4.1 Units

Somewhere near the front of many engineering books there is a tedious and preparatory section, usually ignored by all, about units and dimensions. This book is completely different. That tedious and pedantic section is a whole chapter. Most students are immune to preaching on such topics. They are like spelling and grammar in English. And, as for spelling and grammar, in the end, the only way a student will get good at managing units is by imitation, or when forced to do so in a time of panic, or at a moment of idle curiosity. As for imitation, we have tried to set a good example throughout the book. As for panic and curiosity, this chapter is here for you.

Not everyone takes the care with units that we advise. Yet they still do productive engineering. There are various ways people use, abuse, or don't use units and still get sensible results. We describe some of these here.

Balancing and carrying units

The central rules which we advise that you follow are:

a) Balance your units and
b) Carry your units.

Where do these rules come from?

Physical quantities that are dimensional are represented by a number multiplying a unit.

Thus $d = 7\,\mathrm{m}$ means $7\times$(one meter). The 7 and the 'm' are of equal status in any math you do.

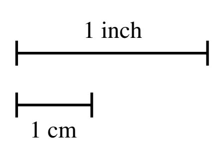

Figure 4.1: Relative size of an inch and a centimeter.

Balance your units

Every line of every calculation should be dimensionally sensible. That is, the dimensions on the left of the equal sign should be consistent with the dimensions on the right, the same way numbers have to balance. Otherwise the equations are not equations.

For example, if two bicycles tied in a race you could say they were in some way equal. But even if you noticed that the weight difference between these equivalent bicycles was 10% more than 2 pounds you would not write

$$8\,\mathrm{kg} = 9\,\mathrm{kg}.$$

The equivalence between the two bikes in race times does not make eight kilograms equal to nine kilograms. In this same way it would be wrong to write

$$1 \in = 1\,\mathrm{s}.$$

if you noticed that it takes a bug about a minute (60 seconds) to walk the length of your body (say about 60 inches). The passing of a second corresponds to the passing of an inch, so for some purposes an inch is equivalent to a second. But that does not mean that an inch *is* a second. An inch has dimensions of length which cannot be equal to a second which has dimensions of time. Length can equal time no more than 8 can equal 9.

But it is correct to write that

$$5.08\,\mathrm{cm} = 2 \in .$$

Both centimeters and inches have dimensions of length and one inch is equivalent to 2.54 centimeters always (fig. 4.1). An equation where the units on both sides of the equation are the same physical quantities (length in the example above) is *balanced* with regard to units.

Carry your units

When you go from one line of a calculation to the next you should carry (keep written track of) the units with as much care as any other numerical or algebraic quantities. When you do arithmetic and don't forget any terms you have 'carried' the numbers from one line of calculation to the next. Similarly, carrying the units just means not forgetting them *in* your calculations①.

①**Caution:** People often put units next to their equations with a vague notion that the units apply to the equation. This sometimes works out in the end and sometimes doesn't. Better to include the units as part of the equations. Because the rules for manipulating units are the same as those for manipulating numbers, things necessarily work out correctly if units are treated with equal status as are the numbers.

Example: Dividing meters by seconds.
A bicycle goes 7 meters in 2 seconds so

$$v = \frac{d}{t} = \frac{7\,\mathrm{m}}{2\,\mathrm{s}} = 3.5\,\mathrm{m/s}.$$

Here we have divided 7 by 2 and also divided m by s. But a meter (m) is not a number, nor is a second (s). So the ratio m/s cannot be reduced more. In particular, the m/s is not sitting next to the equation but is part of the equation: the velocity is not $v = 3.5$ but rather $v = 3.5\,\mathrm{m/s}$.

The rest of this section is, more or less, a discussion of how and why to 'carry your units.'

Dimensions, units and changing units

Distance has dimensions of length $[L]$ that can be measured with various units — centimeters (cm), yards (yd), or furlongs (an obsolete unit equal to 1/8 mile). A meter is the standard unit of length in the SI system. In answer to the question 'What is the length of a bicycle crank ℓ?' we say 'ℓ is seven inches' and write $\ell = 7 \in$ or say 'ℓ is seventeen point seven centimeters' and write $\ell = 17.7\,\mathrm{cm}$. In each case, a number multiplies a dimensional unit.

Force has dimension of mass times acceleration $[m \cdot a]$. Because acceleration itself has dimensions of length over time squared $[L/T^2]$, force also has dimensions of mass times length divided by time squared $[M \cdot L/T^2]$. Because force has such a central role in mechanics, it is often convenient to think of force as having its own units. Force then has dimensions of, simply, force $[F]$. The most common units for force are Newton (N) and the pound force (lbf). The 'f' in the notation for the pound lbf is to distinguish a pound force lbf from the pound mass lbm , $1\,\text{lbf} = \text{lbm} \cdot g \approx 1\,\text{lbm} \cdot 32.2\,\text{ft/s}^2 \approx 32.2\,\text{lbm} \cdot \text{ft/s}^2$. Some people use lb to mean pound force or pound mass, depending on context. To avoid confusion we use lbm for pound mass and lbf for pound force (see box 4.3 on page 11).

Changing units

We can say 'The typical force of a seated racing bicyclist on a bicycle pedal is thirty pounds,' and write any of the following:

$$\begin{aligned}
F &= 30 \text{ lbf} \\
F &= 30 \text{ lbf} \cdot (1) \\
F &= 30 \text{ l}\cancel{\text{b}}\text{f} \cdot \underbrace{\left(\frac{4.45 \text{ N}}{1 \text{ l}\cancel{\text{b}}\text{f}}\right)}_{1} \\
F &= 133.5 \text{ N.}
\end{aligned}$$

Here we have shown one way to change units. Multiply the expression of interest by one (1) and then make an appropriate substitution for one. Any table of units will tell us that 1 lbf is approximately 4.45 N. So we can write 1 = (4.45 Newtons/1 lbf) and multiply any part of an equation by it without affecting the equation's validity. (See fig. 4.2 to get a sense of the relation between a pound force, a Newton, and the less used force units, the poundal and the kilogram-force.)

What if we had made a mistake and instead multiplied the right hand side by the reciprocal expression 1 = (1 lbf /4.45 Newton)? No problem. We would then have

$$F = 30 \text{ lbf} = 30 \text{ lbf} \cdot \frac{1 \text{ lbf}}{4.45 \text{ Newton}} = \frac{30}{4.45} \text{ lbf}^2/\text{N}.$$

This expression is admittedly weird, but it is correct. If you should end up with such a weird but correct answer you can compensate by multiplying by one again and again until the units cancel in a way that you find pleasing. In this case we could get an answer in a more conventional form by multiplying the right hand side by 1^2 using $1 = (4.45\,\text{N}/\,\text{lbf})$:

$$F = \frac{30}{4.45}\frac{\text{lbf}^2}{\text{N}} \cdot 1^2 = \frac{3.0}{4.45}\frac{\text{l}\cancel{\text{b}}\text{f}^2}{\text{N}} \cdot \left(\frac{4.45\,\text{N}}{\text{l}\cancel{\text{b}}\text{f}}\right)^2 = 133.5\,\text{N} \qquad \text{(as expected).}$$

A trivial but surprisingly useful observation is that $F = F$. A quantity is equal to itself no matter how it is represented. That is, 30 lbf = 133.5 N even though $30 \neq 133.5$. To summarize:

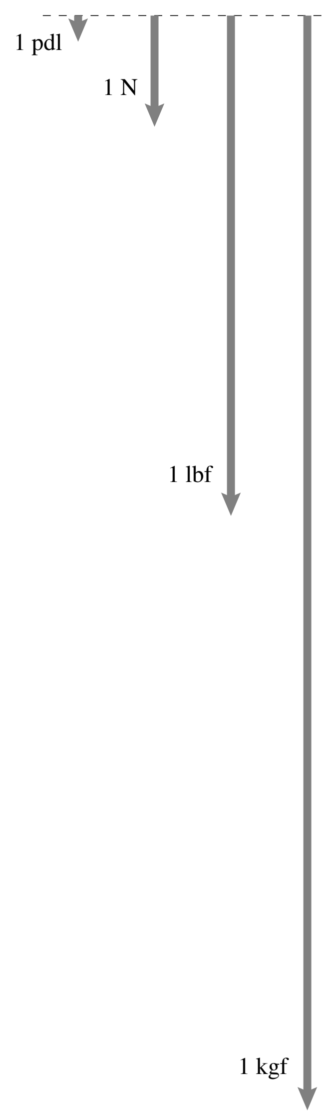

Figure 4.2: Relative size of a poundal pdl, a Newton N, a pound force lbf, and a kilogram force kgf . 1 N = 7.24 pdl, 1 lbf = 4.45 N, and 1 kgf = 1 kilopond = 9.81 N = 2.2 lbf.

Units are manipulated in any and all calculations as if they were numbers or algebraic symbols. For example, canceling equal units from the top and bottom of a fraction is the same as canceling numbers or algebraic symbols.

An advertisement for careful use of units

Units and dimensions are part of scientific notation just as spelling, punctuation, and grammar are parts of English composition. If used properly, they aid both thinking and the communication of these thoughts to others. If units and dimensions are used improperly they can impede communication, even with oneself, and convey the wrong meaning.

Example: Breaking load
A gadget that breaks with a 300 N (300 Newtons) load instead of a needed 300 lbf (300 pounds force) load is exactly as bad as one that breaks with a 67 lbf load instead of a needed 300 lbf load. An unsatisfied consumer will not be placated by learning that the engineer's calculation was 'numerically correct'.

If anybody is ever to use your calculation, giving them the wrong units is just as bad as giving them the wrong numerical value.

Although using units properly often seems annoyingly tedious, it also often pays. If units are carried through honestly, *not just tagged on to the end of an equation for appearance*, you can check your work for dimensional consistency. If you are trying to find a speed and your answer comes out 13 kg· m/ s, you know you have made a mistake — kg· m/ s just isn't a speed. Such dimensional errors in a calculation often reveal corresponding algebraic or conceptual mistakes. Also, if a problem is based on data with mixed units, such as cm and meters, or pound force and pound mass, you may often not know the units of your answer unless you properly 'carry' your units②.

②$(1\,\mathrm{lbf}/1\,\mathrm{pdl})^2 \approx 1000$. You can easily generate errors of approximately a factor of 1000 with English units if you wishfully multiply or divide answers by 32.2 (the value of g in $\mathrm{ft/s^2}$) at the end of a sloppy calculation. If you do it wrong you get an error of a factor of 32.2^2 which is 3% greater than 1000. Following sloppiness with unscrupulousness, some are tempted to then slide a decimal point three places to the right or left to 'fix' things.
The decrepit insecurity that provokes such despicable crimes is avoided by going to a church, temple or mosque regularly, or by making a habit of carrying units.

Three ways to be fussy about units.

People are most pleased if you speak their language, speak correctly, and make sense. Similarly, scientists and engineers with whom you communicate will be most comfortable if you use the units they use and use them with correct notation. But most importantly, you should use units in a way that makes physical sense. Just as the United Nations argues over which language to use for communication, educators, editors, and makers of standards have argued for decades over conventions for units: whether they should come in multiples of 10, whether they should use the standard international scientific conventions, and whether they will be clear to someone who has worked in the stock room of a supplier of $\frac{1}{2}$-inch bolts for 35 years and thinks SI might be a friend of his cousin Amil.

Even if you are not fluent in someone's favorite language, you can still say sensible things. Similarly, no matter what you or your work place's choice of units (SI, English, or hodgepodge), no matter whether you use upper case and lower case correctly, you should make sense. Physically sensible units — that is, balanced units — should be used to make your equations dimensionally correct. Then you should work on refining your notation so as to be more professional.

So, in order of importance,

1. Use balanced units.
2. Use units of the type that are liked by your co-workers.
3. Spell and punctuate these units correctly.

If you are in a situation where your only problem is the third item on the list you are doing fine, unless you are *really* fussy, or work for someone who is *really* fussy. (*e.g.,* the authors of this book only try for the first two items on this list.)

Using units in practice

Units with calculators and computers

Calculators and computers generally do not keep track of units for you. In order for your numerical calculations to make sense you have the following choices.

Use dimensionless variables. Using dimensionless variables is the preferred method of scientists and theoretical engineers. The approach requires that you define a new set of dimensionless variables in terms of your original dimensional variables.

Use a consistent unit system. Express all quantities in terms of units that are consistent[3]. For example, all lengths should be in the same units and the unit of force should equal the unit of mass times the unit of distance divided by the unit of time squared. Each row of the table below defines a consistent set of units for mechanics.

Name	length	mass	force	time	angle
mks	meter	kilogram	Newton	second	radian
cgs	centimeter	gram	dyne	second	radian
English1	foot	lbm	poundal	second	radian
English2	foot	slug	lbf	second	radian

The radian is the unit of angle in all consistent unit systems. Whether or not a radian is a proper unit or not is an issue of some philosophical debate. Practically speaking, you can generally replace 1 radian with the number 1.

[3] **Caution:** Doing a computer calculation using quantities from an inconsistent unit system can easily lead to wrong results. To be safe make sure that all quantities are expressed in terms of only one row of the table shown.

Use numerical equations. If you are using the computer to evaluate a formula that you trust, and you have balanced the units in a way that makes you secure, you can have the computer do the arithmetic part of the calculation. It is easy to make mistakes, however, unless the formula is expressed in consistent units.

Example: Force units conversion
What, in the SI system, is the net braking force when a 2000 lbm car skids to a stop on level ground? For this units problem we skip the careful mechanics and just work with the formula

$$F = \mu m g$$

where m is the mass of the car, g is the local gravitational constant and μ is the coefficient of friction for sliding between the tire and the road. We won't be off by more than a quarter of a percent using the standard rather than the local value of the gravitational constant, $g = 32.2\,\mathrm{ft/s^2}$. The coefficient of friction for rubber and dry road is about one, so we use $\mu = 1$. We proceed by plugging in values into the formula and then multiplying by 1 until things are in standard SI (Systémé Internationale) form. We use a table of units to make the various substitutions for 1. A few of the detailed steps could be contracted. The approach below is only one, albeit an awkward one, of many routes to the answer.

$$\begin{aligned}
F &= \mu m g \\
F &= (1)\cdot(2000\,\mathrm{lbm})\cdot(32.2\,\mathrm{ft/s^2}) \\
&= (2000\cdot 32.2)\frac{\mathrm{lbm}\cdot\mathrm{ft}}{\mathrm{s^2}} \\
&= (2000\cdot 32.2)\frac{\cancel{\mathrm{lbm}}\cdot\cancel{\mathrm{ft}}}{\mathrm{s^2}}\cdot\underbrace{\left(\frac{1\,\mathrm{kg}}{2.2\,\cancel{\mathrm{lbm}}}\right)}_{1}\cdot\underbrace{\left(\frac{30.48\,\cancel{\mathrm{cm}}}{1\,\cancel{\mathrm{ft}}}\right)}_{1}\cdot\underbrace{\left(\frac{1\,\mathrm{m}}{100\,\cancel{\mathrm{cm}}}\right)}_{1} \\
&= 8917\frac{\cancel{\mathrm{kg}}\cdot\cancel{\mathrm{m}}}{\cancel{\mathrm{s}}^2}\cdot\underbrace{\left(\frac{1\,\cancel{\mathrm{N}}}{1\,\cancel{\mathrm{kg}}\cdot\cancel{\mathrm{m}}/\cancel{\mathrm{s}}^2}\right)}_{1}\cdot\underbrace{\left(\frac{1\,\mathrm{kN}}{1000\,\cancel{\mathrm{N}}}\right)}_{1} \\
&= 8.92\ \mathrm{kN}
\end{aligned}$$

The net braking force is 8.92 kN. In each step of the calculation we accumulate what we had from the previous step and then multiply by 1, where 1 is the ratio of two quantities that have the same dimensions but different units.

Repeating, in engineering we do math not just with numbers, but with dimensional quantities. The bad habits of many of us not withstanding, there are good and useful standards for how to deal with units in calculations④.

④An excellent description of good practice is the "Guide for the Use of the International System of Units (SI)" by Barry Taylor, 1998, NIST (National Institute of Standards and Technology) publication # 811.

Use of units in old-style handbooks.

Many standard empirical formulas, formulas based on experience and not theory, are presented in an 'undimensional' or numerical form. The units are not part of the equations. We present the approach here, not because we want to promote it, we don't. But we don't want the more formal approach to units we advocate here to stop you from reading and using empirical sources.

For example, Mark's *Handbook for Mechanical Engineers (8th edition, page 8-138)* presents the following useful formula to describe the working life of commercially manufactured ball bearings:

$$L_{10} = \frac{16,700}{N}\left(\frac{C}{P}\right)^K,$$

where

L_{10}	= the number of hours that pass before 10% of the bearings fail,
N	= the rotational speed in revolutions per minute
C	= the rated load capacity of the bearing in lbf,
P	= the actual load on the bearing in lbf, and
K	= 3 for ball bearings, 10/3 for roller bearings.

In this approach the idea of dimensional consistency has been disguised for the sake of brevity. L_{10}, N, C, and P are just numbers. Such an equation is sometimes called a 'numerical equation'. It is a relation between numerical quantities. If you happen to know the rotation speed of the shaft in radians per second instead of revolutions per minute you will have to first convert before plugging in the formula. Unlike a dimensional formula, the formula does not help you to convert these units. An

4.1 Examples of advised and ill-advised use of units

Good use of units Say a car has a constant speed of $v = 50\ \mathrm{mi/hr}$ for half an hour. The following is true and expressed correctly.

The distance traveled in time t is $x = vt$, so

$$\begin{aligned} x &= vt \\ &= (50\,\mathrm{mi/hr})(30\,\mathrm{min}) = 50\cdot 30\,\mathrm{mi}\cdot\ \mathrm{min/hr} \\ &\qquad\text{(Awkward but true!)} \\ &= 50\cdot 30\,\mathrm{mi}\cdot \cancel{\mathrm{m}}\mathrm{in}/\cancel{\mathrm{h}}\mathrm{r}\underbrace{\left(\frac{1\,\cancel{\mathrm{h}}\mathrm{r}}{60\,\cancel{\mathrm{m}}\mathrm{in}}\right)}_{1} \\ &= 25\,\mathrm{mi} \end{aligned}$$

That is, unsurprisingly, the distance covered in half an hour is 25 mi.

Another good use of units. If we start with the dimensionally correct formula $x = (50\,\mathrm{mi/hr})t$ we can differentiate to get

$$v = \frac{dx}{dt} = 50\,\mathrm{mi/hr}.$$

The answer is dimensionally correct without having to think about the units. v is speed and contains its units, x is distance and contains its units. In any formula that contains t, x or v we can substitute *any* time, distance or speed. How far does the car go in one minute? As in the previous example,

$$\begin{aligned} x &= vt \\ &= (50\,\mathrm{mi/hr})(1\,\mathrm{min}) \\ &= (50\,\mathrm{mi}/\cancel{\mathrm{h}}\mathrm{r})(1\,\cancel{\mathrm{m}}\mathrm{in})\underbrace{\left(\frac{1\,\cancel{\mathrm{h}}\mathrm{r}}{60\,\cancel{\mathrm{m}}\mathrm{in}}\right)}_{1} \\ &= \frac{5}{6}\,\mathrm{mi} \end{aligned}$$

Not such good use of units It is common practice to write sentences like 'the distance the car travels is

$$x = 50t,$$

where x is the distance in miles and t is the time of travel in hours', although we discourage it. Why? Because the variables x and t are ambiguously defined. We would like to use the fact that speed v is the derivative of distance with respect to time:

$$v = \frac{dx}{dt} = \frac{d}{dt}(50t) = 50.$$

But now we have a speed equal to a pure number, 50, rather than a dimensional quantity. In this simple example, common sense tells us that the speed v is measured in mi/hr. But if we want to think of v as a speed, a variable with dimensions of length divided by time, the formula misleads us and requires us to add the units. For this simple example it is not much of a problem to determine what units to add.

But better is if units are included correctly in the equations; then they take care of themselves whenever they are needed. The 'not such good' use of units above is sometimes called using numerical equations, that is equations that have numbers in them only. The good use of units uses quantity equations, that is equations that use dimensional quantities.

alternative to this 'numerical formula' approach for empirical formulas is in box 4.2 on page 8.

Tables of values

How can we write tables of values without the clutter of units in every entry?

The simplest way to use dimensionless variables, though not necessarily the best, is to do something that involves notational compromise. For example, let x represent dimensionless distance rather than distance. That is, x represents distance divided by 1 mi. Similarly, t is time divided by 1 hr. And dx/dt is dimensionless distance differentiated with respect to dimensionless time, which is, evidently, dimensionless speed. In this example, recovering the dimensional speed is common sense: speed is in mi/hr. The notational compromise is that v is being used to represent both dimensional and dimensionless speed, with the precise meaning depending on context

Example: Table of values.
Using notational compromise we can use the formula $x = vt$ with $v = 50\,\mathrm{mi/hr}$ to do a set of calculations. Say we want to know the distance x every quarter of an hour for two hours. So we multiply 50 by .25, .5, .75, ... and thus make a table

4.2 An improvement to the old-style handbook approach

An alternative to the standard approach to empirical formulas is to write a formula that makes sense with any dimensional variables. The bearing life formula would be replaced with the formula below:

$$l_{10} = \frac{16,700}{n}\left(\frac{c}{p}\right)^K \mathrm{hr\cdot rev/min}$$

where
- l_{10} = the time that passes before 10% of the bearings fail,
- n = the rotational speed,
- c = the rated load capacity of the bearing,
- p = the actual load on the bearing, and
- K = 3 for ball bearings, 10/3 for roller bearings.

and the variables l_{10}, n, c, and p are dimensional quantities. One can use any dimensions one wants for all of the variables. For example, using
- n = 50 rev/sec
- c = 1 kN
- p = 100 lbf, and
- K = 3 for the given ball bearing,

we can calculate the life of the bearing by plugging these values in to the formula directly.

$$\begin{aligned}
l_{10} &= \frac{16,700}{50\,\frac{\mathrm{rev}}{\cancel{\mathrm{sec}}}}\left(\frac{1\,\mathrm{kN}}{100\,\mathrm{lbf}}\right)^3 \mathrm{hr\cdot rev/min}\\
&= \frac{16,700}{50\,\mathrm{rev}/\cancel{\mathrm{sec}}}\underbrace{\left(\frac{60\mathrm{sec}}{\mathrm{min}}\right)}_{1}\cdot\\
&\qquad\left(\frac{1\,\cancel{\mathrm{k}}\mathrm{N}}{100\,\cancel{\mathrm{l}}\mathrm{bf}}\underbrace{\left(\frac{1\,\cancel{\mathrm{lbf}}}{4.448\,\cancel{\mathrm{N}}}\right)}_{1}\underbrace{\left(\frac{1\,\cancel{\mathrm{N}}}{1000\,\cancel{\mathrm{k}}\mathrm{N}}\right)}_{1}\right)^3 \mathrm{hr\cdot min/rev}\\
&\approx 45000\,\mathrm{hr}
\end{aligned}$$

This approach has the advantage of precision if mixed units are used. Any of the quantities can be measured with any units and the answer always comes out right. Furthermore, the user is free to measure all the quantities in those units which work out best, in this case using the same units for c and p and measuring n in rev/min. But the user is also free to use any units.

with two columns labeled t (hr) and x (mi).

t (hr)	x (mi)
0	0
.25	12.5
.5	25
.75	37.5

This approach has some ambiguity to some eyes. Here is a more clear way to make the same table.

Example: Less ambiguous table of values.
The exact meaning of the columns in the above example are a little ambiguous. We can make it more precise by labeling the columns as follows

$t/(\mathrm{hr})$	$x/(\mathrm{mi})$
0	0
.25	12.5
.5	25
.75	37.5

That is, the columns of numbers are dimensionless. The first column, is the time divided by one hour the second is distance divided by one mile.

Finally, the way things are most often done in science, and sometimes in engineering practice, is to only use clearly defined and distinct dimensionless variables (*i.e.,* not to use v both for the speed and for the speed as measured in m/s. This approach is more precise, if cumbersome, than using v to be both dimensional and dimensionless depending on context.

Example: Dimensionless table of values.
If we take x to be dimensional distance, t to be dimensional time, and v to be dimensional speed, we can define new dimensionless variables. $t^* = t/(1\,\mathrm{hr})$, $x^* = x/(1\,\mathrm{mi})$, and $v^* = v/(1\,\mathrm{mi/hr})$. Now there is no ambiguity: x is dimensional and x^* is dimensionless. Dividing the equation $x = vt$ on both sides by one mile, and multiplying the right side by 1, in the form of $1 = (1\,\mathrm{hr}/1\,\mathrm{hr})$ we get:

$$\frac{x}{1\,\mathrm{mi}} = \frac{v}{1\,\mathrm{mi/hr}} \cdot \frac{t}{1\,\mathrm{hr}}$$

which is, using the dimensionless variables,

$$x^* = v^* t^*.$$

Because v is 50 mi/ hr, $v^* = 50$ as we can show formally as follows:

$$v^* = v/(1\,\mathrm{mi/hr}) = (50\,\mathrm{mi/hr})/(1\,\mathrm{mi/hr}) = 50.$$

The dimensionless speed v^* is just the dimensionless number 50. Now we can make a table by multiplying 50 by $.25, .5, .75, \ldots$. The columns of the table can be labeled t^* and x^* and all variables are clearly defined.

t^*	x^*
0	0
.25	12.5
.5	25
.75	37.5

In practice most non-theoretiticians will not go to the trouble of defining a whole set of dimensionless variables. But it can be helpful if you have got confused with the difference between a pound force and a pound mass, or from some variables being measured in meters and others in feet, etc.

4.3 Force, mass, weight and the English and SI systems

The simplest way to measure force is to follow the "metric"/SI convention, measure force with Newtons (N) and mass with kilograms (kg).

Newtons. One Newton is defined in terms of a kilogram, meter and second as

$$1N \equiv \frac{1\,\text{kg m}}{\text{s}^2}.$$

Unfortunately, to most everyone's confusion, there are other units of force and mass, for example the kgf, the poundal and the slug.

Pound force. In the USA the unit of force most used is the pound force, lbf. One lbf is the force of gravity (at the earth's surface) on one pound mass. Thus

$$1\,\text{lbf} = g \times 1\,\text{lbm} \approx 32.2\,\text{lbm ft/s}^2.$$

Example: What is the force required to accelerate 10 lbm an amount of 5 ft/s^2?

$$\begin{aligned} F &= ma \\ &= (10\,\text{lbm})(5\,\text{ft/s}^2) \\ &= (10\,\text{lbm})(5\,\text{ft/s}^2) \cdot \underbrace{\left(\frac{1\,\text{lbf}}{1\,\text{lbm} \cdot g}\right)}_{1} \underbrace{\left(\frac{g}{32\,\text{ft/s}^2}\right)}_{1}. \\ &\approx (10 \cdot 5/32)\,\text{lbf} \approx 1.6\,\text{lbf}. \end{aligned} \quad (4.1)$$

Note, the surest way to change units is by systematic multiplication by 1. All of the units in the above expression cancel (appear an equal number of times on the top as the bottom of fractions), except for lbf. If you pick the wrong version of the number 1, say the reciprocal of one of the expressions above, you still get the right answer, but in a strange mixture of units.

The pound force seems silly to those from Europe. But they shouldn't laugh.

Kilogram force. In Europe, outside physics classrooms the most common unit of force, is the kilogram force. It's for force of gravity on a kilogram mass:

$$1\,\text{kgf} = 1\,\text{kg} \cdot g.$$

That's the official value of g. So 1 kgf =1 kilopond=9.80665 N. This is precisely as confusing, no more and no less, than the pound force. If you ask someone's weight in France they won't say, as they should were they to practice what they preach, '490 N'. They will say, 'cinquante kilogrammes', which translates to 'fifty kilograms' by which they mean 50 kgf(hypocrites!). Other names for the kilogram force are kilopond, kp (not to be confused with a kip, which is 1000 lbf).

To stay out of trouble you could avoid use of the pound force and kilogram force altogether. But then how would you measure force in the English system?

The poundal. If the English system imitated the metric system it would have a unit for the force needed to accelerate one pound mass one foot per second squared. And it does.It's called the poundal, abbreviated as pdl.

$$1\,\text{pdl} = 1\,\text{lbm ft/s}^2.$$

Poundals are exactly as sensible and unconfusing in the English system as Newtons are in SI. But because the poundal is unfamiliar, and unfamiliar things are strange, and strange things seem confusing, the poundal is generally catalogued as confusing. But really, the poundal is just as simple as the Newton.

Statics vs Dynamics. The standard official units used in Europe (meters, kilograms, Newtons and seconds) are easy if you are mostly studying dynamics: a unit of mass is accelerated a unit amount with a unit force. This works in Europe with Newtons and kilograms, and in America with pounds mass and poundals. That's basing force on the equation $F = ma$.

But if the mechanics you do is statics, which is most of the mechanics that is done, then the easier system is one based on $F = mg$. The force of gravity on a kg is one kgf. And the force of gravity on a lbm is one lbf.

Why can't we keep it simple? Many students try to avoid this confusion by sticking with SI. That's the real SI that has no such thing as a kilogram force (kgf). But this fights both traditions and gravity-based intuitions. So, if we want to talk with a variety of people we are stuck with the pound force (lbf) and kilogram force (kp, or kilopond). And then, if we want to rationalize the English system, we have to understand poundals too.

How much stuff? Historically, people understood weight before they understood mass: bigger things are harder to hold up so were said to have more weight. And comparisons were made with gravity-based balances. Weight is an easier concept for the pre-Newtonian mind than our modern idea that bigger things are harder to accelerate, i.e., have more mass. So people measured the amount of stuff by weight. 'How much flour?' one would ask. 'A pound of flour,' meaning one pound weight, might be the answer. A one pound weight is pulled with a 1 lbf by gravity, or in the older notation where one did not worry about mass, by 1 lb. People didn't notice that it was a little harder, i.e.. would stretch a given spring more, to hold something up on the north pole than at the top of Mount Everest, so the earth's gravity force on an object was a fine measure of quantity.

When it became important to talk about mass, as opposed to weight, the pound mass was defined as the mass of something that weighed a pound. That is,

$$1\,\text{lbm} \equiv 1\,\text{lbf}/g.$$

Then people thought 'what is the mass that accelerates one foot per second squared if a one-pound force is applied?' They found

$$\begin{aligned} m &= F/a \\ &= (1\,\text{lbf})/(1\,\text{ft/s}^2) = 1\left(\text{lbf/ft/s}^2\right)\underbrace{\left(\frac{32.174\,\text{lbm ft/s}^2}{\text{lbf}}\right)}_{1} \\ &= 32.174\,\text{lbm}. \end{aligned}$$

(continued...)

4.3 Force, mass, weight and the English and SI systems (continued)

The force of gravity on an object is its weight. But a given object has different weight at different places, with up to 0.5% variation on earth. That is, g, the earth's gravitational 'constant,' varies from about $9.78\,\mathrm{m/s^2}$ at the equator to about $9.83\,\mathrm{m/s^2}$ at the North Pole. The official value of the 'constant' g is in between at exactly $9.80665\,\mathrm{m/s^2}$ (this is about $32.1740486\,\mathrm{ft/s^2}$). Multiplying the official g by the mass m will give you almost exactly the force it takes to hold it up if you are in exactly the official place, somewhere in Potsdam. Outside of Potsdam you have to accept an error of up to 0.25% when calculating gravitational forces, unless you happen to know the precise value of g in your neighborhood. The value of a kgf and lbf does not vary from place to place, but the weight of a kg or lbm mass does.

1/32.174 is awkward. It takes 1/32.174 lbf to accelerate one lbm one $\mathrm{ft/s^2}$. As noted, this awkwardness was fixed by the invention of the poundal

$$\begin{aligned} F &= ma \\ &= (1\,\mathrm{lbm})(1\,\mathrm{ft/s^2}) \stackrel{def}{=} 1\,\mathrm{pdl}. \end{aligned}$$

Now things are tidy. It takes one pdl to accelerate one lbm one $\mathrm{ft/s^2}$. But we could also calculate that

$$\begin{aligned} F &= ma \\ &= (1\,\mathrm{lbm})(1\,\mathrm{ft/s^2}) \;=\; 1\left(\cancel{\mathrm{lbm}}\ \cancel{\mathrm{ft}}/\mathrm{s^2}\right)\underbrace{\left(\frac{1\,\mathrm{lbf}}{32.174\,\cancel{\mathrm{lbm}}\ \cancel{\mathrm{ft}}/\mathrm{s^2}}\right)}_{1} \\ &= \left(\frac{1}{32.174}\right)\mathrm{lbf}. \end{aligned}$$

So 1 pdl= 1 lbf/32.174 or 1 lbf = 32.174 pdl.

32.174 is also awkward. A unit of force, the lbf, accelerates a unit of mass, the lbm, $32.174\ \mathrm{ft/s^2}$. People felt that if a unit force causes something to accelerate at a unit rate, that thing should have a unit mass. And the lbf was the natural unit of force. So they invented a new unit of mass, the slug.

$$1\,\mathrm{slug} \equiv 1\,\mathrm{lbf}/(1\,\mathrm{ft/s^2}).$$

What is the mass of something that has an acceleration of $a = 1\,\mathrm{ft/s^2}$ when a force of 1 lbf is applied?

$$\begin{aligned} m &= F/a \\ &= (1\,\mathrm{lbf})/(1\,\mathrm{ft/s^2}) \;=\; 1\left(\mathrm{lbf/ft/s^2}\right)\underbrace{\left(\frac{32.174\,\mathrm{lbm}\ \mathrm{ft/s^2}}{\mathrm{lbf}}\right)}_{1} \\ &\stackrel{def}{=} 1\,\mathrm{slug} \end{aligned}$$

That is, 1 slug accelerates $1\,\mathrm{ft/s^2}$ when 1 lbf is applied. How much does a slug weigh? The force of gravity on a slug, in Potsdam, is 32.174 lbf. It's less on Mount Everest and more on the North pole.

Poundals and slugs. Because English scientists and engineers of old liked the number 1 better than both the number 32.174 and the number 1/32.174 they left us two new units to worry about: the poundal = $1\,\mathrm{lbm\ ft/s^2} = (1/32.174)\,\mathrm{lbf}$, and the slug = $1\,\mathrm{lbf}/(\mathrm{ft/s^2}) = 32.174\,\mathrm{lbm}$. If you are used to the internationally acceptable units for force and mass then you can convert like this

$$1\,\mathrm{pdl} = .138255\,\mathrm{N} \quad \text{and} \quad 1\,\mathrm{slug} = 14.5939\,\mathrm{kg}.$$

Now-a-days there are more people, most likely you are amongst these, who laugh at their confusion about slugs and poundals than there are people who use them seriously.

Europe vs America? Americans and Europeans share the defect of using lbf and kgf, respectively. What about the metric slug. What is the answer to the inevitable question 'What is the mass of a thing which accelerates at $1\ \mathrm{m/s^2}$ when a force of 1 kgf is applied?" That would be the 9.8065 kg, the metric slug, also called the mug.

One lbf isn't the force of gravity on lbf! A kilopond is about the force of gravity on a kilogram and a pound force is about the force of gravity on a pound mass, exactly so somewhere in Potsdam — well, not really. Confusingly, mg is not the force due to gravity for either SI or English units. It is the force of the spring which holds up the mass on a rotating earth! What is called g is the 'effective' gravity which is the acceleration due to gravity minus the centripetal acceleration due to the earth's rotation. In Potsdam. This differs from the acceleration of gravity by almost 0.5%. Never mind how this varies with the phases of the time of day or phases of the moon, which it does (that's why there are tides and why the tides are bigger at full moons).

Engineering accuracy. Most engineering calculations involve various approximations. Most of the time your answers won't be any worse than anyone else's if you use any or all of these approximations:

- $g_{\text{here}} = g_{\text{there}}$ for any two points on the earth.
- g is the force of gravity per unit mass
- the centripetal acceleration from the earth's rotation is negligible.
- $g = 32\,\mathrm{ft/s^2} = 10\,\mathrm{m/s^2} = 10\,\mathrm{N/kg}$.

To some, the list above reads like an invitation to sloppiness. But, being precise about these things is generally either implying greater accuracy than is indeed there, or is more trouble than it is worth. Of course you should be precise in distinguishing a lbf from a lbm and a kgf from a kg. But unless you really have to, maybe just don't worry too much about the (supposedly) exact value of g you use to relate force to mass.