SAMPLE 7.6 A wheeled suitcase of length 60 cm, height 30 cm and 'weighing' 20 kg on the airport check-in counter, has a telescopic handle of length 40 cm. The suitcase is dragged at an angle $\theta = 30°$. Assuming good wheels (negligible friction), find the force applied on the handle in order to wheel the suitcase steadily. (Take $g \approx 10\,\mathrm{m/s^2}$).

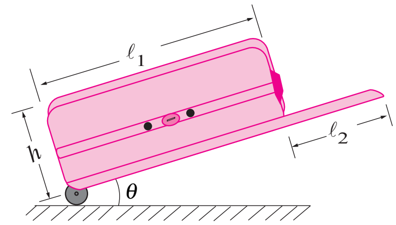

Figure 7.38

Solution The free-body diagram of the suitcase is shown in fig. 7.39. The reaction force at the wheel is almost vertical because of negligible friction. So, we can also assume the force F applied at the handle to be almost vertical. We assume that the center of mass G is located at the geometric center of the rectangular suitcase. Now the moment-balance equation about point A, $\sum M_A = 0$, gives

$$\vec{\boldsymbol{r}}_{G/A} \times (-mg\,\hat{\boldsymbol{\jmath}}) + \vec{\boldsymbol{r}}_{B/A} \times F\,\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{0}}.$$

Substituting $\vec{\boldsymbol{r}}_{G/A} = \frac{\ell_1}{2}\hat{\boldsymbol{\lambda}} + \frac{h}{2}\hat{\boldsymbol{n}}$, $\quad \vec{\boldsymbol{r}}_{B/A} = (\ell_1 + \ell_2)\hat{\boldsymbol{\lambda}}$, and noting that $\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}} = \cos\theta\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}} = -\sin\theta\hat{\boldsymbol{k}}$, we have

$$-\frac{1}{2}mg(\ell_1\cos\theta - h\sin\theta)\hat{\boldsymbol{k}} + F(\ell_1+\ell_2)\cos\theta\hat{\boldsymbol{k}} = 0$$

$$\Rightarrow \quad F = \frac{1}{2}mg\left[\frac{\ell_1}{\ell_1+\ell_2} - \frac{h}{\ell_1+\ell_2}\tan\theta\right].$$

Substituting $\ell_1 = 60\,\mathrm{cm}, \ell_2 = 40\,\mathrm{cm}, h = 30\,\mathrm{cm}, \theta = 30°$ and $mg = 200\,\mathrm{N}$, we get

$$F = \frac{1}{2}\cdot 200\,\mathrm{N}\left[\frac{60\,\mathrm{cm}}{100\,\mathrm{cm}} - \frac{30\,\mathrm{cm}}{100\,\mathrm{cm}}\tan 30°\right] = 42.7\,\mathrm{N}.$$

$F = 42.7\,\mathrm{N}$

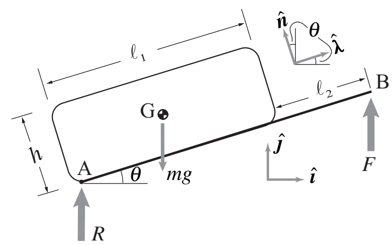

Figure 7.39: Free-body diagram of the suitcase. Unit vectors $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ are along the length and height of the suitcase, respectively. Thus, $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$.

SAMPLE 7.7 The figure shows a basic toggle mechanism. (a) If the applied force is $P = 20\,\mathrm{N}$ and the mechanism is in equilibrium at $\theta = 5°$, find the force applied by the spring. (b) If doubling the load P (to $P = 40\,\mathrm{N}$) causes a decrease of θ by 1° (to $\theta = 4°$), does the spring force at C double too?

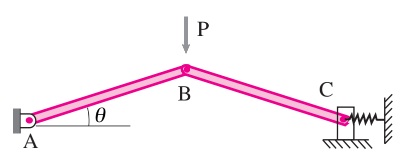

Figure 7.40: Toggle mechanism.

Solution The free-body diagrams of the pin connecting the two rods and the rod BC are shown in fig. 7.41. From the static equilibrium of the pin B, we have

$$\sum F_x = 0 \quad \Rightarrow \quad T_2\cos\theta - T_1\cos\theta = 0 \quad \Rightarrow \quad T_1 = T_2$$

$$\sum F_y = 0 \quad \Rightarrow \quad -(T_1+T_2)\sin\theta - P = 0 \quad \Rightarrow \quad T_2 = -\frac{P}{2\sin\theta}.$$

which follows from setting $T_1 + T_2 = 2T_2$ since $T_1 = T_2$. Now, we consider the free-body diagram of rod BC. The force-balance equation in the x-direction ($\sum F_x = 0$) gives

$$-T_2\cos\theta - F = 0 \quad \Rightarrow \quad F = -T_2\cos\theta = \frac{P\cos\theta}{2\sin\theta}.$$

Since θ is small, we have $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Thus $F = P/2\theta$ where θ is in radians. Substituting $P = 20\,\mathrm{N}$ and $\theta = 5\pi/180$, we get

$$F = \frac{20\,\mathrm{N}}{2\pi/36} = 115\,\mathrm{N}$$

which is almost 6 times P.

(b) If P is doubled, we might expect F to double because $F \approx P/2\theta$. But if θ also decreases to $\theta = 4°$, repeating the calculation above with $P = 40\,\mathrm{N}$, and $\theta = 4°$ we get $F = 286\,\mathrm{N}$ which is 2.5 times the previous spring force.

For $P = 20\,\mathrm{N}, \theta = 5°, F = 115\,\mathrm{N}$; and for $P = 40\,\mathrm{N}, \theta = 4°, F = 286\,\mathrm{N}$

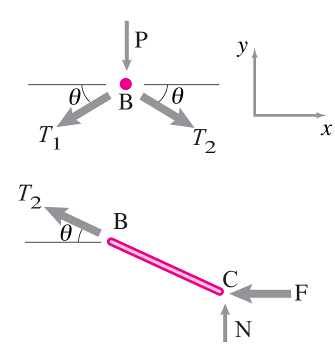

Figure 7.41: Free-body diagrams.

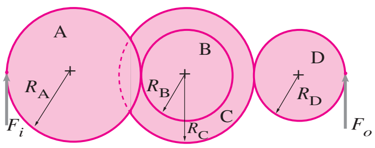

Figure 7.42

SAMPLE 7.8 **A gear train:** In the compound gear train shown in the figure, the various gear radii are: $R_A = 10\,\text{cm}, R_B = 4\,\text{cm}, R_C = 8\,\text{cm}$ and $R_D = 5\,\text{cm}$. The input load $F_i = 50\,\text{N}$. Assuming the gears to be in static equilibrium find the machine load F_o.

Solution You may be tempted to think that a free-body diagram of the entire gear train will do since we only need to find F_o. However, it is not so because there are unknown reactions at the axle of each gear and, therefore, there are too many unknowns. On the other hand, we can find the load F_o easily if we go gear by gear from the left to the right.

The free-body diagram of gear A is shown in Fig. 7.43. Let F_1 be the force at the contact tooth of gear A that meshes with gear B. From the moment balance about the axle-center O, $\sum \vec{\boldsymbol{M}}_{\text{O}} = \vec{\mathbf{0}}$, we have

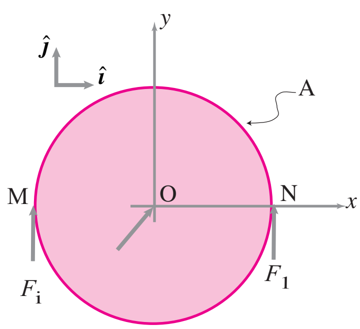

Figure 7.43

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\text{M}} \times \vec{\boldsymbol{F}}_{\text{i}} + \vec{\boldsymbol{r}}_{\text{N}} \times \vec{\boldsymbol{F}}_1 &= \vec{\mathbf{0}} \\
-F_i R_A \hat{\boldsymbol{k}} + F_1 R_A \hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\
\Rightarrow \quad F_1 &= F_i.
\end{aligned}
$$

Similarly, from the free-body diagram of gear B and C (together) we can write the moment balance equation about the axle-center P as

$$
\begin{aligned}
F_1 R_B \hat{\boldsymbol{k}} + F_2 R_C \hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\
\Rightarrow \quad F_2 &= \frac{R_B}{R_C} F_1 \\
&= \frac{R_B}{R_C} F_i.
\end{aligned}
$$

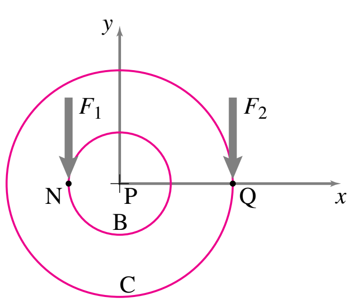

Figure 7.44

Finally, from the free-body diagram of the last gear D and the moment equilibrium about its center R, we get

$$
\begin{aligned}
-F_2 R_D \hat{\boldsymbol{k}} + F_o R_D \hat{\boldsymbol{k}} &= \vec{\mathbf{0}} \\
\Rightarrow \quad F_o &= F_2 \\
&= \frac{R_B}{R_C} F_i \\
&= \frac{4\,\text{cm}}{8\,\text{cm}} \cdot 50\,\text{N} = 25\,\text{N}.
\end{aligned}
$$

$F_o = 25\,\text{N}$

SAMPLE 7.9 Find the force F to hold the 100 kg box shown in the figure in equilibrium. Assume $g \approx 10\,\mathrm{m/s^2}$.

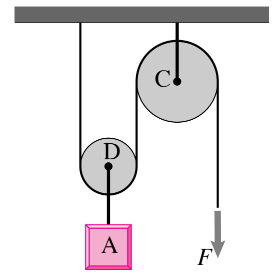

Figure 7.45

Solution The free-body diagrams of the two pulleys are shown in fig. 7.46 where the tension in the rope running over the two pulleys has been assumed as T. For the lower pulley D, the force balance in the y-direction, $\sum F_y = 0$, requires

$$2T - mg = 0 \quad \Rightarrow \quad T = \frac{mg}{2}.$$

The free-body diagram of the upper pulley C contains an unknown reaction force R at the attachment point C. However, if we write moment balance about point C, $\sum M_C = 0$, this unknown force contributes nothing. Let the radius of pulley C be r. Thus, the moment-balance equation about C gives

$$\begin{aligned} Tr - Fr &= 0 \\ \Rightarrow \quad F &= T = \frac{mg}{2} \approx 500\,\mathrm{N}. \end{aligned}$$

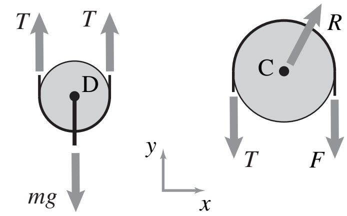

Figure 7.46: Free-body diagram of the two pulleys. We use C and D to designate both the pulleys and their respective centers.

$F \approx 500\,\mathrm{N}$

SAMPLE 7.10 A container box weighing $1\,\mathrm{kN}$ is dragged slowly and steadily along the floor with force F as shown in the figure. The coefficient of friction between the box and the floor is 0.6. Find the force required to pull the box and the force amplification obtained by the pulley arrangement.

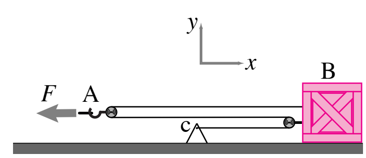

Figure 7.47

Solution It is clear from the figure that the same rope passes over the two pulleys used in the arrangement to pull the box. Let the tension in the rope be T. A partial free-body diagram (that includes forces acting only in the x-direction) of the box along with the pulley attached to it is shown in fig. 7.48. The same figure also shows the free-body diagram of pulley A at the force end. From the force-balance equation for the box in the x-direction, we get

$$f - 3T = 0 \quad \Rightarrow \quad T = \frac{f}{3} = \frac{\mu mg}{3}.$$

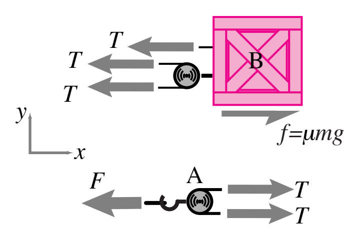

Figure 7.48

Now, from the force balance of pulley A in the x-direction, we get

$$\begin{aligned} 2T - F &= 0 \\ \Rightarrow \quad F &= 2T = \frac{2\mu mg}{3} \\ &= \frac{2 \cdot (0.6) \cdot (1\,\mathrm{kN})}{3} = 400\,\mathrm{N}. \end{aligned}$$

Since the force of friction on the box while sliding is $f = \mu mg = 0.6(1\,\mathrm{kN}) = 600\,\mathrm{N}$ and the force applied at A to overcome this friction is 400 N, the force amplification is 1.5. That is, the pulley arrangement amplifies the input force (400 N) 1.5 times at the output end.

$F = 400\,\mathrm{N}$, Force amplification $= 1.5$

SAMPLE 7.11 A differential hoist is used to lift a crate of mass 500 kg. The hoist pulley uses two discs of radius 30 cm and 25 cm. Find the force F required to lift the crate steadily. Take $g \approx 10\,\mathrm{m/s^2}$.

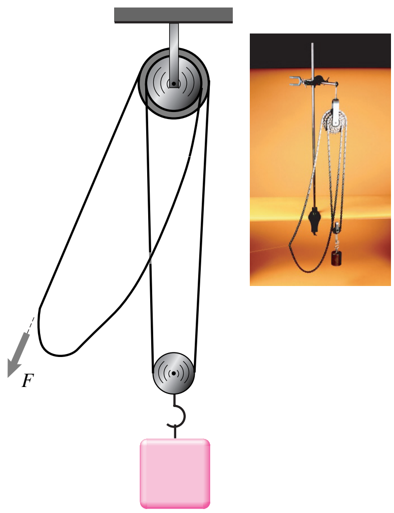

Figure 7.49

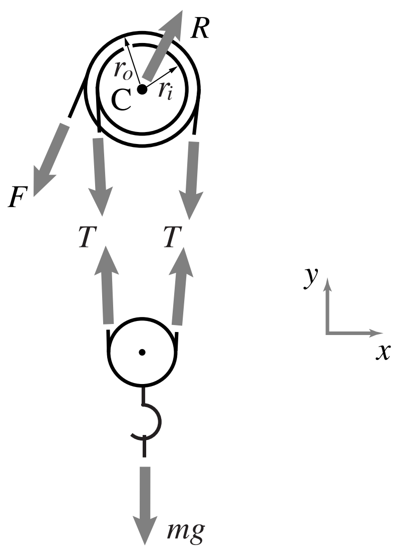

Figure 7.50

Solution The free-body diagrams of the upper pulley and the lower pulley are shown in fig. 7.50. Since the lower pulley is slightly smaller than the upper pulley, the chain passing over the two pulleys is not exactly vertical but makes a small angle with the vertical. Thus the tension forces shown in the free-body diagrams are slightly off from the vertical direction. However, since the angle is very small, we can treat T to be essentially vertical.

For the lower pulley, the force balance in the y direction gives

$$\begin{aligned} 2T - mg &= 0 \\ \Rightarrow \quad T &= \frac{mg}{2}. \end{aligned}$$

Now the moment balance about point C, $\sum M_C = 0$, for the upper pulley gives

$$\begin{aligned} Fr_o + Tr_i - Tr_o &= 0 \\ \Rightarrow \quad F &= \left(\frac{r_o - r_i}{r_o}\right) T \\ &= \left(1 - \frac{r_i}{r_o}\right) \frac{mg}{2} \\ &= \left(1 - \frac{25\,\mathrm{cm}}{30\,\mathrm{cm}}\right) \frac{5000\,\mathrm{N}}{2} \\ &= 417\,\mathrm{N} \end{aligned}$$

Thus the force amplification in this case is about 12 ($5000\,\mathrm{N}/417\,\mathrm{N}$). From the analysis above, it is also clear that the ratio of the radii of the two disks used in the upper pulley decides this force amplification. One can get a big force amplification, at least theoretically, by making $r_i \approx r_o$. In this problem, for example, if $r_i = 29\,\mathrm{cm}$ rather than the given 25 cm, we get $r_i/r_o \approx 0.97$ giving $F \approx 83\,\mathrm{N}$ which corresponds to a force amplification of 60.

$\boxed{F = 417\,\mathrm{N}}$

SAMPLE 7.12 A wedge with friction. Consider the wedge described in the text (page 403), but now with friction μ between the blocks.

Consideration of friction qualitatively changes the behavior of the machine. For simplicity still take the wall and floor interactions to be frictionless.

1. What is the relation between F_A and F_B when block A is sliding down?
2. What is the relation between F_A and F_B when block A is sliding up?
3. Under what conditions is it impossible for F_B to slide block A up, even when F_A is vanishingly small? Such a case is called 'non-back-driveable' or 'self-locking'.

How do your answers simplify for small wedge angle θ and small friction angle ϕ (with $\tan\phi = \mu$).

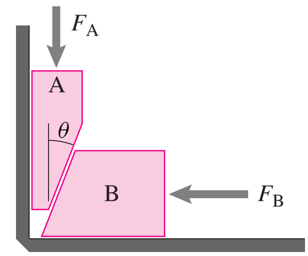

Figure 7.51: A wedge with friction μ between the blocks. Treat the walls as frictionless.

Solution Figure 7.52 shows free-body diagrams of wedge blocks. We draw separate free-body diagrams for the case when (a) block A is sliding down and block B to the right, and (b) block A is sliding up and block B to the left. In both cases the friction resists relative slip and obeys the sliding friction relation

$$F_f = \underbrace{\tan\phi}_{\mu} N,$$

where fig. 7.52 shows the resultant contact force (normal component plus frictional component) and its angle ϕ to the surface normal.

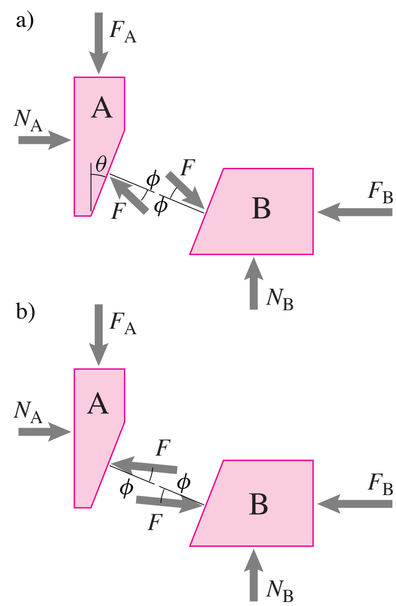

Figure 7.52: Free-body diagrams of a wedge a) assuming A slides down, b) assuming A slides up

1. **Block A sliding down:** Assuming block A is sliding down we get from free-body diagram 7.52a that

$$\begin{array}{llll} \text{For block A}, \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{\jmath} & \Rightarrow & -F_A + F\sin(\theta+\phi) & = 0 \\ \text{For block B}, \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{\imath} & \Rightarrow & -F_B + F\cos(\theta+\phi) & = 0. \end{array}$$

Eliminating F we get,

$$F_B = \frac{1}{\tan(\theta+\phi)} F_A. \tag{7.13}$$

$$\boxed{\text{When A slides down} \quad \Rightarrow \quad F_B = \frac{1}{\tan(\theta+\phi)} F_A.}$$

If we take a taper of 6° and a friction coefficient of $\mu = .3$ ($\Rightarrow \phi \approx 17^\circ$) we get that $F_B/F_A \approx 2.5$ instead of 10 as we got when neglecting friction. The wedge still serves as a way to multiply force, but substantially less so than the frictionless idealization led us to believe.

2. **Block A sliding up:** Now let's consider the case when force F_B pushes block B to the left, pinching block A, and forcing it up. The only change in the calculation is the change in the direction of the friction interaction force. From free-body diagram 7.52b

$$\begin{array}{llll} \text{For block A}, \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{\jmath} & \Rightarrow & -F_A + F\sin(\theta-\phi) & = 0 \\ \text{For block B}, \{\sum \vec{F}_i = \vec{0}\} \cdot \hat{\imath} & \Rightarrow & -F_B + F\cos(\theta-\phi) & = 0 \end{array}$$

Eliminating F we get,

$$F_A = \tan(\theta-\phi) F_B. \tag{7.14}$$

$$\boxed{\text{When A slides up} \quad \Rightarrow \quad F_B = F_A/\tan(\theta-\phi).}$$

Again using $\theta = 6^\circ$ and $\phi = 17^\circ$ we see that if $F_B = 100\,\text{lbf}$ then $F_A = \tan(-11^\circ) \cdot 100\,\text{lbf} \approx -20\,\text{lbf}$. That is, the 100 pounds doesn't push block A up at all, but even with no gravity, you need to pull up with a 20 pound force to get it to move.

If we insist that the downwards force F_A is positive or zero, that the pushing force F_B is positive, and that block A is sliding up then there is no solution to the equilibrium equations whenever $\phi > \theta$. (Actually we didn't need to do this second calculation at all. Eqn 7.13 shows the same paradox when $\theta + \phi > 90^\circ$. Trying to squeeze block B to the right for large θ is exactly like trying to squeeze block A up for small θ.)

3. **Self-locking:** This *self-locking* situation is intuitive. In fact it's hard to picture the contrary, that pushing a block like B would lift block A. If you view this wedge mechanism as a transmission, it is said to be *non-back-drivable* whenever $\phi > \theta$. Even though pushing down on A can 'drive' block B to the right, pushing to the left on block B cannot 'back-drive' block A up. Non-backdrivability is a feature or a defect depending on context(1).

(1) Standard car transmissions are backdriven when they are push-started and when a driver downshifts to slow the car instead of using the brakes. On the other hand, most electric hand-mixers cannot be back-driven; you can't turn the motor by forcing the beater blade (*Unplug before trying.*)

The borderline case of backdrivability is when $\theta = \phi$ and $F_B = F_A / \tan 2\theta$. Assuming θ is a fairly small angle we get

$$F_B = \frac{F_A}{\tan 2\theta} \approx \frac{F_A}{2\theta} \approx \frac{1}{2}\frac{F_A}{\tan\theta}$$
$$\approx \frac{1}{2} \cdot (\text{the value of } F_B \text{ had there been no friction}).$$

Thus the design guideline:

Non-back-driveable transmissions are generally 50% or less efficient, they transmit 50% or less of the force they would transmit if they were frictionless.

To use a wedge in this backwards way requires very low friction. A rare case where a narrow wedge is back drivable is this: a fresh wet watermelon seed squeezed between two pinched fingers.