7.1 Springs

A *spring* is a deformable solid that regains its original shape after being compressed, extended or otherwise deformed. The word spring has a dual personality.

1) **Spring as product.** Springs, in various forms, most characteristically as helices made of steel wire, can be purchased from hardware stores and mechanical parts suppliers (fig. 7.1). Springs are used to

7.1 'Zero-length' springs

Zero rest-length springs

A special case of linear springs, that has remarkable mechanical consequences, is a *zero-rest-length* spring (also called a 'zero-length' spring for short) with relaxed length $\ell_0 = 0$. Zero-rest-length-spring ideas are useful for design, but are not essential for basic understanding of statics.

The defining equations for a zero-rest-length spring, in scalar and vector form, are

$$T = k\ell \quad \text{and} \quad \vec{\boldsymbol{F}}_{\mathrm{B}} = k \cdot \vec{\boldsymbol{r}}_{\mathrm{AB}}.$$

The tension versus length curve for a zero-length spring is shown in fig. 7.4b.

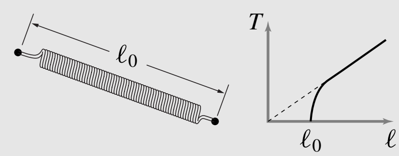
At first blush, such a spring seems *non-physical*; it seems to represent a thing that you can't build. If you take a coil spring and try to collapse it to zero length, when the two ends would touch, all the metal gets in the way and prevents that. In fact, however, there are many ways to make zero-rest-length springs. For example, the tension versus length curve of a rubber band (or piece of surgical tubing) looks something like that shown in fig. 7.4c. Over some portion of the curve, the zero-length spring approximation is reasonable (a sign of this is that the vibration frequency is almost independent of stretch for some range of stretch). For other physical implementations of zero-length springs, read on.

The mathematics in many mechanics problems is simpler for $\ell_0 = 0$ springs than for $\ell_0 \neq 0$ springs.

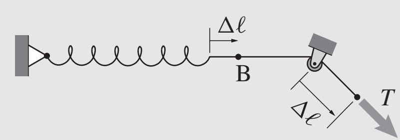
Rubber bands. As shown in fig. 7.4c, straps of rubber behave like zero-length springs over some of their length. If this is the working length of your mechanism, then the zero-length spring approximation may be good.

A stretchy conventional spring. Some springs are stretched much beyond their rest lengths. Thus the approximation that $k(\ell - \ell_0) = k\ell\left(1 - \frac{\ell_0}{\ell}\right) \approx k\ell$ may be reasonable.

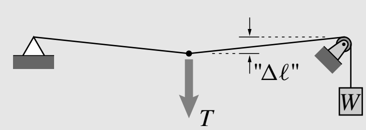
A pre-stressed coil spring. Some door springs and many springs used in desk lamps are made 'close-wound' so that each coil of wire is pressed against the next one. It takes some tension just to start to stretch such a spring. The tension versus length curve for such springs can look very much like a zero-length spring once stretch has started. In fact, the original elegant 1930's patent, which common present-day parallelogram-mechanism lamps imitate, specifies that the spring should behave as a zero-length spring. Such a pre-stressed, zero-length coil spring was a central part of the design of the long period seismometer featured on a 1959 Scientific American cover.



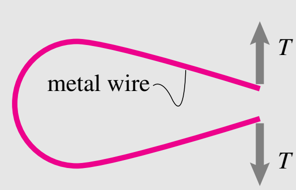
A spring, string, and pulley. If a spring is connected to a string that is wrapped around a pulley then the end of the string can feel like a zero force spring if the attachment point is at the pulley when the spring is relaxed.



A string pulled from the side. If a taut string is pulled from the side, it acts like a zero-length spring in the plane orthogonal to the string.



A 'U' clip. If a springy piece of metal is bent so that its unloaded shape is a pinched 'U' then it acts very much like a zero length spring. This is perhaps the best example in that it needs no anchor (unlike the pulley) and can be relaxed to almost zero length (unlike a pre-stressed coil).



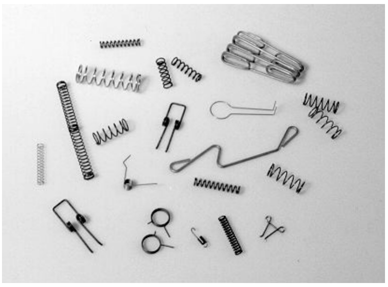


Figure 7.1: Some commercially available springs (Viking Products).

hold things in place (in a clothes pin), to store energy (in a clock or wind-up toy), to reduce contact forces (bumpers), to isolate something from vibrations (a car's suspension), and to modulate the feel for human interaction (under keyboard keys). You will find springs in almost any complicated machine. Take apart a disposable camera, a laser printer, a gas lawn mower, a bicycle, a washing machine or a cruise missile, and you will find springs.

2) **Spring as model.** On the other hand, springs are used in mechanical 'models' of many things that are not, by name, springs (see page 35 for discussion of 'models'). For much of this book we approximate solids as rigid. But sometimes the flexibility or *elasticity* of an object is an important part of its mechanics. The simplest accounting for this is to think of the object as a spring. So a tire may be modeled as a spring as might be the near-contact-point material of a bouncing ball, a strut in a truss, the snapping-back part of the earth's crust in an earthquake, your Achilles tendon, or the give of soil under a concrete slab. Biomechanics Professor Tom McMahon idealized the give of the ground under a person as that of a spring. He used the idea to design the record-breaking running track used in the Harvard stadium.

For simplicity we only concern ourselves with *tension and compression springs* here. These are springs that only have axial loads applied and only at the ends.

If the tension in a spring is a function of its length alone, independent of its rate of lengthening, the spring is said to be 'elastic.' Many materials are well-modeled as elastic for small-enough deformation. If the tension in the spring is proportional to its stretch, the spring is said to be 'linear.' Most elastic materials are close to linear in their behavior. Thus the word *spring* is often short for *linear elastic spring*. The stretch of a spring is the amount by which the spring is longer than when it is relaxed. This relaxed length is also called the *un-stretched* length, the *rest* length, or the *reference* length. If the relaxed length (the length at zero tension) is ℓ_0 and the present length is ℓ, then the stretch of the spring is

$$\Delta\ell = \ell - \ell_0 = \text{Increase in length from rest length}$$

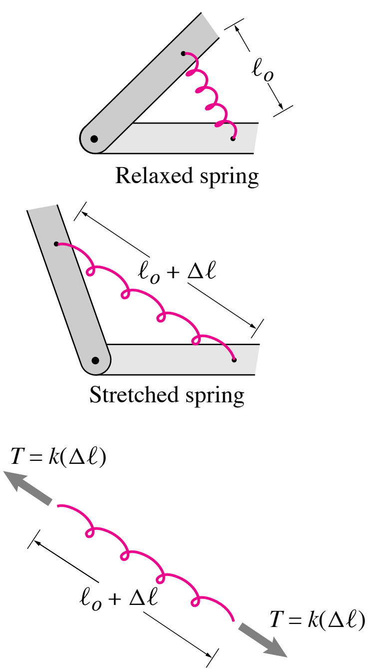


Figure 7.2: **Spring connection.** The tension in a spring is usually assumed to be proportional to its change in length, with proportionality constant k: $T = k\Delta\ell$.

An *ideal spring* is a massless two-force body characterized by its *rest length* ℓ_0 (also called the *relaxed length*, or *reference length*), its *spring constant* k, and the defining equation (or constitutive law), *Hooke's Law*:

$$T = k \cdot (\ell - \ell_0) \quad \text{or} \quad T = k \cdot \Delta\ell$$

where ℓ is the present length and $\Delta\ell$ is the increase in length or *stretch* (see fig. 7.3).

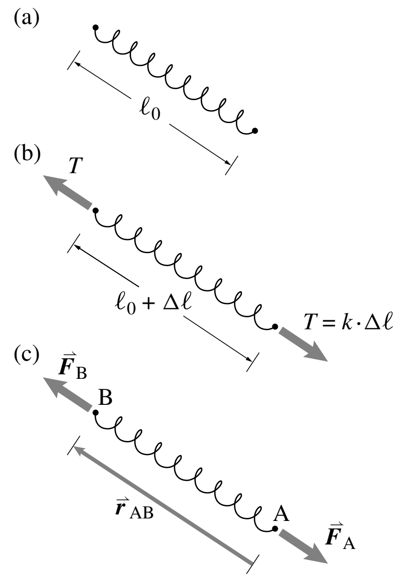


Figure 7.3: An ideal spring with rest length ℓ_0 and stretched length $\ell_o + \Delta\ell$. The tension in the spring is T and the vector forces at the ends are $\vec{F}_A$ and $\vec{F}_B$.

The spring constant k is also sometimes called the spring *rate*, the spring *stiffness* or the spring *proportionality constant*.

The ideal spring is called *linear* because of the formula $k\Delta\ell$ and not, say, $k(\Delta\ell)^3$. The defining spring formula is sometimes, although we don't recommend this, memorized as '$F = kx$'

Note: the formula '$F = kx$' can lead to errors: the direction of the force is not evident, and some people are unclear about the meaning of x in this formula. The safest way to avoid sign errors when dealing with springs is to

- Draw a free-body diagram of the spring;
- Write the *increase* in length $\Delta\ell$ in terms of geometry variables in your problem (even if you know that this increase is going to be a negative number);
- Use $T = k\Delta\ell$ to find the tension in the spring (even if you know the tension will turn out negative); and then
- Use the principle of action and reaction to find the forces on the objects to which the spring is connected.

The main idea is to pick a sign convention (tension and lengthening are positive) and stick with it, accepting the arithmetic of negative numbers if it arises. A plot of tension versus length for an ideal spring is shown in fig. 7.4a.

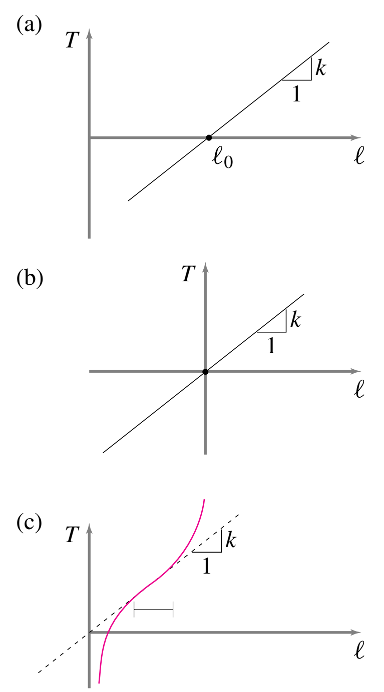


Figure 7.4: a) Tension versus length for an ideal spring, b) for a zero-length spring, and c) for a strip of rubber.

A comment on the notation $\Delta\ell$ Often in engineering we write $\Delta(\textit{something})$ to mean the change of '$\textit{something}$.' Most often one also has in mind a small change. In the context of springs, however, $\Delta\ell$ is allowed to be a rather large change. A useful way to think about springs is that increments of force are proportional to increments of length change, whether the force or length is already large or small:

$$\Delta T = k\Delta\ell \quad \text{or} \quad \frac{dT}{d\ell} = k$$

Compliance. A spring with a large stiffness is called *stiff* or *hard*. The reciprocal of stiffness $\frac{1}{k}$ is called the *compliance*. A spring with a small stiffness and large compliance is called *compliant* or *soft* and has a lot of 'give'.

The force vector on one end of a spring. Because the spring force is along the spring, which is in a known direction, we can write a vector formula for the force on the B (say) end of the spring as (see fig. 7.3)

$$\vec{\boldsymbol{F}}_{\mathrm{B}} = k \cdot \underbrace{\left(|\vec{\boldsymbol{r}}_{\mathrm{AB}}| - \ell_0\right)}_{\Delta\ell} \underbrace{\left(\frac{\vec{\boldsymbol{r}}_{\mathrm{AB}}}{|\vec{\boldsymbol{r}}_{\mathrm{AB}}|}\right)}_{\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}}, \tag{7.1}$$

where $\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}$ is a unit vector along the spring. This explicit formula is useful for, say, numerical calculations. This formula becomes especially simple if the rest-length of the spring is zero ($\ell_0 = 0$) so

$$\vec{\boldsymbol{F}}_{\mathrm{B}} = k\vec{\boldsymbol{r}}_{\mathrm{B/A}}.$$

Absurd as this seems (how could a spring have zero rest length?) the idea is useful both as a model and for engineering design (see box 7.1 on page 1).

Assemblies of springs

Here we see how springs are put together with other springs *in parallel* and *in series*. For starters we'll put together just two springs with rest lengths ℓ_{01} and ℓ_{02}. The extensions and tensions of the two springs are $\Delta\ell_1, \Delta\ell_2, T_1$, and T_2.

The assembly of springs also acts like a single spring. The central issue is finding the properties (the total stiffness and rest length) of the combined spring.

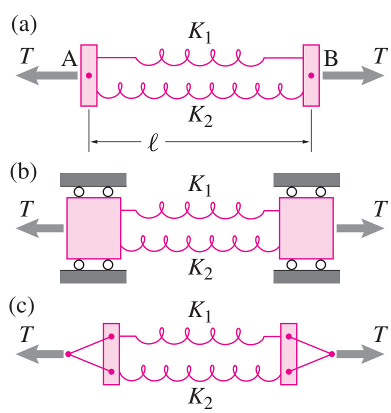


Figure 7.5: a) Schematic of parallel springs, b) genuinely parallel springs, c) a simple implementation that gives a reasonable approximation of parallel springs. The triangle of cables limits the rotation of the end bars if the triangles are long.

Much of what you need to know about the words 'in parallel' and 'in series' is this:

In parallel,	forces	and	stiffnesses	add.
In series,	displacements	and	compliances	add.

which we discuss in detail below.

Springs in parallel

Two springs that share the burden of a load and stretch the same amount are said to be *in parallel*.

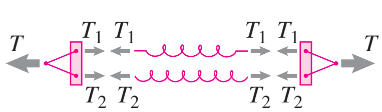


Figure 7.6: Free-body diagrams of the components of a parallel spring arrangement.

Figure 7.5a shows the standard schematic for springs in parallel. This schematic is a non-physical cartoon because the applied tension would likely cause the end-bars to rotate. What is meant by the schematic in fig. 7.5a is the somewhat clumsy constrained mechanism of fig. 7.5b. In engineering practice one rarely builds such a structure. For the purposes of discussion here, we assume that any of fig. 7.5abc represent a situation where the springs both stretch the same amount.

For each spring we have the defining constitutive relation:

$$T_1 = k_1 \Delta\ell_1 \quad \text{and} \quad T_2 = k_2 \Delta\ell_2. \tag{7.2}$$

Using the free-body diagrams in fig. 7.6, force balance for one of the end supports shows that

$$T = T_1 + T_2. \tag{7.3}$$

This is what is meant by the two springs sharing the load. Springs in parallel stretch the same amount thus we have the kinematic relation:

$$\Delta\ell_1 = \Delta\ell_2 = \Delta\ell. \tag{7.4}$$

For simplicity we have assumed that the two springs have the same rest length. Put the two results above together and we have

$$\begin{aligned} T &= T_1 + T_2 \\ &= k_1\Delta\ell_1 + k_2\Delta\ell_2 \\ &= k_1\Delta\ell + k_2\Delta\ell \\ &= \underbrace{(k_1 + k_2)}_{k}\Delta\ell\ . \end{aligned}$$

Thus the effective spring constant of the pair of springs in parallel is, naturally enough:

$$k = k_1 + k_2. \tag{7.5}$$

The loads carried by the springs are

$$T_1 = \frac{k_1}{k_1 + k_2}T \quad \text{and} \quad T_2 = \frac{k_2}{k_1 + k_2}T$$

which add up to T as they must.

Example: Two springs in parallel.
Take $k_1 = 99\,\text{N/cm}$ and $k_2 = 1\,\text{N/cm}$. The effective spring constant of the parallel combination is:

$$k = k_1 + k_2 = 99\,\text{N/cm} + 1\,\text{N/cm} = 100\,\text{N/cm}.$$

Note that $T_1/T = .99$ so even though the two springs share the load, the stiffer one carries 99% of it. For practical purposes, or for the design of this system, it would be reasonable to remove the much less stiff spring.

The reasoning above with two springs in parallel is easy enough to reproduce with 3 or more springs. The result is:

$$k_{\text{tot}} = k_1 + k_2 + k_3 + \ldots \quad \text{and} \quad T_1 = Tk_1/k_{\text{tot}}, \quad T_2 = Tk_2/k_{\text{tot}} \ldots$$

That is,

- The net spring constant is the sum of the constants of the separate springs; and
- The load carried by springs is in proportion to their spring constants.

Some comments on parallel springs

Once you understand the basic ideas and calculations for two side-by-side springs connected to common ends, there are a few things to think about for context.

The simplest redundant truss For the purposes of drawing pictures (*e.g.,* fig. 7.5a) parallel springs are drawn side by side. But, in the mechanics analysis we treated them as if they were on top of each other. A pair of parallel springs is like a two-bar truss where the bars are on top of each other and connected at their ends. With 2 bars ($b = 2$) and 2 joints ($j = 2$), we have $2j < b + 3$, and a redundant truss. This is the simplest redundant truss with one spring (read bar) doing exactly the same job as the other (carries the same loads, resists the same motions). With statics alone, we can not find the tensions in the springs since the statics equation $T_1 + T_2 = T$ has non-unique solutions.

Statically indeterminate problems. Calculating the forces in a set of parallel springs is solving (using more than just statics, namely the spring constitutive law) the simplest statically-indeterminate problem.

Parallel springs and the three pillars of mechanics The laws of statics allow multiple solutions to redundant problems. But a bar in a real physical structure has, at one instant of time, some unique bar tension determined by the deformations and material properties. This is the first, and perhaps most conspicuous, occasion in this book that you see a problem where the three pillars of mechanics (see page 27) are assembled in such clear harmony, namely, material properties (eq. 7.2), the laws of mechanics (eq. 7.3), and the geometry of motion and deformation (eq. 7.4). In strength of materials calculations, where the distribution of stress is not determinable by statics alone, this threesome (geometry of deformation, material properties and statics) clearly comes together in almost every calculation.

Parallel springs are not necessarily geometrically parallel. In the discussion above 'in parallel' corresponded to the springs being geometrically parallel. In common mechanics usage the words 'in parallel' are more general and mean that the net load is the sum of the loads carried by the two springs, and the stretches of the two springs are the same (or in a ratio restricted by kinematics). You will see cases where 'in parallel' springs are not the least bit parallel (*e.g.,* see fig. 7.7).

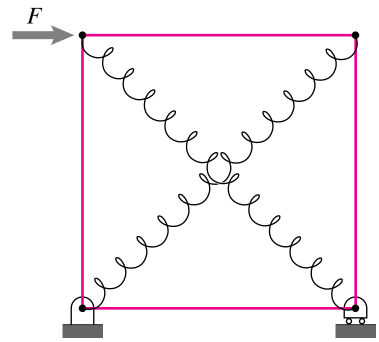


Figure 7.7: Parallel springs are not always geometrically parallel. The deformation of the structure above into a diamond is resisted by the two springs. They share the load and they have stretches that are linked by the kinematics. Thus these two perpendicular springs are 'in parallel.' (Detailed analysis of this structure is a little beyond the coverage here.)

Springs in series

Two springs that share a displacement and carry the same load are *in series*.

A schematic of two springs in series is shown in fig. 7.8a where the springs are aligned serially, one after the other. To determine the net stiffness of this simple spring network we again assemble the three pillars of mechanics, using the free-body diagram of fig. 7.8b.

$$\begin{array}{llll}
\text{Constitutive law:} & T_1 = k_1(\ell_1 - \ell_{1_0}), & T_2 = k_2(\ell_2 - \ell_{2_0}), & \\
\text{Kinematics:} & \ell_0 = \ell_{1_0} + \ell_{2_0}, & \ell = \ell_1 + \ell_2, & (7.6)\\
\text{Force Balance:} & T_1 = T, \quad \text{and} & T_2 = T. &
\end{array}$$

(where,*e.g.,* ℓ_{1_0} is the rest length of spring 1). We can manipulate these equations much as we did for the similar equations for springs in parallel. The manipulation differs in structure the same way the equations do. For springs in parallel the tensions add and the displacements are equal. For springs in series the displacements add and the tensions are equal:

$$\begin{aligned}
\Delta\ell &= \ell - \ell_0 \\
&= (\ell_1 + \ell_2) - (\ell_{1_0} + \ell_{2_0}) \\
&= (\ell_1 - \ell_{1_0}) + (\ell_2 - \ell_{2_0}) \\
&= \Delta\ell_1 + \Delta\ell_2 \\
&= \frac{T_1}{k_1} + \frac{T_2}{k_2} \\
&= \frac{T}{k_1} + \frac{T}{k_2} \\
&= \underbrace{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}_{\frac{1}{k}} T \quad .
\end{aligned}$$

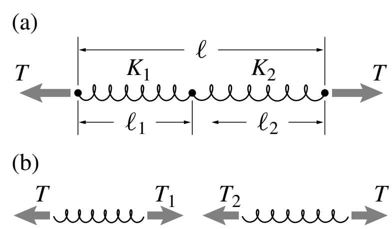


Figure 7.8: Schematic of springs in series.

Thus we get that the net compliance is the sum of the compliances:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2},$$

which you should compare with the case of springs in parallel (Eqn. 7.5). The sharing of the net stretch is in proportion to the compliances:

$$\Delta\ell_1 = \frac{1/k_1}{1/k_1 + 1/k_2}\Delta\ell \quad \text{and} \quad \Delta\ell_2 = \frac{1/k_2}{1/k_1 + 1/k_2}\Delta\ell$$

which add up to $\Delta\ell$ as they must.

Example: Two springs in series.
Take $1/k_1 = 99\,\mathrm{cm/N}$ and $1/k_2 = 1\,\mathrm{cm/N}$. The effective compliance of the parallel combination is:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = 99\,\mathrm{cm/N} + 1\,\mathrm{cm/N} = 100\,\mathrm{cm/N}.$$

Note that $\Delta\ell_1/\Delta\ell = 0.99$ so even though the two springs share the displacement, the more compliant one has 99% of it. For design purposes, or for modeling this system, it would be fair to replace the much stiffer spring with a rigid link.

Consequences of series and parallel springs for modeling

As the previous two examples illustrate, springs can sometimes be replaced with 'air' (nothing) or with rigid links without changing the system

7.2 How stiff a spring is a solid rod

Here we derive the formula for stiffness k of a rod:

$$k = EA/\ell$$

in terms of the material stiffness E, length ℓ and cross-sectional area A. This foreshadowing of Strength of Materials concepts is not central to the study of statics.

Let's take a reference bar with cross sectional area A_0 and rest length ℓ_0 and pull it with tension T and measure the elongation $\Delta\ell_0$ (figure below). The stiffness of this reference rod is $k_0 = T_0/\Delta\ell_0$. Now put two such rods side-by-side and you have parallel springs. You might imagine this sequence: two bars are near each other, then side by side, then touching each other, then glued together, then melted together into one rod with twice the cross section. The same tension in each causes the same elongation, or it takes twice the tension to cause the same elongation when you have twice the cross sectional area. Likewise with three side by side bars and so on, so for bars of equal length

$$k = \frac{A}{A_0} k_0.$$

On the other hand, we could put the reference rods end-to-end in series. Then the same tension causes twice the elongation. We could put three or more rods together in series, thus for bars with equal cross sections:

$$k = \frac{\ell_0}{\ell} k_0.$$

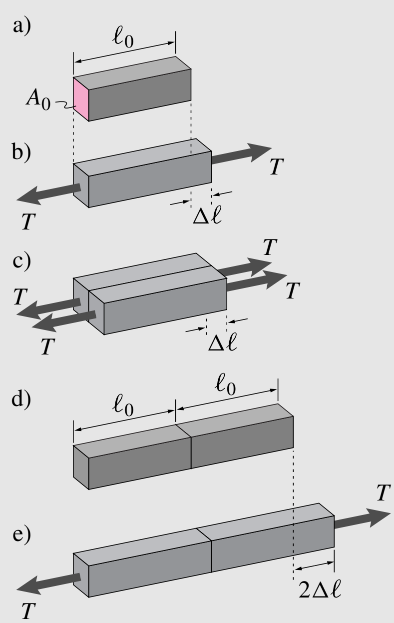
Putting these together we get:

$$k = \left(\frac{A}{A_0}\right)\left(\frac{\ell_0}{\ell}\right) k_0 = \left(\frac{k_0\ell_0}{A_0}\right)\frac{A}{\ell}.$$

Now presumably if we took a rod with a given material, length, and cross section the stiffness would be k, no matter what the dimensions of the reference rod. So $\left(\frac{k_0\ell_0}{A_0}\right)$ has to be a material constant. It is called E, the *modulus of elasticity* or Young's modulus.

For all steels $E \approx 30 \cdot 10^6$ lbf/ $\in^2 \approx 210 \cdot 10^9\,\mathrm{N/m^2}$ (consistent with fig. 7.10c). Aluminum has about a third this stiffness. So, a solid bar is a linear spring, obeying the spring equations:

$$k = \frac{EA}{\ell} \quad \text{or} \quad \Delta\ell = \frac{T\ell}{EA} \quad \text{or} \quad T = \frac{\Delta\ell\, EA}{\ell}.$$



or model behavior much. One way to think about this is that in the limit as $k \to \infty$ a spring becomes a rigid bar and in the limit $k \to 0$ a spring becomes air.

These ideas are used by engineers, often intuitively or even subconsciously and with no substantiating calculations, when making a model of a mechanical system.

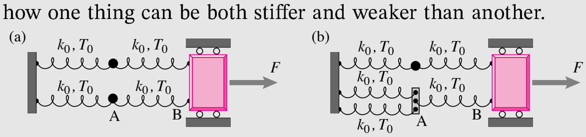
- If one of several pieces in series is much stiffer than the others it is often replaced with a rigid link.
- If one of several pieces in parallel is much more compliant than the others it is often replaced with air (nothing, sailboat fuel).

For example:

- When a coil spring is connected to a linkage, the other pieces in the linkage, though undoubtedly somewhat compliant, are typically modeled as rigid. They are stiffer than the spring and in series with it.
- A single hinge resists rotation about axes perpendicular to the hinge axis. But a door connected at two points along its edge is stiffly prevented against such rotations. Thus the hinge stiffness is in parallel with the greater rotational stiffness of the two connection points and is thus often neglected (see the discussion and figures in section 2.44 starting on page 173).

7.3 Stiffer but weaker

This is an aside for those who wonder how one thing can be both stiffer and weaker than another.



The structure on the left is made with 4 identical springs. The structure on the right is made with 5 of the same springs. All 9 springs have stiffness k_0 and break when the tension in them reaches T_0. We now want to compare the stiffness and strength of the two structures. Because of the mixture of parallel and series springs, the net stiffness of the structure in (a) is

$$k_{\text{net}} = k_0 \qquad \text{and} \qquad \text{strength} = 2T_0$$

because none of the springs reaches its breaking tension until $F = T_0$.

By doubling up one of the springs in (a) to get (b) we get

$$k_{\text{net}} = 7k_0/6 \qquad \text{and} \qquad \text{strength} = 21T_0/12.$$

The structure is made 16% stiffer but spring AB now reaches its breaking point T_0 when the applied load is 12.5% smaller.

What's going on? The second structure is made stiffer by reducing the deflection of point A. But this causes spring AB to stretch more and thus carry more of the total load. In some sense, the load is concentrated in spring AB. This load concentration, where the total load is unevenly carried, is one reason that stiffness and strength need to be considered separately. Load concentration (or *stress concentration*) is a major cause of structural failure.

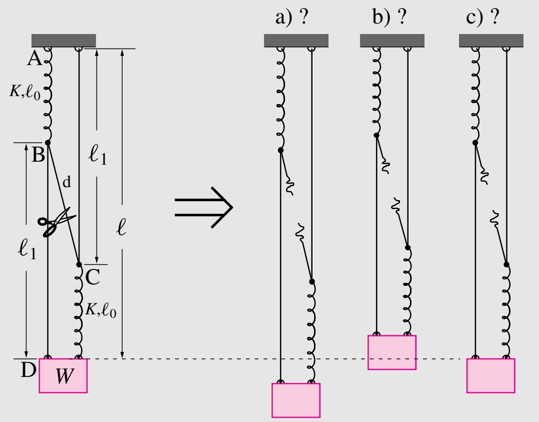
Consider the extreme case: put hundreds of springs in parallel where the pair of springs is now to the left of A in (b). Effectively this welds point A to the left wall. In (b) the load is concentrated in spring AB so (b) is about 50% stiffer than (a) and only has about 75% of (a)'s strength.

In common experience, stiffness and strength *do* correlate. Something that feels rickety (is very compliant) also tends to fail with a small load. But this common experience can be misleading: 1) A stiff structure can be weak because of stress concentrations, and 2) a given material, say baked clay, may be both stiffer and weaker than the other, say tar.

7.4 A puzzle with two springs and three ropes.

This tricky puzzle is an aside.

A weight hangs from 3 strings (BD,BC, and AC) and 2 springs (AB and CD). Point B is above point C and all ropes are taut.



When rope BC is cut does the weight go (a) down?, (b) up?, or (c) stay put? (Three dots represents a time for you to stop and think.)

. . .

Do the experiment. In 15 minutes you can do this experiment with 3 pieces of string, 2 rubber bands and a soda bottle. Hang the partially filled soda bottle from a door knob (or the top corner of a door, or a ruler cantilevered over the top of a refrigerator). Adjust the string lengths and amount of weight so that no strings or rubber bands are slack and make sure point B is above point C. The two points A can coincide as can the two points D. You can separate the strings a little with, say, a small wad of paper so you can see which string is which. Look at your experimental setup, but don't pull and poke at it. Try to predict whether your bottle will go up down or not move when you cut the middle string.

. . .

Spoiler, Answer: *If you did the experiment you know the answer, the weight goes up a little when you cut. That's what really happens.*

This puzzle was published as one for which people have bad intuitions. And that's true, as you probably just found out. Why does the experiment come out the way it does? If you got it wrong (like most people do), can you find the error in your ways?

. . .

Clue: All simple explanations are based on the assumption that the lengths of the two strings AC and BD are constant at ℓ_1.

. . .

Explanation 1: To simplify the reasoning assume that springs AB and CD are identical and carry the same tension T_s and that the ropes AC and BD carry the same tension T_r. As usual, start with free-body diagrams (below). With the symmetry we have assumed diagrams (a) and (c) provide identical information. The three free-body diagrams can be considered before and after the middle string removal by having $T_d > 0$ or $T_d = 0$, respectively. Vertical force balance gives (approximating T_d as vertical):

$$T_s + T_r = W \quad \text{and} \quad 2T_r + T_d = W \quad \Rightarrow \quad T_s = (W + T_d)/2.$$

Because we approximate AC as rigid with length ℓ_1, the downwards position of the weight is the string length ℓ_1 plus the rest length of the spring ℓ_0 plus the stretch of the spring T_s/k:

$$\ell = \ell_1 + \ell_0 + T_s/k = \ell_1 + \ell_0 + (W + T_d)/(2k).$$

In the course of this experiment ℓ_1, ℓ_0, W and k are constants. So as the tension T_d goes from positive to zero (when the rope BC is cut) and ℓ decreases. So the weight goes up.

. . .

Explanation 2: More intuitively, start with the configuration with the rope already cut and apply a small upwards force at C. It has no effect on the tension in spring CD thus the weight does not move. Now apply a small downwards force at B. This does stretch spring AB and thus lower point B, thus lowering the weight since ℓ_1 is constant. Applying both simultaneously is like attaching the middle rope. Thus attaching the middle rope lowers the weight so cutting the middle rope raises the weight.

. . .

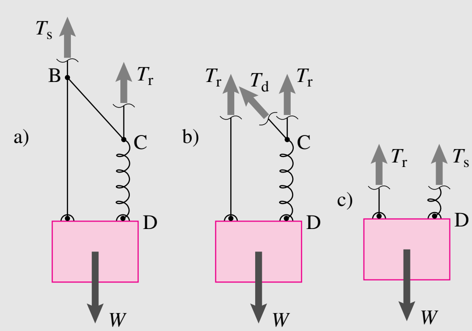
Explanation 3: Here is another intuitive approach. Point C can't move. Point B moves up and down just as much as the weight does. Point B is a distance d above point C. Since the rope BC is taut, releasing it will allow B and C to separate, thus increasing d and raising the weight.

. . .

A wrong explanation: What about springs in parallel and series? Here is a quick but wrong explanation for the experimental result, though it happens to predict the right direction of motion.

> "Before rope BC is cut the two springs are more or less in series because the load is carried from spring through BC to spring. Afterwards they are more or less in parallel because they have the same stretch and share the load. Two springs in parallel have 4 times the stiffness of the same two springs in series. So in the parallel arrangement the deflection is less. So the weight goes up when the springs switch from series to parallel."

What is the error in this thinking? The position of the weight comes from spring deflection added to the position when there is no weight. For the argument just presented to make sense, the rest-position of the mass (with gravity switched off) would have to be the same for the supposed 'series' and 'parallel' cases, which it is not ($\ell_1 + \ell_0 \neq \ell_0 + d + \ell_0$).



Another way to see the fallacy of the 'parallel versus series' argument is that the incremental stiffness of the system is, assuming inextensible ropes, infinite. That is, if you add or subtract a small load, the bottle only moves because of a small stretch of the ropes (which is neglected in the correct simple explanations above). If the springs were in series or parallel we would expect an incremental stiffness that was related to spring stretch not rope stretch.

- Welded joints in a determinate truss are modeled as frictionless pins. The rotational stiffness of the welds is 'in parallel' with the axial stiffness of the bars. To see this, look at two bars welded together at an angle. Imagine trying to break this weld by pulling the two far bar ends apart. Now imagine trying to break the weld if the two far ends are connected to each other with a third bar. The third bar is 'in parallel' with the weld material. See the first few sentences of section 6.1 for a do-it-yourself demonstration of the idea.
- Human bones are often modeled as rigid because, in part, when they interact with the world they are in series with more compliant flesh.

Note, again, that the mechanics usage of the words 'in parallel' and 'in series' don't always correspond to the geometric arrangement. For example the two springs in fig. 7.9a are in series and the two springs in fig. 7.9b are in parallel.

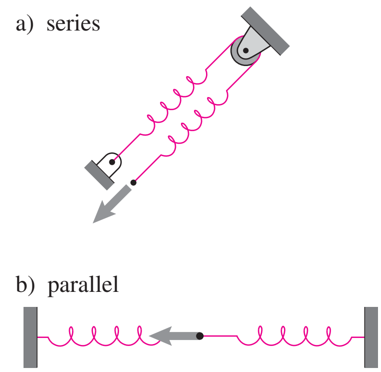


Figure 7.9: a) The two springs shown are in series because they carry the same load and their displacements add. b) These two springs are in parallel because they have a common displacement and their forces add.

Strength and stiffness

Most often when you build a structure you want to make it stiff and strong. The ideas of stiffness and strength are so intimately related that it is sometimes hard to untangle them. For example, you might examine a product in discount store by putting your hand on it, applying small forces and observing the motion. Then you might say: "pretty shaky, I don't think it will hold up" meaning that the stiffness is low so you think the thing may break if the loads get high.

Although stiffness and strength are often correlated, they are distinct concepts. Something is stiff if the force to cause a given motion is high. Something is strong if the force to cause any part of it to break is high. In fact, it is possible for a structure to be made weaker by making it stiffer (see box 7.3 on page 9)

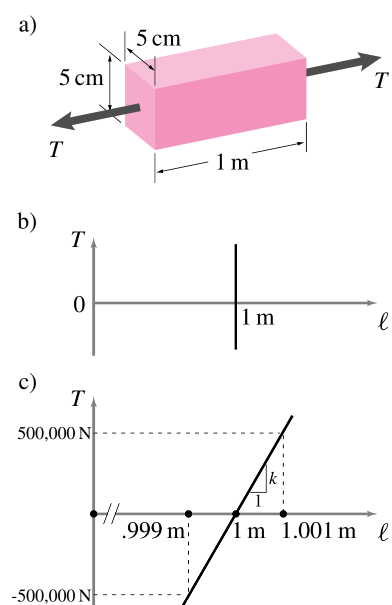


Figure 7.10: a) a steel rod in tension, b) tension versus length curve, c) zoom in on the tension versus length curve.

Why aren't springs in all mechanical models?

All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions that would be little affected by such deformation. In particular, if a problem is statically determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

Linear springs are just one way to model 'give'

If it is important to consider the deformability of an object, the linear spring model is just one simple model. It happens to be a good model for the small deformation of many solids. But the linear spring model is defined by the two words 'linear' and 'elastic'. For some purposes one might want to model the force due to deformation as being *non-linear*, like $T = k_1(\Delta\ell) + k_2(\Delta\ell)^3$. And one may want to take account of the dissipative or *inelastic* nature of something. The most common example is a linear dashpot $T = c\dot{\ell}$. Various mixtures of non-linearity and inelasticity may be needed to model the large deformations of a yielding metal, for example.

Solid bars are linear springs

When a structure or machine is built with literal springs (*e.g.,* a wire helix), it is common to treat the other parts as rigid. But when a structure has no literal springs, the small amount of deformation in rigid-looking objects can be important, especially for determining how loads are shared in redundant structures.

Let's consider a 1 m (about a yard) steel rod with a 5 cm square (about $(2 \in)^2$) cross section (fig. 7.10a). If we plot the tension versus length we get a curve like fig. 7.10b. The length just doesn't visibly change (unless the tension got so large as to damage the rod, not shown). But, when you pull on anything, it does deform at least a little. If we zoom in on the tension versus length plot we get fig. 7.10c. To change the length by one part in a thousand (a millimeter, a twenty-fifth of an inch) we have to apply a tension of about $500,000$ N (about 60 tons). Nonetheless the plot reveals that the solid steel rod behaves like a (very stiff) linear spring.

Surprisingly perhaps, this little bit of compliance is important to structural engineers. Modeling solid metal rods as linear springs is essential for finding internal forces in statically indeterminate structures. Because it is hard to picture steel deforming, your intuition may be helped by exaggerating the deformation. Think of all solids as being rubber. Or, if you want to look inside the solid in your mind, think of every solid as if it was a piece of deforming Jello. ①

How does a solid bar's stiffness depend on its shape and composition? In box 7.2 on page 8 we show that the stiffness of a solid elastic bar is

$$k = \frac{EA}{\ell}$$

where E is a material property called the Young's modulus. It's that E is big that keeps most solids from deforming visibly②.

①Jello is colored sugar water held together, jelled, by long springy gelatin molecules extracted from animal hooves. For their deformation fantasies, vegetarians can use sea-weed based agar jell ('Kosher gelatine').

②**Why aren't springs in all mechanical models?** All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little-affected by such deformation. In particular, if a problem is statically determinate, then very small elastic deformations only have a very small effect on the equilibrium equations and calculated forces. So the deformations are neglected, even though they are there.

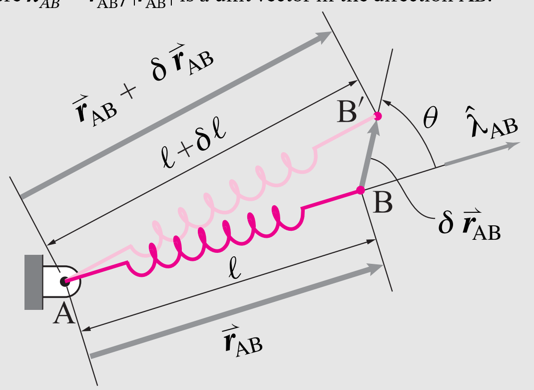
7.5 2D geometry of spring stretch

The material here is used in advanced sample 7.5 on page 401 and some of the later homework problems.

The key result concerns a spring with one end fixed at A and the other at moving point B. When point B moves from $\vec{r}_{\mathrm{B}}$ to $\vec{r}_{\mathrm{B}} + \Delta\vec{r}_{\mathrm{B}}$ then the spring length changes from ℓ to $\ell + \Delta\ell$ with

$$\Delta\ell \approx \hat{\boldsymbol{\lambda}}_{AB} \cdot \Delta\vec{r}_{\mathrm{B}} \qquad (7.7)$$

where $\hat{\boldsymbol{\lambda}}_{AB} = \vec{r}_{\mathrm{AB}}/|\vec{r}_{\mathrm{AB}}|$ is a unit vector in the direction AB.



People use either $\Delta\ell$ or $\delta\ell$ to represent the increase of a spring's length from its rest length. Before we derive the result above a few ways, let's discuss its relevance.

Fixed-configuration statics: the usual approach

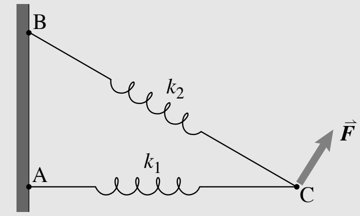
The forces and moments on a system in static equilibrium satisfy force and moment balance. In these equations the force magnitudes and directions, the moments and the locations of points of application of these are those *in the equilibrium configuration*. The equilibrium of the deformed state is expressed in terms of the geometry of that deformed state. Where the structure was before loading doesn't appear in the equilibrium equations.

However, often we know the geometry of a structure before the loads are applied, not after. To avoid calculation and confusion, we assume that the deformations cause negligible changes in positions. This is one reason people mistakenly think of statics as being limited to rigid bodies. Rather, for bodies that don't deform much, we can use the before-load geometry of a structure for reasonably accurate estimation of the deformed geometry.

Statics, taking account of deflection

In principle, the statics of deformable solids is the same as for rigid solids. You just need to use the deformed geometry in the statics calculations. Unfortunately, to find that geometry one needs the forces and their points of application. And one can't find all the locations without finding the deformation which depends on the forces, etc. This dizzying circle is escapable using the 'three pillars' (page 27).

Example: A structure made of springs.



Assume all the lengths and geometry of the two-bar truss are known when there is no load at C. We can find all the tensions and deflections as follows (See page 23 for the general strategy):

1. Assume that the equilibrium loaded location of C is displaced from the rest location by $\delta\vec{r}_{\mathrm{C}} = \delta x_C \hat{\boldsymbol{\imath}} + \delta y_C \hat{\boldsymbol{\jmath}}$, where δx_C and δy_C are *unknowns*;
2. Calculate the lengths of the springs in terms of δx_C and δy_C (this will be a complex expression with squares and square roots);
3. Find the tensions in the springs in terms of their new lengths and thus in terms of δx_C and δy_C;
4. Draw a free-body diagram of C, using the spring orientations and tensions you have found (still in terms of unknowns δx_C and δy_C);
5. Write the force-balance equations. These are two equations for two unknowns δx_C and δy_C.
6. Solve for δx_C and δy_C.
7. Use δx_C and δy_C to find the lengths and thus the tensions in the springs.

What's wrong with taking account of the deflection?

The trap — having to know the deflection to find the tensions but having to know the tensions to find the deflection — is avoided by setting up and solving simultaneous non-linear equations.

Although this non-linear-equation approach is correct, given our spring model, it is generally not used in structural mechanics because:

- **Confusing.** The equations are a mess.
- **Hard.** It is hard to solve non-linear equations, sometimes even hard on a computer.
- **Non-uniqueness.** There may be more than one solution. For example in the math problem above, if F is not too large, there will be two solutions. One solution with C deflected up and to the right, and another with C way to

(continued...)

7.5 2D geometry of spring stretch (continued)

the left of the wall. To get rid of such off-the-wall solutions you need to either use judgment after you find them, or further specify your math problem to eliminate them.

- **Linear equations are good enough.** There are simpler methods that give approximate solutions that are accurate for small-enough loads. That is, if the deflection is small compared to the size of the structure, then linear equations exist which reasonably approximate the non-linear equations above.

Small-deflection theory: the structural-mechanics approach.

So long as F is not too large, the motion of point C will be small compared to the lengths of the springs. Especially since, in practice, those springs are often solid metal rods. The usual *small deformation* assumption is that

- The deflection is small enough so that the spring angle changes have negligible effect on the equilibrium equations, and
- The deflection is small enough for the approximate formula for spring length change, eqn. (7.7), to be adequate.

The recipe for finding the deflection of C in the example above is greatly simplified with these approximations:

1. Assume that the equilibrium loaded location of C is displaced from the rest location by $\delta\vec{r}_{\mathrm{C}} = \delta x_C\hat{\imath} + \delta y_C\hat{\jmath}$, where δx_C and δy_C are *unknowns* (unchanged);
2. Calculate the lengths of the springs in terms of δx_C and δy_C using eqn. (7.7) (simplified);
3. Find the tensions in the springs in terms of their new lengths (unchanged) and thus in terms of δx_C and δy_C (much simpler expressions);
4. Draw a free-body diagram of C, using the original undeflected geometry (much simplified);
5. Write the force-balance equations. These are two equations for two unknowns δx_C and δy_C. (These will now be linear equations instead of a non-linear mess.)
6. Solve for δx_C and δy_C. (This is now the solution of linear instead of non-linear equations.)
7. (simplified) Use δx_C and δy_C to find the lengths and thus the tensions in the springs. (This now uses eqn. (7.7) instead of complicated relations with square roots, etc.)

This simplified recipe depends on the simplified formula for the spring length change eqn. (7.7), derived below four different ways.

Derivations of equation 7.7

Derivation 1 of eqn. (7.7). The law of cosines (page 110) says

$$(\ell + \delta\ell)^2 = \ell^2 + |\delta\vec{r}_{\mathrm{AB}}|^2 + 2\ell|\delta\vec{r}_{\mathrm{AB}}|\cos\theta$$

(θ here is negative of that used in the statement of the law of cosines). Expanding the left side and dropping terms in $\delta\ell^2$ and $|\delta\vec{r}_{\mathrm{AB}}|^2$ on both sides (assuming $\delta\ell/\ell \ll 1$ and $|\delta\vec{r}_{\mathrm{AB}}|/\ell \ll 1$), and dividing both sides by ℓ we get

$$\delta\ell \approx |\delta\vec{r}_{\mathrm{AB}}|\cos\theta = \hat{\boldsymbol{\lambda}}_{AB}\cdot\delta\vec{r}_{\mathrm{AB}}$$

where the last equality comes from the definition of the dot product (Section 1.2).

Derivation 2 of eqn. (7.7). Use the Pythagorean theorem to determine the lengths of $\vec{r}_{\mathrm{AB}}$ and of $\vec{r}_{\mathrm{AB}} + \delta\vec{r}_{\mathrm{AB}}$:

$$\begin{aligned}\ell &= \sqrt{(x_{AB}^2 + y_{AB}^2)}\\ \ell + \delta\ell &= \sqrt{\left((x_{AB}+\delta x_{AB})^2 + (y_{AB}+\delta y_{AB})^2\right)}\end{aligned}$$

Subtracting the first from the second, dividing both sides by ℓ, and expanding the contents of the square root we get

$$\delta\ell/\ell = \sqrt{1 + 2\left(x_{AB}\delta x_{AB} + y_{AB}\delta y_{AB}\right)/\ell^2 + \delta x_{AB}^2\delta x_{AB}^2/\ell^2} - 1.$$

Neglecting δx_{AB}^2 and δy_{AB}^2 (assuming $\delta\ell \ll \ell$) and expanding the square root ($\sqrt{1+\epsilon} \approx 1 + \epsilon/2$), and multiplying through by ℓ, we get

$$\delta\ell \approx (x_{AB}/\ell)\delta x_{AB} + (y_{AB}/\ell)\delta y_{AB})$$

which is eqn. (7.7) because $\hat{\boldsymbol{\lambda}}_{AB} = (x_{AB}/\ell)\hat{\imath} + (y_{AB}/\ell)\hat{\jmath}$.

Derivation 3 of eqn. (7.7). Using vector notation throughout:

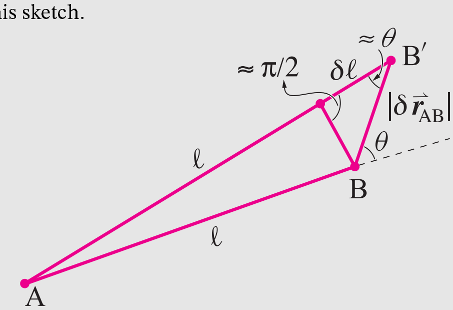
$$\begin{aligned}\ell^2 &= \vec{r}_{\mathrm{AB}}\cdot\vec{r}_{\mathrm{AB}}\\ (\ell+\delta\ell)^2 &= (\vec{r}_{\mathrm{AB}} + \delta\vec{r}_{\mathrm{AB}})\cdot(\vec{r}_{\mathrm{AB}} + \delta\vec{r}_{\mathrm{AB}})\end{aligned}$$

Expanding the second equation, neglecting second order terms and subtracting the first we get

$$\ell\delta\ell \approx \vec{r}_{\mathrm{AB}}\cdot\delta\vec{r}_{\mathrm{AB}}$$

dividing by ℓ and noting that $\hat{\boldsymbol{\lambda}}_{AB} = \vec{r}_{\mathrm{AB}}/\ell$ we again get eqn. (7.7).

Derivation 4 of eqn. (7.7). Finally, and most intuitively, look at this sketch.



The line AB and its deflected self are nearly parallel. Thus the triangle at the end is nearly a right triangle. So, approximately, $\delta\ell = |\delta\vec{r}_{\mathrm{AB}}|\cos\theta$ which is $\delta\vec{r}_{\mathrm{AB}}\cdot\hat{\boldsymbol{\lambda}}_{AB}$, again giving eqn. (7.7).

(continued...)

7.5 2D geometry of spring stretch (continued)

What the pros do

To find the loads in metal structures, the pros treat solid bars as springs, as per box 7.2 on page 8. Then they use eqn. (7.7) in the *small-deflection theory* described above. Generally this is all automated in computer code called a *finite-element* program. Such programs are standard commercial products used daily by hundreds of thousands of engineers around the world.