SAMPLE 2.7 Center of mass in 1-D: Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center-of-mass of the assembly.

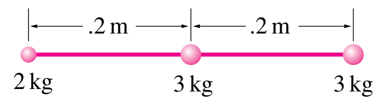

Figure 2.25

Solution Let us select the first mass, $m_1 = 2\,\mathrm{kg}$, to be at the origin of our co-ordinate system with the x-axis along the rod. Since all the three masses lie on the x-axis, the center-of-mass will also lie on this axis. Let the center-of-mass be located at x_{cm} on the x-axis. Then,

$$
\begin{aligned}
m_{\mathrm{tot}} x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 \\
&= m_1(0) + m_2(\ell) + m_3(2\ell) \\
\Rightarrow \quad x_{cm} &= \frac{m_2 \ell + m_3 2\ell}{m_1 + m_2 + m_3} \\
&= \frac{3\,\cancel{\mathrm{k}}\mathrm{g} \cdot 0.2\,\mathrm{m} + 3\,\cancel{\mathrm{k}}\mathrm{g} \cdot 0.4\,\mathrm{m}}{(2+3+3)\,\cancel{\mathrm{k}}\mathrm{g}} \\
&= \frac{1.8\,\mathrm{m}}{8} = 0.225\,\mathrm{m}.
\end{aligned}
$$

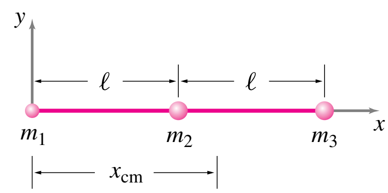

Figure 2.26

$$\boxed{x_{cm} = 0.225\,\mathrm{m}}$$

Alternatively, we could find the center-of-mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center-of-mass of the two equal masses) and then calculate the value of x_{cm} for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. 2.27):

$$x_{cm} = \frac{6\,\mathrm{kg} \cdot 0.3\,\mathrm{m}}{8\,\mathrm{kg}} = \frac{1.8\,\mathrm{m}}{8} = 0.225\,\mathrm{m}.$$

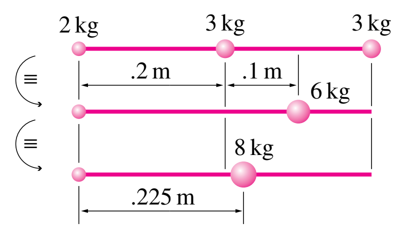

Figure 2.27

SAMPLE 2.8 Center of mass in 2-D: Two particles of mass $m_1 = 1\,\mathrm{kg}$ and $m_2 = 2\,\mathrm{kg}$ are located at coordinates (1m, 2m) and (-2m, 5m), respectively, in the xy-plane. Find the location of their center-of-mass.

Solution Let $\vec{\boldsymbol{r}}_{\mathrm{cm}}$ be the position vector of the center-of-mass. Then,

$$
\begin{aligned}
m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{\mathrm{cm}} &= m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2 \\
\Rightarrow \quad \vec{\boldsymbol{r}}_{\mathrm{cm}} &= \frac{m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2}{m_{\mathrm{tot}}} = \frac{m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2}{m_1 + m_2} \\
&= \frac{1\,\cancel{\mathrm{k}}\mathrm{g}(1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 2\,\mathrm{m}\hat{\boldsymbol{\jmath}}) + 2\,\cancel{\mathrm{k}}\mathrm{g}(-2\,\mathrm{m}\hat{\boldsymbol{\imath}} + 5\,\mathrm{m}\hat{\boldsymbol{\jmath}})}{3\,\cancel{\mathrm{k}}\mathrm{g}} \\
&= \frac{(1\,\mathrm{m} - 4\,\mathrm{m})\hat{\boldsymbol{\imath}} + (2\,\mathrm{m} + 10\,\mathrm{m})\hat{\boldsymbol{\jmath}}}{3} = -1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 4\,\mathrm{m}\hat{\boldsymbol{\jmath}}.
\end{aligned}
$$

Thus the center-of-mass is located at the coordinates(-1m, 4m).

$$\boxed{(x_{cm}, y_{cm}) = (-1\,\mathrm{m}, 4\,\mathrm{m})}$$

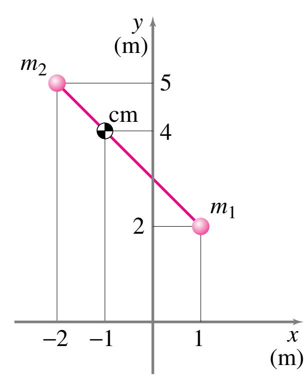

Figure 2.28

Geometrically, this is just a 1-D problem like the previous sample. The center-of-mass has to be located on the straight line joining the two masses. Since the center-of-mass is a point about which the distribution of mass is *balanced*, it is easy to see (see Fig. 2.28) that the center-of-mass must lie one-third way from m_2 on the line joining the two masses so that $2\,\mathrm{kg} \cdot (d/3) = 1\,\mathrm{kg} \cdot (2d/3)$.

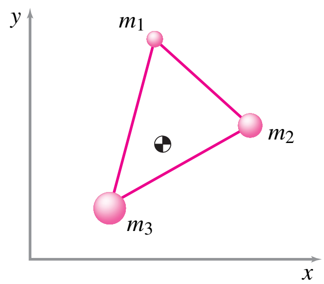

Figure 2.29

SAMPLE 2.9 **Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$, connected rigidly by massless rods. At the instant of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\imath}$, $2\,\text{m/s}(\hat{\imath} - 1.5\hat{\jmath})$ and $1\,\text{m/s}\hat{\jmath}$, respectively. Find the coordinates of the center-of-mass of the structure.

Solution Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center-of-mass, and second, using vector equations for the position of the center-of-mass.

1. **Scalar calculations:** Let (x_{cm}, y_{cm}) be the coordinates of the mass-center. Then from the definition of mass-center,

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{7.5\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}. \end{aligned}$$

Similarly,

$$\begin{aligned} y_{cm} &= \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{8.5\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}. \end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

2. **Vector calculations:** Let $\vec{r}_{\text{cm}}$ be the position vector of the mass-center. Then,

$$\begin{aligned} m_{\text{tot}}\vec{r}_{\text{cm}} &= \sum_{i=1}^{3} m_i\vec{r}_i = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \\ \Rightarrow\quad \vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \end{aligned}$$

Substituting the values of m_1, m_2, and m_3, and $\vec{r}_1 = 1.25\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$, $\vec{r}_2 = 2\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, and $\vec{r}_3 = 0.75\,\text{m}\hat{\imath} + 0.5\,\text{m}\hat{\jmath}$, we get,

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{1\,\text{kg}\cdot(1.25\hat{\imath} + 3\hat{\jmath})\,\text{m} + 2\,\text{kg}\cdot(2\hat{\imath} + 2\hat{\jmath})\,\text{m} + 3\,\text{kg}\cdot(0.75\hat{\imath} + 0.5\hat{\jmath})\,\text{m}}{(1+2+3)\,\text{kg}} \\ &= \frac{(7.5\hat{\imath} + 8.5\hat{\jmath})\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} \\ &= 1.25\,\text{m}\hat{\imath} + 1.42\,\text{m}\hat{\jmath} \end{aligned}$$

which, of course, gives the same location of the mass-center as above.

$$\boxed{\vec{r}_{\text{cm}} = 1.25\,\text{m}\hat{\imath} + 1.42\,\text{m}\hat{\jmath}}$$

SAMPLE 2.10 Center of mass of a bent bar: A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center-of-mass of the bar.

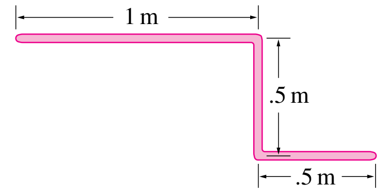

Figure 2.30

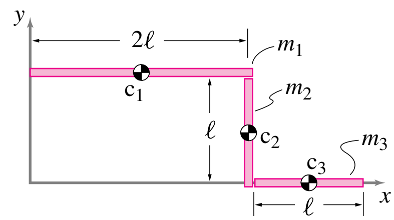

Figure 2.31

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center-of-mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let $m_2 = m_3 = m$, then $m_1 = 2m$; and $m_1 + m_2 + m_3 = 4m = 4\,\text{kg}$ which gives $m = 1\,\text{kg}$. Now, from Fig. 2.31,

$$\begin{aligned}
\vec{r}_1 &= \ell\hat{\imath} + \ell\hat{\jmath} \\
\vec{r}_2 &= 2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \\
\vec{r}_3 &= (2\ell + \frac{\ell}{2})\hat{\imath} = \frac{5\ell}{2}\hat{\imath}
\end{aligned}$$

So,

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_{\text{tot}}} \\
&= \frac{2m(\ell\hat{\imath} + \ell\hat{\jmath}) + m(2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath}) + m(\frac{5\ell}{2}\hat{\imath})}{4m} \\
&= \frac{\cancel{m}\ell(2\hat{\imath} + 2\hat{\jmath} + 2\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{5}{2}\hat{\imath})}{4\cancel{m}} \\
&= \frac{\ell}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= \frac{0.5\,\text{m}}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}}$$

Geometrically, we could find the center-of-mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. 2.32.

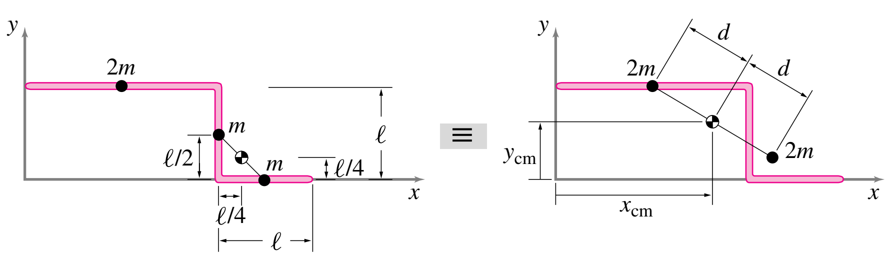

Figure 2.32:

The center-of-mass of m_2 and m_3 (each of mass m) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass $2m$ at their mass-center. Next, we connect this mass-center and m_1 with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center-of-mass of the entire bar.

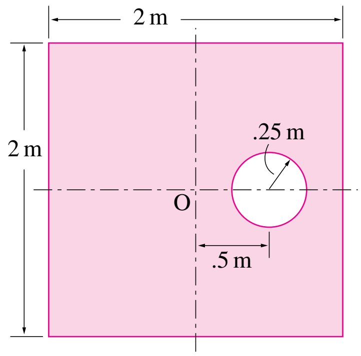

Figure 2.33

SAMPLE 2.11 Shift of mass-center due to cut-outs: A $2\,\mathrm{m} \times 2\,\mathrm{m}$ uniform square plate has mass $m = 4\,\mathrm{kg}$. A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center-of-mass of the plate.

Solution Let us use an xy-coordinate system with its origin at the geometric center of the plate and the x-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the x-axis, the new center-of-mass must lie somewhere on the x-axis. Thus, we only need to find x_{cm} (since $y_{cm} = 0$). Let m_1 be the mass of the plate with the hole, and m_2 be the mass of the circular cut-out. Clearly, $m_1 + m_2 = m = 4\,\mathrm{kg}$. The center-of-mass of the circular cut-out is at A, the center of the circle. The center-of-mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A. \end{aligned}$$

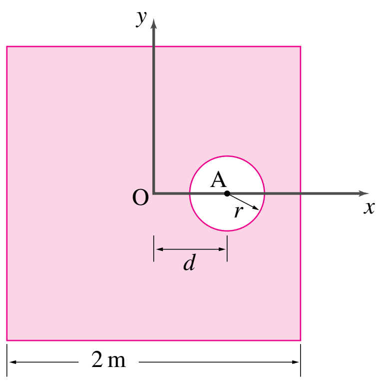

Figure 2.34

Now, since the plate is uniform, the masses m_1 and m_2 are proportional to the surface areas of the geometric objects they represent, *i.e.,*

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi}.$$

Therefore,

$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \qquad (2.5) \\ &= -\frac{\pi}{\left(\frac{2\,\mathrm{m}}{.25\,\mathrm{m}}\right)^2 - \pi} \cdot 0.5\,\mathrm{m} \\ &= -25.81 \times 10^{-3}\,\mathrm{m} = -25.81\,\mathrm{mm} \end{aligned}$$

Thus the center-of-mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

$$\boxed{x_{cm} = -25.81\,\mathrm{mm}}$$

Comments: The advantage of finding the expression for x_{cm} in terms of r and ℓ as in eqn. (2.5) is that you can easily find the center-of-mass of any size circular cut-out located at any distance d on the x-axis. This is useful in design where you like to select the size or location of the cut-out to have the center-of-mass at a particular location.

SAMPLE 2.12 Center of mass of two objects: A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center-of-mass of the combined system.

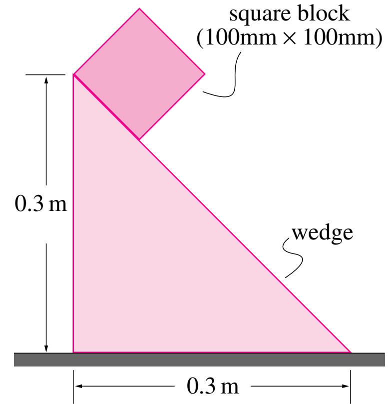

Figure 2.35

Solution The center-of-mass of the triangular wedge is located at $h/3$ above the base and $\ell/3$ to the right of the vertical side. Let m_1 be the mass of the wedge and $\vec{r}_1$ be the position vector of its mass-center. Then, referring to Fig. 2.36,

$$\vec{r}_1 = \frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}.$$

The center-of-mass of the square block is located at its geometric center C_2. From geometry, we can see that the line AE that passes through C_2 is horizontal since $\angle OAB = 45°$ ($h = \ell = 0.3\,\text{m}$) and $\angle DAE = 45°$. Therefore, the coordinates of C_2 are $(d/\sqrt{2}, h)$. Let m_2 and $\vec{r}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$\vec{r}_2 = \frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath}.$$

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2\,\text{kg}$, we find the center-of-mass of the combined system:

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{(m_1 + m_2)} \\
&= \frac{3m(\frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}) + m(\frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath})}{3m + m} \\
&= \frac{\cancel{m}[(\ell + \frac{d}{\sqrt{2}})\hat{\imath} + 2h\hat{\jmath}]}{4\cancel{m}} \\
&= \frac{1}{4}(\frac{d}{\sqrt{2}} + \ell)\hat{\imath} + \frac{h}{2}\hat{\jmath} \\
&= \frac{1}{4}(\frac{0.1\,\text{m}}{\sqrt{2}} + 0.3\,\text{m})\hat{\imath} + \frac{0.3\,\text{m}}{2}\hat{\jmath} \\
&= 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}}$$

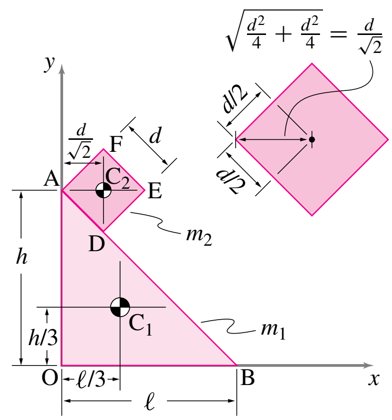

Figure 2.36

Thus, the center-of-mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1(0.1\,\text{m}, 0.1\,\text{m})$. This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at C_1 and m at C_2, connected the two points by a straight line, and located the center-of-mass C on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1(0.1\,\text{m}, 0.1\,\text{m})$ to $C(.093\,\text{m}, 0.15\,\text{m})$ is one third the distance from C to $C_2(.071\,\text{m}, 0.3\,\text{m})$.