. Here's what happens when you multiply the sum of the vectors vA and $\vec{B}$ by the scalar c:

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B},$$

the scalar multiplication 'distributes'.

Unit vectors have magnitude = 1

Unit vectors are vectors with a magnitude of one. Unit vectors are useful for indicating direction. Key examples are the unit vectors pointed in the positive x, y and z directions $\hat{\imath}$ (called 'i hat' or just 'i'), $\hat{\jmath}$, and $\hat{k}$. These three unit vectors are also called $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$, or $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$.

An easy way to find a unit vector in the direction of a vector $\vec{A}$ is to divide $\vec{A}$ by its magnitude. Thus

$$\hat{\lambda}_A \equiv \frac{\vec{A}}{|\vec{A}|}$$

is a unit vector in the $\vec{A}$ direction. We can check that this defines a unit vector by checking the rules for multiplication by a scalar: multiplying $\vec{A}$ by the scalar $1/|\vec{A}|$ gives a new vector with magnitude $|\vec{A}|/|\vec{A}| = 1$.

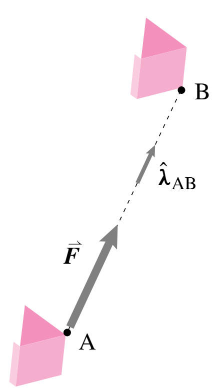

Figure 1.7: The force $\vec{F}$ that points from A to B can be represented as the product of a scalar F and a unit vector $\hat{\lambda}_{AB}$: $\vec{F} = F\hat{\lambda}_{AB}$.

A vector as a scalar times a unit vector. Often we know that a force $\vec{\boldsymbol{F}}$ is a yet unknown scalar F multiplied by a unit vector pointing between known points A and B (fig. 1.7). We can then write $\vec{\boldsymbol{F}}$ as

$$\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}_{\mathrm{AB}} = F\frac{\vec{\boldsymbol{r}}_{\mathrm{AB}}}{|\vec{\boldsymbol{r}}_{\mathrm{AB}}|} = F\frac{\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}}{|\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}|}$$

where we have used $\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}$ as the unit vector pointing from A to B. Note that when we take a vector to be a scalar times a unit vector, a form we will often use, the scalar need not be positive. So the 'scalar part' might be plus or minus the magnitude of the vector⑤.

⑤Many people sloppily, but without harm, call this scalar part the 'magnitude' of the vector, even though, technically, magnitudes are positive and the scalar part might be negative. To be precise, call it 'the scalar part'. To be tolerant, we excuse others who call this 'the magnitude'.

Three notations for vectors in pictures and diagrams.

Some options for drawing vectors are shown in sample 1.1 on page 60. The three notations below are the most common (see fig. 1.8).

Symbolic: labeling an arrow with a vector symbol. Indicate a vector, say a force $\vec{\boldsymbol{F}}$, by drawing an arrow and then labeling it with one of the symbolic notations above as in fig. 1.8a. *In this notation, the arrow is only schematic*, the magnitude and direction are determined by the algebraic symbol $\vec{\boldsymbol{F}}$. This is most clear if you draw the arrow roughly in the vector's direction and roughly to scale, but,

If the symbol and drawing disagree the symbol takes precedence (see sample 1.1j)

Graphical: "scalar times arrow", a scalar multiplies a unit vector in the direction of a drawn arrow (fig. 1.8b). Indicate a vector's direction by drawing an arrow. The direction should be made clear with a marked angle or slope. The length drawn is irrelevant. Write a letter of the alphabet, say F, or a (possibly dimensional number, say 100N) near the vector. The vector indicated is a scalar F (or the number) multiplying a unit vector in the direction of the arrow. Often you know that a force acts along a known line but you don't know which way. This is accommodated by allowing the scalar F to be positive or negative (See examples in sample 1.1.)

Combined: graphical representation used to define a symbolic vector. The symbolic notation can be used with the graphical notation to define the vector symbol. In fig. 1.8c, $\vec{\boldsymbol{r}}$ is being defined (being set equal) to the vector with magnitude 3m and direction 30° CCW from the $+x$ axis.

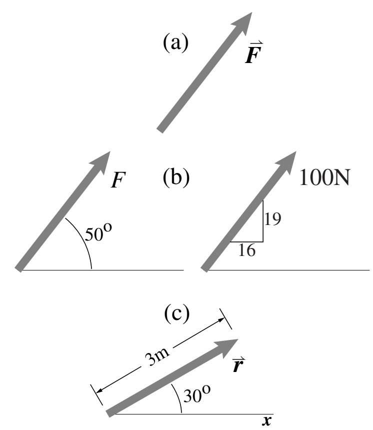

Figure 1.8: Three different ways of drawing a vector **(a) Symbolic.** The magnitude and direction of the vector is given by the symbol $\vec{\boldsymbol{F}}$, the drawn arrow has no quantitative information; **(b) "Scalar times arrow"** shows an arrow with clearly indicated orientation next to the scalar F or the scalar 100N. The vector indicated is the scalar multiplied by a unit vector in the direction drawn; **(c) Combined.** The symbol $\vec{\boldsymbol{r}}$ is *defined* to be (set equal to) a vector with the magnitude and orientation shown.

The cartesian components of a vector

A given vector, say $\vec{\boldsymbol{F}}$, can be described as the sum of vectors each of which is parallel to a coordinate axis. Most often, we use Cartesian axes, with the

x, y, and z axes all orthogonal to each other. Thus $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y$ in 2D and $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y + \vec{\boldsymbol{F}}_z$ in 3D (see fig. 1.9). Each of these vectors can in turn be written as the product of a scalar and a unit vector along the positive axes, *e.g.*, $\vec{\boldsymbol{F}}_x = F_x \hat{\boldsymbol{\imath}}$. So

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} \qquad \text{(2D)}$$

or

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y + \vec{\boldsymbol{F}}_z = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}. \qquad \text{(3D)}$$

The scalars F_x, F_y, and F_z are called the components, or coordinates, of the vector with respect to the axes xyz. The components may also be thought of as the orthogonal projections (the shadows) of the vector onto the respective coordinate axes.

Because the list of components is such a handy way to describe a vector, we have a special notation for it. The bracketed expression $[\vec{\boldsymbol{F}}]_{xyz}$ stands for the list of components of $\vec{\boldsymbol{F}}$ presented as a horizontal or vertical array (depending on context), as shown below.

$$[\vec{\boldsymbol{F}}]_{xyz} = [F_x, \quad F_y, \quad F_z] \quad \text{or} \quad [\vec{\boldsymbol{F}}]_{xyz} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

If you have use for, or facility with, linear algebra, it is safest if you think of this list of components as a vertical, rather than horizontal, list of scalars.

Example: A vector points NE

Assume an xy coordinate system with x pointing East and y pointing North. We can then write the components of a 5 N force pointed Northeast as

$$[\vec{\boldsymbol{F}}]_{xy} = \begin{bmatrix} (5/\sqrt{2})\,\mathrm{N} \\ (5/\sqrt{2})\,\mathrm{N} \end{bmatrix}$$

Note that the components of a vector in some tilted coordinate system $x'y'z'$ are different from the same vector's components in the coordinate system xyz; even though the vector is the same, the projections are different.

Even though

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}$$

it is *not true* that

$$[\vec{\boldsymbol{F}}]_{xyz} = [\vec{\boldsymbol{F}}]_{x'y'z'}$$

(see fig. 1.33 on page 69).

Understanding the relation between $[\vec{\boldsymbol{F}}]_{xyz}$ and $[\vec{\boldsymbol{F}}]_{x'y'z'}$ is especially important in dynamics (see sec. 17.1 on page 840). Because we often make use of multiple coordinate systems, when we define a vector by its components the coordinate system used must be specified.

Rather than using new letters to repeat the same concept we sometimes label the coordinate axes x_1, x_2 and x_3 and the unit vectors along them $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ (thus freeing our minds from silently pronouncing the extra letters y, z, j, and k)⑥.

Study sample 1.1 on page 60 to master the various graphical and component representations of vectors.

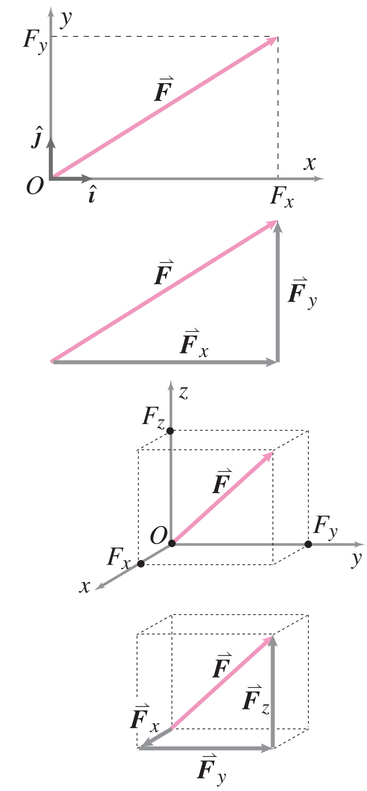

Figure 1.9: A vector can be broken into a sum of vectors, each parallel to the axis of a coordinate system. Each of these is a component multiplied by a unit vector along the coordinate axis, *e.g.*, $\vec{\boldsymbol{F}}_x = F_x\hat{\boldsymbol{\imath}}$.

Manipulating vectors by manipulating components

Once you have a coordinate system in mind, you can represent a vector by its components. Then, we can convert the rules for manipulating geometric vectors into related rules for manipulating vector components. This is important because in practice, when push comes to shove, most calculations with vectors are done with components.

Row vector vs column vector

As noted, the components of a vector are conventionally written as a column, rather than as a row, of numbers. Thus we would write

$$[\vec{\boldsymbol{A}}]_{xyz} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = [2, 4, -5]' = \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}.$$

The $'$ means 'matrix transpose', turning the rows into columns and *vice versa*⑦. If one is mostly representing vectors using lists of components, then a horizontal list and a vertical list are called a 'row vector' and a 'column vector', respectively. They represent the same physical vector, but in linear algebra calculations row vectors are handled differently than column vectors (And, column vectors are usually handled best.).

Adding and subtracting vectors using components

Because a vector can be broken into a sum of orthogonal vectors, because addition is associative, and because each orthogonal vector can be written as a component times a unit vector we get the addition rule:

$$[\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}]_{xyz} = [(A_x + B_x), \quad (A_y + B_y), \quad (A_z + B_z)]'.$$

(Recall, the little apostrophe symbol ' means *transpose*.)

We can describe the rule for component addition with the tricky words

> 'the components of the sum of two vectors are the sums of the corresponding components.' Or, more compactly, 'The components of the sum are the sums of the components'.

⑥Note that non-Cartesian coordinates, most especially polar coordinates, are often useful in dynamics. But we delay discussing non-cartesian coordinates until we need them.

⑦The 'transpose' of a horizontal list of numbers is a vertical list of numbers. Thus writing $[1\,2\,3]'$ is a short hand way of writing

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Similarly for subtraction and for 3D vectors,

$$[\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}]_{xyz} = [(A_x - B_x), \quad (A_y - B_y), \quad (A_z - B_z)]'.$$

Multiplying a vector by a scalar using components

The vector $\vec{\boldsymbol{A}}$ can be decomposed into the sum of three orthogonal vectors. If $\vec{\boldsymbol{A}}$ is multiplied by 7 then so must be each of the component vectors. Thus

$$[c\vec{\boldsymbol{A}}]_{xyz} = [cA_x, \quad cA_y, \quad cA_z]'.$$

The cartesian components of a scaled vector are the corresponding scaled components.

Example: Scalar multiplication of a vector using components
If $c = 3$ and $[\vec{\boldsymbol{A}}]_{xyz} = [2, 4, -5]'$ then

$$[c\vec{\boldsymbol{A}}]_{xyz} = [6, 12, -15]'.$$

We can add vectors and do scalar multiplication using this notation

Example: Scalar multiplication and addition
if $d = -0.5$ and $[\vec{\boldsymbol{B}}]_{xyz} = [100, 200, -300]'$ then

$$[c\vec{\boldsymbol{A}} + d\vec{\boldsymbol{B}}]_{xyz} = c[\vec{\boldsymbol{A}}]_{xyz} + d[\vec{\boldsymbol{B}}]_{xyz} = \begin{bmatrix} cA_x + dB_x \\ cA_y + dBy \\ cA_z + dBz \end{bmatrix} = \begin{bmatrix} -44 \\ -88 \\ 135 \end{bmatrix}$$

Matrix notation for adding vectors each multiplied by a different scalar

Finally, we can use matrix notation and the definition of matrix multiplication to add multiples of vectors[8]

[8] If you are not, or have not, taken linear algebra, you can skip this example.

$$\underbrace{\begin{bmatrix} A_x & B_x \\ A_y & By \\ A_z & Bz \end{bmatrix}}_{\text{A 3 by 2 matrix}} \begin{bmatrix} c \\ d \end{bmatrix} \underbrace{\equiv}_{\boxed{\text{Is defined to mean}}} c\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} + d\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} cA_x + dB_x \\ cA_y + dBy \\ cA_z + dBz \end{bmatrix}.$$

So using the numbers in the example above,

$$\begin{aligned} [c\vec{\boldsymbol{A}} + d\vec{\boldsymbol{B}}]_{xyz} &= \begin{bmatrix} 2 & 100 \\ 4 & 200 \\ -5 & -300 \end{bmatrix} \begin{bmatrix} 3 \\ -0.5 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 2 & + & -0.5 \cdot 100 \\ 3 \cdot 4 & + & -0.5 \cdot 200 \\ 3 \cdot (-5) & + & -0.5 \cdot (-300) \end{bmatrix} = \begin{bmatrix} -44 \\ -88 \\ 135 \end{bmatrix}. \end{aligned}$$

In the language of linear algebra, a matrix [A] multiplied by a column vector [b] is a linear combination of the matrix columns (each is itself a column vector) with weights (coefficients) given by the elements of the column vector [b].

Adding vectors on a computer

Computers deal well with lists of numbers, but not generally with units. So, only the numerical part of a calculation shows in the computer work. For example, when we write on the computer

```
F = [ 3 5 -7]'
```

we take that to be computerese for $[\vec{F}]_{xyz} = [3\,\mathrm{N}, \quad 5\,\mathrm{N}, -7\,\mathrm{N}]'$. To do computer work we have to be clear about what units and what coordinate system we are using. In particular, at this point in the course, we advise you to only use one coordinate system and one consistent set of units in any one problem that uses computer calculations (see the later chapter of this book about Units). With commands something like this, we can add multiples of vectors on a computer

```
A = [   2    4    -5]'
B = [ 100  200  -300]'
c = 3
d = -0.5
C = c*A + d*B
```

or using the matrix notation, like this.

```
A = [   2    4    -5]'
B = [ 100  200  -300]'
M = [A  B]   % matrix M is column A next to column B
c =  3
d = -0.5
v =  [c d]'   % the 'weights' c and d are in the column vector v
C = M*v    % Assuming you are using a computer language, like
           % ... Matlab or Python with numpy,  that has
           % ... built in matrix multiplication.
```

Or, if you like to just put in the numbers and type as little as possible, possibly at the cost of clarity when reading your work later,

```
M = [  2    100
       4    200
      -5   -300]
C = M * [3  -0.5]'.
```

Magnitude of a vector using components

The Pythagorean Theorem for right triangles (‘$A^2+B^2=C^2$’) tells us how to calculate the magnitude of a vector in terms of components

$$|\vec{\boldsymbol{F}}| = \sqrt{F_x^2+F_y^2}, \qquad \text{(2D)}$$

$$|\vec{\boldsymbol{F}}| = \sqrt{F_x^2+F_y^2+F_z^2}. \qquad \text{(3D)} \qquad (1.1)$$

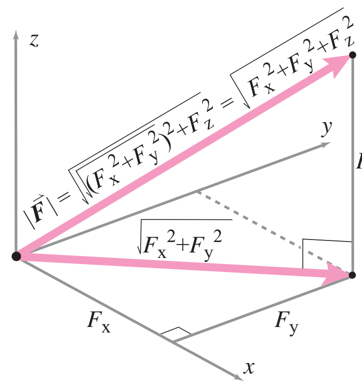

Figure 1.10: The magnitude of the diagonal line on the xy plane is $\sqrt{F_x^2+F_y^2}$. Using the Pythagorean theorem again, gives that $|\vec{\boldsymbol{F}}| = \sqrt{\sqrt{F_x^2+F_y^2}^{\,2}+F_z^2} = \sqrt{F_x^2+F_y^2+F_z^2}$.

To get the result in 3D, the 2D Pythagorean Theorem needs to be applied twice successively, first to get the magnitude of the sum $\vec{\boldsymbol{F}}_x+\vec{\boldsymbol{F}}_y$ and once more to add in $\vec{\boldsymbol{F}}_z$, which is orthogonal to the sum $\vec{\boldsymbol{F}}_x+\vec{\boldsymbol{F}}_y$ (see fig. 1.10).

On a computer one might write something like this

```
  F               = [10 -20 30]
magnitude  =  sqrt( F(1)^2 + F(2)^2 + F(3)^2 )
```

However, this formula is so commonly needed that many computer languages will have a command like `norm` or `mag`. Then, computer code something like `answer = norm(F)` or `answer = mag(F)` might replace the second line in the calculation above.

A given vector can be written as various sums and products

A vector $\vec{\boldsymbol{A}}$ has many representations. The equivalence of different representations of a vector is partially analogous to the case of a dimensional scalar which has the same value no matter what units are used (*e.g.,* the mass $m = 4.41$ lbm is equal to $m = 2$ kg). Here are some common representations of vectors.

Scalar times a unit vector in the vector's direction. $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}$ means the scalar F multiplied by the unit vector $\hat{\boldsymbol{\lambda}}$.

Sum of orthogonal component vectors. $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y$ is a sum of two vectors parallel to the x and y axes, respectively. In three dimensions, $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}_x + \vec{\boldsymbol{F}}_y + \vec{\boldsymbol{F}}_z$.

Components times unit base vectors. $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ or $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$ in three dimensions. One way to think of this sum is to realize that $\vec{\boldsymbol{F}}_x = F_x\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{F}}_y = F_y\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{F}}_z = F_z\hat{\boldsymbol{k}}$.

Components times rotated unit base vectors. $\vec{\boldsymbol{F}} = F'_x\mathbf{i}' + F'_y\mathbf{j}'$ or $\vec{\boldsymbol{F}} = F'_x\mathbf{i}' + F'_y\mathbf{j}' + F'_z\mathbf{k}'$ in three dimensions. Here the base vectors marked with primes, $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$, are unit vectors parallel to some mutually orthogonal x', y', and z' axes. These x', y', and z' axes may be tilted in relation to the x, y, and z axes. That is, the x' axis need not be parallel to the x axis, the y' not parallel to the y axis, and the z' axis not parallel to the z axis.

Components times other unit base vectors. If you use polar or cylindrical coordinates the unit base vectors are $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$, so in 2-D , $\vec{\boldsymbol{F}} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta$ and in 3-D, $\vec{\boldsymbol{F}} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{\boldsymbol{k}}$. If you use 'path' coordinates, you will use the path-defined unit vectors $\hat{\boldsymbol{e}}_t$, $\hat{\boldsymbol{e}}_n$, and $\hat{\boldsymbol{e}}_b$ so in 2-D $\vec{\boldsymbol{F}} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n$. In 3-D $\vec{\boldsymbol{F}} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n + F_b\hat{\boldsymbol{e}}_b$.

A list of components. $[\vec{\boldsymbol{F}}]_{xy} = [F_x, F_y]$ or $[\vec{\boldsymbol{F}}]_{xyz} = [F_x, F_y, F_z]$ in three dimensions. This form coincides best with the way computers handle vectors. The row vector $[F_x, F_y]$ coincides with $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ and the row vector $[F_x, F_y, F_z]$ coincides with $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$.

In summary:

$$
\begin{aligned}
\vec{\boldsymbol{A}} &= \vec{\boldsymbol{A}} \\
&= |\vec{\boldsymbol{A}}|\hat{\boldsymbol{\lambda}}_A = A\hat{\boldsymbol{\lambda}}_A, && \text{where } \hat{\boldsymbol{\lambda}}_A \parallel \vec{\boldsymbol{A}},\ A = |\vec{\boldsymbol{A}}| \text{ and } |\hat{\boldsymbol{\lambda}}_A| = 1\,, \\
&= \vec{\boldsymbol{A}}_x + \vec{\boldsymbol{A}}_y + \vec{\boldsymbol{A}}_z && \text{where } \vec{\boldsymbol{A}}_x, \vec{\boldsymbol{A}}_y, \vec{\boldsymbol{A}}_z \text{ are parallel to the } x, y, z \text{ axes,} \\
&= A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}, && \text{where } \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}} \text{ are parallel to the } x, y, z \text{ axes, and} \\
&= A_{x'}\hat{\boldsymbol{\imath}}' + A_{y'}\hat{\boldsymbol{\jmath}}' + A_{z'}\hat{\boldsymbol{k}}' && \text{where } \hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}', \hat{\boldsymbol{k}}' \text{ are } \parallel \text{ to skewed } x', y', z' \text{ axes} \\
&= A_R\hat{\boldsymbol{e}}_R + A_\theta\hat{\boldsymbol{e}}_\theta + A_z\hat{\boldsymbol{k}}, && \text{using cylindrical coordinate basis vectors.}
\end{aligned}
$$

Also,

$$
\begin{aligned}
[\vec{\boldsymbol{A}}]_{xyz} &= [A_x, A_y, A_z]' && [\vec{\boldsymbol{A}}]_{xyz} \text{ stands for the component list in } xyz \text{ , and} \\
[\vec{\boldsymbol{A}}]_{x'y'z'} &= [A_{x'}, A_{y'}, A_{z'}]' && [\vec{\boldsymbol{A}}]_{x'y'z'} \text{ stands for the component list in } x'y'z'.
\end{aligned}
$$

Note, as emphasized before, that although $[\vec{\boldsymbol{A}}]_{xyz}$ and $[\vec{\boldsymbol{A}}]_{x'y'z'}$ both represent the same vector $\vec{\boldsymbol{A}}$, they are two different lists of numbers, namely:

$$[\vec{\boldsymbol{A}}]_{xyz} \neq [\vec{\boldsymbol{A}}]_{x'y'z'}.$$