8.1 Free body cuts at arbitrary locations

Tension, shear force, and bending moment diagrams

Engineers often want to know how the internal forces vary from point to point in a structure. How do you find these internal forces? You could draw a variety of free-body diagrams with cuts at the points of interest. Another approach, which we present now, is to leave the position, say x, of the free-body-diagram cut a variable, and then calculate the internal forces in terms of x.

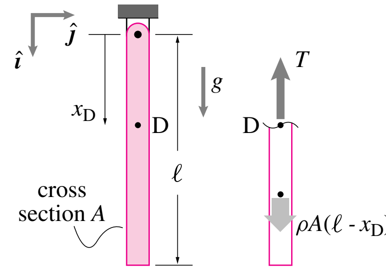

Figure 8.1: a) Rod hanging with gravity. b) free-body diagram with cut at x_D.

Example: Tension in a two-force body
Recall that in the first example of sec. 5.4 we found T without ever using information about the location of the free-body diagram cut. So the location does not affect the tension. For a two-force body the tension is a constant along the length.

Example: Tension in a rod from its own weight.
The uniform $1\,\text{cm}^2$ steel square rod with density $\rho = 7.7\,\text{gm}/\,\text{cm}^3$ and length $\ell = 100\,\text{m}$ has total weight $W = mg = \rho\ell Ag$ (see fig. 8.1). What is the tension a distance x_D from the top? Using the free-body diagram with cut at x_D we get:

$$\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath}$$

$$\begin{aligned}
\Rightarrow \quad T &= \rho Ag(\ell - x_D) \\
&= (7.7\,\text{gm}/\,\text{cm}^3)(1\,\text{cm}^2)(9.8\,\text{N}/\,\text{kg})(100\,\text{m} - x_D) \\
&= 7.7 \cdot 9.8 \frac{\text{gm}\,\text{N}\,\text{m}}{\text{cm}\,\text{kg}} \left(100 - \tfrac{x_D}{\text{m}}\right) \underbrace{\left(\frac{1\,\text{kg}}{1000\,\text{gm}}\right)}_{1} \underbrace{\left(\frac{100\,\text{cm}}{1\,\text{m}}\right)}_{1} \\
&= 7.5\left(100 - \tfrac{x_D}{\text{m}}\right)\,\text{N}.
\end{aligned}$$

So, at the bottom end at $x_D = 100\,\text{m}$ we get $T = 0$ and at the top end where $x_D = 0\,\text{m}$ we get $T = 750\,\text{N}$ and in the middle at $x_D = 50\,\text{m}$ we get $T = 375\,\text{N}$.

Because the free-body diagram cut location is variable, we can plot the internal forces as a function of position. This is most useful in civil engineering where an engineer wants to know the internal forces in a horizontal beam carrying vertical loads. Common examples include bridge platforms and floor joists.

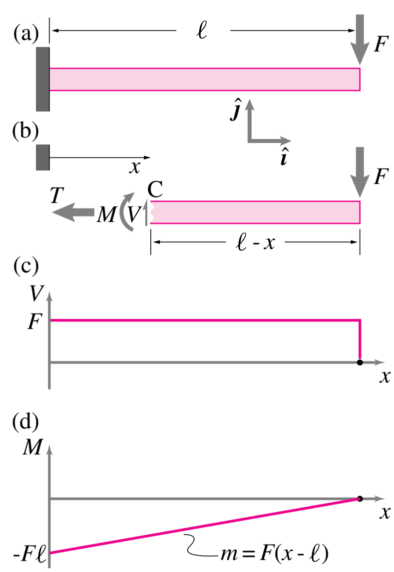

Figure 8.2: a) Cantilever beam, b) free-body diagram, c) Shear force diagram, & d) Bending moment diagram.

Example: Cantilever M and V diagram
A *cantilever* beam is mounted firmly at one end and has various loads orthogonal to its length, in this case a downwards load F at the end (fig. 8.2a). By drawing a free-body diagram with a cut at the arbitrary point C (fig. 8.2b) we can find the internal forces as functions of the position of C.

$$\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} \quad &\Rightarrow \quad & V &= F \\
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad &\Rightarrow \quad & T &= 0 \\
\left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad & M(x) &= F(x - \ell).
\end{aligned}$$

That the axial tension is zero in these problems is so well known that the tension is often not drawn on the free-body diagram, and isn't calculated. We can now plot $V(x)$ and $M(x)$ as in figs. 8.2c and 8.2d. In this case the shear force is a constant and the bending moment varies from its maximum magnitude at the wall ($M = -F\ell$) to 0 at the end. It is the big value of $|M|$ at the fixed support, the maximum over the length, that makes cantilever beams typically break at the ends.

Often one is interested in distributed loads from gravity on the structure itself or from a distribution (say of people on a floor). The method is the same.

Example: Distributed load
A cantilever beam has a downwards uniformly distributed load of w per unit length, *e.g.,* the weight of the beam, (fig. 8.3a). Using the free-body diagram shown (fig. 8.3b) we can find:

$$
\begin{aligned}
\left\{\sum \vec{F}_\mathrm{i} = \vec{0}\right\}\cdot \hat{\jmath} \quad &\Rightarrow \quad \{V(x)\hat{\jmath} \;+\; \textstyle\int d\vec{F}\}\cdot\hat{\jmath} = 0\\
&\Rightarrow \quad V(x) = \textstyle\int_x^\ell w\,dx'\\
&\qquad\qquad\;\; = w\cdot(\ell - x)
\end{aligned}
$$

$$
\begin{aligned}
\left\{\sum \vec{M}_\mathrm{C} = \vec{0}\right\}\cdot \hat{k} \quad &\Rightarrow \quad \left\{M(x)(-\hat{k}) \;+\; \textstyle\int \vec{r}_{/\mathrm{C}}\times d\vec{F}\right\}\cdot\hat{k} = 0\\
&\Rightarrow \quad M(x) = \textstyle\int_x^\ell (x'-x)w\,dx'\\
&\qquad\qquad\;\; = w\cdot\left(x'^2/2 - x'x\right)\Big|_x^\ell\\
&\qquad\qquad\;\; = (\ell^2/2 - \ell x) - (x^2/2 - x^2)\\
&\qquad\qquad\;\; = -w\cdot(\ell - x)^2/2.
\end{aligned}
$$

The integrals were used because of their general applicability for distributed loads. For this problem we could have avoided the integrals by using an equivalent downwards force $w\cdot(\ell - x)$ applied a distance $(\ell - x)/2$ to the right of the cut. Shear and bending moment diagrams are shown in figs. 8.3a and 8.3b.

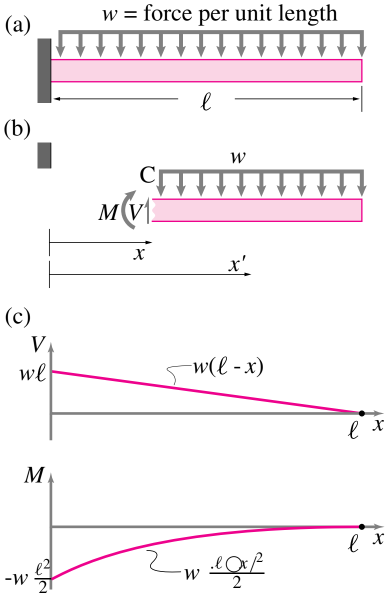

Figure 8.3: a) Cantilever beam with distributed load, b) free-body diagram, c) Shear force diagram, & d) Bending moment diagram.

As for all problems based on the equilibrium equations and a given geometry, the principle of superposition applies.

Example: Superposition
Consider a cantilever beam that simultaneously has both of the loads from the previous two examples. By the principle of superposition:

$$
\begin{aligned}
V &= F + w(\ell - x)\\
M(x) &= F(x-\ell) + -w(\ell - x)^2/2.
\end{aligned}
$$

The shear force at every point is the sum of the shear forces from the previous examples. The bending moment at every point is the sum of the bending moments.

If there are concentrated loads in the middle of the region of interest the calculation gets more elaborate; the concentrated force may or may not show up on the free-body diagram of the cut bar, depending on the location of the cut.

Example: Simply supported beam with point load in the middle

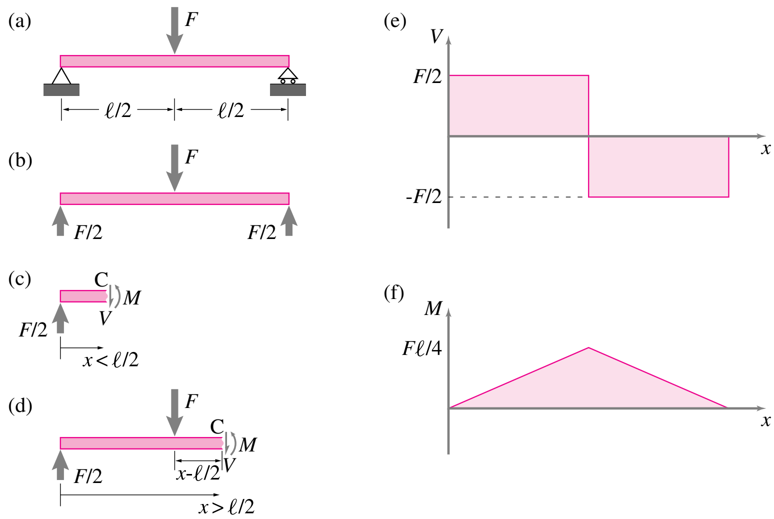

Figure 8.4: a) Simply supported beam, b) free-body diagram of whole beam, c) free-body diagram with cut to the left of the applied force, d) free-body diagram with cut to the right of the applied force e) Shear force diagram, f) Bending moment diagram

A *simply supported* beam is mounted with pivots at both ends (fig. 8.4a). First we draw a free-body diagram of the whole beam (fig. 8.4a) and then two more, one with a cut to the left of the applied force and one with a cut to the right of the applied force (figs. 8.4c and 8.4d). With the free-body diagram 8.4c we can find $V(x)$ and $M(x)$ for $x < \ell/2$ and with the free-body diagram 8.4d we can find $V(x)$ and $M(x)$ for $x > \ell/2$.

$$
\begin{array}{lclcll}
\{\sum \vec{F}_i = \vec{0}\} \cdot \hat{\jmath} & \Rightarrow & V & = & F/2 & \text{for } x < \ell/2 \\
 & & & = & -F/2 & \text{for } x > \ell/2 \\
\{\sum \vec{M}_C = \vec{0}\} \cdot \hat{k} & \Rightarrow & M(x) & = & Fx/2 & \text{for } x < \ell/2 \\
 & & & = & F(\ell - x)/2 & \text{for } x > \ell/2
\end{array}
$$

These relations can be plotted as in figs. 8.4e and 8.4f. Some observations: For this beam the biggest bending moment is in the middle, the place where simply supported beams often break. Instead of the free-body diagrams shown in (c) and (d) we could have drawn a free-body diagram of the bar to the right of the cut and would have gotten the same $V(x)$ and $M(x)$. We avoided drawing a free-body diagram cut *at* the applied load where $V(x)$ has a discontinuity.

How to find T, V, and M

Here are some guidelines for finding internal forces and drawing shear and bending moment diagrams.

- Draw a free-body diagram of the whole bar.
- Using the free-body diagram above, find the reaction forces.
- Draw a free-body diagram(s) of the cut bar.
  - For each region between concentrated loads, draw one free-body diagram.

- Show the piece from the cut to one or the other end (So that all but the internal forces are known).
- Don't make cuts *at* intermediate points of connection or load application.

- Use the equilibrium equations to find T, V, or M (Moment balance about a point at the cut is a good way to find M.)
- Use the results above to plot $V(x)$ and $M(x)$. $T(x)$ is rarely plotted.
  - Use the same x scale for this plot as for the free-body diagram of the whole bar.
  - Put the plots directly under the free-body diagram of the bar (so you can most easily relate features of the loads to features of the V and M diagrams).

Stress is force per unit area

For a given load, if you replace one bar in tension with two bars side-by-side, you would imagine the tension in each bar would go down by a factor of 2. Thus the pair of bars should be twice as strong as a single bar. If you glued these side-by-side bars together you would again have one bar, but it would be twice as strong as the original bar. Why? Because it has twice the cross sectional area.

What makes a solid break is the force per unit area carried by the material. For an applied tension load T, the force per unit area on an interior free-body diagram cut is T/A. The force per unit area, that is normal to an internal free-body diagram cut, is called *tension stress* and denoted σ (lower case 'sigma', the Greek letter s).

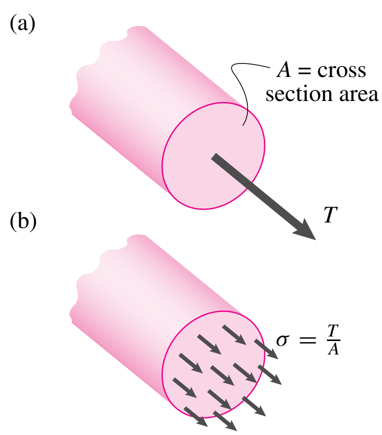

Figure 8.5: a) Tension on a free-body diagram cut is equivalent to b) uniform tension stress.

$$\sigma = T/A$$

Example: Stress in a hanging bar
Look at the hanging bar in the example on page 1. We can find the tension stress in this bar as a function of position along the bar as:

$$\sigma = \frac{T}{A} = \frac{\rho g A(\ell - x)}{A} = \rho g(\ell - x).$$

Note that the stress for this bar doesn't depend on the cross sectional area. The bigger the area the bigger the volume and hence the load. But also, the bigger the area on which to carry it.

For reasons that are beyond this book, the tension stress tends to be uniform in homogeneous (all one material) bars, no matter what their cross sectional shape, so that the average tension stress $\frac{T}{A}$ is actually the tension stress all across the cross section.

We can similarly define the average *shear stress* τ_{ave} ('tau') on a free body diagram cut as the average force per unit area tangent to the cut,

$$\tau_{\text{ave}} = \frac{V}{A}.$$

For reasons you may learn in a strength of materials class, shear stress is not so uniformly distributed across the cross section. But the average shear stress τ_{ave} does give an indication of the actual shear stress in the bar (*e.g.,* for a rectangular elastic bar, the peak shear stress is 50% larger than τ_{ave}).

The biggest stresses typically come from the bending moment. Motivating formulas for these stresses here is too big a digression. The formulas for the stresses due to bending moment are a key part of elementary strength of materials. But just knowing that these stresses tend to be larger, gives you the important notion that bending moment is a common cause of structural failure.

Internal force summary

'Internal forces' are the scalars that describe the force and moment on potential internal free-body diagram cuts. They are found by applying the equilibrium equations to free-body diagrams that have cuts at the points of interest. The internal forces are intimately associated with the internal stresses (force per unit area) and thus are important for determining the strength of structures.