2.1 Equivalent force sets

The definition of the word *equivalent*①, when applied to force sets in mechanics, is:

> Two force systems are said to be *equivalent* if they have the same sum (the same *resultant*, the same net force) and the same resultant moment (net moment) about any one point C.

So that we don't get confused about the system of interest (a collection of material), and the force system (a collection of forces), we will sometimes call the force system a force set.

Why use equivalent force sets?

First, the replacement of one set with an equivalent set is often used to help simplify or solve mechanics problems.

Second, the concept of equivalent force sets allows us to define a *couple*, a concept we will use throughout the book.

And finally, most often one does not want to know the complete details of all the forces acting on a system. When you think of the force of the ground on your bare foot you do not think of the thousands of little forces at each micro-asperity or the billions and billions of molecular interactions between the wood (say) and your skin. Instead you think of some kind of equivalent force.

In what way are two 'equivalent' force systems (or sets) actually equivalent? Well, because all that the equations of mechanics know about forces is their net force and net moment, you have a criterion. You replace the actual force set with a simpler force set, possibly just a single well-placed force, that has the same total force and same total moment with respect to a reference point C②

We have already discussed two important cases of equivalent force sets.

1. On page 50 we stated the mechanics assumption that a set of forces applied at one point is equivalent to a single resultant force, their sum, applied at that point. Thus when doing a mechanics analysis you can replace a collection of forces at a point with their sum. If you think of your whole foot as a 'point', this justifies the replacement of the billions of little atomic ground contact forces with a single force.
2. On page 97 we discovered that a force applied at a different point is equivalent to the same force applied at a point displaced in the direction of the force. You can thus harmlessly slide the point of force application along the line of the force and have its mechanically equivalent effect: One force applied at a point is equivalent to

① Other phrases used to describe the equivalent force sets include:
* *statically equivalent*,
* *mechanically equivalent*, and
* *equipollent*.

②A special point C? No, any point will do. As we will show, equivalent for C implies equivalent for D, E and G *etc.*. (see box 2.2 on page 5 the same force at a different point that is displaced in the direction of the force.

More generally, we can compare two sets of forces. The first set consists of

$$\vec{\boldsymbol{F}}^{(1)}_1, \vec{\boldsymbol{F}}^{(1)}_2, \vec{\boldsymbol{F}}^{(1)}_3, \textit{etc.}$$

applied at positions

$$\vec{\boldsymbol{r}}^{(1)}_{1/C}, \vec{\boldsymbol{r}}^{(1)}_{2/C}, \vec{\boldsymbol{r}}^{(1)}_{3/C}, \textit{etc.}$$

In short hand, these forces are

$$\vec{\boldsymbol{F}}^{(1)}_i \text{ applied at positions } \vec{\boldsymbol{r}}^{(1)}_{i/C},$$

where each value of i describes a different force ($i = 7$ refers to the seventh force in the set). The second set of forces consists of

$$\vec{\boldsymbol{F}}^{(2)}_j \text{ applied at positions } \vec{\boldsymbol{r}}^{(2)}_{j/C},$$

where each value of j describes a different force in the second set.

Now, we compare the net (resultant) force and net moment of the two sets.

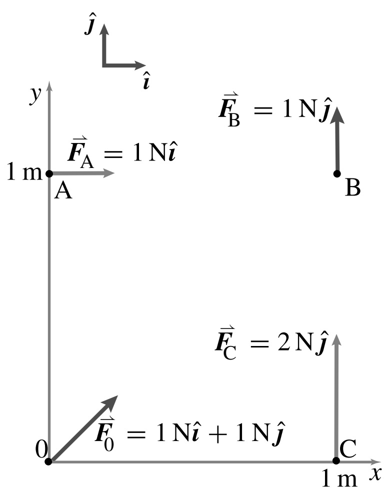

Figure 2.1: The force set $\vec{\boldsymbol{F}}_A, \vec{\boldsymbol{F}}_C$ is equivalent to the force set $\vec{\boldsymbol{F}}_0, \vec{\boldsymbol{F}}_B$.

If,

$$\vec{\boldsymbol{F}}^{(1)}_{\text{tot}} = \vec{\boldsymbol{F}}^{(2)}_{\text{tot}} \quad \text{and} \quad \vec{\boldsymbol{M}}^{(1)}_{C} = \vec{\boldsymbol{M}}^{(2)}_{C}, \tag{2.1}$$

then the two sets are *equivalent*.

Here we have defined the net forces and net moments by

$$\vec{\boldsymbol{F}}^{(1)}_{\text{tot}} = \sum_{\text{all forces } i} \vec{\boldsymbol{F}}^{(1)}_i, \qquad \vec{\boldsymbol{M}}^{(1)}_{C} = \sum_{\text{all forces } i} \vec{\boldsymbol{r}}^{(1)}_{i/C} \times \vec{\boldsymbol{F}}^{(1)}_i, \tag{2.2}$$

$$\vec{\boldsymbol{F}}^{(2)}_{\text{tot}} = \sum_{\text{all forces } j} \vec{\boldsymbol{F}}^{(2)}_j, \quad \text{and} \quad \vec{\boldsymbol{M}}^{(2)}_{C} = \sum_{\text{all forces } j} \vec{\boldsymbol{r}}^{(2)}_{j/C} \times \vec{\boldsymbol{F}}^{(2)}_j. \tag{2.3}$$

If you find the $\sum$ (sum) symbol intimidating see box 2.1 on page 4.

Example:

Consider force set (1) with forces $\vec{\boldsymbol{F}}_A$ and $\vec{\boldsymbol{F}}_C$ and force set (2) with forces $\vec{\boldsymbol{F}}_0$ and $\vec{\boldsymbol{F}}_B$ as shown in fig. 2.1. Are the sets equivalent? First check the sum of forces.

$$\begin{aligned}
\vec{\boldsymbol{F}}^{(1)}_{\text{tot}} &\stackrel{?}{=} \vec{\boldsymbol{F}}^{(2)}_{\text{tot}} \\
\sum \vec{\boldsymbol{F}}^{(1)}_i &\stackrel{?}{=} \sum \vec{\boldsymbol{F}}^{(2)}_j \\
\vec{\boldsymbol{F}}_A + \vec{\boldsymbol{F}}_C &\stackrel{?}{=} \vec{\boldsymbol{F}}_0 + \vec{\boldsymbol{F}}_B \\
1\,\text{N}\hat{\boldsymbol{\imath}} + 2\,\text{N}\hat{\boldsymbol{\jmath}} &\stackrel{\surd}{=} (1\,\text{N}\hat{\boldsymbol{\imath}} + 1\,\text{N}\hat{\boldsymbol{\jmath}}) + 1\,\text{N}\hat{\boldsymbol{\jmath}}
\end{aligned}$$

Then check the sum of moments about C.

$$
\begin{aligned}
\vec{\boldsymbol{M}}_{\mathrm{C}}^{(1)} &\stackrel{?}{=} \vec{\boldsymbol{M}}_{\mathrm{C}}^{(2)} \\
\sum \vec{\boldsymbol{r}}_{i/C}^{(1)} \times \vec{\boldsymbol{F}}_{i}^{(1)} &\stackrel{?}{=} \sum \vec{\boldsymbol{r}}_{j/C}^{(2)} \times \vec{\boldsymbol{F}}_{j}^{(2)} \\
\vec{\boldsymbol{r}}_{\mathrm{A/C}} \times \vec{\boldsymbol{F}}_{\mathrm{A}} + \vec{\boldsymbol{r}}_{\mathrm{C/C}} \times \vec{\boldsymbol{F}}_{\mathrm{C}} &\stackrel{?}{=} \vec{\boldsymbol{r}}_{0/\mathrm{C}} \times \vec{\boldsymbol{F}}_{0} + \vec{\boldsymbol{r}}_{\mathrm{B/C}} \times \vec{\boldsymbol{F}}_{\mathrm{B}} \\
(-1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 1\,\mathrm{m}\,\hat{\boldsymbol{\jmath}}) \times 1\,\mathrm{N}\hat{\boldsymbol{\imath}} + \vec{\boldsymbol{0}} \times 2\,\mathrm{N}\,\hat{\boldsymbol{\jmath}} &\stackrel{?}{=} (-1\,\mathrm{m}\hat{\boldsymbol{\imath}}) \times (1\,\mathrm{N}\hat{\boldsymbol{\imath}} + 1\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}) + 1\,\mathrm{m}\,\hat{\boldsymbol{\jmath}} \times 1\,\mathrm{N}\,\hat{\boldsymbol{\jmath}} \\
-1\,\mathrm{m}\,\mathrm{N}\hat{\boldsymbol{k}} &\stackrel{\surd}{=} -1\,\mathrm{m}\,\mathrm{N}\hat{\boldsymbol{k}}
\end{aligned}
$$

So, the two force sets are equivalent.

What is so special about the point C in the example above? Nothing.

If two force sets are equivalent with respect to some point C, they are equivalent with respect to *any* point, as justified in box 2.2 on page 5.

For example, both of the force sets in the example above have the same moment of $2\,\mathrm{N\,m}\hat{\boldsymbol{k}}$ about the point A. See box 2.2 for the proof of the general case.

Example: Frictionless wheel bearing
If the contact of an axle with a bearing housing is perfectly frictionless then each of the contact forces has no moment about the center of the wheel (fig. 2.2). Thus the whole force set is equivalent to a single force at the center of the wheel.

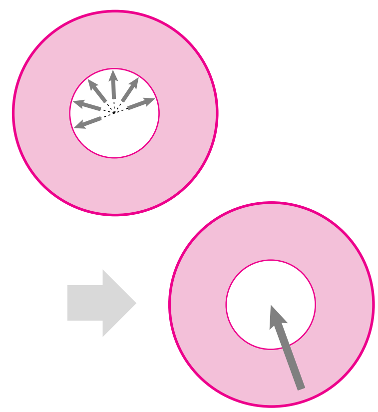

Figure 2.2: Frictionless wheel bearing. All the bearing forces are *equivalent* to a single force acting at the center of the wheel.

Couples

Consider a pair of equal and opposite forces that are not collinear. Such a pair is called a *couple*.The net moment caused by a couple is the size of the force times the perpendicular distance between the two lines of action and doesn't depend on the reference point. In fact, any force system that has $\vec{\boldsymbol{F}}_{\mathrm{tot}} = \vec{\boldsymbol{0}}$ causes the same moment about all different reference points (as shown at the end of box 2.2). So, in modern usage, any force system with any number of forces and with $\vec{\boldsymbol{F}}_{\mathrm{tot}} = \vec{\boldsymbol{0}}$ is called a couple. A couple is described by its net moment.③

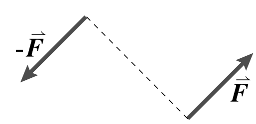

Figure 2.3: One couple. The forces add to zero. The net moment they cause does not.

③People who have been in difficult long-term relationships don't need a mechanics textbook to learn that a couple is a pair of equal and opposite forces that push each other around.

A *couple* is any force system that has a total force of $\vec{\boldsymbol{0}}$. It is described by the net moment $\vec{\boldsymbol{M}}$ that it causes.

We then think of the couple $\vec{\boldsymbol{M}}$ as representing an equivalent force system that contributes $\vec{\boldsymbol{0}}$ to the net force and $\vec{\boldsymbol{M}}$ to the net moment with respect to every reference point.

The concept of a couple (also called an applied moment or an applied torque) is especially useful for representing the net effect of a complicated collection of forces that causes some turning. The complicated set of electromagnetic forces turning a motor shaft can be replaced by a single couple.

Every set of forces is equivalent to a force and a couple

Given any point C, we can calculate the net moment of a system of forces relative to C. We then can replace the sum of forces with a single force at C and the net moment with a couple at C and we have an equivalent force system.

A force system is equivalent to a force $\vec{F} = \vec{F}_{\text{tot}}$ acting at C and a couple M equal to the net moment of the forces about C, *i.e.*, $\vec{M} = \vec{M}_{\text{C}}$.

If instead we want a force system at D, we could recalculate the net moment about D, or just use the translation formula (see box 2.2 on page 5).

$$\begin{aligned} \vec{F}_{\text{tot}} &= \vec{F}_{\text{tot}}, \quad \text{and} \\ \vec{M}_{\text{D}} &= \vec{M}_{\text{C}} + \vec{r}_{\text{C/D}} \times \vec{F}_{\text{tot}}. \end{aligned}$$

2.1 $\sum$ means add

In mechanics, we often need to add up lots of things: all the forces on a body, all the moments they cause, all the mass of a system, *etc.* One notation for adding up all 14 forces on some body is

$$\begin{aligned} \vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6 + \vec{F}_7 \\ &\quad + \vec{F}_8 + \vec{F}_9 + \vec{F}_{10} + \vec{F}_{11} + \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}. \end{aligned}$$

which is a bit long, so we might abbreviate it as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_{14}.$$

But this is definition by pattern recognition. A more explicit statement would be

$$\vec{F}_{\text{net}} = \text{The sum of all 14 forces} \vec{F}_{\text{i}} \text{ where } i = 1 \dots 14$$

which is too space consuming. This kind of summing is so important that mathematicians use up a whole letter of the greek alphabet as a short hand for 'the sum of all'. They use the capital greek 'S' (for Sum) called sigma which looks like this:

$$\sum.$$

When you read $\sum$ aloud you don't say 'S' or'sigma' but rather 'the sum of.' The $\sum$ (sum) notation may remind you of infinite series, and convergence thereof. We will rarely be concerned with infinite sums in this book, and never with convergence issues. So, panic on those grounds is unjustified. We just want to write easily about adding things. For example, we use the $\sum$ (sum) to write the sum of 14 forces $\vec{F}_{\text{i}}$ explicitly and concisely as

$$\sum_{i=1}^{14} \vec{F}_{\text{i}},$$

and say 'the sum of F sub i where i goes from one to fourteen'. Sometimes we don't know, say, how many forces are being added. We just want to add all of them so we write (a little informally)

$$\sum \vec{F}_{\text{i}} \quad \text{meaning} \quad \vec{F}_1 + \vec{F}_2 + \textit{etc.},$$

where the subscript i lets us know that the forces are numbered.

Rather than panic when you see something mathy looking like $\sum_{i=1}^{14}$, just relax and think: oh, we want to add up a bunch of things all of which look like the next thing written. In general,

$$\sum (\text{thing})_i \quad \text{translates to} \quad (\text{thing})_1 + (\text{thing})_2 + (\text{thing})_3 + \textit{etc.}$$

no matter how intimidating the 'thing' is. In time, you can skip the translation into English and will enjoy the concise notation.

The total force $\vec{F}_{\text{net}}$ stays the same and the moment at D is the moment at C plus the moment caused by $\vec{F}_{\text{net}}$ acting at position C relative to D. The net effect of the forces of the ground on a tree, for example, is of a force and a couple acting on the base of the tree.

Equivalent does not mean equivalent for all purposes

We have perhaps oversimplified.

Imagine you stayed up late studying and overslept. Your roommate was not so diligent; woke up on time and went to wake you by gently shaking you. Having read this chapter so far and no further, and being rather literal, your roommate gets down on the floor and presses on the linoleum underneath your bed applying a force that is *equivalent* to pressing on you. Obviously this is not equivalent in the ordinary sense of the word. It isn't even equivalent in all of its mechanics effects. One force moves you even if you don't wake up, and the other doesn't.

2.2 Two force sets that are equivalent for one reference point are equivalent for all reference points.

Consider two sets of forces $\vec{F}_i^{(1)}$ and $\vec{F}_j^{(2)}$ with corresponding points of application $P_i^{(1)}$ and $P_j^{(2)}$ at positions relative to the origin of $\vec{r}_i^{(1)}$ and $\vec{r}_j^{(2)}$. To simplify the discussion let's define the net forces of the two systems as

$$\vec{F}_{\text{tot}}^{(1)} \equiv \sum \vec{F}_i^{(1)} \quad \text{and} \quad \vec{F}_{\text{tot}}^{(2)} \equiv \sum \vec{F}_j^{(2)},$$

and the net moments about the origin as

$$\vec{M}_0^{(1)} \equiv \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} \quad \text{and} \quad \vec{M}_0^{(2)} \equiv \sum \vec{r}_j^{(2)} \times \vec{F}_j^{(2)}.$$

Using point 0 as a reference, the statement that the two sets are equivalent is then $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$ and $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$. Now consider point C with position $\vec{r}_{\text{C}} = \vec{r}_{\text{C}/0} = -\vec{r}_{0/\text{C}}$. What is the net moment of force set (1) about point C?

$$\begin{aligned}
\vec{M}_{\text{C}}^{(1)} &\equiv \sum \vec{r}_{i/\text{C}}^{(1)} \times \vec{F}_i^{(1)} \\
&= \sum \left(\vec{r}_i^{(1)} - \vec{r}_{\text{C}}\right) \times \vec{F}_i^{(1)} \\
&= \sum \left(\vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_{\text{C}} \times \vec{F}_i^{(1)}\right) \\
&= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \sum \vec{r}_{\text{C}} \times \vec{F}_i^{(1)} \\
&= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_{\text{C}} \times \left(\sum \vec{F}_i^{(1)}\right) \\
&= \vec{M}_0^{(1)} - \vec{r}_{\text{C}} \times \vec{F}_{\text{tot}}^{(1)}. \\
&= \vec{M}_0^{(1)} + \vec{r}_{0/\text{C}} \times \vec{F}_{\text{tot}}^{(1)}.
\end{aligned}$$

[**Note.** The calculation above uses the 'move' (A math operation is kind of analogous to a chess move) of factoring a constant vector out of a sum. This move, of using the distributive rule backwards (undistributing), is used often in mechanics theory.]

Similarly, for force set (2):

$$\vec{M}_{\text{C}}^{(2)} = \vec{M}_0^{(2)} + \vec{r}_{0/\text{C}} \times \vec{F}_{\text{tot}}^{(2)}.$$

If the two force sets are equivalent for reference point 0 then $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$ and $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$ and the expressions above imply that $\vec{M}_{\text{C}}^{(1)} = \vec{M}_{\text{C}}^{(2)}$. Because we specified nothing special about the point C, the systems are equivalent for *any* reference point. Thus, to demonstrate equivalence we need to use a reference point, but once equivalence is demonstrated we need not name the point since the equivalence holds for all points.

By the same reasoning we find that once we know the net force and net moment of a force system ($\vec{F}_{\text{tot}}$) relative to some point C (call it $\vec{M}_{\text{C}}$), we know the net moment relative to point D as

$$\vec{M}_{\text{D}} = \vec{M}_{\text{C}} + \vec{r}_{\text{C}/\text{D}} \times \vec{F}_{\text{tot}}.$$

Note that if the net force is $\vec{0}$ (and the force set is then called a couple) that $\vec{M}_{\text{D}} = \vec{M}_{\text{C}}$ so the net moment is the same for all reference points.

Some force systems that are ‘equivalent’ but different *do* have different mechanical effects. In what sense are two force systems that have the same net force and the same net moment really equivalent?

> ‘Equivalent’ force systems are equivalent in their contributions to the equations of mechanics (equations 0-II on the inside cover) for any system to which they are both applied.

But full mechanical analysis of a situation requires looking at the mechanics equations of many subsystems. In the mechanics equations for each subsystem, two ‘equivalent’ force systems are equivalent if they are both applied to that subsystem.

For the analysis of the subsystem that is you sleeping, the force of your roommate’s hand on the floor isn’t applied to you, so it doesn’t show up in the mechanics equations for you, and doesn’t have the same effect as a force on you.

In summary, repeating:

> Two force systems are called mechanically ‘equivalent’, *with regard to a particular system on which they both act*, if they have the same net force and also, the same net moment about some common point.

Two force sets can be equivalent, but not have all of the same mechanical effects, if they are not equivalent for all of the relevant subsystems. That is, you can only replace a force set with an equivalent force set after you have chosen the system of study[4].

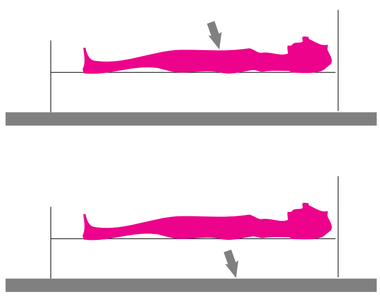

Figure 2.4: It feels different if someone presses on you or presses on the floor underneath you with an ‘equivalent’ force. The equivalence of ‘equivalent’ force systems depends on them both being applied to the same system.

[4] Or, at least you know that both sets act, or not, on all subsystems you will study

2.3 The tidiest representation of a force system: A “wrench”

This is an advanced aside.

Any force system can be represented by an equivalent force and a couple at any point. But force systems can be reduced to simpler forms. That this is so is of more theoretical than practical import. We state the results here without proof.

In 2D one of these two things is always true:

- The system is equivalent to a couple, or most often
- There is a line parallel to the force along which the system can be described by an equivalent force with no couple.

In 3D one of these three things is true:

- The system is equivalent to a couple (applied anywhere), or
- The system is equivalent to a force (applied on a given line parallel to the force), or most often
- There is a line for which the system can be reduced to a force and a couple where the force, couple, and line are all parallel. The representation of the system of forces as a force and a parallel moment is called a *wrench*.