SAMPLE 7.13 **A slider crank:** A torque $M = 20\,\text{N·m}$ is applied at the bearing end A of the crank AD of length $\ell = 0.2\,\text{m}$. If the mechanism is in static equilibrium in the configuration shown, find the load F on the piston.

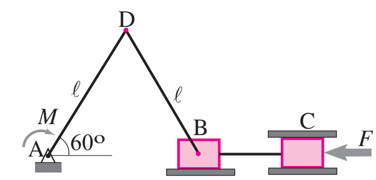

Figure 7.58

Solution The free-body diagram of the whole mechanism is shown in Fig. 7.59. From the moment equilibrium about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned}
\vec{\boldsymbol{M}} + \vec{\boldsymbol{r}}_{\text{B/A}} \times (\vec{\boldsymbol{B}} + \vec{\boldsymbol{F}}) &= \vec{\boldsymbol{0}} \\
-M\hat{\boldsymbol{k}} + 2\ell\cos\theta\hat{\boldsymbol{\imath}} \times (B_y\hat{\boldsymbol{\jmath}} - F\hat{\boldsymbol{\imath}}) &= \vec{\boldsymbol{0}} \\
(-M + 2B_y\ell\cos\theta)\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad B_y &= \frac{M}{2\ell\cos\theta}.
\end{aligned}$$

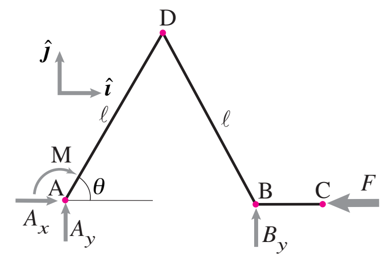

Figure 7.59

The force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$\begin{aligned}
(A_x - F)\hat{\boldsymbol{\imath}} + (A_y + B_y)\hat{\boldsymbol{\jmath}} &= 0 \\
A_x &= F \\
A_y &= -B_y
\end{aligned}$$

Note that we still need to find F or A_x. So far, we have had only three equations in four unknowns (A_x, A_y, B_y, F). To solve for the unknowns, we need one more equation. We now consider the free-body diagram of the mechanism without the crank, that is, the connecting rod DB and the piston BC together. See Fig. 7.60. Unfortunately, we introduce two more unknowns (the reactions) at D. However, we do not care about them. Therefore, we can write the moment equilibrium equation about point D, $\sum \vec{\boldsymbol{M}}_{\text{D}} = \vec{\boldsymbol{0}}$ and get the required equation without involving D_x and D_y.

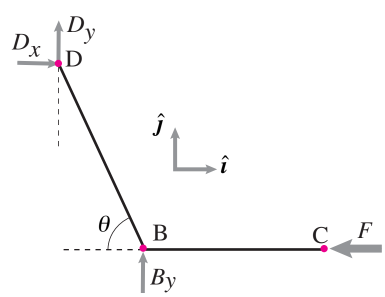

Figure 7.60

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{B/D}} \times (-F\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}} \\
\ell(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}) \times (-F\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}} \\
B_y\ell\cos\theta\hat{\boldsymbol{k}} - F\ell\sin\theta\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}}.
\end{aligned}$$

Dotting the last equation with $\hat{\boldsymbol{k}}$ we get

$$\begin{aligned}
F &= B_y\frac{\cos\theta}{\sin\theta} \\
&= \frac{M}{2\ell\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} \\
&= \frac{M}{2\ell\sin\theta} \\
&= \frac{20\,\text{N·m}}{2 \cdot 0.2\,\text{m} \cdot \sqrt{3}/2} \\
&= 57.74\,\text{N}.
\end{aligned}$$

$$\boxed{F = 57.74\,\text{N}}$$

Note that the force equilibrium carried out above is not really useful since we are not interested in finding the reactions at A. We did it above to show that just one free-body diagram of the whole mechanism was not sufficient to find F. On the other hand, writing moment equations about A for the whole mechanism and about D for the connecting rod plus the piston is enough to determine F.

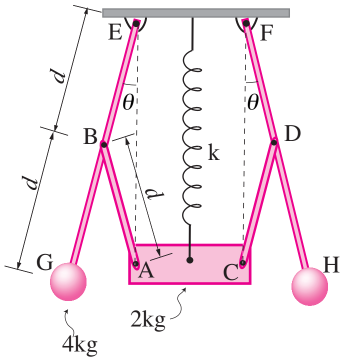

Figure 7.61

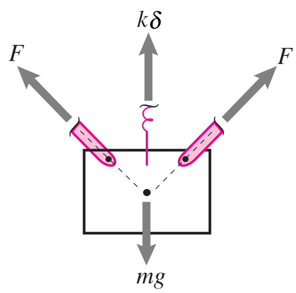

Figure 7.62

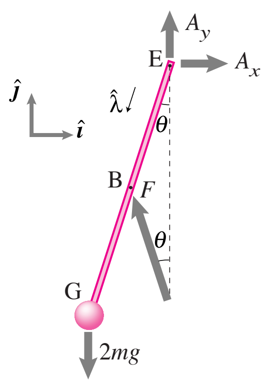

Figure 7.63

SAMPLE 7.14 A flyball governor: A flyball governor, usually considered in a dynamics context, is shown in static equilibrium. The relaxed length of the spring is 0.15 m and its stiffness is 500 N/m.

1. Find the static equilibrium position of the center collar.
2. Find the force in the strut AB.
3. How does the spring force required to hold the collar depend on θ?

Solution Let $\ell_0 (= 0.15\,\mathrm{m})$ denote the relaxed length of the spring and let ℓ be the stretched length in the static equilibrium configuration of the flyball, *i.e.*, the collar is at a distance ℓ from the fixed support EF. Then the net stretch in the spring is $\delta \equiv \Delta\ell = \ell - \ell_0$. We need to determine ℓ, the spring force $k\delta$, and its dependence on the angle θ of the ball-arm.

The free-body diagram of the collar is shown in fig. 7.62. Note that the struts AB and CD are two-force bodies (forces act only at the two end points on each strut). Therefore, the force at each end must act along the strut. From geometry (AB = BE = d), then, the strut force F on the collar must act at angle θ from the vertical. Now, the force balance in the vertical direction, *i.e.*, $[\sum \vec{F} = \vec{0}] \cdot \hat{\jmath}$, gives

$$-2F\cos\theta + k\delta = mg. \tag{7.15}$$

Thus to find δ we need to find F and θ. Now we draw the free-body diagram of arm EBG as shown in fig. 7.63. From the moment balance about point E, we get

$$\begin{aligned}
\vec{r}_{G/E} \times (-2mg\hat{\jmath}) + \vec{r}_{B/E} \times \vec{F} &= \vec{0} \\
2d\hat{\lambda} \times (-2mg\hat{\jmath}) + d\hat{\lambda} \times F(-sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) &= \vec{0} \\
-4mgd(\underbrace{\hat{\lambda} \times \hat{\jmath}}_{-\sin\theta\hat{k}}) + Fd[-\sin\theta(\underbrace{\hat{\lambda} \times \hat{\imath}}_{\cos\theta\hat{k}}) + \cos\theta(\underbrace{\hat{\lambda} \times \hat{\jmath}}_{-\sin\theta\hat{k}})] &= \vec{0} \\
4mgd\sin\theta\hat{k} + Fd(-\sin\theta\cos\theta\hat{k} - \cos\theta\sin\theta\hat{k}) &= \vec{0} \\
(4mgd\sin\theta - 2Fd\sin\theta\cos\theta)\hat{k} &= \vec{0}.
\end{aligned}$$

Dotting this equation with $\hat{k}$ and assuming that $\theta \neq 0$, we get

$$2F\cos\theta = 4mg. \tag{7.16}$$

Substituting eqn. (7.16) in eqn. (7.15) we get

$$\begin{aligned}
k\delta &= mg + 2F\cos\theta = mg + 4mg = 5mg \\
\Rightarrow \quad \delta &= \frac{5mg}{k} = \frac{5 \cdot 2\,\mathrm{kg} \cdot 9.81\,\mathrm{m/s^2}}{500\,\mathrm{N/m}} = 0.196\,\mathrm{m}.
\end{aligned}$$

1. The equilibrium configuration is specified by the stretched length ℓ of the spring (which specifies θ). Thus,

   $$\ell = \ell_0 + \delta = 0.15\,\mathrm{m} + 0.196\,\mathrm{m} = 0.346\,\mathrm{m}.$$

   Now, from $\ell = 2d\cos\theta$, we find that $\theta = 30.12°$.
2. The force in strut AB is

   $$F = 2mg/\cos\theta = 45.36\,\mathrm{N}.$$

3. The force in the spring $k\delta = 5mg$ as shown above and thus, it does not depend on θ! In fact, the angle θ is determined by the relaxed length of the spring.

(a) $\ell = 0.346\,\mathrm{m}$, (b) $F = 45.36\,\mathrm{N}$, (c) $k\delta \neq f(\theta)$

SAMPLE 7.15 : A motor housing support: A slotted arm mechanism is used to support a motor housing that has a belt drive as shown in the figure. The motor housing is bolted to the arm at B and the arm is bolted to a solid support at A. The two bolts are tightened enough to be modeled as welded joints (*i.e.,* they can also take some torque). Find the support reactions at A.

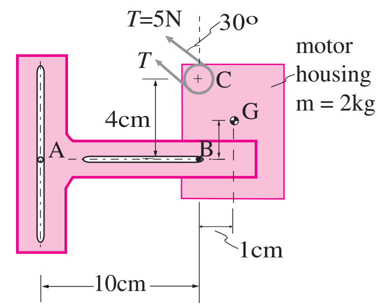

Figure 7.64

Solution Although the mechanism looks complicated, the problem is straightforward. We cut the bolt at A and draw the free-body diagram of the motor housing plus the slotted arm. Since the bolt, modeled as a welded joint, can take some torque, the unknowns at A are $\vec{\boldsymbol{A}}(=A_x\hat{\boldsymbol{\imath}}+A_y\hat{\boldsymbol{\jmath}})$ and $\vec{\boldsymbol{M}}_{\mathrm{A}}$. The free-body diagram is shown in fig. 7.65. Note that we have replaced the tension at the two belt ends by a single equivalent tension $2T$ acting at the center of the axle. Now taking moments about point A, we get

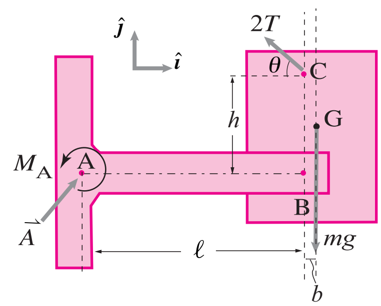

Figure 7.65

$$\vec{\boldsymbol{M}}_{\mathrm{A}}+\vec{\boldsymbol{r}}_{\mathrm{C/A}}\times 2\vec{\boldsymbol{T}}+\vec{\boldsymbol{r}}_{\mathrm{G/A}}\times m\vec{\boldsymbol{g}}=\vec{\boldsymbol{0}}$$

where

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{C/A}}\times 2\vec{\boldsymbol{T}} &= (\ell\hat{\boldsymbol{\imath}}+h\hat{\boldsymbol{\jmath}})\times 2T(-\cos\theta\hat{\boldsymbol{\imath}}+\sin\theta\hat{\boldsymbol{\jmath}})\\
&= 2T(\ell\sin\theta+h\cos\theta)\hat{\boldsymbol{k}}\\
\vec{\boldsymbol{r}}_{\mathrm{G/A}}\times m\vec{\boldsymbol{g}} &= [(\ell+d)\hat{\boldsymbol{\imath}}+(\text{anything})\hat{\boldsymbol{\jmath}}]\times(-mg\hat{\boldsymbol{\jmath}})\\
&= -mg(\ell+d)\hat{\boldsymbol{k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{\boldsymbol{M}}_{\mathrm{A}} &= -\vec{\boldsymbol{r}}_{\mathrm{C/A}}\times 2\vec{\boldsymbol{T}}-\vec{\boldsymbol{r}}_{\mathrm{G/A}}\times m\vec{\boldsymbol{g}}\\
&= -2T(\ell\sin\theta+h\cos\theta)\hat{\boldsymbol{k}}+mg(\ell+d)\hat{\boldsymbol{k}}\\
&= -2(5\,\mathrm{N})(0.1\,\mathrm{m}\cdot\sin 60^\circ+0.04\,\mathrm{m}\cdot\cos 60^\circ)\hat{\boldsymbol{k}}\\
&\quad +2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot(0.1+0.01)\,\mathrm{m}\hat{\boldsymbol{k}}\\
&= 1.092\,\mathrm{N{\cdot}m}\hat{\boldsymbol{k}}.
\end{aligned}$$

The reaction force $\vec{\boldsymbol{A}}$ can be determined from the force balance, $\sum\vec{\boldsymbol{F}}=\vec{\boldsymbol{0}}$ as follows.

$$\begin{aligned}
\vec{\boldsymbol{A}}+2\vec{\boldsymbol{T}}+m\vec{\boldsymbol{g}} &= \vec{\boldsymbol{0}}\\
\Rightarrow\quad \vec{\boldsymbol{A}} &= -2\vec{\boldsymbol{T}}-m\vec{\boldsymbol{g}}\\
&= -10\,\mathrm{N}(-\frac{1}{2}\hat{\boldsymbol{\imath}}+\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}})-(-19.62\,\mathrm{N}\hat{\boldsymbol{\jmath}})\\
&= 5\,\mathrm{N}\hat{\boldsymbol{\imath}}+10.96\,\mathrm{N}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{M}}_{\mathrm{A}}=1.092\,\mathrm{N{\cdot}m}\hat{\boldsymbol{k}}\qquad\text{and}\qquad\vec{\boldsymbol{A}}=5\,\mathrm{N}\hat{\boldsymbol{\imath}}+10.96\,\mathrm{N}\hat{\boldsymbol{\jmath}}}$$

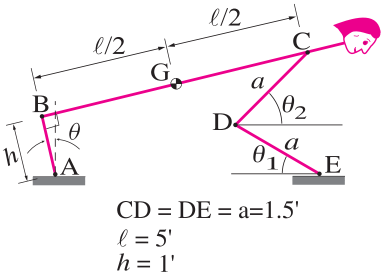

Figure 7.66

SAMPLE 7.16 Push-up mechanics: During push-ups, the body including the legs, usually moves as a single rigid unit; the ankle is almost locked, and the push-up is powered by the shoulder and the elbow muscles. A simple model of the body during push-ups is a four-bar linkage ABCDE shown in the figure. In this model, each link is a rigid rod, joint B is rigid (thus ABC can be taken as a single rigid rod), joints C, D, and E are hinges, but there is a motor at D that can supply torque. The weight of the person, $W = 150\,\text{lbf}$, acts through G. Find the torque at D for $\theta_1 = 30°$ and $\theta_2 = 45°$.

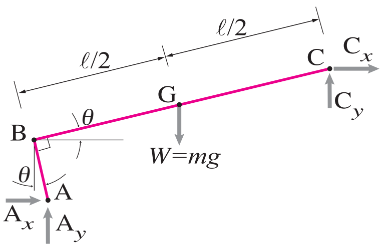

Figure 7.67

Solution The free-body diagram of part ABC of the mechanism is shown in fig. 7.67. Writing the moment-balance equation about point A, $\sum \vec{M}_A = \vec{0}$, we get

$$\vec{r}_C \times \vec{C} + \vec{r}_G \times \vec{W} = \vec{0}.$$

Let $\vec{r}_C = r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}$ and $\vec{r}_G = r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}$ for now (we can figure it out later). Then, the moment equation becomes

$$\begin{aligned}
(r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}) \times (C_x\hat{\imath} + C_y\hat{\jmath}) + (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) \times (-W\hat{\jmath}) &= \vec{0} \\
[(C_y r_{C_x} - C_x r_{C_y})\hat{\boldsymbol{k}} - W r_{G_x}\hat{\boldsymbol{k}} &= \vec{0}] \\
[\;]\cdot\hat{\boldsymbol{k}} \quad \Rightarrow \qquad C_y r_{C_x} - C_x r_{C_y} &= W r_{G_x}.
\end{aligned} \tag{7.17}$$

We now draw free-body diagrams of the links CD and DE separately (fig. 7.68) and write the moment- and force-balance equations for them.

For link CD, the force equilibrium $\sum \vec{F} = \vec{0}$ gives

$$(-C_x + D_x)\hat{\imath} + (D_y - C_y)\hat{\jmath} = \vec{0}.$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$ gives

$$\begin{aligned} D_x &= C_x \\ D_y &= C_y \end{aligned} \tag{7.18}$$

and the moment equilibrium about point D, gives

$$\begin{aligned}
M\hat{\boldsymbol{k}} - a(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath}) \times (-C_x\hat{\imath} - C_y\hat{\jmath}) &= \vec{0} \\
M\hat{\boldsymbol{k}} + (C_y a\cos\theta_2 - C_x a\sin\theta_2)\hat{\boldsymbol{k}} &= \vec{0}.
\end{aligned} \tag{7.19}$$

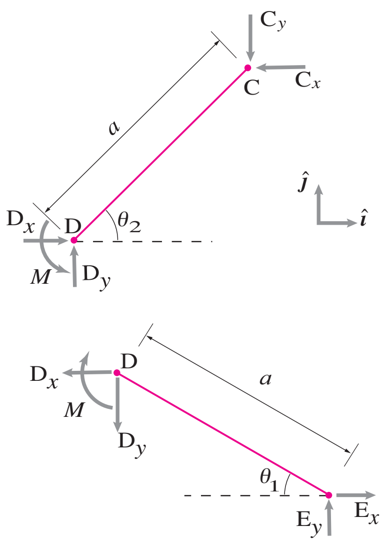

Figure 7.68

Similarly, the force equilibrium for link DE requires that

$$\begin{aligned} E_x &= D_x \\ E_y &= D_y \end{aligned} \tag{7.20}$$

and the moment equilibrium of link DE about point E gives

$$-M + D_x a\sin\theta_1 + D_y a\cos\theta_1 = 0. \tag{7.21}$$

Now, from eqns. (7.18) and (7.21)

$$-M + C_x a\sin\theta_1 + C_y a\cos\theta_1 = 0. \tag{7.22}$$

Adding eqns. (7.19) and (7.22) and solving for C_x, we get

$$C_x = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1} C_y.$$

For simplicity, let

$$f(\theta_1, \theta_2) = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1}$$

so that

$$C_x = f(\theta_1, \theta_2) C_y. \tag{7.23}$$

Now substituting eqn. (7.23) in (7.17), we get

$$C_y = \frac{r_{G_x}}{r_{C_x} - r_{C_y} f} W.$$

Now substituting C_y and C_x into eqn. (7.22) we get

$$M = \frac{r_{G_x} a(\cos\theta_1 + f\sin\theta_1)}{r_{C_x} - r_{C_y} f} W$$

where

$$\begin{aligned} r_{G_x} &= (\ell/2)\cos\theta - h\sin\theta \\ r_{C_x} &= \ell\cos\theta - h\sin\theta \\ r_{C_y} &= \ell\sin\theta + h\cos\theta. \end{aligned}$$

Now plugging in all the given values: $W = 160\,\text{lbf}, \theta_1 = 30°, \theta_2 = 45°, \ell = 5\,\text{ft}, h = 1\,\text{ft}, a = 1.5\,\text{ft}$, and, from simple geometry, $\theta = 9.49°$,

$$\begin{aligned} f &= 7.60 \\ r_{C_x} &= 4.77\,\text{ft}, \\ r_{C_y} &= 1.81\,\text{ft}, \\ r_{G_x} &= 2.30\,\text{ft} \\ \Rightarrow \quad M &= -269\,\text{lb}\cdot\text{ft}. \end{aligned}$$

$$M = -269\,\text{lb}\cdot\text{ft}$$

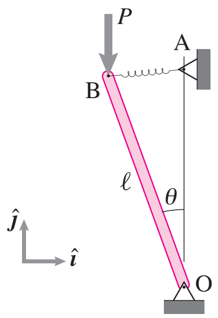

Figure 7.69

SAMPLE 7.17 **A spring and rod buckling model:** A simple model of sideways buckling of a flexible (elastic) rod can be constructed with a spring and a *rigid* rod as shown in the figure. The solution comes out most beautifully if we assume that the rod is held in place with a zero-rest-length spring that has no tension when B is on top of A. If $P = 0$ the system is in equilibrium with the rod vertical, $\theta = 0$. We find the so-called 'buckling' load by finding a load P at which there are solutions with $\theta \neq 0$. So, we assume the rod to be in static equilibrium at some angle $\theta \neq 0$ from the vertical. The vertical load is P, spring stiffness k, and bar length ℓ. For this problem, with a zero-rest-length spring we need not assume small angles.

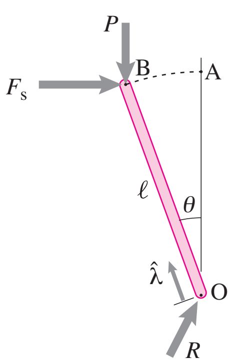

Figure 7.70

Solution When the rod is displaced from its vertical position, the zero-rest-length spring gets stretched, *i.e.*, , it is now under tension. The spring then exerts a force on the rod in the opposite direction of the tilt. The free-body diagram of the rod with a counterclockwise tilt θ is shown in Fig. 7.70. From the moment balance $\sum \vec{\boldsymbol{M}}_O = \vec{\boldsymbol{0}}$ (about the bottom support point O of the rod, which is also used as the origin for finding position vectors in the calculations below), we have

$$\vec{\boldsymbol{r}}_B \times \vec{\boldsymbol{P}} + \vec{\boldsymbol{r}}_B \times \vec{\boldsymbol{F}}_s = \vec{\boldsymbol{0}}.$$

Noting that

$$\begin{aligned} \vec{\boldsymbol{r}}_B &= \ell\hat{\boldsymbol{\lambda}}, \\ \vec{\boldsymbol{P}} &= -P\hat{\boldsymbol{\jmath}}, \\ \text{and} \quad \vec{\boldsymbol{F}}_s &= k(\vec{\boldsymbol{r}}_A - \vec{\boldsymbol{r}}_B) \quad \text{(For a zero-rest-length spring)} \\ &= k(\ell\hat{\boldsymbol{\jmath}} - \ell\hat{\boldsymbol{\lambda}}), \end{aligned}$$

we get

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (P\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{\lambda}} \times k\ell(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\lambda}}) &= \vec{\boldsymbol{0}} \\ -P\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + k\ell^2(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{0}}. \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$ we get

$$\begin{aligned} -P\ell + k\ell^2 &= 0 \\ \Rightarrow \quad P &= k\ell. \end{aligned}$$

Thus the equilibrium only requires that P be equal to $k\ell$ and it is independent of θ! That is, the system will be in static equilibrium at any θ as long as $P = k\ell$.

If $P = k\ell$, any θ is an equilibrium position. This is the 'buckling' load.