SAMPLE 9.1 A uniform solid cylinder of mass $m = 12\,\mathrm{kg}$, diameter $d = 0.1$ m and height $h = 2$ m floats in water (density $\rho = 1000\,\mathrm{kg/\,m^3}$).

1. Assuming the cylinder floats vertically, find the submerged height of the cylinder.
2. If the cylinder floats longitudinally (its longitudinal axis parallel to the water surface), what will be the submerged section of the cylinder?

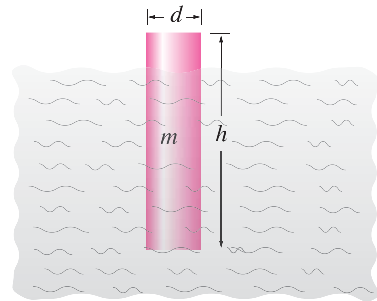

Figure 9.6

Solution

1. **Cylinder floating vertically:** Let h_s be the submerged height of the cylinder and $r = d/2$ be its radius. Then the force of buoyancy F_B is equal to the weight of water replaced by the submerged volume of the cylinder. Thus,

$$\vec{F}_\mathrm{B} = \overbrace{\pi r^2 h_s}^{\text{volume}} \gamma\, \hat{\boldsymbol{\jmath}}.$$

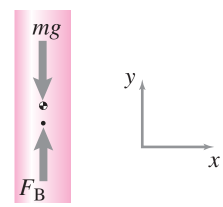

Figure 9.7

From the force balance on the cylinder (see the free-body diagram in fig. 9.7),

$$\begin{aligned}
\vec{F}_\mathrm{B} - mg\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\
\Rightarrow \quad (\pi r^2 h_s \gamma - mg)\hat{\boldsymbol{\jmath}} &= \vec{\mathbf{0}} \\
\Rightarrow \quad h_s &= \frac{mg}{\pi r^2 \gamma} = \frac{m}{\pi r^2 \rho} \\
&= \frac{12\,\mathrm{kg}}{\pi \cdot (0.05\,\mathrm{m})^2 \cdot 1000\,\mathrm{kg/\,m^3}} = 1.53\,\mathrm{m}.
\end{aligned}$$

$\boxed{h_s = 1.53\,\mathrm{m}}$

2. **Cylinder floating horizontally:** No matter how the cylinder floats, the force of buoyancy has to equal the weight of the cylinder. This force is equal to the weight of the displaced water. Thus, the volume of displaced water has to be the same no matter what the orientation of the cylinder is with respect to the water surface. Therefore, the submerged volume of the cylinder while floating longitudinally must equal the volume submerged while floating vertically. That is (see fig. 9.8),

$$\text{area of BCD} \cdot h = \pi r^2 h_s \qquad \Rightarrow \qquad \text{area of BCD} = \pi r^2 (h_s/h) = 0.006\,\mathrm{m}^2.$$

From the mass and volume we can calculate that the density of the cylinder is .764 that of water. Now we can figure out what d_s should be so that the submerged cross-sectional area is 76% of the total cross-sectional area. This is an exercise in geometry. Since, area of BCD $= \pi r^2 -$ area of ABD, area of ABD $= \pi r^2 -$ area of BCD $= 0.018\,\mathrm{m}^2$. But the area of ABD is the area of the circular sector OBAD ($r^2\theta$) minus the area of triangle OBD ($\frac{1}{2} \cdot r\cos\theta \cdot 2r\sin\theta$). Thus,

$$\begin{aligned}
\text{area of ABD} \equiv r^2\theta - \frac{1}{2} r^2 \sin 2\theta &= 0.018\,\mathrm{m}^2 \\
\Rightarrow \quad \theta - \frac{1}{2}\sin 2\theta - 0.738 &= 0.
\end{aligned}$$

We need to solve this nonlinear equation. Using trial and error or root finding on a computer or a graphical method, we find $\theta = 1.126$ rad $= 64.5°$. Using this value, we get, $d_s = r + r\cos\theta = 0.07$ m.

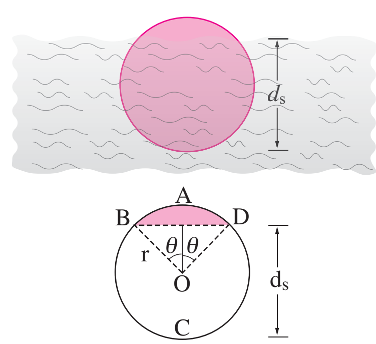

Figure 9.8

$\boxed{d_s = 0.07\,\mathrm{m}}$

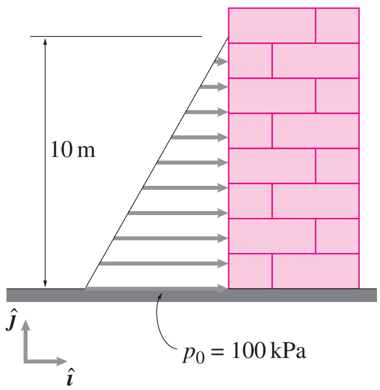

Figure 9.9

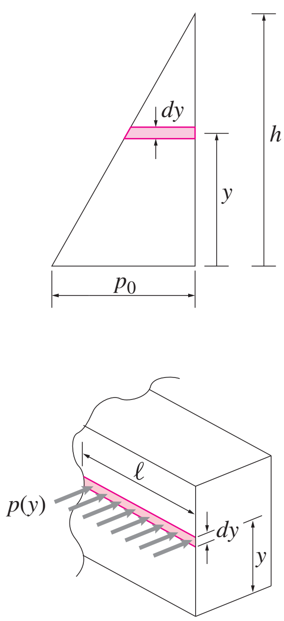

Figure 9.10

SAMPLE 9.2 The force due to varying hydrostatic pressure: The hydrostatic pressure distribution on the face of a wall submerged in water up to a height $h = 10\,\text{m}$ is shown in the figure. Find the net force on the wall from water. Take the length of the wall (into the page) to be 1 m.

Solution Since the pressure varies across the height of the submerged part of the wall, let us take an infinitesimal strip of height dy along the full length ℓ of the wall as shown in fig. 9.10. Since the height of the strip is infinitesimal, we can treat the water pressure on this strip to be essentially constant and equal to $p_0 \frac{y}{h}$. Then the force on the strip (of area $\ell\,dy$) due to the constant water pressure $p(y) = p_0 y/h$ is

$$d\vec{F} = (p(y) \cdot \overbrace{\ell\,dy}^{\text{area}})\,\hat{\imath} = p_0 \frac{y}{h}\,\ell\,dy\,\hat{\imath}.$$

The net force due to the pressure distribution on the whole wall can now be found by integrating $d\vec{F}$ along the height of the wall.

$$\begin{aligned}
\vec{F} &= \int d\vec{F} = \int_0^h p_0 \frac{y}{h}\,\ell\,dy\,\hat{\imath} \\
&= \left(p_0 \frac{\ell}{h} \int_0^h y\,dy\right)\hat{\imath} = p_0 \frac{\ell}{h}\frac{h^2}{2}\hat{\imath} \\
&= \frac{1}{2} p_0 h \ell\,\hat{\imath} \\
&= \frac{1}{2} \cdot (100\frac{\text{kN}}{\text{m}^2}) \cdot (10\,\text{m}) \cdot (1\,\text{m})\hat{\imath} \\
&= (500\,\text{kN})\,\hat{\imath}.
\end{aligned}$$

$$\boxed{\vec{F} = 500\,\text{kN}\,\hat{\imath}}$$

Alternatively, the net force can be computed by calculating the area of the pressure triangle and multiplying by the unit length ($\ell = 1\,\text{m}$), *i.e.*,

$$\begin{aligned}
\vec{F} &= \overbrace{\frac{1}{2} \cdot h \cdot p_0}^{\text{triangle area}}\ \ell\,\hat{\imath} \\
&= \left(\frac{1}{2} \cdot 10\,\text{m} \cdot 100\frac{\text{kN}}{\text{m}^2} \cdot 1\,\text{m}\right)\hat{\imath} \\
&= 500\,\text{kN}\,\hat{\imath}.
\end{aligned}$$

SAMPLE 9.3 **Forces on a submerged sluice gate:** A rectangular plate is used as a gate in a tank to prevent water from draining out. The plate is hinged at A and rests on a frictionless surface at B. Assume the width of the plate to be 1 m. The height of the water surface above point A is h. Ignoring the weight of the plate, find the forces on the hinge at A as a function of h. In particular, find the vertical pull on the hinge for $h = 0$ and $h = 2\,\text{m}$.

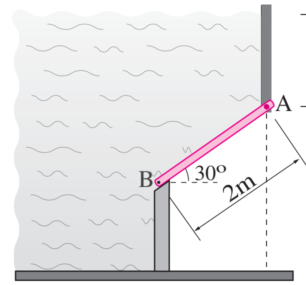

Figure 9.11

Solution Let $\gamma = \rho g$ be the weight density (weight per unit volume) of water. Then the pressure due to water at point A is $p_A = \gamma h$ and at point B is $p_B = \gamma(h + \ell \sin\theta)$. The pressure acts perpendicular to the plate and varies linearly from p_A at A to p_B at B. The free-body diagram of the plate is shown in fig. 9.12. Let $\hat{\boldsymbol{\lambda}}$ be a unit vector along BA and $\hat{\boldsymbol{n}}$ be a unit vector normal to BA. For computing the reaction forces on the plate at points A and B, we first replace the distributed pressure on the plate by two equivalent concentrated forces F_1 and F_2 by dividing the pressure distribution into a rectangular and a triangular region and finding their resultants.

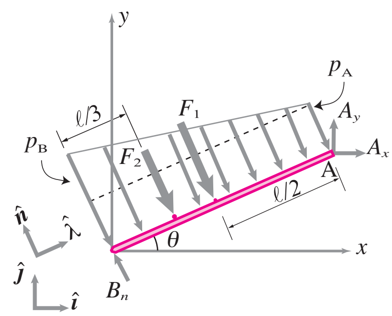

Figure 9.12: The free-body diagram of the gate (plate AB) with distributed water pressure.

$$F_1 = p_A \ell = \gamma h \ell, \quad F_2 = (p_B - p_A)\frac{\ell}{2} = \frac{1}{2}\gamma \ell^2 \sin\theta.$$

Now, we carry out moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{B/A}} \times \vec{\boldsymbol{B}} + \vec{\boldsymbol{r}}_{\text{D/A}} \times \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{r}}_{\text{C/A}} \times \vec{\boldsymbol{F}}_1 &= \vec{\boldsymbol{0}} \\
-\ell\hat{\boldsymbol{\lambda}} \times B_n \hat{\boldsymbol{n}} - \frac{2\ell}{3}\hat{\boldsymbol{\lambda}} \times (-F_1\hat{\boldsymbol{n}}) - \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times (-F_2\hat{\boldsymbol{n}}) &= \vec{\boldsymbol{0}} \\
-B_n \ell \hat{\boldsymbol{k}} + F_1 \frac{2\ell}{3}\hat{\boldsymbol{k}} + F_2 \frac{\ell}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad B_n = \frac{2F_1}{3} + \frac{F_2}{2} = \gamma\ell\left(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta\right)
\end{aligned}$$

and, from force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned}
\vec{\boldsymbol{A}} &= -B_n\hat{\boldsymbol{n}} + F_1\hat{\boldsymbol{n}} + F_2\hat{\boldsymbol{n}} \\
&= \left(-\gamma\ell\left(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta\right) + \gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} \\
&= \left(\frac{1}{3}\gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} = \gamma\ell\left(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta\right)\hat{\boldsymbol{n}}.
\end{aligned}$$

The force $\vec{\boldsymbol{A}}$ computed above is the force exerted by the hinge at A on the plate. Therefore, the force on the hinge, exerted by the plate, is $-\vec{\boldsymbol{A}}$ as shown in fig. 9.13. From the expression for this force, we see that it varies linearly with h.

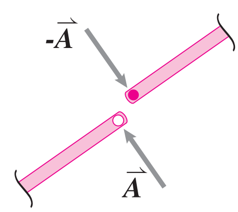

Figure 9.13: The free-body diagram of the pin at A.

Let the vertical pull on the hinge be A_{hinge_y}. Then

$$A_{\text{hinge}_y} = -\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\jmath}} = -\gamma\ell\left(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta\right)\overbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\jmath}}}^{\cos\theta} = \frac{1}{4}\gamma\ell\sin 2\theta + \left(\frac{1}{3}\gamma\ell\cos\theta\right)h.$$

Now, substituting $\gamma = 9.81\,\text{kN/m}^3$, $\ell = 2\,\text{m}$, $\theta = 30°$, the two specified values of h, and multiplying the result (which is force per unit length) with the width of the plate (1 m) we get,

$$\boxed{A_{\text{hinge}_y}\big|_{h=0} = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}\big|_{h=2\,\text{m}} = 15.58\,\text{kN}}$$

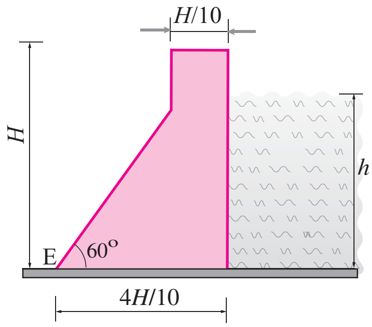

Figure 9.14

SAMPLE 9.4 **Tipping of a dam:** The cross-section of a concrete dam is shown in the figure. Take the weight-density $\gamma(=\rho g)$ of water to be $10\,\mathrm{kN/m^3}$ and that of concrete to be $25\,\mathrm{kN/m^3}$. For the given design of the cross-section, find the ratio h/H that is safe enough for the dam to not tip over (about the downstream edge E).

Solution Let us imagine the critical situation when the dam is just about to tip over about edge E. In such a situation, the dam bottom would almost lose contact with the ground except along edge E. In that case, there is no force along the bottom of the dam from the ground except at E.① With this assumption, the free-body diagram of the dam is shown in fig. 9.15.

① This assumption is valid only if water does not leak through the lower right of the damn through to the bottom of the dam. If it did, there would be a force on the bottom due to the water pressure. See the following sample where we include the water pressure at the bottom in the analysis.

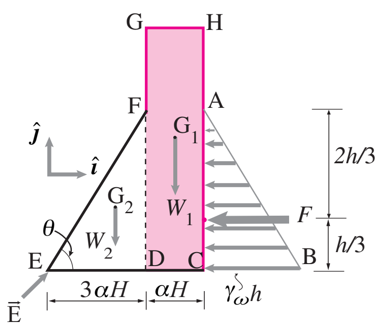

Figure 9.15

To compute all the forces acting on the dam, we assume the width w (into the paper) to be unity (*i.e.,* $w = 1$ m). Let γ_w and γ_c denote the weight-densities of water and concrete, respectively. Then the resultant force from the water pressure is

$$F = \frac{1}{2}\gamma_w h \cdot h \cdot w = \frac{1}{2}\gamma_w h^2 w.$$

This is the horizontal force (in the -$\hat{\boldsymbol{\imath}}$ direction) that acts through the centroid of triangle ABC.

To compute the weight of the dam, we divide the cross-section into two sections — the rectangular section CDGH and the triangular section DEF. We compute the weight of these sections separately by computing their respective volumes (we use the shape constant $\alpha = 1/10$):

$$\begin{aligned} W_1 &= \underbrace{\alpha H^2 \cdot w}_{\text{volume}} \cdot \gamma_c = \gamma_c \alpha H^2 w \\ W_2 &= \underbrace{\frac{1}{2}\cdot 3\alpha H \cdot 3\alpha H \tan\theta \cdot w}_{\text{volume}} \cdot \gamma_c = \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta. \end{aligned}$$

Now we apply moment balance about point E, $\sum \vec{\boldsymbol{M}}_{\mathrm{E}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned} \vec{\boldsymbol{r}}_{G_1} \times \vec{\boldsymbol{W}}_1 + \vec{\boldsymbol{r}}_{G_2} \times \vec{\boldsymbol{W}}_2 + \vec{\boldsymbol{r}}_{G_3} \times \vec{\boldsymbol{F}} &= \vec{\boldsymbol{0}} \\ -(3\alpha H + \frac{1}{2}\alpha H) W_1 \hat{\boldsymbol{k}} - \frac{2}{3}(3\alpha H) W_2 \hat{\boldsymbol{k}} + \frac{h}{3} F \hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}}. \end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{k}}$, we get

$$\begin{aligned} \frac{h}{3}F &= (3\alpha H + \frac{1}{2}\alpha H)\cdot \gamma_c \alpha H^2 w + \frac{2}{3}(3\alpha H)\cdot \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= 9\gamma_c \alpha^3 H^3 \tan\theta + \frac{7}{2}\gamma_c \alpha^2 H^3 \\ \Rightarrow \quad \left(\frac{h}{H}\right)^3 &= \frac{\gamma_c}{\gamma_w}(54\alpha^3 \tan\theta + 21\alpha^2) \\ &= 2.5(54\cdot 0.1^3 \cdot \sqrt{3} + 21\cdot 0.1^2) = 0.7588 \\ \Rightarrow \quad \frac{h}{H} &= 0.91. \end{aligned}$$

Thus, for the dam to not tip over, $h \le 0.91H$ or 91% of H.

$\frac{h}{H} \le 0.91$

SAMPLE 9.5 **Dam design:** You are to design a dam of rectangular cross-section ($b \times H$), ensuring that the dam does not tip over even when the water level h reaches the top of the dam ($h = H$). Take the specific weight of concrete to be 3. Consider the following two scenarios for your design.

1. The downstream bottom edge of the dam is plugged so that there is no leakage underneath.
2. The lower right edge is not plugged and water leaks under the dam bottom has full pressure across the bottom.

Solution Let γ_c and γ_w denote the weight densities of concrete and water, respectively. We are given that $\gamma_c/\gamma_w = 3$. Also, let $b/H = \alpha$ so that $b = \alpha H$. We consider both scenarios to find a safe cross-sectional shape and size for the dam. In the calculations below, we consider unit length (into the paper) of the dam.

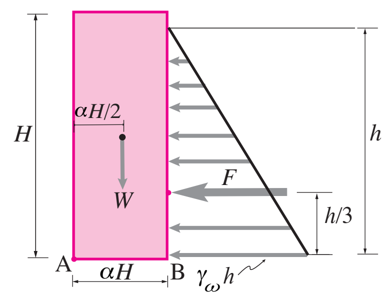

Figure 9.16

1. *No water pressure on the bottom:* When there is no water pressure on the bottom of the dam, then the water pressure acts only on the downstream side of the dam. The free-body diagram of the dam, considering critical tipping (just about to tip), is shown in fig. 9.16 in which F is the resultant force of the triangular water pressure distribution. The known forces acting on the dam are $W = \gamma_c \alpha H^2$, and $F = (1/2)\gamma_w h^2$. The moment balance about point A gives

$$\begin{aligned} F \cdot \frac{h}{3} &= W \cdot \frac{\alpha H}{2} \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= \gamma_c \frac{\alpha^2 H^3}{2} \\ \Rightarrow \quad \alpha^2 &= (1/3)(\gamma_w/\gamma_c)(h/H)^3. \end{aligned}$$

Considering the case of critical water level up to the height of the dam, *i.e.*, $h/H = 1$, and substituting $\gamma_c/\gamma_w = 3$, we get

$$\alpha^2 = 1/9 \quad \Rightarrow \quad \alpha = 1/3 = 0.333.$$

Thus the width of the cross-section needs to be at least one-third of the height. For example, if the height of the dam is 9 m then it needs to be at least 3 m wide.

$$\boxed{b/H = 0.33}$$

2. *Full water pressure on the bottom:* In this case, the water pressure on the bottom is uniformly distributed and its intensity is the same as the lateral pressure at B, *i.e.*, $p = \gamma_w h$. The free-body diagram diagram is shown in fig. 9.17 where the known forces are $W = \gamma_c \alpha H^2$, $F = (1/2)\gamma_w h^2$, and $R = \gamma_w \alpha h H$. Again, we carry out moment balance about point A to get

$$\begin{aligned} F \cdot \frac{h}{3} &= (W - R) \cdot \frac{\alpha h}{2} \\ \gamma_w h^3 &= 3(\gamma_c \alpha H^2 - \gamma_w \alpha h H)\alpha H \\ \alpha^2 &= \frac{(h/H)^3}{3(\gamma_c/\gamma_w - h/H)}. \end{aligned}$$

Once again, substituting the given values and $h/H = 1$, we get

$$\alpha^2 = 1/6 \quad \Rightarrow \quad \alpha = 0.408.$$

Thus the width in this case needs to be at least 0.41 times the height H, slightly wider than the previous case.

$$\boxed{b/H \geq 0.41}$$

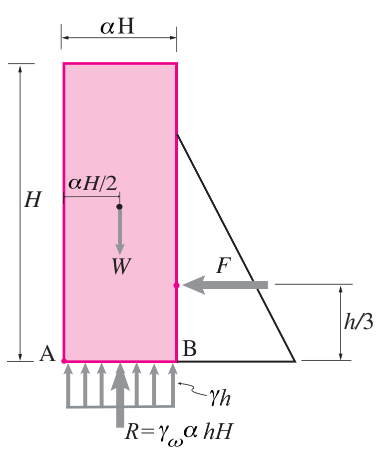

Figure 9.17