9.1 Fluid pressure

Besides the basic laws of mechanics that you already know, elementary hydrostatics is based on the following two constitutive assumptions (see page 27):

1) The force of water on a surface is perpendicular to the surface; and
2) The density of water, ρ (pronounced 'row') is a constant (doesn't vary with depth or pressure),

Sometimes we use the weight density $\gamma = g\rho$ (pronounced 'gammuh equals gee row'), the weight per unit volume . The first assumption, that all static water forces are perpendicular to surfaces on which they act, can be restated:

Still water cannot carry any shear stress.

For near-still water, this constitutive assumption is abnormally accurate (compared to most constitutive assumptions for materials), approximately as good as the laws of mechanics.

The assumption of constant density is called *incompressibility* because it corresponds to the idea that water does not change its volume (compress) much under pressure. This assumption is reasonable for most purposes. At the bottom of the deepest oceans, for example, the extreme pressure (about 800 atmospheres) causes water to increase its density only about 4% from that of water at the surface[1]

[1] That fluid density does depend on salinity, temperature and pressure is sometimes important in hydrostatics, in particular for determining which water floats on which other water. This density variation is important in the ecology of lakes, the effects of the oceans on climate, and in air for the stability of the atmosphere, and the mechanics of fireplace chimneys.

We also assume that the direction and magnitude of the local gravitational 'constant' is, well, constant. This assumption becomes inaccurate when considering, say, the hydrostatics of whole oceans (the direction of the gravity force changes as you go around the world, this helps keep the Australians in place). When considering the upper atmosphere one might consider that the magnitude of the gravity decays with distance from the center of the earth. And it's more subtle than that if you care about fine measurement (see page 236). However, for most purposes taking gravity as constant in magnitude and direction is good enough.

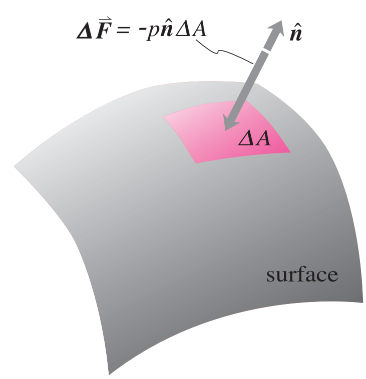


Figure 9.1: A bit of area ΔA on a surface on which pressure p acts. The outward (into the water) normal of the surface is $\hat{\boldsymbol{n}}$ so the force is $\Delta\vec{F} = -p\hat{\boldsymbol{n}}\Delta A$.

Surface area A, outward normal $\hat{\boldsymbol{n}}$, pressure p, and force $\vec{F}$

We will generalize the high-school physics fact

$$\text{force} = \text{pressure} \times \text{area}$$

to take account that force is a vector, that pressure varies with position, and that not all surfaces are flat. So we need a clear notation and sign convention. The area of a surface is A which we can think of as being the sum of the bits of area ΔA that compose it:

$$A = \sum \Delta A = \int dA.$$

Every bit of surface area has an *outer* normal $\hat{\boldsymbol{n}}$ that points from the surface out into the fluid. The (scalar) force per unit area on the surface is called the pressure p, so that the force on a small bit of surface is

$$\Delta\vec{\boldsymbol{F}} = p\,(-\hat{\boldsymbol{n}})\,(\Delta A)$$

pointing into the surface, assuming positive pressure, and with magnitude proportional to both pressure and area. Thus the total force and moment due to pressure forces on a surface :

$$\begin{aligned} \vec{\boldsymbol{F}} &= \int d\vec{\boldsymbol{F}} &&= -\int_A p\,\hat{\boldsymbol{n}}\; dA \\ \vec{\boldsymbol{M}}_{\mathrm{C}} &= \int_A d\vec{\boldsymbol{M}}_{/\mathrm{C}} &&= -\int_A \vec{\boldsymbol{r}}_{/\mathrm{C}} \times (p\,\hat{\boldsymbol{n}})\; dA \end{aligned} \tag{9.1}$$

Hydrostatics is the evaluation of the (intimidating-at-first-glance) integrals 9.1 and their role in equilibrium equations. In the rest of this section we consider various important special cases.

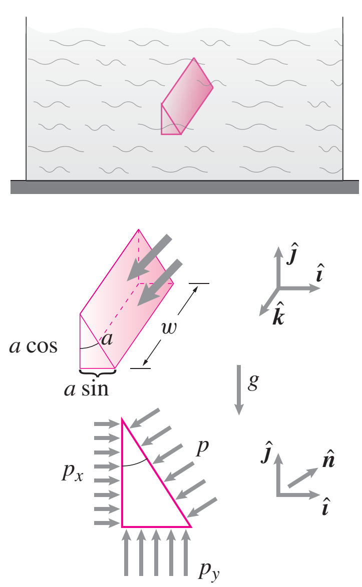


Figure 9.2: A small prism of water in equilibrium is isolated from other water. The free-body diagram does not show the forces in the z direction. Force balance applied to this free-body diagram shows that $p_x = p_y = p$, pressure is the same in all directions.

Water in equilibrium with itself

Before we worry about how water pushes on other things, let's first understand what it means for water to be in static equilibrium. These first important facts about hydrostatics follow from drawing free-body diagrams of various chunks of water and assuming static equilibrium (see box 9.2 on page 5).

1. Pressure is the same in every direction, $p_x = p_y = p$.
2. Pressure doesn't vary with side to side position, $p(x, y, z) = p(y)$.
3. Pressure varies linearly with depth, $p = \rho g h = \gamma h$.

The buoyant force of water on water.

Imagine a chunk of water the shape of a sea monster. Where? In a place under water in a calm swimming pool where there is nothing but water.

Now draw a free-body diagram of that water. Because your sea monster is in equilibrium, force balance and moment balance must apply. The only forces are the complicated distribution of pressure forces and the weight of water. The pressure forces must exactly cancel the weight of the water and, to satisfy moment balance, must pass through the center-of-mass of the water monster. So, in static equilibrium:

> The pressure forces acting on a surface enclosing a volume of water are equivalent to the negative weight passing through the center-of-mass of the water.

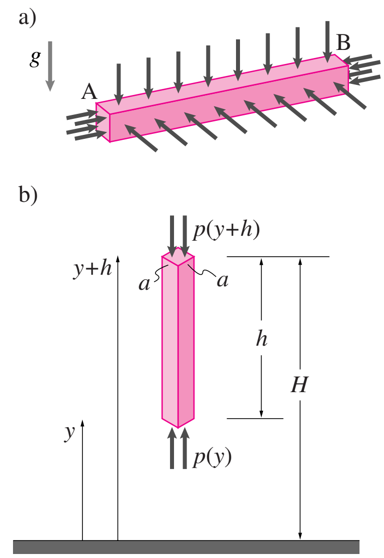


Figure 9.3: Free-body diagrams of aligned boxes of water cut out of a bigger body of water. a) a horizontally aligned box, b) a vertically aligned box. (To avoid visual clutter the self-cancelling horizontal forces are not shown on the second free-body diagram.) Force balance applied to these free-body diagrams shows that $p = p(y)$.

The force of water on submerged and floating objects

The net pressure force and moment on a still object surrounded by still water can be found by a clever argument credited to Archimedes. The pressure at any one point on the outside of the object does not depend on what's inside. The pressure is determined by how far the point of interest is below the surface by eqn. 9.2[2]. So if you can find the resultant force on any object that is the shape of the submerged object, but replacing the submerged object, it tells you what you want to know.

The clever idea is to replace your object with water. In this new system the water is in equilibrium, so the pressure forces exactly balance the weight. We thus obtain Archimedes' Principle:

> The resultant of all pressure forces on a totally submerged object is an upwards force with the same magnitude as the weight of the displaced water. The resultant acts at the centroid of the displaced volume:
>
> $$\vec{\boldsymbol{F}}_{\text{buoyancy}} = \gamma V \hat{\boldsymbol{\jmath}} \qquad \text{acting at} \qquad \vec{\boldsymbol{r}} = \frac{\int \vec{\boldsymbol{r}}_{/0}\, dV}{V}.$$

[2] If there is no column of water from the point up to the surface it is still true that the pressure is γh, as you can figure out by tracking the pressure changes along on a staircase-like path from the surface to that point.

The result can also be found by adding the effects of all the pressure forces on the outside surface (see box 9.1 on page 4).

For floating objects, the same argument can be carried out, but since the replaced fluid has to be in equilibrium we cannot replace the whole object with fluid, but only the part which is below the level of the water surface.

Displaced fluid

Sometimes people discuss Archimedes' principle in terms of the displaced fluid. A floating object in equilibrium displaces an amount of fluid with the same weight as the object; this is also the amount of volume of the floating object that is below the water level. On the other hand an object

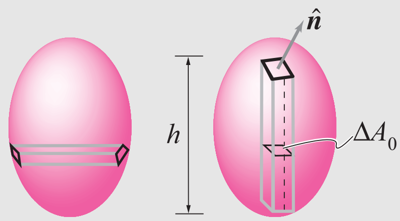
9.1 Adding forces to derive Archimedes' principle

(One can do most hydrostatics calculations, say typical homework problems, without being able to reproduce the derivations here.)

Archimedes' principle follows from adding up all the pressure forces on the outer surfaces of an arbitrarily shaped, submerged solid, say something potato-shaped.

First we find the answer by cutting the potato into french fries. This approach is effectively a derivation of a theorem in vector calculus. After that, for those who have the appropriate math background, we quote the vector calculus directly.

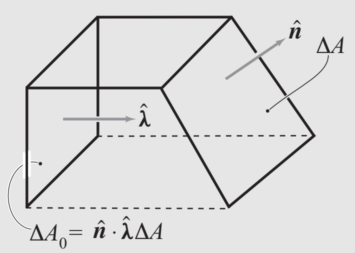
First cut the potato into horizontal french-fries (horizontal prisms) and look at the forces on the end caps (there are no water forces on the sides since those are inside the potato).



The pressure on two ends is the same (because they have the same water depth). The areas on the two ends are probably different because your potato is probably not box shaped. But the area is bigger at one end if the normal to the surface is more oblique compared to the axis of the prism. If the cross sectional area of the prism is ΔA_0 then the area of one of the prism caps is

$$\Delta A = \Delta A_0/(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}})$$

where $\hat{\boldsymbol{\lambda}}$ is along the axis of the prism and $\hat{\boldsymbol{n}}$ is the outer unit normal to the end cap (Note $\Delta A \geq \Delta A_0$ because $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}} \leq 1$).

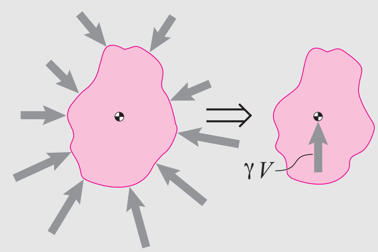


So the net force on the cap is $-p\Delta A_0\hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}})$. The component of the force along the prism is $[-p\Delta A_0\hat{\boldsymbol{n}}/(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}})]\cdot\hat{\boldsymbol{\lambda}}$ which is $-p\Delta A_0$. An identical calculation at the other end of the french fry gives minus the same answer. So the net force of the water pressure along the prism is zero for this and every prism and thus the whole potato. Likewise for prisms with any horizontal orientation. Thus the net sideways force of water on any submerged object is zero.

To find the net vertical force on the potato we cut it into vertical french fries. The net forces on the end caps are calculated just as in the above paragraph but taking account that the pressure on the bottom of the french fry is bigger than at the top. The sum of the forces of the top and bottom caps is an upwards force that is

$$\begin{aligned}\text{net upwards force on vertical french fry} &= \Delta p\Delta A_0\\ &= (\gamma h)\Delta A_0\\ &= \gamma(h\Delta A_0)\\ &= \gamma\,\Delta V_0\end{aligned}$$

where ΔV_0 is the volume of the french fry. Adding up over all the french fries that make up the potato one gets that the net upwards force is γV. The net result, summarized by the figure below, is that the resultant of the pressure forces on a submerged solid is an upwards force whose magnitude is the weight of the displaced water. The location of the force is the centroid of the displaced volume. (Note that the centroid of the displaced volume is not necessarily at the center of mass of the submerged object.)



. . .

A vector calculus derivation

Here is a derivation of Archimedes' principle, at least the net force part, using multi-variable integral calculus. Only read on if you have taken a math class that covers the divergence theorem. The net pressure force on a submerged object is

$$\begin{aligned}\vec{\boldsymbol{F}}_{\text{buoyancy}} &= -\int_A p\,\hat{\boldsymbol{n}}\,dA\\ &= -\int_S p\,\hat{\boldsymbol{n}}\,dS\\ &= -\int_S (H-z)\gamma\,\hat{\boldsymbol{n}}\,dS\\ &= -\int_V \vec{\nabla}((H-z)\gamma)\,dV\\ &= -\int_V (-\hat{\boldsymbol{k}})\gamma\,dV\\ &= \int_V \gamma\,dV\,\hat{\boldsymbol{k}}\\ &= (\text{weight of displaced water})\,\hat{\boldsymbol{k}}.\end{aligned}$$

In this derivation we first changed from calling bits of surface area dA to dS because that is a common notation in calculus books. The depth from the surface, of a point with vertical component z from the bottom, is $H-z$. The $\vec{\nabla}$ symbol indicates the gradient and its place in this equation is from the divergence theorem:

$$\int_S (\text{any scalar})\hat{\boldsymbol{n}}\,dS = \int_V \vec{\nabla}(\text{the same scalar})\,dV.$$

The gradient of $(H-z)\gamma$ is $-\hat{\boldsymbol{k}}\gamma$ because H and γ are constants. Note, where we write $\int_S$ some books would write $\iint_S$, and where we write $\int_V$ some books would write $\iiint_V$.

that is totally under water, for whatever reason (it is resting on the bottom, or it is being held underwater by a string, etc), displaces as much fluid as the space it occupies. Putting these two ideas together one can remember

9.2 Pressure doesn't depend on direction or horizontal position and increases linearly with depth

We assume that the pressure p does not vary too wildly from point to point, thus if we look at a small enough region we can think of the pressure as constant in that region. If we draw a free-body diagram of a little triangular prism of water the net forces on the prism must add to zero (see fig. 9.2 on page 2). For each surface the magnitude of the force is the pressure times the area of the surface and the direction is minus the outward normal of the surface. We assume, for the time being, that the pressure is different on the differently oriented surfaces. So, for example, because the area of the left surface is $a\cos\theta w$ and the pressure on the surface is p_x, the net force is $a\cos\theta w p_x \hat{\boldsymbol{\imath}}$. Calculating similarly for the other surfaces:

$$\begin{aligned}
\vec{\mathbf{0}} &= \sum \vec{\boldsymbol{F}}_i \\
&= \underbrace{(a\cos\theta)w\,p_x\,\hat{\boldsymbol{\imath}} + (a\sin\theta)w\,p_y\,\hat{\boldsymbol{\jmath}} - a\,w\,p\,\hat{\boldsymbol{n}}}_{\text{pressure terms}} \\
&\qquad - \underbrace{\frac{a^2\cos\theta\sin\theta w}{2}\rho g\,\hat{\boldsymbol{\jmath}}}_{\text{weight}} \\
&= aw\left(\cos\theta\,p_x\,\hat{\boldsymbol{\imath}} + \sin\theta\,p_y\,\hat{\boldsymbol{\jmath}} - p\,\underbrace{(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}})}_{\hat{\boldsymbol{n}}}\right. \\
&\qquad \left. - \tfrac{a\cos\theta\sin\theta}{2}\rho g\,\hat{\boldsymbol{\jmath}}\right).
\end{aligned}$$

If a is arbitrarily small, the weight term drops out compared to the pressure terms. Dividing through by aw we get

$$\vec{\mathbf{0}} = \cos\theta p_x \hat{\boldsymbol{\imath}} + \sin\theta\,p_y\,\hat{\boldsymbol{\jmath}} - p\,(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}).$$

Taking the dot product of both sides of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ gives that

$$p = p_x = p_y.$$

Since θ could be anything, force balance for the free-body diagram of a small prism tells us that for a fluid in static equilibrium

pressure is the same in every direction.

[Other free-body diagrams can be used. That pressure has to be the same in any pair of directions could also be found by drawing a prism with a cross section which is an isosceles triangle. The prism is oriented so that two surfaces of the prism have equal area and have the desired orientations. Force balance along the base of the triangle gives that the pressures on the equal area surfaces are equal. The argument that pressure must not depend on direction in 3D is generally based on equilibrium of a small tetrahedron.]

Pressure doesn't vary with side-to-side position Consider the equilibrium of a horizontally aligned box of water cut out of a bigger body of water (fig. 9.3a on page 3). The forces on the end caps at A and B are the only forces along the box. Therefore they must cancel. Since the areas at the two ends are the same, the pressure must be also. This box could be anywhere and at any length and any horizontal orientation. Thus for a fluid in static equilibrium

pressure doesn't depend on horizontal position.

If we take the $\hat{\boldsymbol{\jmath}}$ or y direction to be up (fig. 9.3 on page 3), then we have

$$p(x, y, z) = p(y).$$

Pressure increases linearly with depth Consider the vertically aligned box of fig. 9.3b.

$$\begin{aligned}
\left\{\sum \vec{\boldsymbol{F}}_i = \vec{\mathbf{0}}\right\}\cdot\hat{\boldsymbol{\jmath}} \quad &\Rightarrow \quad \underbrace{p(y)a^2 - p(y+h)a^2}_{\text{pressure terms}} - \underbrace{\rho g a^2 h}_{\text{weight}} = 0 \\
&\Rightarrow \quad p_{\text{bottom}} - p_{\text{top}} = \rho g h.
\end{aligned}$$

So the pressure increases linearly with depth. If the top of a lake, say, is at atmospheric pressure p_a then we have that

$$p = p_a + \rho g h = p_a + \gamma h = p_a + (H - y)\gamma$$

where h is the distance down from the surface, H is the depth to some reference point underwater and y is the distance up from that reference point (so that $h = H - y$). Neglecting atmospheric pressure at the top surface, we have the useful and easy to remember formula:

$$p = \gamma h. \tag{9.2}$$

Because the pressure at equal depths must be equal and because the pressure at the top surface must be equal to atmospheric pressure, the top surface must be flat and level. Thus waves and the like are a definite sign of static disequilibrium as are any bumps on the water surface even if they don't seem to move (as for a bump in the water where a stream goes steadily over a rock).

that

A floating object displaces its weight, a submerged object displaces its volume.

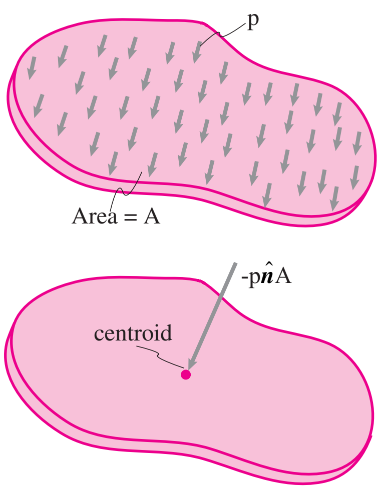


Figure 9.4: The resultant force from a constant pressure p on a flat plate is $\vec{\boldsymbol{F}} = -pA\hat{\boldsymbol{n}}$ acting at the centroid of the plate.

The force of constant pressure on a totally immersed object

When there is no gravity, or gravity is neglected, the pressure in a static fluid is the same everywhere. Exactly the same argument we have just used shows that the resultant of the pressure forces is zero. We could derive this result just by setting $\gamma = 0$ in the formulas above. That is, it is not pressure that causes buoyancy, but the gravity-induced gradient of pressure. If there is no gravity there is no buoyancy.

The force of constant pressure on a flat surface

The net force of constant pressure on a flat surface (not all the way around a submerged volume) is the pressure times the area acting normal to the surface at the centroid of the surface:

$$\begin{aligned}\vec{\boldsymbol{F}}_{\text{net}} &= \int_A -p\,\hat{\boldsymbol{n}}\,dA \\ &= -pA\hat{\boldsymbol{n}}.\end{aligned}$$

That this force acts at the centroid can be checked by calculating the moment of the pressure forces relative to the centroid C,

$$\begin{aligned}\vec{\boldsymbol{M}}_{/\text{C,net}} &= \int_A \vec{\boldsymbol{r}}_{/\text{C}} \times (-p\,\hat{\boldsymbol{n}}\,dA) \\ &= \underbrace{\left(\int_A \vec{\boldsymbol{r}}_{/\text{C}}\,dA\right)}_{0} \times (-p\,\hat{\boldsymbol{n}}) \\ &= 0\ ,\end{aligned}$$

where the zero is because the position of the center-of-mass, relative to the center-of-mass, is zero.

The force of water on a rectangular plate

Consider a rectangular plate with width (into the page) w and length ℓ. There is water on one side and nothing on the other. Assume the water-side normal to the plate is $\hat{\boldsymbol{n}}$ and that the top edge of the plate is horizontal. Take $\hat{\boldsymbol{\jmath}}$ to be the up direction with y being distance up from the bottom and the total depth of the water is H. Thus the area of the plate is $A = \ell w$. If the bottom and top of the plate are at y_1 and y_2 the net force on the plate can be found as:

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -\int_A p\hat{\boldsymbol{n}}\,dA \\
&= -\int_A \gamma(H-y)\hat{\boldsymbol{n}}\,dA \\
&= -w\int_0^{\ell} \gamma(H-y(s))\hat{\boldsymbol{n}}\,ds \\
&= -w\int_0^{\ell} \gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,s)\right)\hat{\boldsymbol{n}}\,ds \\
&= -w\gamma\left(H\ell - y_1\ell - \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell^2/2\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+(y_2-y_1)/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\left(\gamma(H-y_1)/2+\gamma(H-y_2)/2)\right)\hat{\boldsymbol{n}}
\end{aligned}
$$

So

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -w\ell\,\frac{p_1+p_2}{2}\,\hat{\boldsymbol{n}}. \\
&= -(\text{area})(\text{average pressure})(\text{outwards normal direction}).
\end{aligned}
$$

The net water force is the same as that of the average pressure acting on the whole surface. To find where it acts, it is easiest to think of the pressure distribution as the sum of two different pressure distributions. One is a constant over the plate at the pressure of the top of the plate. The other varies linearly from zero at the top to $\gamma(y_2-y_1)$ at the bottom.

$$
p = \gamma(H-y) = \underbrace{\gamma(H-y_2)}_{\text{Constant pressure, the pressure at the top edge.}} + \underbrace{\gamma(y_2-y)}_{\text{Varies linearly from 0 at the top to } \gamma(y_2-y_1) \text{ at the bottom.}}
$$

The first corresponds to a force of $w\ell\gamma(H-y_2)$ acting at the middle of the plate. The second corresponds to a force of $w\ell\gamma\frac{y_2-y_1}{2}$ acting a third of the way up from the bottom of the plate.

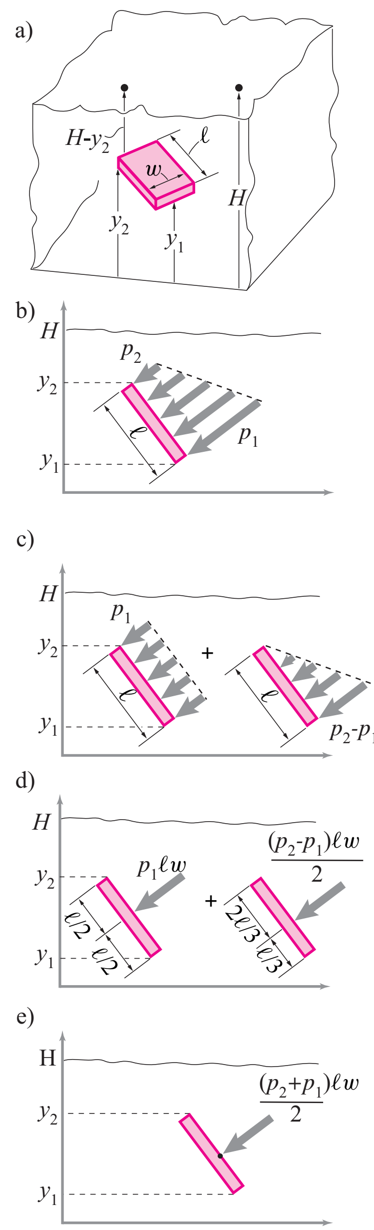


Figure 9.5: The resultant force from a constant depth-increasing pressure on a rectangular plate.