7.2 Force amplification devices: Levers, wedges, toggles, gears, and pulleys

Simple objects can be connected in various arrangements for various purposes. Here we describe 5 machine fragments that can be used to amplify force. Most machines use these ideas in combination. It might help intuitive understanding of machines to recognize one of these methods in use, although precise categorization of every machine part as one or another of these devices is not possible.

A lever

One of the simplest machines, long understood, and even longer used by humans, is a lever (*e.g.,* , fig. 7.26 and the top of fig. 7.27). Although now we think of statics as a special case of dynamics, statics is, and was, a well-founded subject, independent of dynamics. The statics of a lever was well understood 50 generations before Newton discovered modern dynamics.

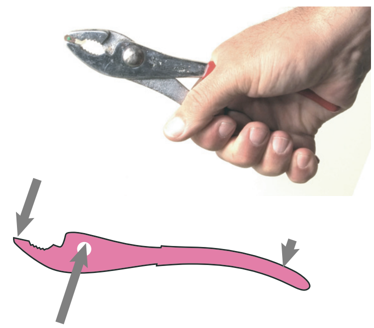


Figure 7.26: One half of a pair of pliers is one of many classic lever examples.

> An ideal lever is a rigid body held in place with a frictionless hinge and with two other applied loads.

The free-body diagram fig. 7.27 is the same whether the hinge is at point A, B or C①.

Lots of things can be viewed as levers including, for example, a wheelbarrow, a hammer extracting a nail, a boat oar, one half of a pair of tweezers, a brake lever, a gear, and, most generally, any three-force body. Using the equilibrium relations on the free-body diagram in fig. 7.27 you get that

$$\frac{F_A}{a} = \frac{F_B}{b} = \frac{F_C}{c}$$

from which you could find the relation between any pair of the forces. In practice it is easier to use moment balance about an appropriate point than to memorize and recall this formula.

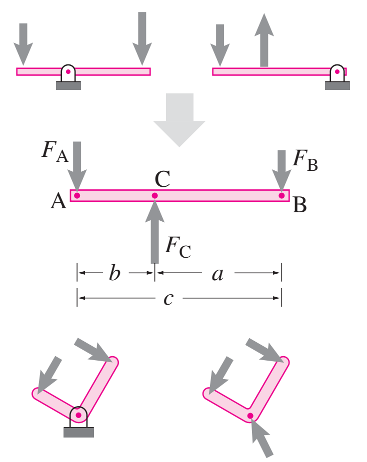


Figure 7.27: A lever can have the pivot in various places. The free-body diagram looks the same whether the pivot is at the left, middle, or right.

①People studied levers before mechanics was a quantitative subject. So they have a specialized antiquated vocabulary, classifying them according to which of A, B, or C is the pivot point and which of the other two forces you think of as input and which as the output: A 'class one' lever has the pivot in between; a 'class two' lever has the pivot at one end and the input force at the other; and a 'class three' lever has the pivot at one end and the input force in the middle.

An ideal wedge

Wedges are a kind of machine. You see them used in exactly their wedge configuration, to split and separate things (see fig. 7.28). But screws are also, effectively wedges, with torque replacing the downwards force. For an ideal wedge, one neglects friction, effectively replacing sliding contact with rolling contact (see fig. 7.29ab). Although this approximation may not be accurate, it is helpful for building intuition. Because the key idea depends on force balance and not moment balance, for the free-body diagrams of fig. 7.29c, we have not specified the exact location of the contact

Figure 7.28: A hatchet splitting wood is a classic wedge. In this case it is used dynamically, with the force coming from the deceleration of mass. But other more complex wood splitting devices push the wedge in slowly.

forces . Neglecting gravity,

$$\begin{aligned} &\text{For block A,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} && \Rightarrow && -F_A + F\sin\theta && = 0 \\ &\text{For block B,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} && \Rightarrow && -F_B + F\cos\theta && = 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan\theta} F_A.$$

Using the small-angle approximation that $\tan\theta \approx \theta$, we have that

> The force amplification of a wedge is roughly the reciprocal of the wedge angle (in radians).

To multiply the force F_A by 10 takes a wedge with a taper of $\theta = \tan^{-1} 0.1 \approx 6^{\circ}$. With this taper, an ideal wedge could also be viewed as a device to attenuate the force F_B by a factor of 10, although wedges are never used for force attenuation in practice, because friction often causes them to bind. A wedge with friction is considered in 7.12 on page 413.

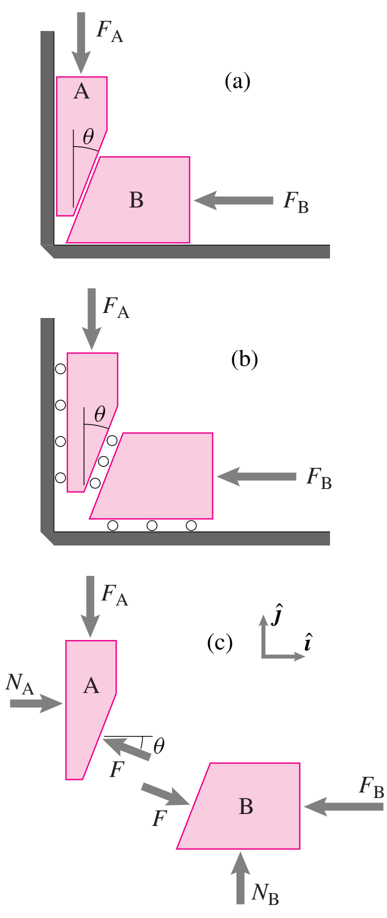


Figure 7.29: a) A wedge, b) treated as frictionless in the ideal case, c) free-body diagrams (assuming no friction)

A toggle

The classic *toggle* mechanism for amplifying force tends to have a 'snap-through' or bi-stable aspect which is used in the design of some electrical switches. Hence, perhaps, the two dictionary meanings of the word toggle: 1) a force amplifying mechanism, 2) a switch between two states (see fig. 7.32). The simplest version of the toggle mechanism is shown in fig. 7.30. The force amplification is $N/F = 1/\tan\theta$.

Usually the toggle concept is not used with a wall but with a pair of bars (fig. 7.31) Simple truss analysis shows the bar compressions are $-T = N/2\sin\theta$ and $N = F/2\tan\theta$. The toggle-like force amplification occurs for tension as well as compression. But, because of the oft-desirable snap-through and because the amplification increases as the applied force F moves down, the toggle is most often used in compression.

A toggle as lever and wedge. The distinction between toggles and wedges and levers is not precise. On the one hand the toggle is a lever where the lever arm of F is $\ell\cos\theta$ and the lever arm of N is $\ell\sin\theta$. On the other hand the toggle is sort of a rotary wedge with wedge angle θ.

Pulleys and Gears

Here we discuss a few more common machine components which are used to transmit and amplify or attenuate a force or moment.

Gears

One type of transmission is based on gears (fig. 7.34a). If we think of the input and output as the moments on the two gears, we find from the free-body diagram in fig. 7.34b that

$$
\begin{aligned}
&\text{For gear A,} \left\{\sum \vec{\boldsymbol{M}}_{\mathrm{i/A}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} && \Rightarrow && -R_{\mathrm{A}}F + M_{\mathrm{A}} &= 0\\
&\text{For gear B,} \left\{\sum \vec{\boldsymbol{M}}_{\mathrm{i/B}} = \vec{\boldsymbol{0}}\right\} \cdot \hat{\boldsymbol{k}} && \Rightarrow && -R_{\mathrm{B}}F + M_{\mathrm{B}} &= 0
\end{aligned}
$$

$$
\text{eliminating } F \quad \Rightarrow \quad M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B
$$

depending on which you want to think of input and which as output. The force amplification or attenuation ratio is just the radius ratio, just like for a lever.

Because the spacing of gear teeth for both of a meshed pair of gears is the same, a gear's circumference, and hence its radius is proportional to the number of teeth. And formulas involving radius ratios can just as well be expressed in terms of ratios of numbers of teeth. The tooth ratio is not just used as an approximation to the radius ratio. Averaged over the passage of several teeth, it is exactly the reciprocal ratio of the turning rates of the meshed gears.

Two gears pulled out of a bigger transmission are shown in fig. 7.34c. Gear A has an inner part with radius R_{A_i} welded to an outer part with radius R_{A_o}. Gear B also has an inner part welded to an outer part.

Moment balance about A in the first free-body diagram in fig. 7.34d gives that $R_{A_i} F_A = R_{A_o} F$. You can think of the one gear as a lever (see fig. 7.33). Moment balance about B in the second free-body diagram gives that $R_{B_i} F = R_{B_o} F_B$. Combining, we get

$$
F_B = \frac{R_{A_i}}{R_{A_o}} \frac{R_{B_i}}{R_{B_o}} F_A \quad \text{or} \quad F_A = \frac{R_{A_o}}{R_{A_i}} \frac{R_{B_o}}{R_{B_i}} F_B
$$

depending on which force you want to find in terms of the other. The transmission attenuates the force if you think of F_A as the input and amplifies the force if you think of F_B as the input. If the inner gears have one tenth the radius of the outer gears then the multiplication or attenuation is a factor of 100.

Trains of gears can build up large net gear ratios. The ratio of the fastest to slowest gear in a common clock or mechanical watch is on the order of 10,000.

In some gear trains, like the example above, large torque amplification comes from a large ratio of concentrically welded gears. A large amplification can also come from differences rather than ratios. The designs based primarily on differences rather than ratios are called 'differentials', 'harmonic drives', or 'planetary gears'.

Example: Planetary gear with a large ratio

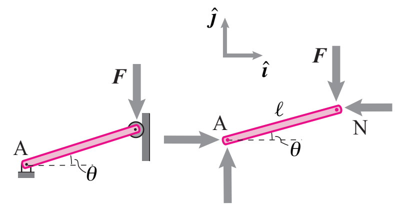


Figure 7.30: **A simple toggle mechanism.** Moment balance about point A gives that $N = F/\tan\theta$ so if θ is small N is large.

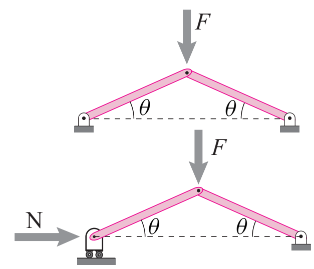


Figure 7.31: Two bars in a toggle mechanism.

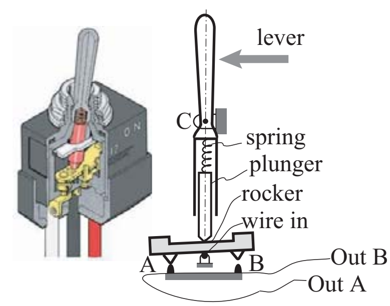


Figure 7.32: Cutaway drawing of a toggle switch and a simplified schematic of the idea. The lever, spring and plunger rotate about C. The spring is pre-stressed so that the plunger presses hard against the rocker. The plunger slides on the rocker (input line) which is pressed against the contact at outputs A or B. The relevant idea here is that the force of the plunger on the rocker can be much bigger than the input force on the lever (cutaway from NKK Switches online catalogue).

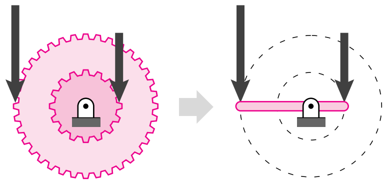


Figure 7.33: One gear may be thought of as a lever.

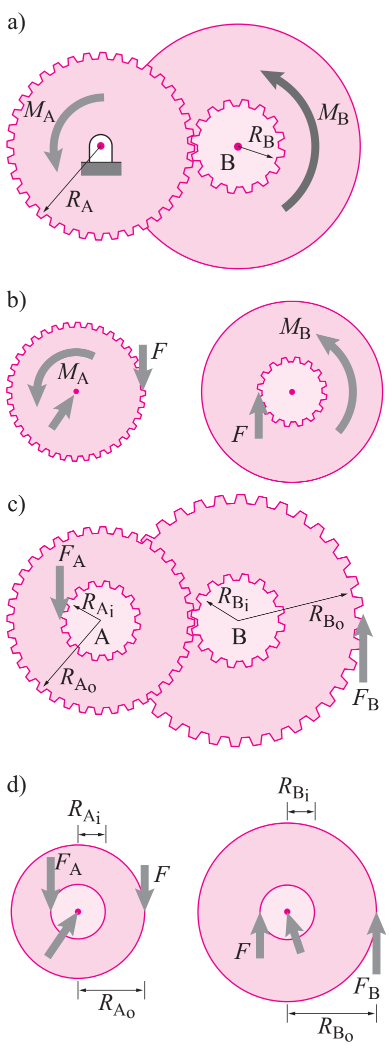


Figure 7.34: a) Two gear pairs pulled out of a transmission with forces on the teeth, b) Free-body diagrams, c) The same gear pair, but loaded with tooth-forces from unseen gears, d) the consequent free-body diagrams.

fig. 7.36 shows a gear design where the ratio of the input torque on the drive gear, to the output torque, on the spider can be huge. In particular, for the design shown, the torque ratio is approximately:

$$\frac{M_{out}}{M_{in}} \approx \frac{2}{R_D/R_R - 1}$$

where R_D is the ratio of the outer drive-gear to inner drive-gear radius and R_R is the ratio of the outer ring-gear to inner ring gear radius. Thus if the inner and outer drive gears have 49 and 50 teeth, respectively, and the inside and outside of the ring gear have 50 and 51 teeth, then the torque multiplication is nearly 5000. (See homework 7.2.17).

Pulleys

We have already studied a pulley as a single object (see page 261). Now we show, as you probably have learned a few times before in school, how to use pulleys to amplify or attenuate force. We assume pulleys are round, massless, and have frictionless bearings.

The key fact for statics analysis is

> For an ideal round pulley with negligible mass (or negligible angular acceleration) the tension on the cable is the same on both sides of the pulley:
>
> $$T_1 = T_2$$

The classic problem is shown in fig. 7.35a where you would like to use a pulley to make the task easier. Figures 7.35b-d show three possible uses of pulleys. If, at a glance, you can't see that these three designs are quite different in their effects you should puzzle them out slowly now.

Because the two tensions in the rope that wraps around the pulley are the same on both sides, the central rope has twice the tension. Design (b) gives no mechanical advantage but does allow one to pull down in order to lift the weight. Design (c) halves the required pulling force. Design (d), which might look superficially similar to (c) doubles the required pulling force, requiring 4 times the force of (c).

By using pulleys in combination, one can get various force attenuations and gains. The 10-pulley design in fig. 7.37 multiplies the force by about 1000.

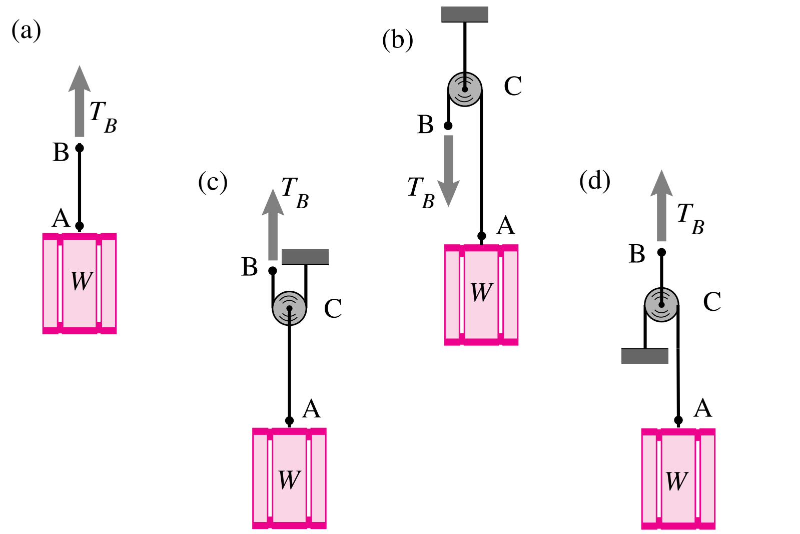


Figure 7.35: **Simple pulley arrangements. a)** Lifting a weight with no pulley, **b)** a pulley lets you pull down instead of up, **c)** a pulley halves the needed pull, **d)** a pulley doubles the needed pull.

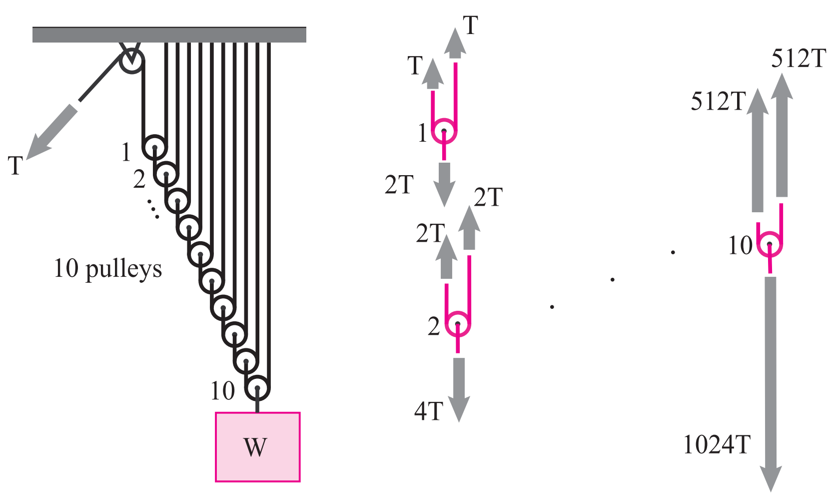


Figure 7.37: **Archimedes pulley.** A pulley arrangement sometimes commonly attributed to Archimedes. With 10 pulleys, he could lift a weight of W with a pull of about $T = 2^{-10}W \approx W/1000$. With 20 pulleys the force amplification is by more than a million. With 67 pulleys he, assuming he was as strong as the average person, could hoist a rock the size of the moon.

General force amplification concepts

If a mechanism generates a large force ratio (output/input), this usually corresponds to a large ratio in some geometric quantities. For a lever we have the ratio of two lever arms. For a wedge, the small wedge angle, and for a toggle also a small angle.

More precisely

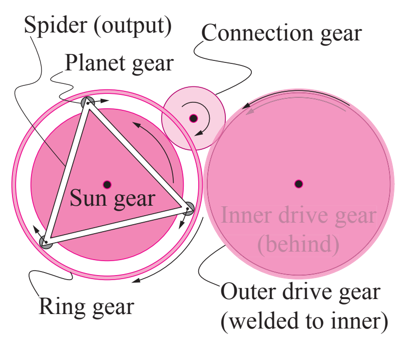


Figure 7.36: **A high-gain planetary gear system**, so named because the 'planet' gears go around the 'sun' gear. Often planetary gears depend on the sun or the spider or the ring not rotating. In this design, all three rotate. A CCW torque is applied to the inner drive gear which is welded to the outer drive gear (they are concentric and have nearly the same radius). The outer drive gear turns the ring gear CW. The inner drive gear turns the connection gear CW which turns the sun gear CCW. The three planet gears spin between the inside of the ring gear and the outside of the sun gear. If the inside of the ring gear moves a little faster down than the outside of the sun gear moves up, then the spider turns CW with the center of the planet gears. As drawn, this motion would be *much* slower than the motion of the drive gears.

For a frictionless transmission, the ratio of the input force to the output force is the reciprocal of the ratio of input motion to output motion.

For a high-gain lever, the handle moves much further than the load. For a narrow wedge, the slip distance is much bigger than the spreading distance. For a toggle the motion of the compressed end is much smaller than that of the applied load. That the force amplification is identical to the motion attenuation follows from energy conservation: the work into the mechanism equals the work out.

So, when Archimedes pulls in a kilometer of rope while lifting a rock, the moon, with 67 pulleys, that huge rock moves 2^{-67} km or about one hundred millionth of a nanometer. A big force, and a small motion. This is as it must be, because Archimedes only supplies so much work. If the transmission greatly amplifies force it must greatly attenuate motion. Similarly with levers. When Archimedes famously said:

> *Give me a lever long enough and a fulcrum on which to place it,*
> *and I shall move the world,*

he was careful not to say how far he would move it. People might have been less impressed if they knew that the distance of movement was far below the resolving power of all past, present and future microscopes.

SAMPLE 7.6 A wheeled suitcase of length 60 cm, height 30 cm and 'weighing' 20 kg on the airport check-in counter, has a telescopic handle of length 40 cm. The suitcase is dragged at an angle $\theta = 30°$. Assuming good wheels (negligible friction), find the force applied on the handle in order to wheel the suitcase steadily. (Take $g \approx 10\,\mathrm{m/s^2}$).

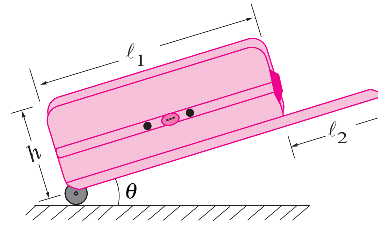


Figure 7.38

Solution The free-body diagram of the suitcase is shown in fig. 7.39. The reaction force at the wheel is almost vertical because of negligible friction. So, we can also assume the force F applied at the handle to be almost vertical. We assume that the center of mass G is located at the geometric center of the rectangular suitcase. Now the moment-balance equation about point A, $\sum M_A = 0$, gives

$$\vec{\boldsymbol{r}}_{\mathrm{G/A}} \times (-mg\hat{\boldsymbol{\jmath}}) + \vec{\boldsymbol{r}}_{\mathrm{B/A}} \times F\hat{\boldsymbol{\jmath}} = \vec{\mathbf{0}}.$$

Substituting $\vec{\boldsymbol{r}}_{\mathrm{G/A}} = \frac{\ell_1}{2}\hat{\boldsymbol{\lambda}} + \frac{h}{2}\hat{\boldsymbol{n}}$, $\quad \vec{\boldsymbol{r}}_{\mathrm{B/A}} = (\ell_1 + \ell_2)\hat{\boldsymbol{\lambda}}$, and noting that $\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}} = \cos\theta\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}} = -\sin\theta\hat{\boldsymbol{k}}$, we have

$$-\frac{1}{2}mg(\ell_1\cos\theta - h\sin\theta)\hat{\boldsymbol{k}} + F(\ell_1 + \ell_2)\cos\theta\hat{\boldsymbol{k}} = 0$$

$$\Rightarrow \quad F = \frac{1}{2}mg\left[\frac{\ell_1}{\ell_1 + \ell_2} - \frac{h}{\ell_1 + \ell_2}\tan\theta\right].$$

Substituting $\ell_1 = 60\,\mathrm{cm}$, $\ell_2 = 40\,\mathrm{cm}$, $h = 30\,\mathrm{cm}$, $\theta = 30°$ and $mg = 200\,\mathrm{N}$, we get

$$F = \frac{1}{2}\cdot 200\,\mathrm{N}\left[\frac{60\,\mathrm{cm}}{100\,\mathrm{cm}} - \frac{30\,\mathrm{cm}}{100\,\mathrm{cm}}\tan 30°\right] = 42.7\,\mathrm{N}.$$

$F = 42.7\,\mathrm{N}$

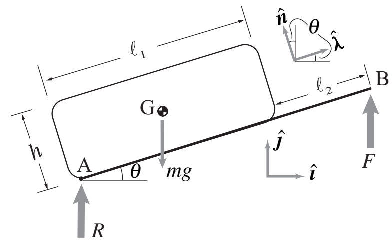


Figure 7.39: Free-body diagram of the suitcase. Unit vectors $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ are along the length and height of the suitcase, respectively. Thus, $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

SAMPLE 7.7 The figure shows a basic toggle mechanism. (a) If the applied force is $P = 20\,\mathrm{N}$ and the mechanism is in equilibrium at $\theta = 5°$, find the force applied by the spring. (b) If doubling the load P (to $P = 40\,\mathrm{N}$) causes a decrease of θ by 1° (to $\theta = 4°$), does the spring force at C double too?

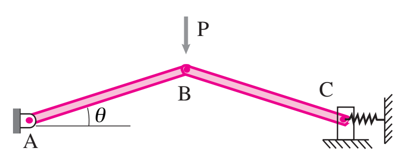


Figure 7.40: Toggle mechanism.

Solution The free-body diagrams of the pin connecting the two rods and the rod BC are shown in fig. 7.41. From the static equilibrium of the pin B, we have

$$\sum F_x = 0 \quad \Rightarrow \quad T_2\cos\theta - T_1\cos\theta = 0 \quad \Rightarrow \quad T_1 = T_2$$

$$\sum F_y = 0 \quad \Rightarrow \quad -(T_1 + T_2)\sin\theta - P = 0 \quad \Rightarrow \quad T_2 = -\frac{P}{2\sin\theta}.$$

which follows from setting $T_1 + T_2 = 2T_2$ since $T_1 = T_2$. Now, we consider the free-body diagram of rod BC. The force-balance equation in the x-direction ($\sum F_x = 0$) gives

$$-T_2\cos\theta - F = 0 \quad \Rightarrow \quad F = -T_2\cos\theta = \frac{P\cos\theta}{2\sin\theta}.$$

Since θ is small, we have $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Thus $F = P/2\theta$ where θ is in radians. Substituting $P = 20\,\mathrm{N}$ and $\theta = 5\pi/180$, we get

$$F = \frac{20\,\mathrm{N}}{2\pi/36} = 115\,\mathrm{N}$$

which is almost 6 times P.

(b) If P is doubled, we might expect F to double because $F \approx P/2\theta$. But if θ also decreases to $\theta = 4°$, repeating the calculation above with $P = 40\,\mathrm{N}$, and $\theta = 4°$ we get $F = 286\,\mathrm{N}$ which is 2.5 times the previous spring force.

For $P = 20\,\mathrm{N}, \theta = 5°, F = 115\,\mathrm{N}$; and for $P = 40\,\mathrm{N}, \theta = 4°, F = 286\,\mathrm{N}$

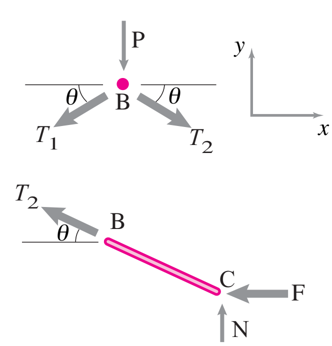


Figure 7.41: Free-body diagrams.

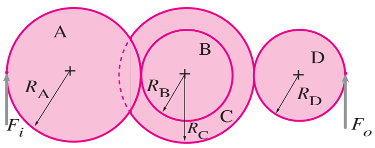


Figure 7.42

SAMPLE 7.8 **A gear train:** In the compound gear train shown in the figure, the various gear radii are: $R_A = 10\,\mathrm{cm}, R_B = 4\,\mathrm{cm}, R_C = 8\,\mathrm{cm}$ and $R_D = 5\,\mathrm{cm}$. The input load $F_i = 50\,\mathrm{N}$. Assuming the gears to be in static equilibrium find the machine load F_o.

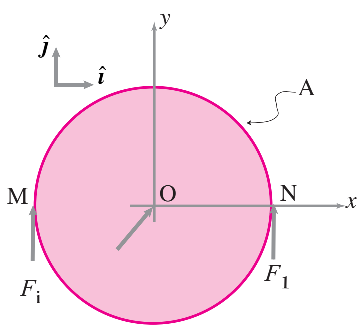


Figure 7.43

Solution You may be tempted to think that a free-body diagram of the entire gear train will do since we only need to find F_o. However, it is not so because there are unknown reactions at the axle of each gear and, therefore, there are too many unknowns. On the other hand, we can find the load F_o easily if we go gear by gear from the left to the right.

The free-body diagram of gear A is shown in Fig. 7.43. Let F_1 be the force at the contact tooth of gear A that meshes with gear B. From the moment balance about the axle-center O, $\sum \vec{\boldsymbol{M}}_O = \vec{\mathbf{0}}$, we have

$$\begin{aligned} \vec{\boldsymbol{r}}_M \times \vec{\boldsymbol{F}}_i + \vec{\boldsymbol{r}}_N \times \vec{\boldsymbol{F}}_1 &= \vec{\mathbf{0}} \\ -F_i R_A \hat{\boldsymbol{k}} + F_1 R_A \hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad F_1 &= F_i. \end{aligned}$$

Similarly, from the free-body diagram of gear B and C (together) we can write the moment balance equation about the axle-center P as

$$\begin{aligned} F_1 R_B \hat{\boldsymbol{k}} + F_2 R_C \hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad F_2 &= \frac{R_B}{R_C} F_1 \\ &= \frac{R_B}{R_C} F_i. \end{aligned}$$

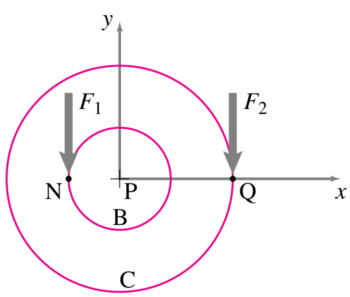


Figure 7.44

Finally, from the free-body diagram of the last gear D and the moment equilibrium about its center R, we get

$$\begin{aligned} -F_2 R_D \hat{\boldsymbol{k}} + F_o R_D \hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\ \Rightarrow \quad F_o &= F_2 \\ &= \frac{R_B}{R_C} F_i \\ &= \frac{4\,\mathrm{cm}}{8\,\mathrm{cm}} \cdot 50\,\mathrm{N} = 25\,\mathrm{N}. \end{aligned}$$

$\boxed{F_o = 25\,\mathrm{N}}$

SAMPLE 7.9 Find the force F to hold the 100 kg box shown in the figure in equilibrium. Assume $g \approx 10\,\mathrm{m/s^2}$.

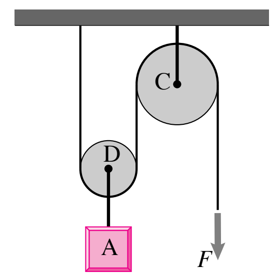


Figure 7.45

Solution The free-body diagrams of the two pulleys are shown in fig. 7.46 where the tension in the rope running over the two pulleys has been assumed as T. For the lower pulley D, the force balance in the y-direction, $\sum F_y = 0$, requires

$$2T - mg = 0 \quad \Rightarrow \quad T = \frac{mg}{2}.$$

The free-body diagram of the upper pulley C contains an unknown reaction force R at the attachment point C. However, if we write moment balance about point C, $\sum M_C = 0$, this unknown force contributes nothing. Let the radius of pulley C be r. Thus, the moment-balance equation about C gives

$$\begin{aligned} Tr - Fr &= 0 \\ \Rightarrow \quad F &= T = \frac{mg}{2} \approx 500\,\mathrm{N}. \end{aligned}$$

$\boxed{F \approx 500\,\mathrm{N}}$

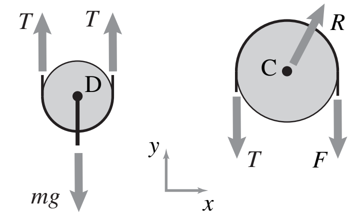


Figure 7.46: Free-body diagram of the two pulleys. We use C and D to designate both the pulleys and their respective centers.

SAMPLE 7.10 A container box weighing 1 kN is dragged slowly and steadily along the floor with force F as shown in the figure. The coefficient of friction between the box and the floor is 0.6. Find the force required to pull the box and the force amplification obtained by the pulley arrangement.

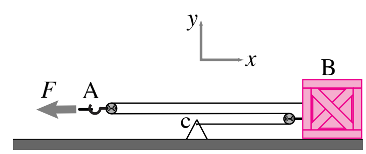


Figure 7.47

Solution It is clear from the figure that the same rope passes over the two pulleys used in the arrangement to pull the box. Let the tension in the rope be T. A partial free-body diagram (that includes forces acting only in the x-direction) of the box along with the pulley attached to it is shown in fig. 7.48. The same figure also shows the free-body diagram of pulley A at the force end. From the force-balance equation for the box in the x-direction, we get

$$f - 3T = 0 \quad \Rightarrow \quad T = \frac{f}{3} = \frac{\mu mg}{3}.$$

Now, from the force balance of pulley A in the x-direction, we get

$$\begin{aligned} 2T - F &= 0 \\ \Rightarrow \quad F &= 2T = \frac{2\mu mg}{3} \\ &= \frac{2 \cdot (0.6) \cdot (1\,\mathrm{kN})}{3} = 400\,\mathrm{N}. \end{aligned}$$

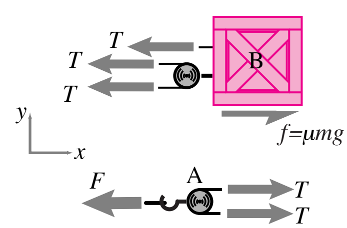


Figure 7.48

Since the force of friction on the box while sliding is $f = \mu mg = 0.6(1\,\mathrm{kN}) = 600\,\mathrm{N}$ and the force applied at A to overcome this friction is 400 N, the force amplification is 1.5. That is, the pulley arrangement amplifies the input force (400 N) 1.5 times at the output end.

$\boxed{F = 400\,\mathrm{N},\ \text{Force amplification} = 1.5}$

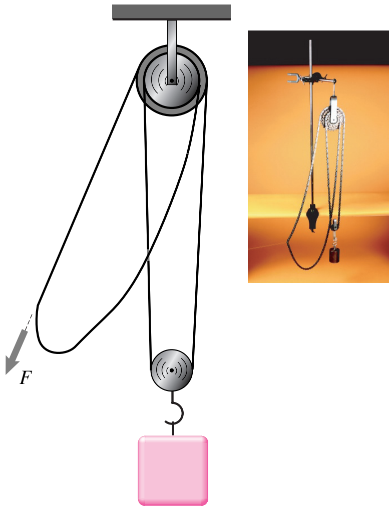


Figure 7.49

SAMPLE 7.11 A differential hoist is used to lift a crate of mass 500 kg. The hoist pulley uses two discs of radius 30 cm and 25 cm. Find the force F required to lift the crate steadily. Take $g \approx 10\,\text{m/s}^2$.

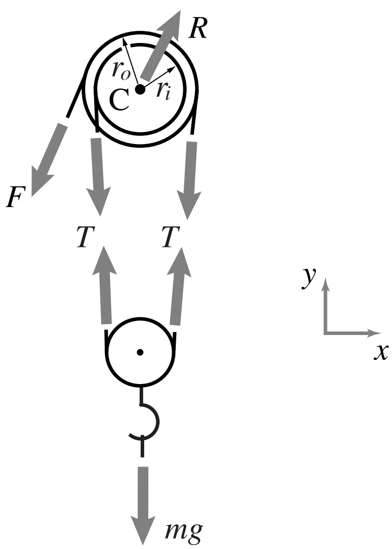


Figure 7.50

Solution The free-body diagrams of the upper pulley and the lower pulley are shown in fig. 7.50. Since the lower pulley is slightly smaller than the upper pulley, the chain passing over the two pulleys is not exactly vertical but makes a small angle with the vertical. Thus the tension forces shown in the free-body diagrams are slightly off from the vertical direction. However, since the angle is very small, we can treat T to be essentially vertical.

For the lower pulley, the force balance in the y direction gives

$$\begin{aligned} 2T - mg &= 0 \\ \Rightarrow \quad T &= \frac{mg}{2}. \end{aligned}$$

Now the moment balance about point C, $\sum M_C = 0$, for the upper pulley gives

$$\begin{aligned} Fr_o + Tr_i - Tr_o &= 0 \\ \Rightarrow \quad F &= \left(\frac{r_o - r_i}{r_o}\right)T \\ &= \left(1 - \frac{r_i}{r_o}\right)\frac{mg}{2} \\ &= \left(1 - \frac{25\,\text{cm}}{30\,\text{cm}}\right)\frac{5000\,\text{N}}{2} \\ &= 417\,\text{N} \end{aligned}$$

Thus the force amplification in this case is about 12 (5000 N/417 N). From the analysis above, it is also clear that the ratio of the radii of the two disks used in the upper pulley decides this force amplification. One can get a big force amplification, at least theoretically, by making $r_i \approx r_o$. In this problem, for example, if $r_i = 29\,\text{cm}$ rather than the given 25 cm, we get $r_i/r_o \approx 0.97$ giving $F \approx 83\,\text{N}$ which corresponds to a force amplification of 60.

$\boxed{F = 417\,\text{N}}$

SAMPLE 7.12 A wedge with friction. Consider the wedge described in the text (page 403), but now with friction μ between the blocks.

Consideration of friction qualitatively changes the behavior of the machine. For simplicity still take the wall and floor interactions to be frictionless.

1. What is the relation between F_A and F_B when block A is sliding down?
2. What is the relation between F_A and F_B when block A is sliding up?
3. Under what conditions is it impossible for F_B to slide block A up, even when F_A is vanishingly small? Such a case is called 'non-back-driveable' or 'self-locking'.

How do your answers simplify for small wedge angle θ and small friction angle ϕ (with $\tan\phi = \mu$).

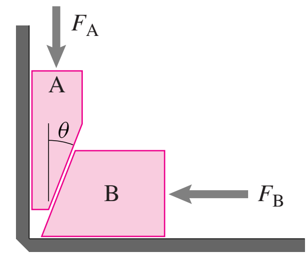


Figure 7.51: A wedge with friction μ between the blocks. Treat the walls as frictionless.

Solution Figure 7.52 shows free-body diagrams of wedge blocks. We draw separate free-body diagrams for the case when (a) block A is sliding down and block B to the right, and (b) block A is sliding up and block B to the left. In both cases the friction resists relative slip and obeys the sliding friction relation

$$F_f = \underbrace{\tan\phi}_{\mu} N,$$

where fig. 7.52 shows the resultant contact force (normal component plus frictional component) and its angle ϕ to the surface normal.

1. **Block A sliding down:** Assuming block A is sliding down we get from free-body diagram 7.52a that

$$\begin{array}{llll} \text{For block A}, \{\sum \vec{\boldsymbol{F}}_\mathrm{i} = \vec{\boldsymbol{0}}\} \cdot \hat{\boldsymbol{\jmath}} & \Rightarrow & -F_A + F\sin(\theta+\phi) & = 0 \\ \text{For block B}, \{\sum \vec{\boldsymbol{F}}_\mathrm{i} = \vec{\boldsymbol{0}}\} \cdot \hat{\boldsymbol{\imath}} & \Rightarrow & -F_B + F\cos(\theta+\phi) & = 0. \end{array}$$

   Eliminating F we get,

$$F_B = \frac{1}{\tan(\theta+\phi)} F_A. \qquad (7.13)$$

$$\boxed{\text{When A slides down} \quad \Rightarrow \quad F_B = \frac{1}{\tan(\theta+\phi)} F_A.}$$

   If we take a taper of 6° and a friction coefficient of $\mu = .3$ ($\Rightarrow \phi \approx 17^\circ$) we get that $F_B/F_A \approx 2.5$ instead of 10 as we got when neglecting friction. The wedge still serves as a way to multiply force, but substantially less so than the frictionless idealization led us to believe.

2. **Block A sliding up:** Now let's consider the case when force F_B pushes block B to the left, pinching block A, and forcing it up. The only change in the calculation is the change in the direction of the friction interaction force. From free-body diagram 7.52b

$$\begin{array}{llll} \text{For block A}, \{\sum \vec{\boldsymbol{F}}_\mathrm{i} = \vec{\boldsymbol{0}}\} \cdot \hat{\boldsymbol{\jmath}} & \Rightarrow & -F_A + F\sin(\theta-\phi) & = 0 \\ \text{For block B}, \{\sum \vec{\boldsymbol{F}}_\mathrm{i} = \vec{\boldsymbol{0}}\} \cdot \hat{\boldsymbol{\imath}} & \Rightarrow & -F_B + F\cos(\theta-\phi) & = 0 \end{array}$$

   Eliminating F we get,

$$F_A = \tan(\theta-\phi) F_B. \qquad (7.14)$$

$$\boxed{\text{When A slides up} \quad \Rightarrow \quad F_B = F_A/\tan(\theta-\phi).}$$

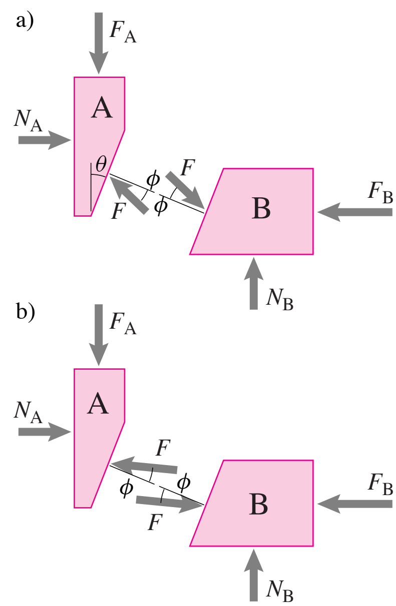


Figure 7.52: Free-body diagrams of a wedge a) assuming A slides down, b) assuming A slides up

Again using $\theta = 6^\circ$ and $\phi = 17^\circ$ we see that if $F_B = 100\,\text{lbf}$ then $F_A = \tan(-11^\circ) \cdot 100\,\text{lbf} \approx -20\,\text{lbf}$. That is, the 100 pounds doesn't push block A up at all, but even with no gravity, you need to pull up with a 20 pound force to get it to move.

If we insist that the downwards force F_A is positive or zero, that the pushing force F_B is positive, and that block A is sliding up then there is no solution to the equilibrium equations whenever $\phi > \theta$. (Actually we didn't need to do this second calculation at all. Eqn 7.13 shows the same paradox when $\theta + \phi > 90^\circ$. Trying to squeeze block B to the right for large θ is exactly like trying to squeeze block A up for small θ.)

3. **Self-locking:** This *self-locking* situation is intuitive. In fact it's hard to picture the contrary, that pushing a block like B would lift block A. If you view this wedge mechanism as a transmission, it is said to be *non-back-drivable* whenever $\phi > \theta$. Even though pushing down on A can 'drive' block B to the right, pushing to the left on block B cannot 'back-drive' block A up. Non-backdrivability is a feature or a defect depending on context①.

   The borderline case of backdrivability is when $\theta = \phi$ and $F_B = F_A / \tan 2\theta$. Assuming θ is a fairly small angle we get

$$F_B = \frac{F_A}{\tan 2\theta} \approx \frac{F_A}{2\theta} \approx \frac{1}{2}\frac{F_A}{\tan\theta}$$
$$\approx \frac{1}{2} \cdot \text{(the value of } F_B \text{ had there been no friction)}.$$

Thus the design guideline:

> Non-back-driveable transmissions are generally 50% or less efficient, they transmit 50% or less of the force they would transmit if they were frictionless.

To use a wedge in this backwards way requires very low friction. A rare case where a narrow wedge is back drivable is this: a fresh wet watermelon seed squeezed between two pinched fingers.

①Standard car transmissions are backdriven when they are push-started and when a driver downshifts to slow the car instead of using the brakes. On the other hand, most electric hand-mixers cannot be back-driven; you can't turn the motor by forcing the beater blade (*Unplug before trying.*)

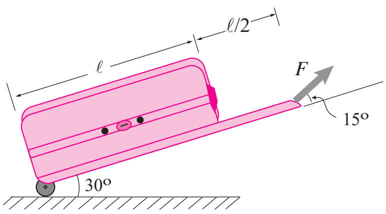
7.2 Levers, Wedges, Toggles, Gears and Pulleys

7.2.1 A suitcase with bad wheels has length $\ell = 0.5\,\text{m}$ and thickness (height) of $\ell = 0.25\,\text{m}$ is pulled along steadily with a force $F = 100\,\text{N}$ as shown in the figure.

a) Find the weight W of the suitcase.

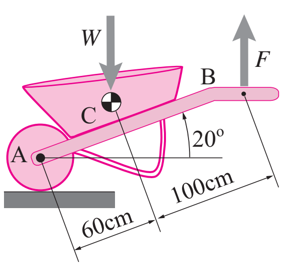
b) Find the ground force on the wheel (both magnitude and direction).

c) What is force amplification if you consider F as the input and W as the output.



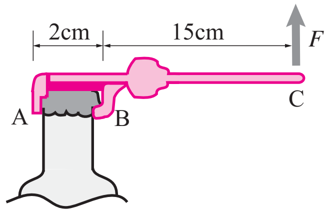
Problem 7.2.1

7.2.2 A wheelbarrow containing 100 kg of this-n-that is wheeled steadily with a force F as shown in the figure. For the given geometry and $g \approx 10\,\text{m/s}^2$, find the required force F.



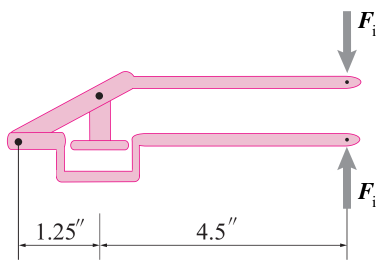
Problem 7.2.2

7.2.3 A bottle-opener ABC contains a cut-out AB of approximate diameter 2 cm that clamps on the bottle cap. The arm BC is approximately 15 cm long. If the cap is opened by applying a vertical force $F = 10\,\text{N}$ at C, find the force on the cap at B.



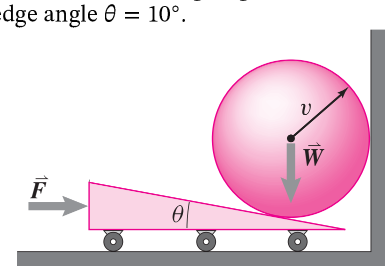
Problem 7.2.3

7.2.4 A cut-out view of a garlic press is shown in the figure. For an input force $F_i = 10\,\text{lbf}$, find the output force F_o at the site of the press. What is the force amplification?



Problem 7.2.4

7.2.5 Assuming all frictionless contacts, find the force F on the wedge required to lift the sphere weighing 500 N if the wedge angle $\theta = 10°$.

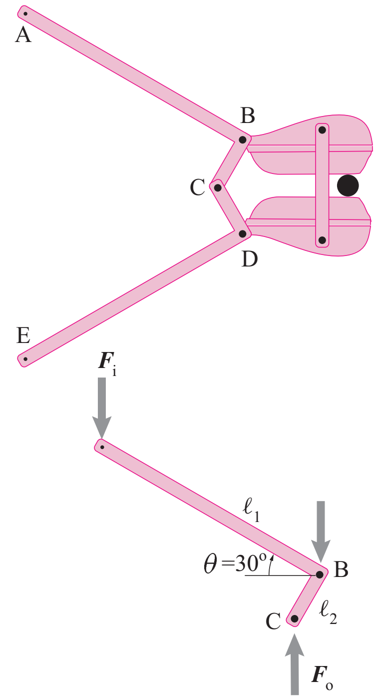


Problem 7.2.5

7.2.6 A cutter, shown in the figure, uses a toggle mechanism BCD to get a big force amplification at the cutting edge. A partial free-body diagram of one of the arms of the cutter is shown in the figure. Assuming an input force of $F_i = 20\,\text{N}$ at A, find the intermediate output force F_o at C when

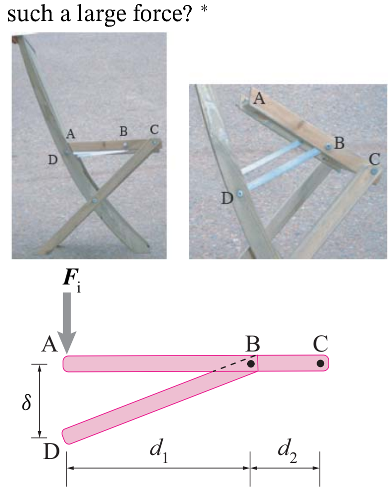
a) $\theta = 30°$,

b) $\theta = 10°$.



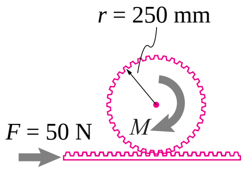
Problem 7.2.6

7.2.7 A toggle-like mechanism is used in a folding chair shown in the pictures here. The metallic link DB gets almost parallel to the seating plank AC when the chair is open. Given the dimensions $d_1 = 30\,\text{cm}, d_2 = 10\,\text{cm}, \delta = 2\,\text{cm}$ and the force at A, $F_i = 500\,\text{N}$, find the force in the link DB. Why is this force so big or small? In practice why would you never see such a large force? *



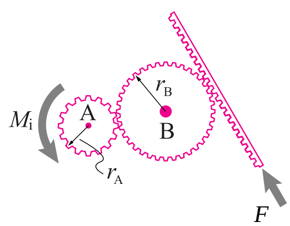
Problem 7.2.7

7.2.8 A gear of radius 250 mm is meshed in with a rack that carries a horizontal load $F = 50\,\text{N}$. Find the torque M on the gear that is required for equilibrium.



Problem 7.2.8

7.2.9 The input gear A of radius r_A = 10 cm drives gear B that is one and a half times bigger than gear A. Gear B, in turn, drives a rack. If the input torque on gear A is $M_i = 30\,\text{N·m}$, find the load F on the rack.

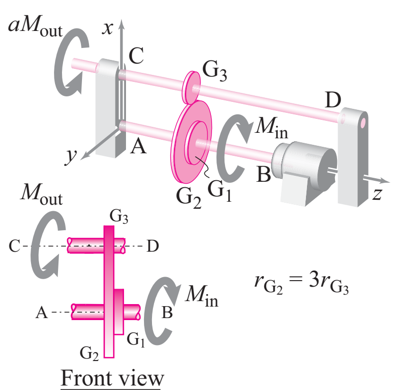


Problem 7.2.9

7.2.10 In the gear arrangement shown, gears G_1 and G_2 are welded together. The output gear G_3 is one third the size of gear G_2.

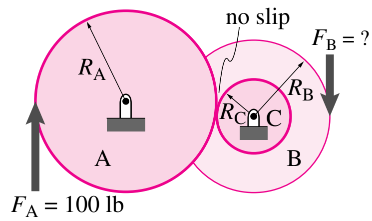
a) Is this gear train for torque amplification or for torque reduction?

b) If the input torque M_{in} on gear G_1 is 300 N·m, find the output torque M_{out}.



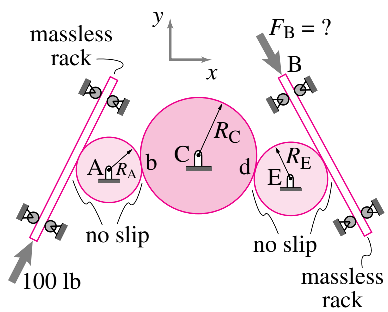
Problem 7.2.10

7.2.11 At the input to a gear box, a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of F_B to the output gear. Assume the system of gears is at rest. What is F_B?



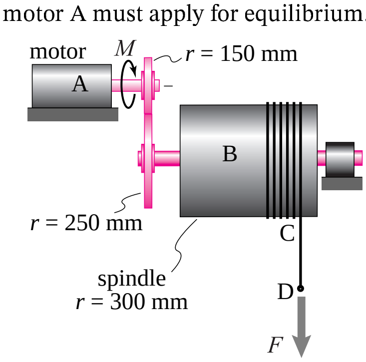
Problem 7.2.11

7.2.12 A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack. Assume the gear-train is at rest. What is F_B?



Problem 7.2.12

7.2.13 The gear train and spindle shown in the figure are used for hoisting heavy loads. For the dimensions given, if the load $F = 2\,\text{kN}$, find the torque M that the motor A must apply for equilibrium.



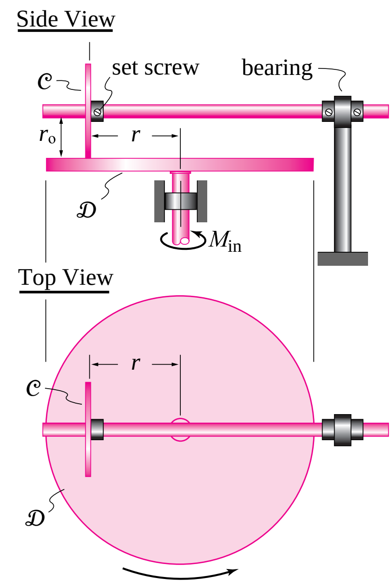
Problem 7.2.13

7.2.14 The figure shows a brush gear (also called a crown wheel) where wheel $\mathcal{C}$ of radius r_o, rolls on the surface of wheel $\mathcal{D}$ without slipping. In addition, the position r of wheel $\mathcal{C}$ from the center of wheel D can be varied. Let the input torque on wheel $\mathcal{D}$ be M_i.

a) Find the output torque M_o on wheel $\mathcal{C}$ as a function of r.

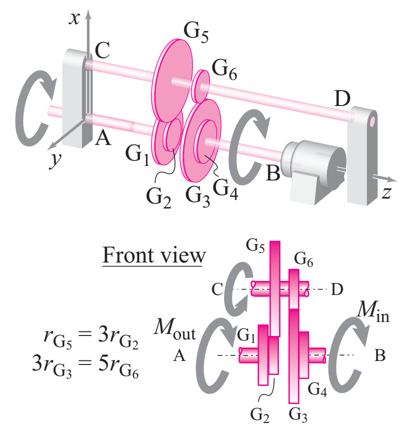
b) Find the output torque M_o when $r = r_o$ and when $r = r_o/4$.

c) If the output torque were not to exceed 100 times the input torque, where will you put safety latches on the axle of wheel $\mathcal{C}$?



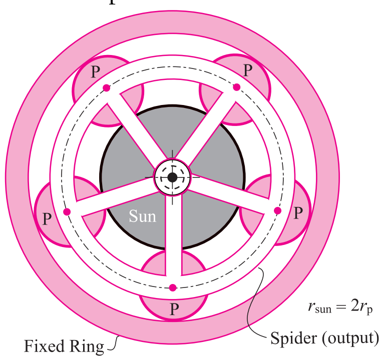
Problem 7.2.14

7.2.15 For the gear train shown in the figure, find the torque amplification $M_{\text{out}}/M_{\text{in}}$.



Problem 7.2.15

7.2.16 A torque amplifying planetary gear is shown in the figure where the sun-gear is free to rotate but the ring-gear is fixed. The sun-gear drives five planet-gears that drive the spider-gear through their axles housed in bearings in the spider. The radius of the planet-gears $r_p = 50\,\text{mm}$ and the radius of the sun-gear is twice as big. If the input torque on the sun-gear is 2000 N·m, find the output torque on the spider.

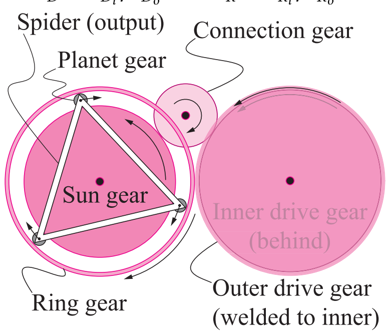


Problem 7.2.16

7.2.17 Consider the high gear ratio planetary gear discussed on page 406 of the text. Let r_{D_i} and r_{D_o} be the inner and outer radii, respectively, of the drive gear, and r_{R_i} and r_{R_o} be the inner and outer radii, respectively, of the ring-gear. Let r_s and r_p denote the radii of the sun and the planet-gear respectively. Show that the ratio of the output torque M_{out} on the spider-gear to the input torque M_{in} on the drive gear is approximately given by

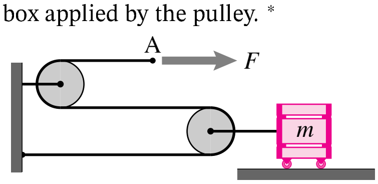
$$\frac{M_{out}}{M_{in}} \approx \frac{2}{\frac{R_D}{R_R} - 1}$$

where $R_D = r_{D_i}/r_{D_o}$ and $R_R = r_{R_i}/r_{R_o}$.



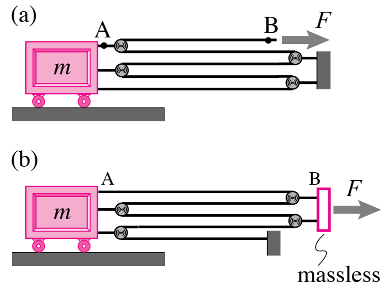
Problem 7.2.17

7.2.18 A force $F = 100\,\text{N}$ acts at A. The pulleys are frictionless. Find the force on the box applied by the pulley. *



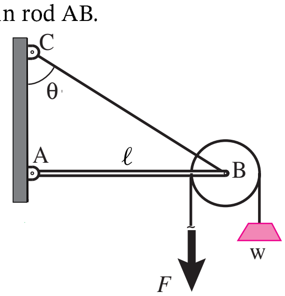
Problem 7.2.18

7.2.19 A force F is applied as shown in the pulley arrangements shown in (a) and (b). Which arrangement gives a bigger force amplification on the box?



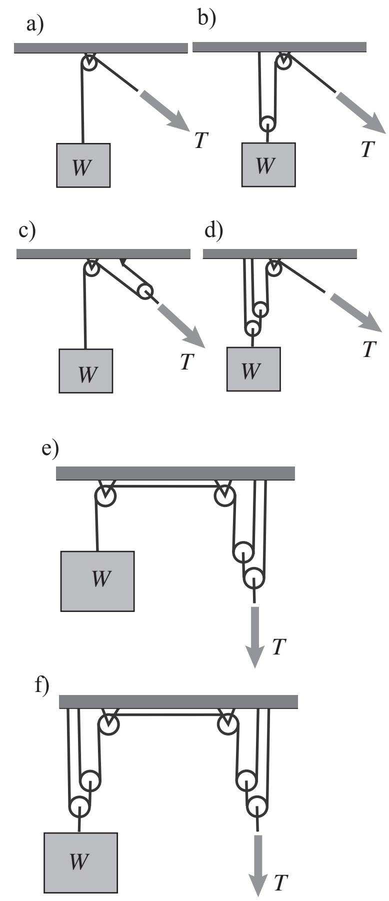
Problem 7.2.19

7.2.20 A weight W is held in place with a force $F = 100\,\text{N}$ applied through a massless pulley as shown in the figure. The pulley is attached to a rod AB which, in turn, is held horizontal with the help of a string CB. Find the tension (or compression) in rod AB.



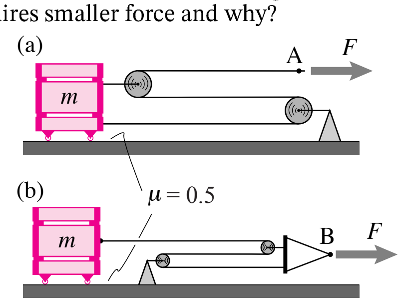
Problem 7.2.20

7.2.21 Given W and the frictionless pulleys shown find the tension T needed to lift the weight in the situations shown.



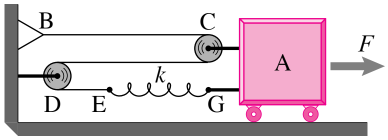
Problem 7.2.21

7.2.22 In the two cases shown in (a) and (b), find the maximum force F that can be applied before the box starts skidding on the ground. Take $m = 50\,\text{kg}$ and $g \approx 10\,\text{m/s}^2$. Which arrangement requires smaller force and why?



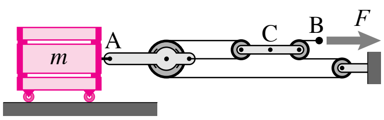
Problem 7.2.22

7.2.23 The pulley arrangement shown in the figure uses a spring EG of stiffness $k = 200\,\text{N/cm}$. If the spring is stretched by 1.5 cm under the application of force F for equilibrium, find F.



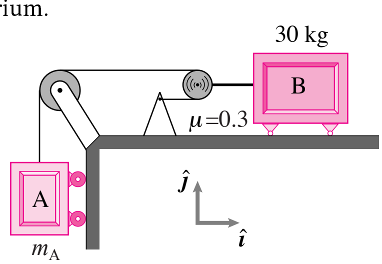
Problem 7.2.23

7.2.24 Find the force on the mass at A in terms of F and thus find the force amplification provided by the pulley arrangement used.



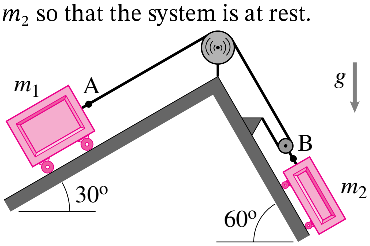
Problem 7.2.24

7.2.25 In the figure shown, there is no friction between block A and the vertical wall but there is friction ($\mu = 0.3$) between block B and the floor. If $m_B = 30\,\text{kg}$, find the mass of block A for equilibrium.



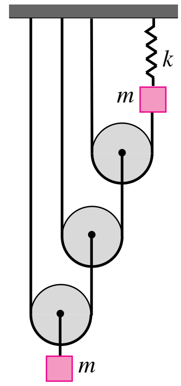
Problem 7.2.25

7.2.26 Find the ratio of the masses m_1 and m_2 so that the system is at rest.



Problem 7.2.26

7.2.27 If the mass and pulley system shown in the figure is in equilibrium when the spring is stretched by 3 cm, find m, given $k = 500\,\text{N/m}$ and $g \approx 10\,\text{m/s}^2$.



Problem 7.2.27