3.1 Basic summary of ODEs for mechanics

This section is *not* an aside. Know it well.

Here are some of the simplest useful ordinary differential equations (ODEs) and their general solutions. Think of u as the distance an object has moved to the right of its 'home' at $u = 0$ in time t. The velocity and acceleration to the right are the derivatives of u.

$$\begin{array}{rr} \text{position} & u \\ \text{velocity} & du/dt = \dot{u} \\ \text{acceleration} & d^2u/dt^2 = \ddot{u}. \end{array} \tag{3.1}$$

If $\dot{u} < 0$ the particle is moving to the left. If $\ddot{u} < 0$ the particle is accelerating to the left.

In all cases below A and B are constants and λ is a positive constant. C_1, C_2, C_3, and C_4 are arbitrary (undetermined) constants in the solutions that get pinned down (determined) by fixing the initial conditions. This section is not about how to derive the solutions, it is about how to *know* them.

Constant position: $\dot{u} = 0 \ \Rightarrow\ u = C_1$

$\dot{u} = 0$ means that the velocity is zero. This equation would arise in dynamics if a particle has no initial velocity and no force is applied to it. The particle doesn't move. Its position must be constant. But it could be anywhere, say at position C_1. Hence the general solution $u = C_1$, as can be found by direct integration.

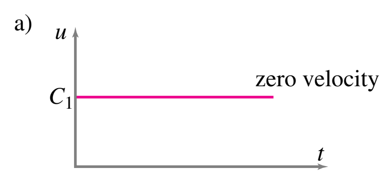

Figure 3.1: Constant position.

Constant velocity: $\dot{u} = A \ \Rightarrow\ u = At + C_1$

$\dot{u} = A$ means the object has constant speed. This equation describes the motion of a particle that starts with speed $v_0 = A$ and because it has no force acting on it continues to move at constant speed. How far does it go in time t? It goes $v_0 t$. Where was it at time $t = 0$? It could have been anywhere then, say C_1. So where is it at time t? It's at its original position plus how far it has moved, $u = v_0 t + C_1$, as can also be found by direct integration.

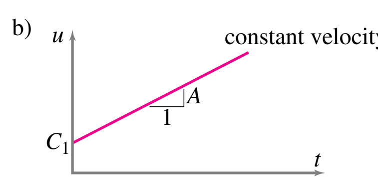

Figure 3.2: Constant velocity.

Zero acceleration: $\ddot{u} = 0 \ \Rightarrow\ u = C_1 + C_2 t$

$\ddot{u} = 0$ means the acceleration is zero. That is, the rate of change of velocity is zero. This constant-velocity motion is the general equation for a particle with no force acting on it. The velocity, if not changing, must be constant. What constant? It could be anything, say C_2. Now we have the same situation as in case (b). So the position as a function of time is anything consistent with an object moving at constant velocity: $u = C_1 + C_2 t$, where the constants C_1 and C_2 depend on the initial position and initial velocity.

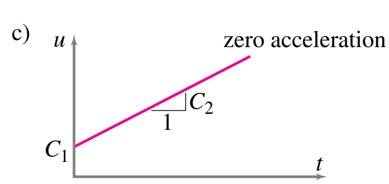

Figure 3.3: Zero acceleration.

If you know that the position at $t = 0$ is u_0 and the velocity at $t = 0$ is v_0, then the position is $u = u_0 + v_0 t$.

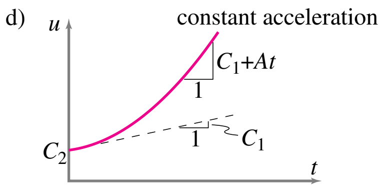

Figure 3.4: Constant acceleration.

Constant acceleration: $\ddot{u} = A \Rightarrow \quad u = At^2/2 + C_1 t + C_2$

This constant acceleration A, constant rate of change of velocity, is the classic (all-too-often studied) case. This situation arises for vertical motion of an object in a constant gravitational field as well as in problems of constant acceleration or deceleration of vehicles. The velocity increases in proportion to the time that passes. The change in velocity in a given time is thus At and the velocity is $v = \dot{u} = v_0 + At$ (given that the velocity was v_0 at $t = 0$). Because the velocity is increasing constantly over time, the average velocity in a trip of length t occurs at $t/2$ and is $v_0 + At/2$. The distance traveled is the average velocity times the time of travel so the distance of travel is $t \cdot (v_0 + At/2) = v_0 t + At^2/2$. The position is the position at $t = 0$, u_0, plus the distance traveled since time zero. So $u = u_0 + v_0 t + At^2/2 = C_2 + C_1 t + At^2/2$. This solution can also be found by direct integration.

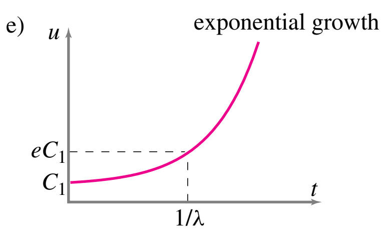

Figure 3.5: Exponential growth.

Exponential growth: $\dot{u} = \lambda u \;\Rightarrow\; u = C_1 e^{\lambda t}$

The displacement u grows in proportion to its present size. This equation describes the initial falling of an inverted pendulum in a thick viscous fluid. The bigger the u, the faster it moves. Such situations are called exponential growth (as in population growth or monetary inflation) for a good mathematical reason. The solution u is an exponential function of time: $u(t) = C_1 e^{\lambda t}$, as can be found by separating variables or guessing.

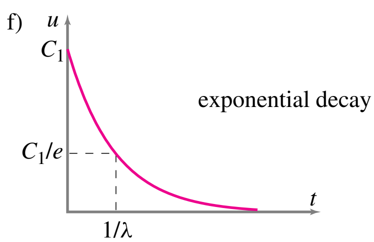

Figure 3.6: Exponential decay.

Exponential decay: $\dot{u} = -\lambda u \;\Rightarrow\; u = C_1 e^{-\lambda t}$

The smaller u is, the more slowly it gets smaller. u gradually tapers towards nothing: u decays exponentially. The solution to the equation is: $u(t) = C_1 e^{-\lambda t}$. This expression is essentially the same equation as in (e) above.

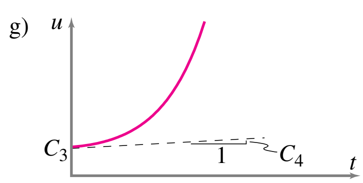

Figure 3.7: Acceleration proportional to position.

Acceleration proportional to position: $\ddot{u} = \lambda^2 u$

$$\Rightarrow u = C_1 e^{\lambda\, t} \;+\; C_2 e^{-\lambda\, t}$$

$$\textbf{or} \Rightarrow u = C_3 \cosh(\lambda\, t) \;+\; C_4 \sinh(\lambda\, t)$$

Note, sinh and cosh are just combinations of exponentials. For $\ddot{u} = \lambda^2 u$, the point accelerates more and more away from the origin in proportion to the distance from the origin. This equation describes the initial falling of a nearly vertical inverted pendulum when there is no friction. Most often, the solution of this equation gives roughly exponential growth. The pendulum accelerates away from being upright. The reason there is also an exponentially decaying solution to this equation is a little more subtle to understand intuitively: if a not quite upright pendulum is given just the right initial velocity it will slowly approach becoming just upright with an exponentially decaying displacement. This decaying solution is not easy to see experimentally because, without the perfect initial condition, the exponentially growing part of the solution eventually dominates and the pendulum accelerates away from being just upright.

Harmonic oscillator: $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ $\Rightarrow$ $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$.

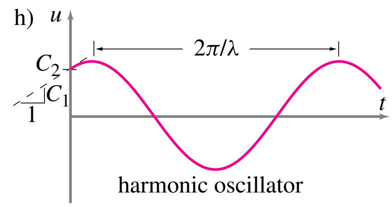

Figure 3.8: Harmonic oscillator.

This equation describes a mass that is restrained by a spring which is relaxed when the mass is at $u = 0$. When u is positive, $\ddot{u}$ is negative. That is, if the particle is on the right side of the origin it accelerates to the left. Similarly, if the particle is on the left it accelerates to the right. In the middle, where $u = 0$, it has no acceleration, so it neither speeds up nor slows down in its motion whether it is moving to the left or the right. So the particle goes back and forth: its position oscillates. A function that correctly describes this oscillation is $u = \sin(\lambda t)$, that is, sinusoidal oscillations. The oscillations are faster if λ is bigger. Another solution is $u = \cos(\lambda t)$. The general solution is $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$. A plot of this function reveals a sine wave shape for any value of C_1 or C_2, although the phase depends on the relative values of C_1 and C_2. The equation $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ is called the 'harmonic oscillator' equation and is important in almost all branches of science. The solution may be found by guessing or other means (which are usually guessing in disguise). In the context of this equation, λ is called the (angular) frequency of oscillation.

More ODEs

Besides the ODEs listed here there are a few others that are often solved by hand rather than with numerical simulation. Most famously there is the damped oscillator equation $A\ddot{u} + B\dot{u} + Cu = 0$ discussed in Box 11.3 on page 563.

With the exception of the damped or forced oscillator, most engineers now-a-days will use numerical integration if they want to solve an ODE not in this appendix.

Learn, memorize, know, whatever

Conversely,

> Anyone competent at dynamics knows all the equations and solutions on these few pages outside, inside, and inside-out.

Whether you have or have not learned these in a calculus course you should learn them now every way possible.