2.2 Center of mass and gravity

The center-of-mass of a system is the point at the position $\vec{\boldsymbol{r}}_{\mathrm{cm}}$ defined by

$$\begin{aligned} \vec{\boldsymbol{r}}_{\mathrm{cm}} &= \frac{\sum \vec{\boldsymbol{r}}_{\mathrm{i}} m_i}{m_{\mathrm{tot}}} \quad \text{for discrete systems} \\ &= \frac{\int \vec{\boldsymbol{r}} dm}{m_{\mathrm{tot}}} \quad \text{for continuous systems} \end{aligned} \tag{2.4}$$

where $m_{\mathrm{tot}} = \sum m_i$ for discrete systems and $m_{\mathrm{tot}} = \int dm$ for continuous systems (see boxes 2.1 and 2.4 on pages 142 and 2 for a discussion of the $\sum$ and $\int$ sum notations).

For every system, and at every instant in time, this is a unique location in space that is the average position of the system's mass. The *center of mass* is commonly designated by

- cm
- c.o.m.,
- COM,
- G,
- c.g., or
- with a circle with diagonally-opposite quadrants shaded

One of the important (if routine) tasks of many real engineers is to find the center-of-mass of a complex machine.

What is the center of mass good for? Just knowing the location of the center-of-mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The center-of-mass of a boat must be low enough for the boat to be stable. Any propulsive force on a space craft must be directed towards the center-of-mass in order to not induce rotations. Tracking the trajectory of the center-of-mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center-of-mass on the axis of rotation if it is not to cause much vibration.

Also, many calculations in mechanics are greatly simplified by making use of a system's center-of-mass. In particular,

> the whole complicated distribution of near-earth gravity forces on a body is equivalent to a single force at the body's center-of-mass.

Many of the important quantities in dynamics are similarly simplified using the center-of-mass.

The following rearranged definition of center of mass, might be easier to remember, and is sometimes more directly useful:

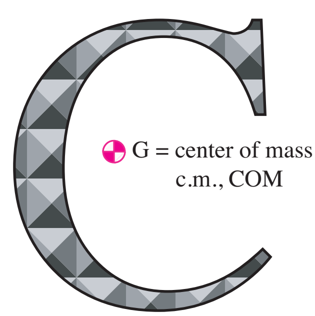


Figure 2.17: The center of mass of the ink making up the letter C is approximately at G (marked with a circle with the diagonally-opposite quadrants shaded). The various notations for center of mass include: G and COM.

$$m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} = \sum m_i \vec{\boldsymbol{r}}_{\text{i}} \quad \text{or} \quad m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} = \int \vec{\boldsymbol{r}}\, dm.$$

That is, the center of mass is the position that, when multiplied by the total mass, gives the same result as the sum of all the mass bits each multiplied by their individual positions.

For theoretical purposes, we rarely need to evaluate these sums and integrals. And, for simple problems there are sometimes shortcuts that reduce the calculation to a matter of observation. However, for complex machines, one or both of the formulas 2.4 must be evaluated in detail ①.

①This routine work is generally done with CAD (computer aided design) software. But an engineer still needs to know the basic calculation skills in order to make sense of, and roughly check, the computer calculations and in order to think of appropriate design changes.

Example: Center of mass of two point masses
Intuitively, the center-of-mass of the two masses shown in fig. 2.18 is between the two masses and closer to the larger one. Referring to equation 2.4,

2.4 Like $\sum$, the symbol $\int$ also means add

We often add things up in mechanics. For example, the total mass of some particles is

$$m_{\text{tot}} = m_1 + m_2 + m_3 + \cdots = \sum m_i$$

or more specifically the mass of 137 particles is, say, $m_{\text{tot}} = \sum_{i=1}^{137} m_i$.

And the total mass of a bicycle is:

$$m_{\text{bike}} = \sum_{i=1}^{100{,}000{,}000{,}000{,}000{,}000{,}000{,}000} m_i$$

where m_i are the masses of each of the 10^{23} (or so) atoms of metal, rubber, plastic, cotton, and paint. But atoms are so small and there are so many of them. Instead we often think of a bike as built of macroscopic parts. The total mass of the bike is then the sum of the masses of the tires, the tubes, the wheel rims, the spokes and nipples, the ball bearings, the chain pins, and so on. And we would write:

$$m_{\text{bike}} = \sum_{i=1}^{2{,}000} m_i$$

where now the m_i are the masses of the 2,000 or so bike parts. This sum is more manageable, but still too detailed for some purposes.

An approach that avoids attending to atoms or ball bearings, is to think of sending the bike to a big shredding machine that cuts it up into very small bits. Now we write

$$m_{\text{bike}} = \sum m_i$$

where the m_i are the masses of the very small bits. We don't fuss over whether one bit is a piece of ball bearing or fragment of cotton from the tire walls. We just chop the bike into bits and add up the contribution of each bit. If you take the letter S, as in SUM, and distort it and you get a big old fashioned German 'S' used in calculus as the integral sign

S S ʃ ∫ ∫

Evolution.
From left to right, the letter S is distorted into the integral sign $\int$.

So, instead of writing, like Lagrange would have (if he knew about bicycles and cared about their mass), $m_{\text{bike}} = \mathbf{S}\, dm$, we write

$$m_{\text{bike}} = \int dm.$$

This is the $\int$um of all the teeny bits of mass dm. More formally, we mean the value of that sum in the limit that all the bits are infinitesimal (not minding the technical fine point that it's hard to chop atoms into infinitesimal pieces).

The mass is one of many things we would like to add up. Many of the other things also involve mass. For example, to find the center-of-mass we add up the positions 'weighted' by mass.

$$\int \vec{\boldsymbol{r}}\, dm \quad \text{which means} \sum_{\lim m_i \to 0} \vec{\boldsymbol{r}}_{\text{i}} m_i.$$

That is, to find the center of mass, you take your object of interest and chop it into a billion pieces. Then you re-assemble it. For each piece, you make the vector which is the position vector of the piece multiplied by ('weighted by') its mass. Then you add up the billion vectors. Well, really you chop the thing into a trillion trillion ... pieces, but a billion gives the idea.

$$
\begin{aligned}
\vec{r}_{\rm cm} &= \frac{\sum \vec{r}_i m_i}{m_{tot}} \\
&= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2} \\
&= \frac{\vec{r}_1(m_1+m_2) - \vec{r}_1 m_2 + \vec{r}_2 m_2}{m_1+m_2} \\
&= \vec{r}_1 + \underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{\text{the fraction of the distance that the cm is from } \vec{r}_1 \text{ to } \vec{r}_2} \underbrace{(\vec{r}_2 - \vec{r}_1)}_{\text{the vector from } \vec{r}_1 \text{ to } \vec{r}_2.}.
\end{aligned}
$$

The math agrees with common sense: the center-of-mass is on the line connecting the masses. If $m_2 \gg m_1$, then the center-of-mass is near m_2. If $m_1 \gg m_2$, then the center-of-mass is near m_1. If $m_1 = m_2$ the center-of-mass is right in the middle at $(\vec{r}_1 + \vec{r}_2)/2$.

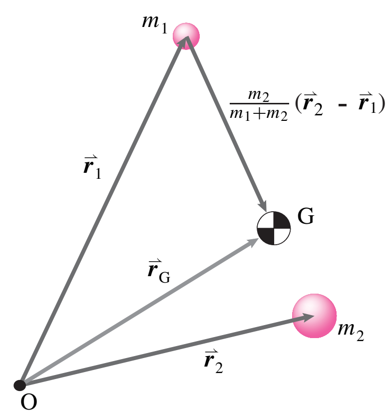


Figure 2.18: Center of mass of a system consisting of two points.

Continuous systems

How do we evaluate integrals like $\int$(something) dm? In center-of-mass calculations, (something) is position, but we will evaluate similar integrals where (something) is some other scalar or vector-valued function of position. Most often, we label the material by its spatial position, and evaluate dm in terms of increments of position. For 3D solids $dm = \rho dV$ where ρ is density (mass per unit volume). So $\int$(something)dm turns into a standard volume integral $\int_V$(something)$\rho\, dV$②.

② Note: writing $\int_{m_1}^{m_2}$(something) dm is nonsense because m is not a scalar parameter which labels points in a material (*i.e.,* there is no point at m = 3 kg).

Planar objects. For thin flat things like metal sheets we often take ρ to mean mass per unit area A so then $dm = \rho dA$ and $\int$(something)dm = $\int_A$(something)$\rho\, dA$.

1D objects. For mass distributed along a line or curve we take ρ to be the mass per unit length or arc length s and so $dm = \rho ds$ and $\int$(something)$dm = \int_{curve}$(something)$\rho\, ds$.

CoM of uniform rod. The center-of-mass of a uniform rod is naturally in the middle, as the calculations here show (see fig. 2.19a). Assume the rod has length L = 3 m and mass m = 7 kg.

$$
\vec{r}_{\rm cm} = \frac{\int \vec{r}\, dm}{m_{\rm tot}} = \frac{\int_0^L x\hat{\imath}\ \overbrace{\rho dx}^{dm}}{\int_0^L \rho\, dx} = \frac{\rho(x^2/2)|_0^L}{\rho(1)|_1^L}\hat{\imath} = \frac{\rho(L^2/2)}{\rho L}\hat{\imath} = (L/2)\hat{\imath}
$$

So $\vec{r}_{\rm cm} = (L/2)\hat{\imath}$, or by dotting with $\hat{\imath}$ (taking the x component) we get that the center-of-mass is on the rod a distance $d = L/2$ = 1.5 m from the end.

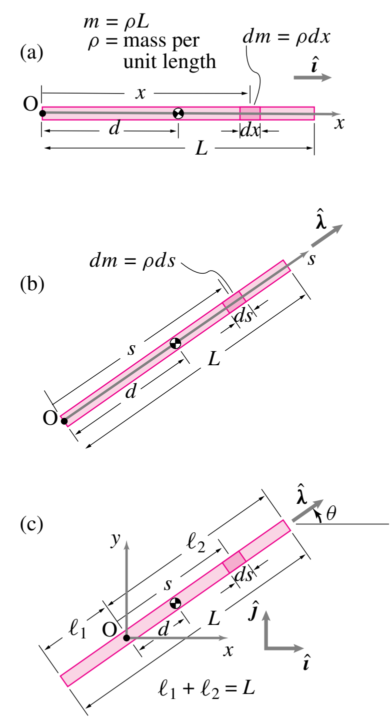


Figure 2.19: Where is the center-of-mass of a uniform rod? In the middle, as you can find calculating a few ways, or, more simply, by symmetry.

The center-of-mass calculation is *objective*. It describes something about the object that does not depend on the coordinate system. In different coordinate systems the center-of-mass for the rod above will have different coordinates, but it will always be at the middle of the rod.

Example. Find the center-of-mass using the coordinate system with s & $\hat{\boldsymbol{\lambda}}$ in fig. 2.19b:

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} = \frac{\int_0^L s\hat{\boldsymbol{\lambda}}\, \rho ds}{\int_0^L \rho\, ds}\hat{\boldsymbol{\lambda}} = \frac{\rho(s^2/2)|_0^L}{\rho(1)|_0^L}\hat{\boldsymbol{\lambda}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\lambda}} = (L/2)\hat{\boldsymbol{\lambda}},$$

again showing that the center-of-mass is in the middle.

Note, one can treat the center-of-mass vector calculations as separate scalar equations, one for each component. For example:

$$\hat{\boldsymbol{\imath}} \cdot \left\{ \vec{\boldsymbol{r}}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} \right\} \quad \Rightarrow \quad r_{x\text{cm}} = x_{\text{cm}} = \frac{\int x\, dm}{m_{\text{tot}}}.$$

Finally, there is no law that says you have to use the best coordinate system. One is free to make trouble for oneself and use an inconvenient coordinate system.

CoM using awkward coordinates. Use the xy coordinates of fig. 2.19c to find the center-of-mass of the rod.

$$x_{\text{cm}} = \frac{\int x dm}{m_{\text{tot}}} = \frac{\int_{-\ell_1}^{\ell_2} \overbrace{s\cos\theta}^{x}\, \rho ds}{\int_0^L \rho\, ds} = \frac{\rho\cos\theta \frac{s^2}{2}|_{-\ell_1}^{\ell_2}}{\rho(1)|_{-\ell_1}^{\ell_2}} = \frac{\rho\cos\theta \frac{(\ell_2^2 - \ell_1^2)}{2}}{\rho(\ell_1 + \ell_2)}$$

$$= \frac{\cos\theta(\ell_2 - \ell_1)}{2}$$

Similarly $y_{\text{cm}} = \sin\theta(\ell_2 - \ell_1)/2$ so

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\ell_2 - \ell_1}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}),$$

which is at the middle of the rod.

The most commonly needed center-of-mass that can be found analytically but not so-simply from symmetry is that of a triangle (see box 2.6 on page 11). There are more examples of using integration to find the center-of-mass in your multi-variable text book.

Center of mass and centroid

For objects with uniform material density we have

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}} dm}{m_{\text{tot}}} = \frac{\int_V \vec{\boldsymbol{r}} \rho dV}{\int_V \rho dV} = \frac{\rho \int_V \vec{\boldsymbol{r}} dV}{\rho \int_V dV} = \frac{\int_V \vec{\boldsymbol{r}} dV}{V}.$$

The last expression is just the formula for geometric centroid. Analogous calculations hold for 2D and 1D geometric objects.

For objects with density that does not vary from point to point, the center-of-mass is at the same place as the geometric centroid.

Center of mass and symmetry

The center-of-mass respects any symmetry in the mass distribution of a system. If the word ‘middle’ has unambiguous meaning in English, then that is the location of the center-of-mass, as for the rod of fig. 2.19 and the other examples in fig. 2.20.

Example: Center of mass of a semicircle.
(see fig. 2.21) The center of mass of a semicircular arc of mass m and radius r is on the y axis at $x = 0$, by symmetry. The location of y_G is found by

$$
\begin{aligned}
y_G &= \frac{\int y\, dm}{m_{\text{tot}}} = \frac{1}{m}\int_0^{\pi} \overbrace{r\sin\theta}^{y}\; \overbrace{\rho\, ds}^{dm} = \frac{1}{m}\int_0^{\pi} r\sin\theta\; \overbrace{\frac{m}{\pi r}}^{\rho}\; \overbrace{r\, d\theta}^{ds} \\
&= \frac{r}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{2}{\pi} r \approx 0.64\, r
\end{aligned}
$$

The center of mass of a semicircle is almost 2/3 of the way towards the perimeter from the center of the circle. You can see that G has to be above the halfway point by noticing how much more mass is near to $y = r$ (where the circular arc is nearly horizontal) than to $y = 0$ (where the circular arc is running away from the x axis).

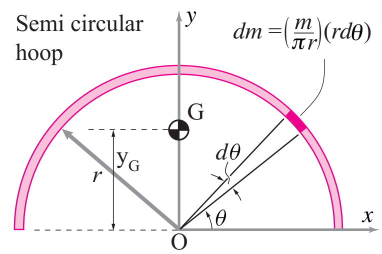


Figure 2.21: The center of mass of a semicircular arc is about 0.6 r from the center of the circle.

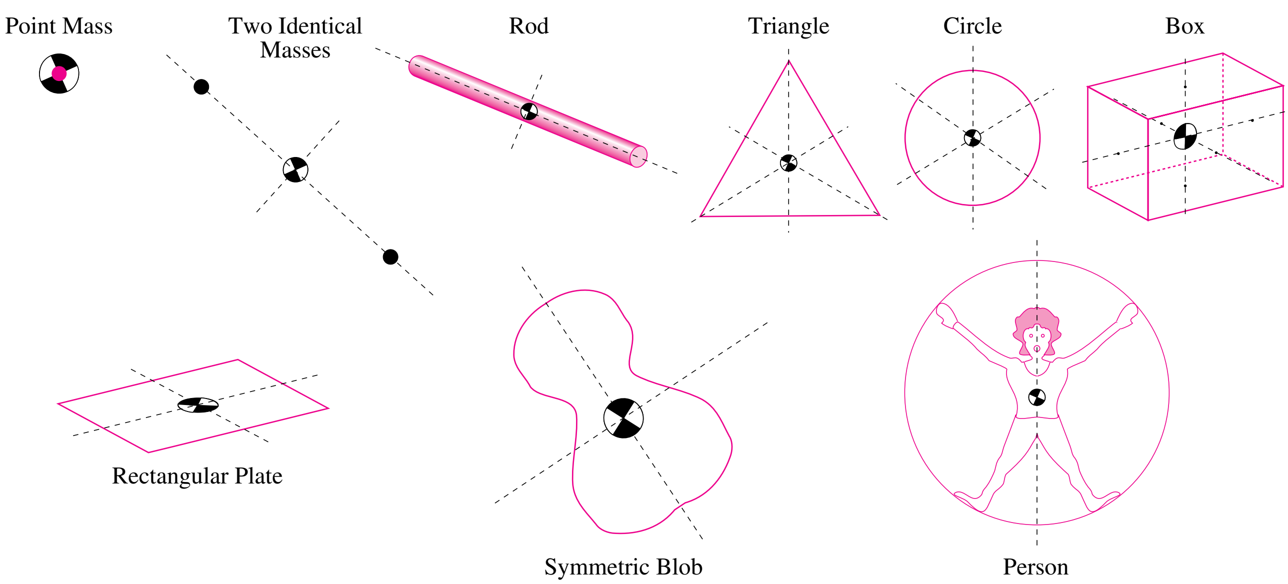


Figure 2.20: Both the center of mass and the geometric centroid share the symmetries of the object.

Systems of systems and composite objects

Another way of interpreting the formula

$$\vec{\boldsymbol{r}}_{\mathrm{cm}} = \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

is that the m's are the masses of subsystems, not just points, and that the $\vec{\boldsymbol{r}}_i$ are the positions of the centers of mass of these systems. This subdivision, follows from the associated law of addition, and is justified in box 2.5 on page 7.

> The center-of-mass of a single complex shaped object can be found by treating it as an assembly of simpler objects.

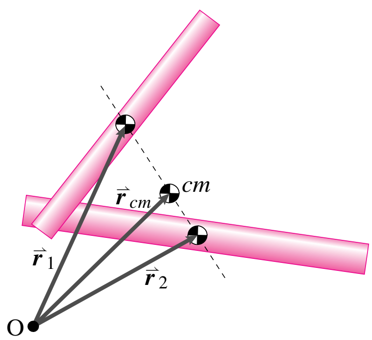


Figure 2.22: Center of mass of two rods

Example: Two rods
The center-of-mass of two rods shown in fig. 2.22 can be found as

$$\vec{\boldsymbol{r}}_{\mathrm{cm}} = \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2}$$

where $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$ are the positions of the centers of mass of each rod and m_1 and m_2 are the masses.

Example: 'L' shaped plate
Consider the plate with uniform mass per unit area ρ.

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{G}} &= \frac{\vec{\boldsymbol{r}}_{\mathrm{I}} m_I + \vec{\boldsymbol{r}}_{\mathrm{II}} m_{II}}{m_I + m_{II}} \\
&= \frac{(\frac{a}{2}\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}})(2\rho a^2) + (\frac{3}{2}a\hat{\boldsymbol{\imath}} + \frac{a}{2}\hat{\boldsymbol{\jmath}})(\rho a^2)}{(2\rho a^2) + (\rho a^2)} \\
&= \frac{5}{6}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

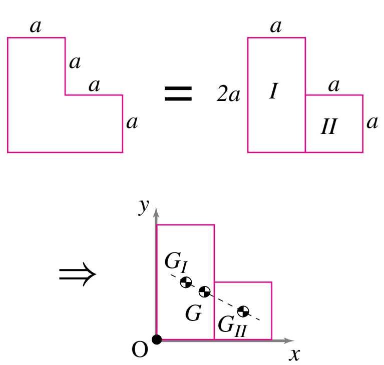


Figure 2.23: The center-of-mass of the 'L' shaped object can be found by thinking of it as a rectangle plus a square.

Composite objects using subtraction

It is sometimes useful to think of an object as composed of pieces, some of which have negative mass.

Example: 'L' shaped plate, again
Reconsider the plate from the previous example.

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{G}} &= \frac{\vec{\boldsymbol{r}}_{\mathrm{I}} m_I + \vec{\boldsymbol{r}}_{\mathrm{II}} m_{II}}{m_I + m_{II}} \\
&= \frac{(a\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}})(\rho(2a)^2) + (\frac{3}{2}a\hat{\boldsymbol{\imath}} + \frac{3}{2}a\hat{\boldsymbol{\jmath}})\overbrace{(-\rho a^2)}^{m_{II}}}{(\rho(2a)^2) + \underbrace{(-\rho a^2)}_{m_{II}}} \\
&= \frac{5}{6}a(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

Center of gravity

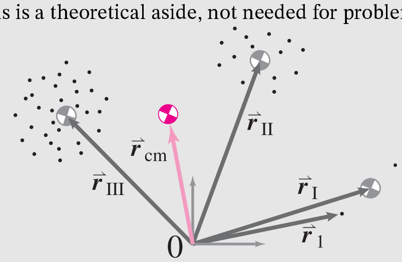
The force of gravity on each little bit of an object is gm_i where g is the local gravitational 'constant' and m_i is the mass of the bit. For objects that are small compared to the radius of the earth (a reasonable assumption for all but a few special engineering calculations) the gravity constant is indeed constant from one point on the object to another (see box 4.3 on page 4.3 for a discussion of the meaning and history of g.)

Not only that, all the gravity forces point in the same direction, down. (For engineering purposes, the two intersecting lines that go from your two hands to the center of the earth are parallel.) Let's call this the $-\hat{\boldsymbol{k}}$ direction. So the net force of gravity on an object is:

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= \sum \vec{\boldsymbol{F}}_{\text{i}} = \sum m_i g(-\hat{\boldsymbol{k}}) = -mg\hat{\boldsymbol{k}} \quad \text{(discrete systems)} \\
&= \int d\vec{\boldsymbol{F}} = \int \underbrace{-g\hat{\boldsymbol{k}}\,dm}_{d\vec{\boldsymbol{F}}} = -mg\hat{\boldsymbol{k}} \quad \text{(continuous systems)}
\end{aligned}
$$

2.5 Why can subsystems be treated like particles when finding the center-of-mass?

This is a theoretical aside, not needed for problem solving.



Look at the collection of 47 particles above and then think of it as a set of three subsystems: I, II, and III with 2, 14, and 31 particles, respectively. Masses 1 and 2 as subsystem I with center-of-mass $\vec{\boldsymbol{r}}_{\text{I}}$ and total mass m_I. Similarly, we call subsystem II masses m_3 to m_{16}, and subsystem III, masses m_{17} to m_{47}. We can calculate the center-of-mass of the system by treating it as 47 particles, or we can re-arrange the sum as follows:

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2 + \cdots + \vec{\boldsymbol{r}}_{46} m_{46} + \vec{\boldsymbol{r}}_{47} m_{47}}{m_1 + m_2 + \cdots + m_{47}} \\
&= \frac{\frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2}(m_1 + m_2)}{m_1 + m_2 + \cdots + m_{47}} \\
&\quad + \frac{\frac{\vec{\boldsymbol{r}}_3 m_3 + \cdots + \vec{\boldsymbol{r}}_{16} m_{16}}{m_3 + \cdots + m_{16}}(m_3 + \cdots + m_{16})}{m_1 + m_2 + \cdots + m_{47}} \\
&\quad + \frac{\frac{\vec{\boldsymbol{r}}_{17} m_{17} + \cdots + \vec{\boldsymbol{r}}_{47} m_{47}}{m_{17} + \cdots + m_{47}}(m_{17} + \cdots + m_{47})}{m_1 + m_2 + \cdots + m_{47}} \\
&= \frac{\vec{\boldsymbol{r}}_{\text{I}} m_I + \vec{\boldsymbol{r}}_{\text{II}} m_{II} + \vec{\boldsymbol{r}}_{\text{III}} m_{III}}{m_I + m_{II} + m_{III}}, \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
\text{where } \vec{\boldsymbol{r}}_{\text{I}} &= \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2}, \\
m_I &= m_1 + m_2 \\
\vec{\boldsymbol{r}}_{\text{II}} &= \textit{etc.}
\end{aligned}
$$

That is, the center of mass of the 47 particles is the same as the center of mass of three particles, where each of the three particles has the total mass of its subsystem located at its subsystem's center of mass.

The reduction of subsystem of particles to one particle is easily generalized to the integral formulae as well like this.

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\int \vec{\boldsymbol{r}}\,dm}{\int dm} \\
&= \frac{\int_{\text{region 1}} \vec{\boldsymbol{r}}\,dm + \int_{\text{region 2}} \vec{\boldsymbol{r}}\,dm + \int_{\text{region 3}} \vec{\boldsymbol{r}}\,dm + \cdots}{\int_{\text{region 1}} dm + \int_{\text{region 2}} dm + \int_{\text{region 3}} dm + \cdots} \\
&= \frac{\vec{\boldsymbol{r}}_{\text{I}} m_I + \vec{\boldsymbol{r}}_{\text{II}} m_{II} + \vec{\boldsymbol{r}}_{\text{III}} m_{III} + \cdots}{m_I + m_{II} + m_{III} + \cdots}.
\end{aligned}
$$

The general idea of the calculations above is that center-of-mass calculations are basically big sums (addition), and addition is 'associative.'

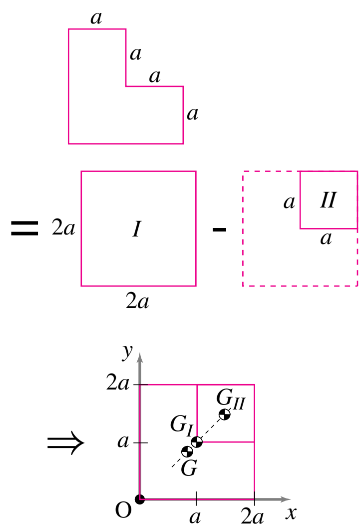


Figure 2.24: Another way of looking at the ‘L’ shaped object is as a square minus a smaller square in its upper right-hand corner.

③We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.

That’s easy, the billions of gravity forces on an objects microscopic constituents add up to mg pointed down. What about the net moment of the gravity forces? The answer turns out to be simple. The top line of the calculation below poses the question, the last line gives the lucky answer.③

$$
\begin{aligned}
\vec{\boldsymbol{M}}_{\mathrm{C}} &= \int \vec{\boldsymbol{r}} \times d\vec{\boldsymbol{F}} && \text{The net moment with respect to C.}\\
&= \int \vec{\boldsymbol{r}}_{/\mathrm{C}} \times (-g\hat{\boldsymbol{k}}\, dm) && \text{A force bit is gravity acting on a mass bit.}\\
&= \left(\int \vec{\boldsymbol{r}}_{/\mathrm{C}} dm\right) \times (-g\hat{\boldsymbol{k}}) && \text{Distributive law (g \& } \hat{\boldsymbol{k}} \text{ are constants).}\\
&= (\vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} m) \times (-g\hat{\boldsymbol{k}}) && \text{Definition of center-of-mass.}\\
&= \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} \times (-mg\hat{\boldsymbol{k}}) && \text{Re-arranging terms.}\\
&= \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{C}} \times \vec{\boldsymbol{F}}_{\mathrm{net}} && \text{Express in terms of net gravity force.}
\end{aligned}
$$

Thus the net moment is the same as for the total gravity force acting at the center-of-mass.

> The near-earth gravity forces acting on a system are *equivalent* to a single force, mg, acting at the system’s center-of-mass.

For the purposes of calculating the net force and moment from near-earth (constant g) gravity forces, a system can be replaced by a point mass at the center of gravity. The words ‘center-of-mass’ and ‘center of gravity’ both describe the same point in space.

Although the result we have just found seems plain enough, here are two things to ponder about gravity when viewed as an inverse square law (and thus not constant like we have assumed) that may make the result above seem less obvious.

- The net gravity force on a sphere is indeed equivalent to the force of a point mass at the center of the sphere. It took the genius Isaac Newton 3 years to deduce this result and the reasoning involved is too advanced for this book.
- The net gravity force on systems that are not spheres is generally *not* equivalent to a force acting at the center-of-mass (this is important for the understanding of tides as well as the orientational stability of satellites).

How to find the center-of-mass of a complex system

You find the center-of-mass of a complex system by knowing the masses and mass centers of its components. You find each of these centers of mass by

- Treating it as a point mass, or

- Treating it as a symmetric body and locating the center-of-mass in the middle, or
- Using integration, or
- Using the result of an experiment (which we will discuss in statics), or
- Treating the component as a complex system itself and applying this very recipe.

The recipe is just an application of the basic definition of center-of-mass (eqn. 2.4) but with our accumulated wisdom that the locations and masses in that sum can be the centers of mass and total masses of complex subsystems.

One way to arrange one's data is in a table or spreadsheet, like below.

Center of mass spreadsheet

Subsys#	1	2	3	4	5	6	7
Subsys 1	x_1	y_1	z_1	m_1	$m_1 x_1$	$m_1 y_1$	$m_1 z_1$
Subsys 2	x_2	y_2	z_2	m_2	$m_2 x_2$	$m_2 y_2$	$m_2 z_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Subsys N	x_N	y_N	z_N	m_N	$m_N x_N$	$m_N y_N$	$m_N z_N$
Row $N+1$ =sums				$m_{\text{tot}} = \sum m_i$	$\sum m_i x_i$	$\sum m_i y_i$	$\sum m_i z_i$
divide row $N+1$ by m_{tot} $\Rightarrow$ Result					$x_{\text{cm}} = \dfrac{\sum m_i x_i}{m_{\text{tot}}}$	$y_{\text{cm}} = \dfrac{\sum m_i y_i}{m_{\text{tot}}}$	$z_{\text{cm}} = \dfrac{\sum m_i z_i}{m_{\text{tot}}}$

1. The first four columns are the basic data. They are the x, y, and z coordinates of the subsystem center-of-mass locations (relative to some clear reference point), and the masses of the subsystems, one row for each of the N subsystems.
2. One next calculates three new columns (5,6, and 7) which come from each coordinate multiplied by its mass. For example the entry in the 6th row and 7th column is the z component of the 6th subsystem's center-of-mass multiplied by the mass of the 6th subsystem.
3. Then one sums columns 4 through 7. The sum of column 4 is the total mass, the sums of columns 5 through 7 are the total mass-weighted positions.
4. Finally the result, the system center of mass coordinates, are found by dividing columns 5-7 of row N+1 by column 4 of row N+1.

Of course, there are multiple ways of systematically representing the data. The spreadsheet-like calculation above is just one organization scheme.

Summary of center-of-mass

All discussions in mechanics make frequent reference to the concept of center of mass

$$
\begin{aligned}
m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} &= \sum \vec{\boldsymbol{r}}_{\text{i}} m_i && \text{for discrete systems or systems of systems} \\
&= \int \vec{\boldsymbol{r}}\, dm && \text{for continuous systems}
\end{aligned}
$$

where

$$
\begin{aligned}
m_{\text{tot}} &= \sum \mathrm{m}_i && \text{for discrete systems or systems of systems} \\
&= \int dm && \text{for continuous systems.}
\end{aligned}
$$

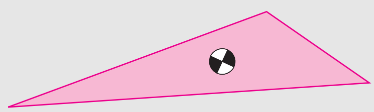
Who cares about the center of mass? We showed that the gravity moment is calculated correctly by applying the net gravity force at the center-of-mass. These other facts about center-of-mass will be needed for dynamics.

For non-point-mass systems, the expressions for linear momentum, angular momentum, and kinetic energy are all simplified by using the center-of-mass.

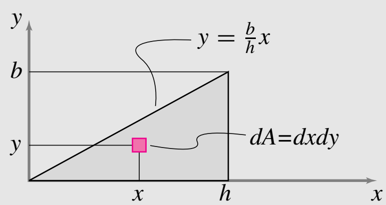
Simple center-of-mass calculations also can serve as a check of a more complicated analysis. For example, after a computer simulation of a system with many moving parts is complete, one way of checking the calculation is to see if the whole system's center of mass moves as would be expected by applying the net external force to the system.

2.6 The COM of a uniform triangle is $h/3$ up from the base

The theory here is for the curious. One can just remember the result: The center-of-mass of a 2D uniform triangular region is the centroid of the area.



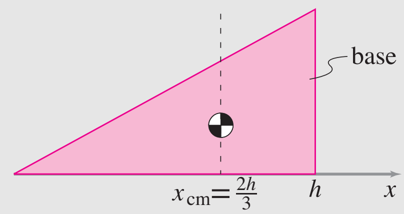
First, we consider a right triangle with perpendicular sides b and h



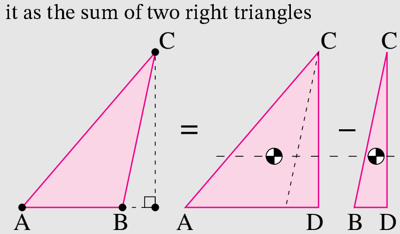
and find the x coordinate of the centroid as

$$
\begin{aligned}
x_{\mathrm{cm}} A &= \int x\, dA \\
&= \int_0^h \left[\int_0^{\frac{b}{h}x} x\, dy \right] dx = \int_0^h [xy] \Big|_{y=0}^{y=\frac{b}{h}x} dx \\
x_{\mathrm{cm}} \left(\frac{bh}{2} \right) &= \int_0^h x \left(\frac{b}{h} x \right) dx = \frac{b}{h} \frac{x^3}{3} \Big|_0^h = \frac{bh^2}{3}
\end{aligned}
$$

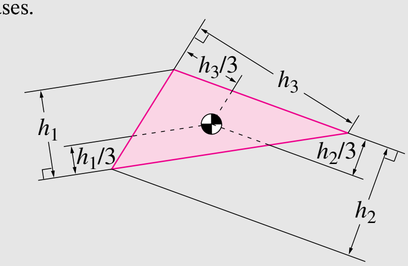
$\Rightarrow \quad x_{\mathrm{cm}} = \frac{2h}{3}$, a third of the way to the left of the vertical base on the right. By similar reasoning, but in the y direction, the centroid is a third of the way up from the base.



The center-of-mass of an arbitrary triangle can be found by treating it as the sum of two right triangles



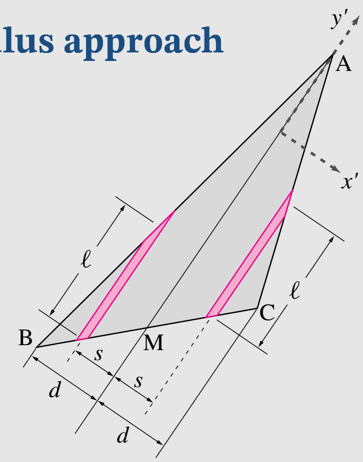
so the centroid is a third of the way up from the base of any triangle. Finally, the result holds for all three bases. Summarizing, the centroid of a triangle is at the point one third up from each of the bases.



(continued...)

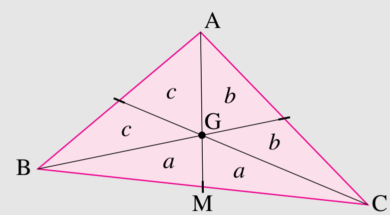
2.6 The COM of a uniform triangle is $h/3$ up from the base (continued)

Non-calculus approach



M is the midpoint of the line segment BC. Divide triangle ABC into equal width strips that are parallel to AM. Group these strips into pairs, each a distance s from AM. Because M is the midpoint of BC, by proportions each of these strips has the same length ℓ. What is the distance of the center-of-mass from the line AM? Because the strips are of equal area and equal distance from AM but on opposite sides, contributions to the sum come in canceling pairs. So the centroid is on AM. Likewise for all three sides. So the triangle's centroid is at the intersection of the three side bisectors.

Why do the three side bisectors intersect a third of the way up each base? Look at the 6 triangles formed by the side bisectors.



The two triangles marked a and a have the same area (call it a) because they have the same height and bases of equal length (BM and CM). Likewise for the other side bisectors, so that the pairs marked b have equal area as do the pairs c. Triangle ABM has the same base and height and thus the same area as the triangle ACM. So $a + b + b = a + c + c$. Thus $b = c$ and similarly $a = b$: all six little triangles have equal area. Thus the area of ABC is 3 times the area of GBC. Because ABC and GBC share the base BC, ABC must have 3 times the height as GBC, and point G is thus a third of the way up from the base.

Where is the middle of a triangle?

We just showed that the centroid of a triangle is at the point that is at the intersection of: the three side bisectors; the three area bisectors (which are the side bisectors); and the three lines one third of the way up from the three bases.

And, if the triangle had three equal point masses on its vertices, and no mass anywhere else, the center of mass also lands on that same place. Thus the 'middle' of a triangle seems pretty well defined. *Yet*, there is ambiguity. If the triangle were made of bars along each edge, each with equal cross sections, the center-of-mass would be in a different location (except for equilateral triangles). Also, the three angle bisectors of a triangle do not intersect at the centroid. Unless we *define* the middle to mean centroid, the "middle" of a triangle is not well defined.

SAMPLE 2.7 **Center of mass in 1-D:** Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center-of-mass of the assembly.

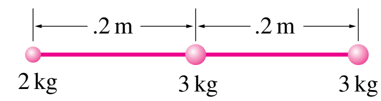


Figure 2.25

Solution Let us select the first mass, $m_1 = 2\,\text{kg}$, to be at the origin of our co-ordinate system with the x-axis along the rod. Since all the three masses lie on the x-axis, the center-of-mass will also lie on this axis. Let the center-of-mass be located at x_{cm} on the x-axis. Then,

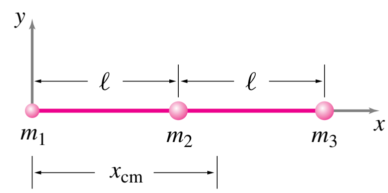


Figure 2.26

$$
\begin{aligned}
m_{\text{tot}}x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1x_1 + m_2x_2 + m_3x_3 \\
&= m_1(0) + m_2(\ell) + m_3(2\ell) \\
\Rightarrow \quad x_{cm} &= \frac{m_2\ell + m_3 2\ell}{m_1 + m_2 + m_3} \\
&= \frac{3\,\cancel{k}g \cdot 0.2\,\text{m} + 3\,\cancel{k}g \cdot 0.4\,\text{m}}{(2+3+3)\,\cancel{k}g} \\
&= \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}.
\end{aligned}
$$

$$\boxed{x_{cm} = 0.225\,\text{m}}$$

Alternatively, we could find the center-of-mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center-of-mass of the two equal masses) and then calculate the value of x_{cm} for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. 2.27):

$$x_{cm} = \frac{6\,\text{kg} \cdot 0.3\,\text{m}}{8\,\text{kg}} = \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}.$$

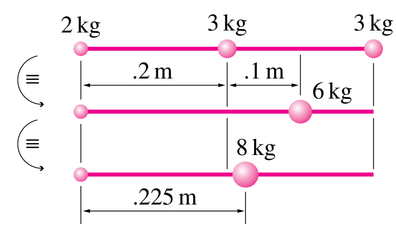


Figure 2.27

SAMPLE 2.8 **Center of mass in 2-D:** Two particles of mass $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$ are located at coordinates (1m, 2m) and (-2m, 5m), respectively, in the xy-plane. Find the location of their center-of-mass.

Solution Let $\vec{\boldsymbol{r}}_{\text{cm}}$ be the position vector of the center-of-mass. Then,

$$
\begin{aligned}
m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} &= m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 \\
\Rightarrow \quad \vec{\boldsymbol{r}}_{\text{cm}} &= \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2}{m_{\text{tot}}} = \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2}{m_1 + m_2} \\
&= \frac{1\,\cancel{k}g(1\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}) + 2\,\cancel{k}g(-2\,\text{m}\hat{\boldsymbol{\imath}} + 5\,\text{m}\hat{\boldsymbol{\jmath}})}{3\,\cancel{k}g} \\
&= \frac{(1\,\text{m} - 4\,\text{m})\hat{\boldsymbol{\imath}} + (2\,\text{m} + 10\,\text{m})\hat{\boldsymbol{\jmath}}}{3} = -1\,\text{m}\hat{\boldsymbol{\imath}} + 4\,\text{m}\hat{\boldsymbol{\jmath}}.
\end{aligned}
$$

Thus the center-of-mass is located at the coordinates(-1m, 4m).

$$\boxed{(x_{cm}, y_{cm}) = (-1\,\text{m}, 4\,\text{m})}$$

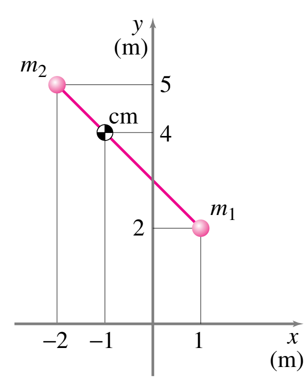


Figure 2.28

Geometrically, this is just a 1-D problem like the previous sample. The center-of-mass has to be located on the straight line joining the two masses. Since the center-of-mass is a point about which the distribution of mass is *balanced*, it is easy to see (see Fig. 2.28) that the center-of-mass must lie one-third way from m_2 on the line joining the two masses so that $2\,\text{kg} \cdot (d/3) = 1\,\text{kg} \cdot (2d/3)$.

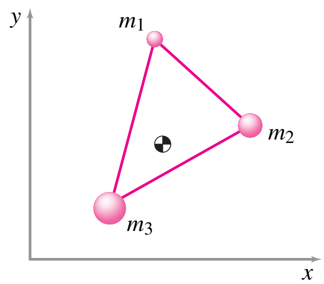


Figure 2.29

SAMPLE 2.9 **Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$, connected rigidly by massless rods. At the instant of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively. Find the coordinates of the center-of-mass of the structure.

Solution Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center-of-mass, and second, using vector equations for the position of the center-of-mass.

1. **Scalar calculations:** Let (x_{cm}, y_{cm}) be the coordinates of the mass-center. Then from the definition of mass-center,

$$\begin{aligned}
x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\
&= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\
&= \frac{7.5\,\cancel{\text{k}}\text{g}\cdot\text{m}}{6\,\cancel{\text{k}}\text{g}} = 1.25\,\text{m}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_{cm} &= \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \\
&= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\
&= \frac{8.5\,\cancel{\text{k}}\text{g}\cdot\text{m}}{6\,\cancel{\text{k}}\text{g}} = 1.42\,\text{m}.
\end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

2. **Vector calculations:** Let $\vec{\boldsymbol{r}}_{\text{cm}}$ be the position vector of the mass-center. Then,

$$\begin{aligned}
m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} &= \sum_{i=1}^{3} m_i\vec{\boldsymbol{r}}_i = m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + m_3\vec{\boldsymbol{r}}_3 \\
\Rightarrow \quad \vec{\boldsymbol{r}}_{\text{cm}} &= \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2 + m_3\vec{\boldsymbol{r}}_3}{m_1 + m_2 + m_3}
\end{aligned}$$

Substituting the values of m_1, m_2, and m_3, and $\vec{\boldsymbol{r}}_1 = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{r}}_2 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{r}}_3 = 0.75\,\text{m}\hat{\boldsymbol{\imath}} + 0.5\,\text{m}\hat{\boldsymbol{\jmath}}$, we get,

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{1\,\text{kg}\cdot(1.25\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})\,\text{m} + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{m} + 3\,\text{kg}\cdot(0.75\hat{\boldsymbol{\imath}} + 0.5\hat{\boldsymbol{\jmath}})\,\text{m}}{(1+2+3)\,\text{kg}} \\
&= \frac{(7.5\hat{\boldsymbol{\imath}} + 8.5\hat{\boldsymbol{\jmath}})\,\cancel{\text{k}}\text{g}\cdot\text{m}}{6\,\cancel{\text{k}}\text{g}} \\
&= 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}}
\end{aligned}$$

which, of course, gives the same location of the mass-center as above.

$$\boxed{\vec{\boldsymbol{r}}_{\text{cm}} = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}}}$$

SAMPLE 2.10 **Center of mass of a bent bar:** A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center-of-mass of the bar.

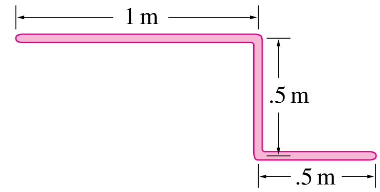


Figure 2.30

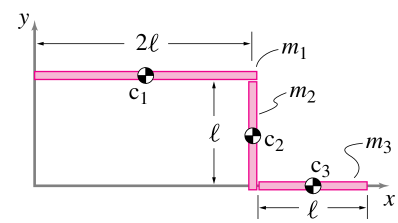


Figure 2.31

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center-of-mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let $m_2 = m_3 = m$, then $m_1 = 2m$; and $m_1 + m_2 + m_3 = 4m = 4\,\text{kg}$ which gives $m = 1\,\text{kg}$. Now, from Fig. 2.31,

$$\begin{aligned}
\vec{r}_1 &= \ell\hat{\imath} + \ell\hat{\jmath} \\
\vec{r}_2 &= 2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \\
\vec{r}_3 &= (2\ell + \frac{\ell}{2})\hat{\imath} = \frac{5\ell}{2}\hat{\imath}
\end{aligned}$$

So,

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_{\text{tot}}} \\
&= \frac{2m(\ell\hat{\imath} + \ell\hat{\jmath}) + m(2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath}) + m(\frac{5\ell}{2}\hat{\imath})}{4m} \\
&= \frac{\cancel{m}\ell(2\hat{\imath} + 2\hat{\jmath} + 2\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{5}{2}\hat{\imath})}{4\cancel{m}} \\
&= \frac{\ell}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= \frac{0.5\,\text{m}}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}}$$

Geometrically, we could find the center-of-mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. 2.32.

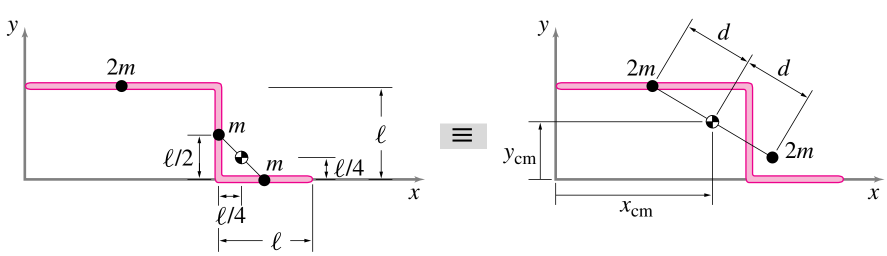


Figure 2.32:

The center-of-mass of m_2 and m_3 (each of mass m) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass $2m$ at their mass-center. Next, we connect this mass-center and m_1 with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center-of-mass of the entire bar.

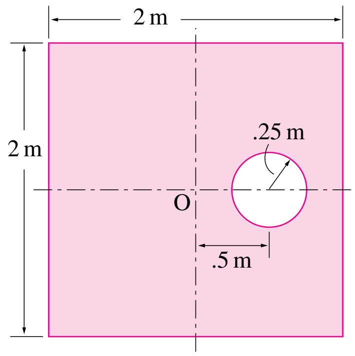


Figure 2.33

SAMPLE 2.11 **Shift of mass-center due to cut-outs:** A $2\,\text{m} \times 2\,\text{m}$ uniform square plate has mass $m = 4\,\text{kg}$. A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center-of-mass of the plate.

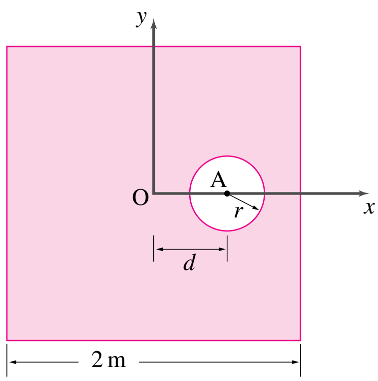


Figure 2.34

Solution Let us use an xy-coordinate system with its origin at the geometric center of the plate and the x-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the x-axis, the new center-of-mass must lie somewhere on the x-axis. Thus, we only need to find x_{cm} (since $y_{cm} = 0$). Let m_1 be the mass of the plate with the hole, and m_2 be the mass of the circular cut-out. Clearly, $m_1 + m_2 = m = 4\,\text{kg}$. The center-of-mass of the circular cut-out is at A, the center of the circle. The center-of-mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A. \end{aligned}$$

Now, since the plate is uniform, the masses m_1 and m_2 are proportional to the surface areas of the geometric objects they represent, *i.e.*,

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi}.$$

Therefore,

$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \qquad (2.5) \\ &= -\frac{\pi}{\left(\frac{2\,\text{m}}{.25\,\text{m}}\right)^2 - \pi} \cdot 0.5\,\text{m} \\ &= -25.81 \times 10^{-3}\,\text{m} = -25.81\,\text{mm} \end{aligned}$$

Thus the center-of-mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

$$\boxed{x_{cm} = -25.81\,\text{mm}}$$

Comments: The advantage of finding the expression for x_{cm} in terms of r and ℓ as in eqn. (2.5) is that you can easily find the center-of-mass of any size circular cut-out located at any distance d on the x-axis. This is useful in design where you like to select the size or location of the cut-out to have the center-of-mass at a particular location.

SAMPLE 2.12 Center of mass of two objects: A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center-of-mass of the combined system.

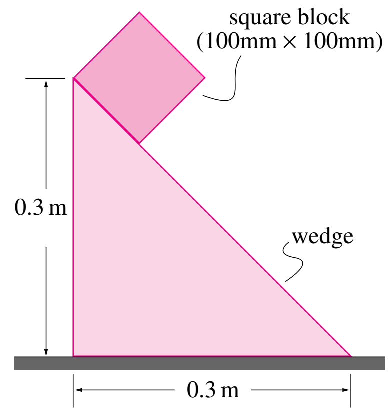


Figure 2.35

Solution The center-of-mass of the triangular wedge is located at $h/3$ above the base and $\ell/3$ to the right of the vertical side. Let m_1 be the mass of the wedge and $\vec{r}_1$ be the position vector of its mass-center. Then, referring to Fig. 2.36,

$$\vec{r}_1 = \frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}.$$

The center-of-mass of the square block is located at its geometric center C_2. From geometry, we can see that the line AE that passes through C_2 is horizontal since $\angle OAB = 45°$ ($h = \ell = 0.3\,\text{m}$) and $\angle DAE = 45°$. Therefore, the coordinates of C_2 are ($d/\sqrt{2}, h$). Let m_2 and $\vec{r}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$\vec{r}_2 = \frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath}.$$

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2\,\text{kg}$, we find the center-of-mass of the combined system:

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{(m_1 + m_2)} \\
&= \frac{3m(\frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}) + m(\frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath})}{3m + m} \\
&= \frac{\cancel{m}[(\ell + \frac{d}{\sqrt{2}})\hat{\imath} + 2h\hat{\jmath}]}{4\cancel{m}} \\
&= \frac{1}{4}(\frac{d}{\sqrt{2}} + \ell)\hat{\imath} + \frac{h}{2}\hat{\jmath} \\
&= \frac{1}{4}(\frac{0.1\,\text{m}}{\sqrt{2}} + 0.3\,\text{m})\hat{\imath} + \frac{0.3\,\text{m}}{2}\hat{\jmath} \\
&= 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}}$$

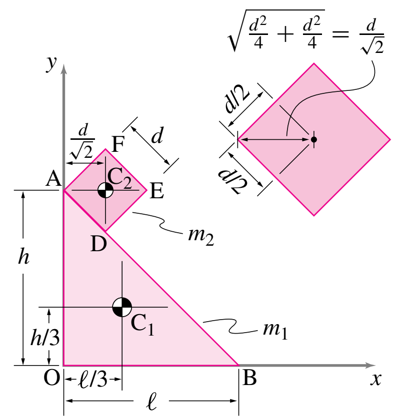


Figure 2.36

Thus, the center-of-mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1(0.1\,\text{m}, 0.1\,\text{m})$. This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at C_1 and m at C_2, connected the two points by a straight line, and located the center-of-mass C on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1(0.1\,\text{m}, 0.1\,\text{m})$ to $C(.093\,\text{m}, 0.15\,\text{m})$ is one third the distance from C to $C_2(.071\,\text{m}, 0.3\,\text{m})$.

2.2 Center of mass and center of gravity

2.2.1 An otherwise massless structure is made of four point masses, m, $2m$, $3m$ and $4m$, located at coordinates (0, 1 m), (1 m, 1 m), (1 m, −1 m), and (0, −1 m), respectively. Locate the center of mass of the structure. *

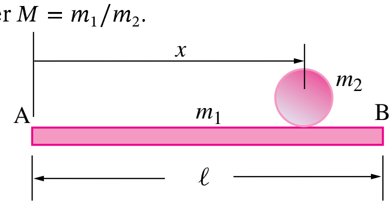
2.2.2 3-D: The following data is given for a structural system modeled with five point masses in 3-D-space:

mass	**coordinates (in m)**
0.4 kg	(1,0,0)
0.4 kg	(1,1,0)
0.4 kg	(2,1,0)
0.4 kg	(2,0,0)
1.0 kg	(1.5,1.5,3)

Locate the center of mass of the system.

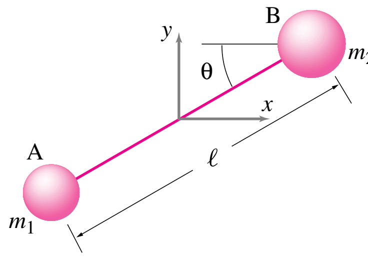
2.2.3 Write a computer program to find the center of mass of a point-mass-system. The input to the program should be a table (or matrix) containing individual masses and their coordinates. (It is possible to write a single program for both 2-D and 3-D cases, write separate programs for the two cases if that is easier for you.) Check your program on Problems 2.2.1 and 2.2.2.

2.2.4 A cylinder of mass m_2 and radius R rolls on a flat circular plate of mass m_1 and length ℓ. Let the position of the cylinder from the left edge of the plate be x. Find the horizontal position of the center of mass of the system as a function of x and a non-dimensional mass parameter $M = m_1/m_2$.



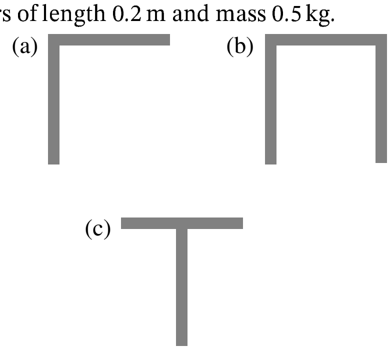
Problem 2.2.4

2.2.5 Two masses m_1 and m_2 are connected by a massless rod AB of length ℓ. In the position shown, the rod is inclined to the horizontal axis at an angle θ. Find the position of the center of mass of the system as a function of angle θ and the other given variables. Check if your answer makes sense by setting appropriate values for m_1 and m_2.



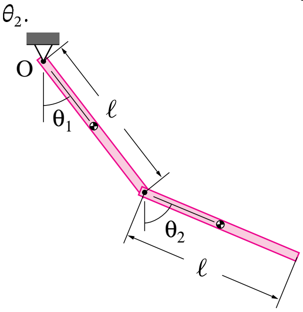
Problem 2.2.5

2.2.6 Find the center of mass of the following composite bars. Each composite shape is made of two or more uniform bars of length 0.2 m and mass 0.5 kg.



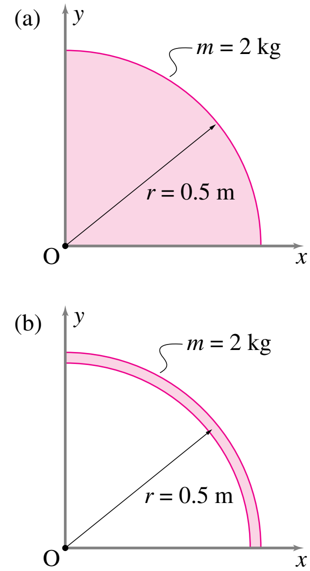
Problem 2.2.6

2.2.7 A double pendulum consists of two uniform bars of length ℓ and mass m each. The pendulum hangs in the vertical plane from a hinge at point O. Taking O as the origin of a xy coordinate system, find the location of the center of gravity of the pendulum as a function of angles θ_1 and θ_2.



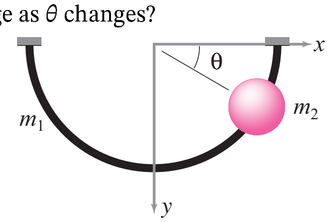
Problem 2.2.7

2.2.8 Find the center of mass of the following two objects [Hint: set up and evaluate the needed integrals.]



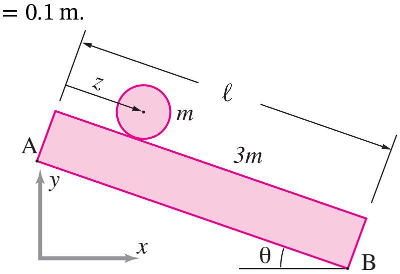
Problem 2.2.8

2.2.9 A semicircular ring of radius R = 1 m and mass m_1 = 0.1 kg rests in the vertical plane. A bead of mass m_2 = 0.25 kg slides on the ring. Find the position of the center of mass of the ring-bead-system at an instant when $\theta = 30^\circ$. How does the center of mass position change as θ changes?



Problem 2.2.9

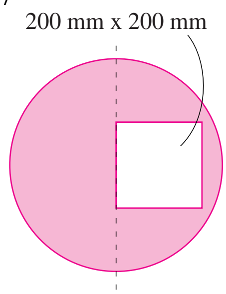
2.2.10 A uniform circular disk of mass m and radius R rolls on an inclined rectangular plate of mass $3m$ and dimensions $2R \times \ell$. Point A is on the y axis and point B is on the x axis. Find the coordinates of the center of mass of the system for m = 1 kg, ℓ = 1 m, z = 0.2 m, and R = 0.1 m.



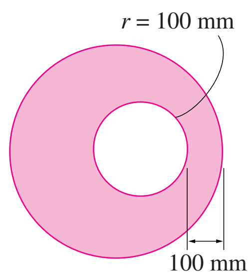
Problem 2.2.10

2.2.11 Find the center of mass of the following plates obtained from cutting out a section from a uniform circular plate of mass 1 kg (prior to removing the cutout) and radius 1/4 m.

(a)



(b)



Problem 2.2.11

2.2.1 An otherwise massless structure is made of four point masses, m, $2m$, $3m$ and $4m$, located at coordinates (0, 1 m), (1 m, 1 m), (1 m, −1 m), and (0, −1 m), respectively. Locate the center of mass of the structure.

Answer:
(0.5 m, −0.4 m)

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*