8.2 Singularity functions: an algorithm for V and M diagrams

This whole section is an advanced aside. It is a method for calculations of shear force and bending moment diagrams for straight beams. It reduces that problem to a simple recipe. Nothing in the book depends on this material. And, many competent engineers do not know or use it. On the other hand, some people find the method cute and fun because it is so straightforward (once you learn it).

For loads $w(x)$ that are *not* smooth, singularity functions make it easier to find the shear force V, bending moment M, slope u' and displacement u. Problems with multiple supports or multiple concentrated loads are especially simplified by use of singularity functions.

Context

In general if you know the downwards load per unit length $w(x)$ on a beam you can integrate 4 times to find

$$
\begin{aligned}
V(x) &= -\int w(x)\,dx + C_1 \\
EIu'' = M(x) &= \int V(x)\,dx + C_2, \\
u' &= \int u''\,dx + C_3 \qquad \text{and} \\
u &= \int u'\,dx + C_4. \qquad (8.13)
\end{aligned}
$$

These apply for any distributed loading $w(x)$.

But what kind of function $w(x)$ do you use for a concentrated load P; or a reaction force F; or for a load that is constant in some regions and zero in others; or when there is an applied couple? What are the functions $w(x)$ for such loads and how do you integrate them? The answers are: *singularity functions*.

Without singularity functions you have to find separate expressions for V, M, *etc.* for the regions to the left and to the right of such discontinuities. And then, as described in most books about beams, you have to pick integration constants in the two regions so that there is an appropriate jump, or not, of the V, M, u' or u. Using singularity functions you can skip all this matching. You just follow the integration rules and the solution is properly expressed on both sides of the discontinuity. That is, singularity functions allow you to do ordinary calculus even with functions that are discontinuous. Sometimes this is called *operational* calculus.

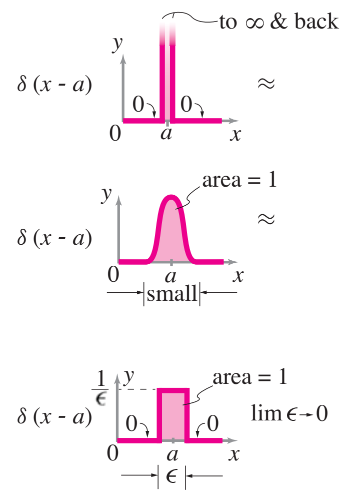

Figure 8.39: **The Dirac delta function** $\delta(x-a)$. By definition, the Dirac delta function is the bracket function with a superscript -1:
$< x - a >^{-1} \equiv \delta(x-a)$.
You can think of $\delta(x-a)$ as very tall and very narrow with a total area underneath of 1. In the third case above you have a box with width ϵ and height $1/\epsilon$.

Delta function and step function

Dirac delta function. The most famous(1) singularity function is called

(1)Note, the most famous singularity function, the delta function, isn't the most singular one, at least not in the mathematical sense of the word 'singular'.

the Dirac delta function, or the 'impulse' function. If it applies at a it is written:

$$\delta(x-a) = \text{'delta of x minus a'}.$$

What function is it? In the classical mathematics sense the delta function $\delta(x-a)$ isn't a function. Mathematicians were upset about this for a while. There are lots of ways to think about this function (See fig. 8.39). One way to think of $\delta(x-a)$ is as a function of x that is zero except for very close to $x = a$. Near $x = a$ it is as tall as it is narrow (*i.e.,* it's very tall and very narrow). So the area underneath is one. Actually $\delta(x-a)$ is the limit of infinitely tall and infinitely narrow, but the area underneath is still equal to one.

Concentrated load P. You can replace a downwards force P with a very large force per unit length $w(x)$ acting on a small region near $x = a$ but that has total force P:

$$\int_{\text{left of } a}^{\text{right of } a} w(x)dx = P.$$

But if the concentrated load is very high in value and very narrow in spatial extent, it's just like the delta function. Thus for a concentrated load P at a the associated distributed load w is

$$w(x) = P\delta(x-a).$$

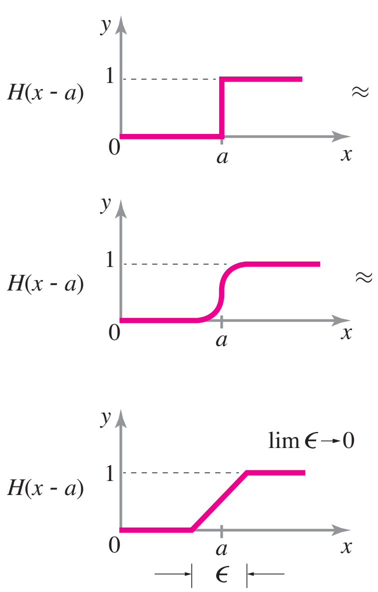

Figure 8.40: **Heaviside step function.** Compare these three curves for the Heaviside step function with the three ways of thinking about the delta function.

Heaviside step function. There are two primary ways to think of the Heaviside step function:

1. The step function is the integral of the delta function, informally:

$$\int \delta(x-a)\,dx = H(x-a)$$

   or more formally

$$\int_{-\infty}^{x} \delta(x'-a)\,dx' = H(x-a).$$

   That is, the Heaviside step function is the cumulative area under the delta function curve.

2. The Heaviside step function is that function of x that is zero to the left of a and one to the right of a,

$$H(x-a) \equiv \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x \geq a. \end{cases}$$

This is pictured in the first of fig. 8.40.

Shear V for a concentrated load. If there is various loading on a beam, part of which is a concentrated load P at a then the whole $w(x)$ function is several terms of which we only write out the term of interest here, $P\delta(x-a)$.

$$w(x) = \ldots P\delta(x-a)\ldots.$$

So, integrating,

$$\begin{aligned} V(x) &= -\int w(x)\,dx \\ &= -\int \overbrace{\ldots \qquad \underbrace{P\delta(x-a)}_{P\text{ times the delta function}} \qquad \ldots}^{w(x)} dx \\ &= \ldots \qquad \underbrace{-PH(x-a)}_{-P\text{ times the step function}} \qquad \ldots. \end{aligned}$$

If there is a concentrated load P at a then the shear force function $V(x)$ has a step down of size P at a.

The ramp function. See fig. 8.41. This one is not so famous[2]. The ramp function is $(x-a)$ to the right of a (for $x > a$), and zero to the left of a (for $x < a$). So we define the ramp function $R(x-a)$ as

$$\underbrace{R(x-a)}_{\text{Unit ramp function}} \equiv < x-a >^1 \equiv \begin{cases} (x-a) & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$

The brackets <> mean that you should think of the whole expression as being zero if $x < a$. For $x > a$ the brackets are like ordinary parentheses (at least for $n \geq 0$). It's easy to see that

$$\int H(x-a)\,dx = R(x-a)$$

or more formally, $$\int_{-\infty}^{x} H(x'-a)\,dx' = R(x-a)$$

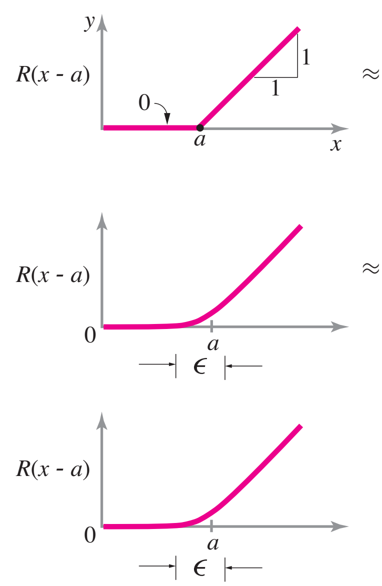

Figure 8.41: **The unit ramp function.**

[2]The ramp function is actually the Macauley ramp function. But Macauley isn't as famous as Dirac or Heaviside so most people just call it the ramp function.

The switched on parabola, *etc.* We can keep defining new functions this way with higher and higher powers. These functions are zero for $x < a$ and then switch on at $x = a$, *e.g.*,

$$< x-a >^2 \equiv \begin{cases} (x-a)^2 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$

and

$$< x-a >^3 \equiv \begin{cases} (x-a)^3 & \text{if } x \geq a, \\ 0 & \text{if } x < a. \end{cases}$$

In the same way that the step function integrates to the ramp we have

$$\int < x-a >^1 \, dx =< x-a >^2 /2, \qquad \text{and}$$
$$\int < x-a >^2 \, dx =< x-a >^3 /3.$$

Moment, slope and deflection due to a concentrated load. A concentrated load P at a contributes $P\delta(x-a)$ to $w(x)$. We already integrated to get shear, now let's keep going,

$$\begin{aligned} w(x) &= \ldots P\delta(x-a) + \ldots \\ V = -\int w \quad \Rightarrow \quad V(x) &= \cdots - PH(x-a) + \ldots \\ M = \int V \quad \Rightarrow \quad EIu''(x) = M(x) &= \cdots - PR(x-a) + \ldots \\ u' = \int u'' \quad \Rightarrow \quad EIu'(x) &= \cdots - P < x-a >^2 /2 + \ldots \\ u = \int u' \quad \Rightarrow \quad EIu(x) &= \cdots - P < x-a >^3 /6 + \ldots \end{aligned} \tag{8.14}$$

Every concentrated load leads to a displacement term that is cubic in x.

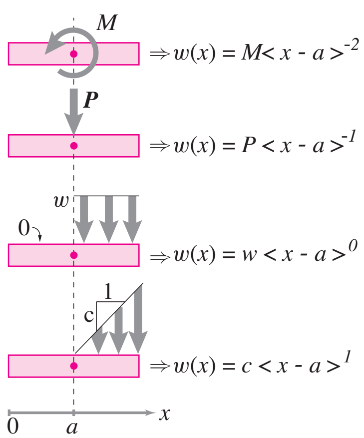

Figure 8.42: **How to represent loads with singularity functions.** From top to bottom: an applied couple at $x = a$, a concentrated load at $x = a$, a step in the distributed load at $x = a$, and a ramp function starting at $x = a$. The most important case is the concentrated load P.

More brackets. So that we don't have to remember all those names (Dirac, Heaviside and Macauley) nor the names of their functions (δ, H, R) we invent a notation that covers all cases, the last of which we already defined.

$$\begin{aligned} < x-a >^{-1} &\equiv \delta(x-a), \\ < x-a >^{0} &\equiv H(x-a), \qquad \text{and} \\ < x-a >^{1} &\equiv R(x-a). \end{aligned}$$

We can now write eqn. (8.14) with this notation.

$$\begin{aligned} \text{Given that} \quad w(x) &= \ldots \quad P < x-a >^{-1} + \ldots \\ \Rightarrow \quad V(x) &= \cdots - P < x-a >^{-0} + \ldots \\ \Rightarrow \quad \underbrace{M(x)}_{EIu''} &= \cdots - P < x-a >^{1} + \ldots \\ \Rightarrow \quad EIu'(x) &= \cdots - P < x-a >^2 /2 + \ldots \\ \Rightarrow \quad EIu(x) &= \cdots - P < x-a >^3 /6 + \ldots \end{aligned} \tag{8.15}$$

Note, if the superscript n is positive, think of it as an exponent. If the superscript is negative it is *not* an exponent. For negative n the exponent is just a label marking the degree of singularity: -1 is singular, -2 is more so, etc.

Applied couples. If a couple is applied to a beam it is like two big equal and opposite forces next to each other. You can think of this as the derivative of the delta function. For an applied counterclockwise couple M applied at a we write

$$w(x) = M < x - a >^{-2}$$

and use the integration rule $\int < x - a >^{-2} dx =< x - a >^{-1}$. Remember, for negative n you don't think of the superscript n as a power, but just as a label.

The general case

Here is the full definition of the singularity functions and the general formulas for their integration.

$$\text{If} \quad n < 0 : < x - a >^n \equiv \begin{cases} 0 & \text{if } x < a, \\ \text{undefined} & \text{if } x = a, \\ 0 & \text{if } x > a, \end{cases}$$

$$\text{If} \quad n \geq 0 : < x - a >^n \equiv \begin{cases} 0 & \text{if } x < a, \\ (x-a)^n & \text{if } x \geq a, \end{cases} \tag{8.16}$$

$$\text{If} \quad n \leq 0 : \quad \int < x - a >^n \ dx \equiv < x - a >^{n+1}$$

$$\text{If} \quad n \geq 0 : \quad \int < x - a >^n \ dx \equiv < x - a >^{n+1} /(n+1) \tag{8.17}$$

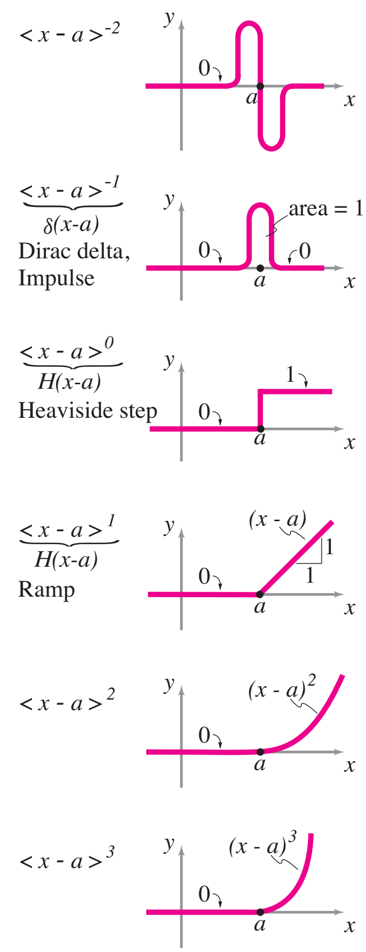

Figure 8.43: Graphical representation of the singularity functions $< x - a >^n$ for $-2 \leq n \leq 3$. The most famous and most used of these, used in many science and engineering disciplines, are **the Dirac delta function** $\delta(x-a) =< x - a >^{-1}$ and **the Heaviside step function** $H(x-a) =< x - a >^0$. The delta function is also sometimes called the impulse function. The ramp, quadratic functions and so on are useful for beam problems, but not so useful in the rest of science and engineering. The derivative of the delta function, $< x - a >^{-2}$ is mathematically even more singular than the delta function. It has some uses outside of beam deflection problems, but is much more rarely seen than the delta function or the Heaviside step function.